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Global classical solution of Muskat free boundary problem ☆

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Abstract

In this paper the Muskat problem which describes a two-phase flow of two fluids, for example, oil and water, in porous media is discussed. The problem involves in seeking two time-dependent harmonic functions $u_1(x, y, t)$ and $u_2(x, y, t)$ in oil and water regions, respectively, and the interface between oil and water, i.e., the free boundary $\Gamma\colon\ y = \rho(x, t)$, such that on the free boundary

$$u_1 = u_2, \qquad V_n = -k_1 \frac{\partial u_1}{\partial n} = -k_2 \frac{\partial u_2}{\partial n},$$

where n the unit normal vector on the free boundary toward oil region, V_n is the normal velocity of the free boundary Γ, k_1 and k_2 are positive constants satisfying $k_1 > k_2$. We prove the existence of classical solution globally in time under some reasonable assumptions. The argument developed in this paper can be used in any multidimensional case.

Keywords: Global existence; Free boundary; Muskat problem

1. Introduction

In order to demonstrate the problem and argument more clearly, in this paper we confine ourself to discuss the two-dimensional problem with a free boundary $y = \rho(x, t)$ for $x \in \mathbb{R}$, $t > 0$. In fact the argument which we developed in this paper can be used readily to any multidimensional and bounded domain.

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The goal of two-dimensional Muskat problem is to find $(u_1(x,y,t), u_2(x,y,t), \rho(x,t))$ such that

$$\Delta u_1(x,y,t) := \partial_x^2 u_1 + \partial_y^2 u_1 = 0, \quad x \in \mathbb{R},\ \rho(x,t) < y < a,\ t \geqslant 0, \tag{1.1}$$

$$\Delta u_2(x,y,t) = 0, \quad x \in \mathbb{R},\ -a < y < \rho(x,t),\ t \geqslant 0, \tag{1.2}$$

$$\partial_y u_1(x,y,t) = g_1(x,t), \quad x \in \mathbb{R},\ y = a,\ t \geqslant 0, \tag{1.3}$$

$$u_2(x,y,t) = g_2(x,t), \quad x \in \mathbb{R},\ y = -a,\ t \geqslant 0, \tag{1.4}$$

$$u_1 = u_2, \quad x \in \mathbb{R},\ y = \rho(x,t),\ t \geqslant 0, \tag{1.5}$$

$$V_n = -k_1 \frac{\partial u_1}{\partial n} = -k_2 \frac{\partial u_2}{\partial n}, \quad x \in \mathbb{R},\ y = \rho(x,t),\ t \geqslant 0, \tag{1.6}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}, \tag{1.7}$$

where $y = \rho(x,t)$ is (unknown) free boundary, $\{x \in \mathbb{R},\ \rho(x,t) < y < a,\ t \geqslant 0\}$ is oil region, $\{x \in \mathbb{R}, -a < y < \rho(x,t), t \geqslant 0\}$ is water region, n is the unit normal vector at a point of $y = \rho(x,t)$ toward oil region from water region. V_n represents the normal velocity of the free boundary in the direction of n, u_1 and u_2 represent the pressures of oil and water, respectively. $k_1 = \mu_1^{-1}$ and $k_2 = \mu_2^{-1}$, where μ_1 and μ_2 are positive constants representing viscosities of oil and water, respectively. a is a positive constant, g_1, g_2 and ρ_0 are known functions. To avoid a trouble for x at ∞, we suppose that all the functions are periodic with respect to x with a period $p > 0$.

Equations (1.1) and (1.2) come from Darcy's law neglecting gravity. (1.3) and (1.4) are boundary conditions on fixed boundaries, (1.4) represents the supply of water. (1.5) and (1.6) are free boundary conditions, (1.6) describes the law of energy conservation on the unknown boundary.

Muskat problem was proposed by Muskat in 1934 (see [1]). This problem describes the flows of two fluids in porous media. Its weak formulation was obtained by Jiang and Chen in 1987 (see [2]). But, as we know, so far the existence of weak solution is still open. One of the interesting result in this aspect is that in 2000, Schroll and Tveito proved local classical solvability for weak formulation (see [3]). Kametaka and Radkevich in 2000 made a smooth approximation as people did in phase field models (see [4,5]), in spherical case they proved that for some initial data the solution of smooth approximation converges to a weak solution of Muskat problem.

As for classical solvability, the existence of the solution locally in time was proved by author in 1996 (see [6]) by employing the Newton's iteration method. Xu and Zhang in 2002 also considered the existence of solution locally in time with kinetic free boundary condition (see [7]).

If (1.1) and (1.2) are replaced by heat equations, then the problem is called Verigin problem, its classical existence locally in time was shown in 1993 (see [8]) by Radkevich.

In this paper we consider the solvability of the problem (1.1)–(1.7) globally in time. Considering

$$n = \frac{(-\rho_x, 1)}{\sqrt{1+\rho_x^2}}, \qquad V_n = \frac{\partial_t \rho}{\sqrt{1+\rho_x^2}},$$

(1.6) becomes

$$\partial_t \rho = -k_1(\partial_y u_1 - \partial_x \rho \partial_x u_1) = -k_2(\partial_y u_2 - \partial_x \rho \partial_x u_2). \tag{1.8}$$

It is well known that in general cases evolutionary free boundary problems usually do not have global classical solution. To ensure the solvability some restrictions in some sense on known data must be prescribed.

First consider the special one-dimensional problem in which $g_1(x,t)$, $g_2(x,t)$ and $\rho_0(x)$ are independent of x. Given

$$g_1 = g_1(t), \qquad g_2 = g_2(t), \qquad \rho_0 = 0, \tag{1.9}$$

one can easily find a unique solution $(u_1^*(y,t), u_2^*(y,t), \rho^*(t))$ for the problem (1.1)–(1.5), (1.7), (1.8) and (1.9) as follows:

$$u_1^*(y,t) = g_1(t)y + c(t), \tag{1.10}$$

$$u_2^*(y,t) = \frac{k_1}{k_2} g_1(t)(y+a) + g_2(t), \tag{1.11}$$

$$\rho^*(t) = -\int_0^t k_1 g_1(\tau)\,d\tau. \tag{1.12}$$

where $c(t) = (k_1/k_2)g_1(t)(\rho^*(t)+a) + g_2(t) - g_1(t)\rho^*(t)$ and we suppose

$$\left|\int_0^T k_1 g_1(t)\,dt\right| \leqslant \frac{a}{12}, \tag{1.13}$$

$$k_1 > k_2. \tag{1.14}$$

The condition (1.13) means $|\rho^*(t)| \leqslant a/12$, (1.14) means that $\mu_1 < \mu_2$ and the viscosity of water is greater than that of oil. In fact it is true in the oil production where oil is competed by water which posses higher viscosity.

Assume

$$g_1(x,t) = g_1(t) + \varepsilon f_1(x,t), \tag{1.15}$$

$$g_2(x,t) = g_2(t) + \varepsilon f_2(x,t), \tag{1.16}$$

$$\rho_0(x) = \varepsilon \sigma_0(x), \tag{1.17}$$

where ε is a sufficiently small positive constant. The global solution will have the form

$$u_1(x,y,t) = u_1^*(y,t) + \varepsilon v_1(x,y,t), \tag{1.18}$$

$$u_2(x,y,t) = u_2^*(y,t) + \varepsilon v_2(x,y,t), \tag{1.19}$$

$$\rho(x,t) = \rho^*(t) + \varepsilon \sigma(x,t), \tag{1.20}$$

with suitable function v_1, v_2 and σ which depend on ε. Substituting (1.18)–(1.20) into (1.1)–(1.5), (1.7) and (1.8), using (1.10)–(1.12) and (1.15)–(1.17) yield the problem for (v_1, v_2, σ):

$$\Delta v_1 = 0, \quad x \in \mathbb{R},\ \rho(x,t) < y < a,\ t > 0, \tag{1.21}$$

$$\Delta v_2 = 0, \quad x \in \mathbb{R},\ -a < y < \rho(x,t),\ t > 0, \tag{1.22}$$

$$\partial_y v_1 = f_1(x,t), \quad x \in \mathbb{R},\ y = a,\ t > 0, \tag{1.23}$$

$$v_2 = f_2(x,t), \quad x \in \mathbb{R},\ y = -a,\ t > 0, \tag{1.24}$$

$$\left(\frac{k_1}{k_2} - 1\right) g_1(t)\sigma = v_1 - v_2, \quad x \in \mathbb{R},\ y = \rho(x,t),\ t > 0, \tag{1.25}$$

$$\begin{aligned} \partial_t \sigma &= -k_1(\partial_y v_1 - \varepsilon \partial_x \sigma \partial_x v_1) \\ &= -k_2(\partial_y v_2 - \varepsilon \partial_x \sigma \partial_x v_2), \quad x \in \mathbb{R},\ y = \rho(x,t),\ t > 0, \end{aligned} \tag{1.26}$$

$$\sigma(x,0) = \sigma_0(x), \quad x \in \mathbb{R}. \tag{1.27}$$

We will prove that if ε is small enough then for any $T > 0$ problem (1.21)–(1.27) has a classical solution (v_1, v_2, σ).

In the next section we will prove a fundamental theorem (Theorem 2.5) which is useful in section three where we prove that the problem (1.21)–(1.27) has a global classical solution.

2. Elliptic diffraction problem with time derivative on the interface

In this section we consider the problem

$$\begin{aligned} &a_{11}^{(k)} \partial_x^2 u_k + 2a_{12}^{(k)} \partial_x \partial_y u_k + a_{22}^{(k)} \partial_y^2 u_k + b_1^{(k)} \partial_x u_k + b_2^{(k)} \partial_y u_k + c_k u_k = f_k, \\ &\quad (x,y) \in \Omega_k,\ t \geqslant 0,\ k = 1,2, \end{aligned} \tag{2.1}$$

$$\partial_y u_1 = h_1(x,t) \quad \text{on } y = a, \tag{2.2}$$

$$u_2 = h_2(x,t) \quad \text{on } y = -a, \tag{2.3}$$

$$\beta \cdot (\lambda_1 \nabla u_1 - \lambda_2 \nabla u_2) = g_1(x,t) \quad \text{on } y = 0, \tag{2.4}$$

$$\partial_t(u_1 - u_2) - \gamma \cdot \nabla u_1 + d_1 u_1 + d_2 u_2 = g_2(x,t) \quad \text{on } y = 0, \tag{2.5}$$

$$u_1(x,0,0) - u_2(x,0,0) = u_0(x), \tag{2.6}$$

where $\Omega_1 = \{(x,y) \in \mathbb{R}^2;\ x \in \mathbb{R},\ 0 < y < a\}$, $\Omega_2 = \{(x,y) \in \mathbb{R}^2;\ x \in \mathbb{R},\ -a < y < 0\}$. Suppose

$$\sigma_1 |\xi|^2 \leqslant \sum a_{ij}^{(k)} \xi_i \xi_j \leqslant \sigma_2 |\xi|^2, \quad \sigma_1, \sigma_2 > 0,\ \forall \xi \in \mathbb{R}^2, \tag{2.7}$$

$$\begin{aligned} &a_{ij}^{(k)}, b_i^{(k)}, c_k, f_k \in C\big([0,T]; C^\alpha(\overline{\Omega}_k)\big), \qquad c \leqslant 0, \\ &a_{ij}^{(1)}(x,0,t) = a_{ij}^{(2)}(x,0,t), \end{aligned} \tag{2.8}$$

$$h_1 \in C\big([0,T]; C^{1+\alpha}(\mathbb{R})\big), \qquad h_2 \in C\big([0,T]; C^{2+\alpha}(\mathbb{R})\big), \tag{2.9}$$

$$\begin{aligned} &\beta, \gamma, d_k, g_k \in C\big([0,T]; C^{1+\alpha}(\mathbb{R})\big), \\ &\beta = (\beta_1, \beta_2), \quad \gamma = (\gamma_1, \gamma_2), \quad \beta_2 > 0,\ \gamma_2 > 0, \end{aligned} \tag{2.10}$$

λ_1, λ_2 are positive constants, (2.11)

$$u_0(x) \in C^{2+\alpha}(\mathbb{R}), \tag{2.12}$$

all the functions are periodic with respect to x with a period $p > 0$. (2.13)

First we consider a simple situation

$$\Delta u_k = f_k(x, y, t), \quad (x, y) \in \Omega_k,\ t \geqslant 0,\ k = 1, 2, \tag{2.14}$$

$$\partial_y u_1 = h_1(x, t) \quad \text{on } y = a, \tag{2.15}$$

$$u_2 = h_2(x, t) \quad \text{on } y = -a, \tag{2.16}$$

$$\lambda_1 \partial_y u_1 - \lambda_2 \partial_y u_2 = g_1(x, t) \quad \text{on } y = 0, \tag{2.17}$$

$$\partial_t (u_1 - u_2) - \lambda_3 \partial_y u_1 = g_2(x, t) \quad \text{on } y = 0, \tag{2.18}$$

$$u_1(x, 0, 0) - u_2(x, 0, 0) = u_0(x), \tag{2.19}$$

where λ_i, $i = 1, 2, 3$, are positive constants. We observe that in this problem $(u_1(x, y, 0), u_2(x, y, 0))$ is determined by

$$\Delta u_k(x, y, 0) = f_k(x, y, 0), \quad (x, y) \in \Omega_k,\ k = 1, 2,$$

$$\partial_y u_1 = h_1(x, 0) \quad \text{on } y = a,$$

$$u_2 = h_2(x, 0) \quad \text{on } y = -a,$$

$$\lambda_1 \partial_y u_1 - \lambda_2 \partial_y u_2 = g_1(x, 0) \quad \text{on } y = 0,$$

$$u_1(x, 0, 0) - u_2(x, 0, 0) = u_0(x).$$

This diffraction problem has unique solution (see [9, Lemma 1.5]) and the following estimate holds:

$$\sum_{k=1}^{2} \left|u(x, y, 0)\right|_{C^{2+\alpha}(\overline{\Omega}_k)} \leqslant C\left[\sum_{k=1}^{2}\left(\left|f_k(x, y, 0)\right|_{C^{\alpha}(\overline{\Omega}_k)} + \left|h_k(x, 0)\right|_{C^{k+\alpha}(\mathbb{R})}\right) + \left|g_1(x, 0)\right|_{C^{1+\alpha}(\mathbb{R})} + |u_0|_{C^{2+\alpha}(\mathbb{R})}\right]. \tag{2.20}$$

Theorem 2.1. *Problem* (2.14)–(2.19) *has a unique solution* $(u_1, u_2) \in C([0, T]; C^{2+\alpha}(\overline{\Omega}_1)) \times C([0, T]; C^{2+\alpha}(\overline{\Omega}_2))$ *with* $\partial_t(u_1(x, 0, t) - u_2(x, 0, t)) \in C([0, T]; C^{1+\alpha}(\mathbb{R}))$. *Moreover,*

$$\sum_{k=1}^{2} |u_k|_{C([0,T]; C^{2+\alpha}(\overline{\Omega}_k))} + \left|\partial_t\left(u_1(x, 0, t) - u_2(x, 0, t)\right)\right|_{C([0,T]; C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant C\left[\sum_{k=1}^{2}\left(|f_k|_{C([0,T]; C^{\alpha}(\overline{\Omega}_k))} + |g_k|_{C([0,T]; C^{1+\alpha}(\mathbb{R}))} + |h_k|_{C([0,T]; C^{k+\alpha}(\mathbb{R}))}\right) + |u_0|_{C^{2+\alpha}(\mathbb{R})}\right], \tag{2.21}$$

where C *is a positive constant.*

Proof. First we define $(v_1(x,y,t), v_2(x,y,t)) \in C([0,T]; C^{2+\alpha}(\overline{\Omega}_k))$ with $\partial_t(v_1(x,0,t) - v_2(x,0,t)) \in C([0,T]; C^{1+\alpha}(\mathbb{R}))$ which satisfies

$$\begin{aligned}
&\Delta v_k = f_k, \quad (x,y) \in \Omega_k,\ t \geqslant 0,\ k = 1,2,\\
&\partial_y v_1 = h_1(x,t) \quad \text{on } y = a,\\
&v_2 = h_2(x,t) \quad \text{on } y = -a,\\
&\lambda_1 \partial_y v_1 - \lambda_2 \partial_y v_2 = g_1(x,t) \quad \text{on } y = 0,\\
&v_1(x,0,t) - v_2(x,0,t) = u_0(x) + v^*(x,t),
\end{aligned}$$

where $v^*(x,t)$ satisfies

(1) $v^*(x,t) \in C([0,T]; C^{2+\alpha}(\mathbb{R}))$ and $\partial_t v^*(x,t) \in C([0,T]; C^{1+\alpha}(\mathbb{R}))$;
(2) $v^*(x,0) = 0$ and $\partial_t v^*(x,0) = \partial_t(u_1(x,0,t) - u_2(x,0,t))|_{t=0} = g_2(x,0) + \lambda_3 \times \partial_y u_1(x,y,0)|_{y=0}$.

For example, $v^*(x,t)$ can be constructed as the solution of parabolic problem

$$\begin{aligned}
&\partial_t v^* - \Delta v^* = g_2(x,0) + \lambda_3 \partial_y u_1(x,y,0)|_{y=0}, \quad x \in \mathbb{R},\ t > 0,\\
&v^*(x,0) = 0.
\end{aligned}$$

Since $g_2(x,0) + \lambda_3 \partial_y u_1(x,y,0)|_{y=0} \in C^{1+\alpha}(\mathbb{R})$, we have $v^*(x,t) \in C^{3+\alpha,(3+\alpha)/2}(\mathbb{R} \times [0,T])$. Moreover,

$$\begin{aligned}
&\left|v^*(x,t)\right|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + \left|\partial_t v^*(x,t)\right|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}\\
&\quad \leqslant \left|v^*(x,t)\right|_{C^{3+\alpha,(3+\alpha)/2}(\mathbb{R}\times[0,T])} \leqslant C\left|g_2(x,0) + \lambda_3 \partial_y u_1(x,y,0)\big|_{y=0}\right|_{C^{1+\alpha}(\mathbb{R})}\\
&\quad \leqslant C\Bigg[\sum_{k=1}^{2}\Big(\left|f_k(x,y,0)\right|_{C^{\alpha}(\overline{\Omega}_k)} + \left|g_k(x,0)\right|_{C^{1+\alpha}(\mathbb{R})} + \left|h_k(x,0)\right|_{C^{k+\alpha}(\mathbb{R})}\Big)\\
&\qquad + |u_0|_{C^{2+\alpha}(\mathbb{R})}\Bigg].
\end{aligned}$$

Here we have used inequality (2.20). In this way (v_1, v_2) admits an estimate

$$\begin{aligned}
&\sum_{k=1}^{2} |v_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))}\\
&\quad \leqslant C\Bigg[\sum_{k=1}^{2}\Big(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))} + \left|g_k(x,t)\right|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\Big)\\
&\qquad + |u_0|_{C^{2+\alpha}(\mathbb{R})}\Bigg].
\end{aligned}$$

Noting that $\partial_t(v_1 - v_2)(x,0,t) = \partial_t v^*(x,t)$, it reduces to

$$\sum_{k=1}^{2}|v_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))}+\big|\partial_t\big(v_1(x,0,t)-v_2(x,0,t)\big)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant C\Bigg[\sum_{k=1}^{2}\big(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))}+\big|g_k(x,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}+|h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\big)$$
$$+|u_0|_{C^{2+\alpha}(\mathbb{R})}\Bigg]. \tag{2.22}$$

Set $w=u-v$; then w satisfies

$$\Delta w_k=0,\quad (x,y)\in\Omega_k,\ t\geqslant 0,\ k=1,2, \tag{2.23}$$
$$\partial_y w_1=0 \quad \text{on } y=a, \tag{2.24}$$
$$w_2=0 \quad \text{on } y=-a, \tag{2.25}$$
$$\lambda_1\partial_y w_1-\lambda_2\partial_y w_2=0 \quad \text{on } y=0, \tag{2.26}$$
$$\partial_t(w_1-w_2)-\lambda_3\partial_y w_1=G(x,t) \quad \text{on } y=0, \tag{2.27}$$
$$w_1(x,0,0)-w_2(x,0,0)=0, \tag{2.28}$$

where

$$G(x,t)=g_2(x,t)-\partial_t(v_1-v_2)(x,0,t)+\lambda_3\partial_y v_1(x,0,t)$$
$$=g_2(x,t)-\partial_t v^*(x,t)+\lambda_3\partial_y v_1(x,0,t)$$

which satisfies $G(x,0)=0$ since $v_k(x,y,0)=u_k(x,y,0)$. It is reduced by (2.22) that

$$\big|G(x,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant\big|g_2(x,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}+\big|\partial_t v^*(x,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$+\big|v_1(x,y,t)\big|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_1))}$$
$$\leqslant C\Bigg[\sum_{k=1}^{2}\big(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))}+|g_k|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}+|h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\big)$$
$$+|u_0|_{C^{2+\alpha}(\mathbb{R})}\Bigg]. \tag{2.29}$$

The estimates (2.22) and (2.29) imply that (2.21) is equivalent to

$$\sum_{k=1}^{2}|w_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))}+\big|\partial_t(w_1-w_2)(x,0,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}. \tag{2.30}$$

On the other hand, if we can prove

$$\sum_{k=1}^{2}\big|w_k(x,0,t)\big|_{C([0,T];C^{2+\alpha}(\mathbb{R}))}\leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}, \tag{2.31}$$

then by applying the theory of the Dirichlet problem of elliptic equation we obtain

$$\sum_{k=1}^{2} |w_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))} \leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}.$$

This estimate and boundary condition (2.27) will reduce to

$$\left|\partial_t(w_1 - w_2)(x,0,t)\right|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} \leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}.$$

Hence next step what we need is to prove the existence to the problem (2.23)–(2.28) and establish the estimate (2.31).

In order to prove the existence of the problem (2.23)–(2.28) we construct an approximating problem

$$\Delta w_k = 0, \quad (x,y) \in \Omega_k,\ t \geqslant 0,\ k = 1,2, \tag{2.32}$$

$$\partial_y w_1 = 0 \quad \text{on } y = a, \tag{2.33}$$

$$w_2 = 0 \quad \text{on } y = -a, \tag{2.34}$$

$$\lambda_1 \partial_y w_1 - \lambda_2 \partial_y w_2 = 0 \quad \text{on } y = 0, \tag{2.35}$$

$$\partial_t(w_1 - w_2) - \varepsilon \partial_{xx}(w_1 - w_2) = \lambda_3 \partial_y w_1 + G(x,t) \quad \text{on } y = 0, \tag{2.36}$$

$$w_1(x,0,0) - w_2(x,0,0) = 0, \tag{2.37}$$

where ε is a small positive constant. Now we prove that the problem (2.32)–(2.37) has a global solution for fixed $\varepsilon > 0$.

Define a function set

$$\mathcal{D} = \left\{v(x,t) \in C\left([0,T]; C^{2+\alpha}(\mathbb{R})\right),\ v(x,0) = 0\right\}.$$

For given $v \in \mathcal{D}$, let $(w_1, w_2) \in C([0,T]; C^{2+\alpha}(\overline{\Omega}_1)) \times C([0,T]; C^{2+\alpha}(\overline{\Omega}_2))$ be the solution of elliptic diffraction problem (2.32)–(2.35) and

$$w_1 - w_2 = v(x,t) \quad \text{on } y = 0.$$

In view of conditions (2.36) and (2.37) we define $\bar{v}(x,t)$ is the solution of parabolic problem

$$\partial_t \bar{v} - \varepsilon \partial_x^2 \bar{v} = \lambda_3 \partial_y w_1(x,0,t) + G(x,t), \quad x \in \mathbb{R},\ t > 0,$$

$$\bar{v}(x,0) = 0, \quad x \in \mathbb{R}.$$

Since $\lambda_3 \partial_y w_1(x,0,t) \in C([0,T]; C^{1+\alpha}(\mathbb{R}))$, we have $\bar{v}(x,t) \in C([0,T]; C^{3+\alpha}(\mathbb{R}))$ and $\partial_t \bar{v}(x,t) \in C([0,T]; C^{1+\alpha}(\mathbb{R}))$ (see Theorem 5.14 of [10]). Define a mapping $\mathcal{F}$ by $\mathcal{F}v = \bar{v}$, which is continuous from $\mathcal{D}$ to $\mathcal{D}$. Noting that

$$C\left([0,T]; C^{3+\alpha}(\mathbb{R})\right) \cap C^1\left([0,T]; C^{1+\alpha}(\mathbb{R})\right) \to C\left([0,T]; C^{2+\alpha}(\mathbb{R})\right)$$

is compact (see [11, Corollary 4, p. 85]). So if T is small enough, the mapping $\mathcal{F}$ will have a fixed point, i.e., the linear problem (2.32)–(2.37) has a (local) classical solution. We can continue this procedure step by step to obtain the global solution.

Next we prove that the solution $(w_1, w_2) = (w_1^\varepsilon, w_2^\varepsilon)$ of problem (2.32)–(2.37) admits a uniform estimate analogous to the type of (2.31),

$$\sum_{k=1}^{2}\left|w_k^\varepsilon(x,0,t)\right|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} \leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}, \tag{2.38}$$

where C is independent of ε.

In fact, by using a partition of unity we need merely to prove that the solution of

$$\Delta w_1^\varepsilon = 0, \quad x\in\mathbb{R},\ y>0,\ t\geqslant 0, \tag{2.39}$$

$$\Delta w_2^\varepsilon = 0, \quad x\in\mathbb{R},\ y<0,\ t\geqslant 0, \tag{2.40}$$

$$\lambda_1\partial_y w_1^\varepsilon - \lambda_2\partial_y w_2^\varepsilon = 0 \quad \text{on } y=0, \tag{2.41}$$

$$\partial_t\big(w_1^\varepsilon - w_2^\varepsilon\big) - \varepsilon\partial_{xx}\big(w_1^\varepsilon - w_2^\varepsilon\big) - \lambda_3\partial_y w_1^\varepsilon = G(x,t) \quad \text{on } y=0, \tag{2.42}$$

$$w_1^\varepsilon(x,0,0) - w_2^\varepsilon(x,0,0) = 0, \tag{2.43}$$

$$\lim_{y\to+\infty} w_1^\varepsilon(x,y,t) \text{ is bounded}, \qquad \lim_{y\to-\infty} w_2^\varepsilon(x,y,t) \text{ is bounded} \tag{2.44}$$

satisfies estimate (2.38). To do this, making Fourier transformation with respect to x in the system (2.39)–(2.44) we have

$$-|\xi|^2\widetilde{w}_1^\varepsilon(\xi,y,t) + \frac{d^2}{dy^2}\widetilde{w}_1^\varepsilon(\xi,y,t) = 0, \quad \xi\in\mathbb{R},\ y>0,\ t\geqslant 0, \tag{2.45}$$

$$-|\xi|^2\widetilde{w}_2^\varepsilon(\xi,y,t) + \frac{d^2}{dy^2}\widetilde{w}_2^\varepsilon(\xi,y,t) = 0, \quad \xi\in\mathbb{R},\ y<0,\ t\geqslant 0, \tag{2.46}$$

$$\lambda_1\partial_y \widetilde{w}_1^\varepsilon - \lambda_2\partial_y \widetilde{w}_2^\varepsilon = 0 \quad \text{on } y=0, \tag{2.47}$$

$$\partial_t\big(\widetilde{w}_1^\varepsilon - \widetilde{w}_2^\varepsilon\big) + \varepsilon|\xi|^2\big(\widetilde{w}_1^\varepsilon - \widetilde{w}_2^\varepsilon\big) - \lambda_3\partial_y \widetilde{w}_1^\varepsilon = \widetilde{G}(\xi,t) \quad \text{on } y=0, \tag{2.48}$$

$$\widetilde{w}_1^\varepsilon(\xi,0,0) - \widetilde{w}_2^\varepsilon(\xi,0,0) = 0, \tag{2.49}$$

$$\lim_{y\to+\infty} \widetilde{w}_1^\varepsilon(\xi,y,t) \text{ is bounded}, \qquad \lim_{y\to-\infty} \widetilde{w}_2^\varepsilon(\xi,y,t) \text{ is bounded}, \tag{2.50}$$

where $\widetilde{w}_k^\varepsilon$ and $\widetilde{G}$ are Fourier transformations of w_k^ε and G, respectively. The solutions of (2.45), (2.46) and (2.50) are expressed as

$$\widetilde{w}_1^\varepsilon(\xi,y,t) = \widetilde{w}_1^\varepsilon(\xi,0,t)e^{-|\xi|y}, \tag{2.51}$$

$$\widetilde{w}_2^\varepsilon(\xi,y,t) = \widetilde{w}_2^\varepsilon(\xi,0,t)e^{|\xi|y}. \tag{2.52}$$

Substituting (2.51) and (2.52) into (2.47)–(2.49) we obtain

$$\widetilde{w}_1^\varepsilon(\xi,0,t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^t \exp\left\{-\left(\varepsilon|\xi|^2 + \frac{\lambda_2\lambda_3}{\lambda_1+\lambda_2}|\xi|\right)(t-\tau)\right\}\widetilde{G}(\xi,\tau)\,d\tau, \tag{2.53}$$

$$\widetilde{w}_2^\varepsilon(\xi,0,t) = -\frac{\lambda_1}{\lambda_2}\widetilde{w}_1^\varepsilon(\xi,0,t). \tag{2.54}$$

Lemma 2.2. *Suppose that w_1^ε, w_2^ε are defined by* (2.53) *and* (2.54). *Then*

$$\left|w_k^\varepsilon(x,0,t)\right|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} \leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}, \quad k=1,2, \tag{2.55}$$

where C is constant which is independent of ε.

Proof. (2.54) means that $w_2^\varepsilon(x,0,t) = -(\lambda_1/\lambda_2)w_1^\varepsilon(x,0,t)$, so we just prove (2.55) with $k=1$ only. We estimate $\widetilde{w}_1^\varepsilon(\xi,0,t)$ by the method of ring partition which is used in the theory of pseudodifferential operators. Define

$$A(\xi) = \varepsilon|\xi|^2 + \frac{\lambda_2\lambda_3}{\lambda_1+\lambda_2}|\xi|, \qquad K(\xi,t) = \exp\{-A(\xi)t\}.$$

Let $\varphi(\xi) \in C_0^\infty(\mathbb{R})$ satisfying

$$\varphi(\xi) = \begin{cases} 1, & \text{when } |\xi| \leqslant 1, \\ 0, & \text{when } |\xi| \geqslant 2. \end{cases}$$

Denote

$$\psi(\xi) = \varphi(\xi) - \varphi(2\xi),$$
$$\psi_j(\xi) = \psi(2^{-j}\xi), \quad j = 0,1,2,\ldots,$$
$$\psi_{-1}(\xi) = \varphi(2\xi).$$

It is clear that supp $\psi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}$, $j = 0,1,2,\ldots,$ where $B_r = \{\xi \in \mathbb{R}, |\xi| \leqslant r\}$. It may be seen that

$$\sum_{j=-1}^{\infty} \psi_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R},$$

which is called a Littlewood–Paley ring partition. We reduce from (2.53) that

$$\left[\sum_{j=-1}^{\infty} \psi_j(\xi)\right]\widetilde{w}_1^\varepsilon(\xi,0,t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^t \left[\sum_{j=-1}^{\infty} \psi_j(\xi)\right] K(\xi,t-\tau)\widetilde{G}(\xi,\tau)\,d\tau.$$

We introduce $w_{1j}^\varepsilon(\omega,t)$, $j=-1,0,1,\ldots,$ as

$$\widetilde{w}_{1j}^\varepsilon(\xi,t) = \psi_j(\xi)\widetilde{w}_1^\varepsilon(\xi,0,t).$$

We will use the following equivalent norm of the usual C^β norm on $\mathbb{R}^m$, $m \geqslant 1$ (see p. 2 of [12]):

$$\bigl|u(x,t)\bigr|_{C_x^\beta} = \sup_{-1\leqslant j<+\infty}\bigl\{2^{\beta j}\bigl|u_j(x,t)\bigr|_{L_x^\infty}\bigr\} \quad \text{for } 0 \leqslant t \leqslant T, \tag{2.56}$$

where $\beta > 0$, $\beta \neq 1,2,\ldots,$ and $\widetilde{u}_j(\xi,t) = \psi_j(\xi)\widetilde{u}(\xi,t)$. Considering

$$\sum_{i=-1}^{1} \psi_{i+j}(\xi) = 1 \quad \text{on supp}\,\psi_j,\ \ j = 0,1,2,\ldots,$$

for $j \geqslant 0$ we have

$$w_{1j}^\varepsilon(x,t) = \frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^t F^{-1}\left\{K(\xi,t-\tau)\psi_j(\xi)\widetilde{G}(\xi,\tau)\left[\sum_{i=-1}^{1}\psi_{i+j}(\xi)\right]\right\}d\tau$$

$$= \sum_{i=-1}^{1} \frac{\lambda_2}{\lambda_1+\lambda_2} \int_0^t \bigl\{F^{-1}\bigl[K(\xi,t-\tau)\psi_j(\xi)\bigr] * F^{-1}\bigl[\widetilde{G}(\xi,\tau)\psi_{i+j}(\xi)\bigr]\bigr\}\,d\tau$$

$$= \sum_{i=-1}^{1} \frac{\lambda_2}{\lambda_1+\lambda_2} \int_0^t \int_{\mathbb{R}_z} \Biggl[\int_{\mathbb{R}_\xi} e^{i(x-z)\xi} K(\xi,t-\tau)\psi_j(\xi)\,\delta\xi\Biggr] G_{i+j}(z,\tau)\,dz\,d\tau$$

$$:= \sum_{i=-1}^{1} S_{ij}(x,t),$$

where $\delta\xi = (2\pi)^{-1/2}d\xi$, F^{-1} denotes the inverse Fourier transformation, i in the term $e^{i(x-z)\xi}$ is the unit of imaginary number, $\widetilde{G}_{i+j}(\xi,\tau) = \psi_{i+j}(\xi)\widetilde{G}(\xi,\tau)$ and by the equivalent norm (2.56), for $0 \leqslant t \leqslant T$, $-1 \leqslant i \leqslant 1$ and $j = 0, 1, 2, \ldots,$ we have

$$\bigl|G_{i+j}(z,t)\bigr|_{L_z^\infty} \leqslant C2^{-(1+\alpha)(i+j)}\bigl|G(z,t)\bigr|_{C_z^{1+\alpha}}.$$

For fixed $-1 \leqslant i \leqslant 1$, we have

$$S_{ij}(x,t) = \int_{|z-x|\leqslant 2^{-j}} + \int_{|z-x|\geqslant 2^{-j}} = I_1 + I_2; \tag{2.57}$$

$$|I_1| \leqslant |G_{i+j}|_{L^\infty} \sup_{2^{j-1}\leqslant|\xi|\leqslant 2^{j+1}} \int_0^t K(\xi,t-\tau)\,d\tau \int_{|z-x|\leqslant 2^{-j}} dz \int_{\mathbb{R}_\xi} \psi(2^{-j}\xi)\,\delta\xi$$

$$\leqslant C2^{-(1+\alpha)(i+j)}|G|_{C_x^{1+\alpha}} \frac{1}{A(2^{j-1})} 2^{-j}2^j \int_{\mathbb{R}} \psi(\eta)\,\delta\eta$$

$$\leqslant C2^{-(2+\alpha)j}|G|_{C([0,T];C^{1+\alpha})},$$

where C is a constant. Here we have used inequality $1/A(2^{j-1}) \leqslant C2^{-j}$.

$$I_2 = \int_{|z-x|\geqslant 2^{-j}} \int_{\mathbb{R}_\xi} \biggl[\frac{-i\partial_\xi}{(x-z)}\biggr]^2 e^{i(x-z)\xi}\psi_j(\xi)\Biggl[\int_0^t K(\xi,t-\tau)G_{i+j}(z,\tau)\,d\tau\Biggr]\delta\xi\,dz.$$

Integrating by parts repeatedly with respect to ξ, we have

$$|I_2| \leqslant C|G_{i+j}|_{L^\infty} \int_{|z-x|\geqslant 2^{-j}} \frac{1}{|z-x|^2}\,dz$$

$$\times \Biggl\{\sum_{\nu=0}^{2} \int_{\mathbb{R}_\xi} \bigl|\partial^{2-\nu}\psi_j(\xi)\bigr| \Biggl[\int_0^t \bigl|\partial_\xi^\nu K(\xi,t-\tau)\bigr|\,d\tau\Biggr]\delta\xi\Biggr\}.$$

Since $|\partial_\xi^\nu K(\xi,t-\tau)| \leqslant C|\xi|^{-\nu}\sum_{l=0}^{\nu}[A(\xi)]^l(t-\tau)^l|K(\xi,t-\tau)|$, and $|\xi|^{-\nu} \leqslant C2^{-\nu j}$ when $\xi \in \operatorname{supp}\psi_j$, we conclude

$$
\begin{aligned}
|I_2| &\leqslant C2^{-2j}|G_{i+j}|_{L^\infty} \int\limits_{|z-x|\geqslant 2^{-j}} \frac{dz}{|z-x|^2}\left[\sum_{\beta=0}^{2}\int\limits_{\mathbb{R}_\xi}\left|\partial^\beta\psi(2^{-j}\xi)\right|\delta\xi\right] \\
&\quad\times \frac{1}{A(2^{j-1})}\int\limits_0^\infty \sum_{l=0}^{2}\lambda^l e^{-\lambda}\,d\lambda \quad \bigl(\lambda = A(\xi)(t-\tau)\bigr) \\
&\leqslant C2^{-(2+\alpha)j}|G|_{C([0,T];C^{1+\alpha})},
\end{aligned}
$$

where C is a constant which is independent of ε.

We have thus established for $j = 0, 1, 2, \ldots$

$$
\left|w_{1j}^\varepsilon\right|_{L^\infty} \leqslant C2^{-(2+\alpha)j}|G|_{C([0,T];C^{1+\alpha})}.
$$

For $j = -1$, repeating the proof above but not multiplying with $\sum_{i=-1}^{1}\psi_{i+j}(\xi)$, we also see

$$
\left|w_{1j}^\varepsilon\right|_{L^\infty} \leqslant C2^{-(2+\alpha)j}|G|_{C([0,T];C^{1+\alpha})}, \quad j = -1.
$$

In view of the equivalent norm (2.56) we have already proved for fixed $0 \leqslant t \leqslant T$,

$$
\left|w_1^\varepsilon(\cdot,0,t)\right|_{C^{2+\alpha}(\mathbb{R})} \leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}, \tag{2.58}
$$

where C is independent of ε.

We next prove

$$
\lim_{\Delta t\to 0}\left|w_1^\varepsilon(\cdot,0,t+\Delta t) - w_1^\varepsilon(\cdot,0,t)\right|_{C^{2+\alpha}(\mathbb{R})} = 0, \quad \text{uniformly for } \varepsilon. \tag{2.59}
$$

In fact, for any Δt, extend w_1^ε, w_2^ε and G to $t < 0$ as follows:

$$
w_1^\varepsilon(x,y,t) = w_2^\varepsilon(x,y,t) = G(x,t) = 0.
$$

Recalling the system (2.39)–(2.44) we see that $U_1^\varepsilon(x,y,t) := w_1^\varepsilon(x,y,t+\Delta t) - w_1^\varepsilon(x,y,t)$, $U_2^\varepsilon(x,y,t) := w_2^\varepsilon(x,y,t+\Delta t) - w_2^\varepsilon(x,y,t)$ satisfies

$$
\begin{aligned}
&\Delta U_1^\varepsilon = 0, \quad x\in\mathbb{R},\ y>0,\ t\geqslant -|\Delta t|, \\
&\Delta U_2^\varepsilon = 0, \quad x\in\mathbb{R},\ y<0,\ t\geqslant -|\Delta t|, \\
&\lambda_1\partial_y U_1^\varepsilon - \lambda_2\partial_y U_2^\varepsilon = 0 \quad \text{on } y=0, \\
&\partial_t\bigl(U_1^\varepsilon - U_2^\varepsilon\bigr) - \varepsilon\partial_{xx}\bigl(U_1^\varepsilon - U_2^\varepsilon\bigr) - \lambda_3\partial_y U_1 = G(x,t+\Delta t) - G(x,t) \quad \text{on } y=0, \\
&U_1^\varepsilon\bigl(x,0,-|\Delta t|\bigr) - U_2^\varepsilon\bigl(x,0,-|\Delta t|\bigr) = 0, \\
&\lim_{y\to+\infty} U_1^\varepsilon(x,y,t) \text{ is bounded}, \qquad \lim_{y\to-\infty} U_2^\varepsilon(x,y,t) \text{ is bounded}.
\end{aligned}
$$

From the result of (2.58), we have

$$
\left|U_1^\varepsilon(\cdot,0,t)\right|_{C^{2+\alpha}(\mathbb{R})} \leqslant C\left|G(x,t+\Delta t) - G(x,t)\right|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}.
$$

This completes the proof of (2.59). Combining (2.58) and (2.59) leads to (2.55) for $k = 1$. We thus complete the proof of Lemma 2.2. □

Proof Theorem 2.1 (continued). Going back to the system (2.32)–(2.37),

$$\Delta w_k^\varepsilon = 0, \quad (x, y) \in \Omega_k, \tag{2.60}$$

$$\partial_y w_1^\varepsilon = 0 \quad \text{on } y = a, \qquad w_2^\varepsilon = 0 \quad \text{on } y = -a, \tag{2.61}$$

$$\lambda_1 \partial_y w_1^\varepsilon - \lambda_2 \partial_y w_2^\varepsilon = 0, \tag{2.62}$$

$$\partial_t\big(w_1^\varepsilon - w_2^\varepsilon\big) - \varepsilon\partial_x^2\big(w_1^\varepsilon - w_2^\varepsilon\big) - \lambda_3 \partial_y w_1^\varepsilon = G(x,t) \quad \text{on } y = 1, \tag{2.63}$$

$$w_1^\varepsilon(x,0,0) - w_2^\varepsilon(x,0,0) = 0, \tag{2.64}$$

(2.60), (2.61) and (2.55) tell us that w_k^ε are uniformly bounded in $C([0,T]; C^{2+\alpha}(\overline{\Omega}_k))$, meanwhile (2.55) shows that $\partial_{xx} w_k^\varepsilon(x,0,t)$ are uniformly bounded in $C([0,T]; C^\alpha(\mathbb{R}))$. Notice that (2.54) means $w_2^\varepsilon = -(\lambda_1/\lambda_2)w_1^\varepsilon$, so from condition (2.63) we know that $\partial_t w_k^\varepsilon(x,0,t)$ are uniformly bounded in $C([0,T]; C^\alpha(\mathbb{R}))$. Applying the compactness theorem again (see [11, p. 85]), we know that $C([0,T]; C^{2+\alpha}(\mathbb{R})) \cap C^1([0,T]; C^\alpha(\mathbb{R})) \to C([0,T]; C^{2+\beta}(\mathbb{R}))$, $0 < \beta < \alpha$, is compact, hence there exist w_k^ε, $k = 1, 2$, $w_k^\varepsilon(x,0,t) \in C([0,T]; C^{2+\alpha}(\mathbb{R}))$ such that as $\varepsilon \to 0$, possibly subsequences,

$$w_k^\varepsilon(x,0,t) \to w_k(x,0,t) \quad \text{in } C\big([0,T]; C^{2+\beta}(\mathbb{R})\big),\ 0 < \beta < \alpha.$$

Letting $\varepsilon \to 0$, we consequently get system (2.23)–(2.28) and the inequality (2.31).

We finally establish the uniqueness of problem (2.14)–(2.19). In fact suppose (u_1, u_2) and (v_1, v_2) are two solutions of the problem, then $w_k = u_k - v_k$ satisfies

$$\Delta w_k = 0, \quad (x, y) \in \Omega_k,\ t \geqslant 0, \tag{2.65}$$

$$\partial_y w_1 = 0 \quad \text{on } y = a, \tag{2.66}$$

$$w_2 = 0 \quad \text{on } y = -a, \tag{2.67}$$

$$\lambda_1 \partial_y w_1 - \lambda_2 \partial_y w_2 = 0, \tag{2.68}$$

$$\partial_t(w_1 - w_2) - \lambda_3 \partial_y w_1 = 0 \quad \text{on } y = 1, \tag{2.69}$$

$$w_1(x,0,0) - w_2(x,0,0) = 0. \tag{2.70}$$

Multiplying Eq. (2.65) by $\lambda_k w_k$ and integrating over $D_1 = \{0 < x < p,\ 0 < y < a,\ 0 < t < T\}$ and $D_2 = \{0 < x < p,\ -a < y < 0,\ 0 < t < T\}$, respectively, then applying boundary conditions (2.66)–(2.70), we conclude

$$\sum_{k=1}^{2} \iiint_{D_k} \lambda_k \big|\nabla w_k(x,y,t)\big|^2\, dx\, dy\, dt + \frac{\lambda_1}{2\lambda_3} \int_0^p (w_1 - w_2)^2(x,0,T)\, dx = 0.$$

This implies $w_k = 0$, i.e., $u_k = v_k$. This completes the proof of Theorem 2.1. □

The method in Theorem 2.1 can be easily applied to the analysis of somewhat more general problem with constant coefficients and oblique derivative boundary condition at $y = 0$:

$$a_{11}^{(k)} \partial_x^2 u_k + 2a_{12}^{(k)} \partial_x \partial_y u_k + a_{22}^{(k)} \partial_y^2 u_k = f_k, \quad (x, y) \in \Omega_k,\ t \geqslant 0,\ k = 1, 2, \tag{2.71}$$

$$\partial_y u_1 = h_1(x,t) \quad \text{on } y = a, \tag{2.72}$$

$$u_2 = h_2(x,t) \quad \text{on } y = -a, \tag{2.73}$$

$$\beta \cdot (\lambda_1 \nabla u_1 - \lambda_2 \nabla u_2) = g_1(x,t) \quad \text{on } y = 0, \tag{2.74}$$

$$\partial_t(u_1 - u_2) - \gamma \cdot \nabla u_1 = g_2 \quad \text{on } y = 0, \tag{2.75}$$

$$u_1(x,0,0) - u_2(x,0,0) = u_0(x), \tag{2.76}$$

where $(a_{ij}^{(k)})_{2\times 2}$ is a positively definite constant matrix satisfying $a_{ij}^{(1)} = a_{ij}^{(2)}$; λ_1, λ_2 are positive constants; $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ are constant vectors with $\beta_2 > 0$, $\gamma_2 > 0$.

Lemma 2.3. *If $u_k \in C([0,T]; C^{2+\alpha}(\overline{\Omega}_k))$, $\partial_t(u_1(x,0,t) - u_2(x,0,t)) \in C([0,T]; C^{1+\alpha}(\mathbb{R}))$ and (u_1,u_2) is a solution of the problem* (2.71)–(2.76)*, then (u_1,u_2) admits a priori estimate*

$$\sum_{k=1}^{2} |u_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))} + \bigl|\partial_t\bigl(u_1(x,0,t) - u_2(x,0,t)\bigr)\bigr|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant C\Biggl[\sum_{k=1}^{2}\bigl(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))} + |g_k|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\bigr)$$
$$+ |u_0|_{C^{2+\alpha}(\mathbb{R})}\Biggr], \tag{2.77}$$

where C is a positive constant.

Proof. Note that in this lemma we are not concerned with the existence. Without loss of generality we may suppose $\beta = (0,1)$. In fact, if we make a transformation of variables

$$x = x' + \beta_1 y', \qquad y = \beta_2 y',$$

then

$$\partial_{y'} = \beta_1 \partial_x + \beta_2 \partial_y = \beta \cdot \nabla.$$

After this transformation of variables, the equations remain elliptic one and the vector γ remains toward outside Ω_2. So we just suppose $\beta = (0,1)$.

Corresponding to the system (2.39)–(2.44) with $\varepsilon = 0$, we have

$$a_{11}^{(k)}\partial_x^2 w_k + 2a_{12}^{(k)}\partial_x\partial_y w_k + a_{22}^{(k)}\partial_y^2 w_k = 0, \quad x \in \mathbb{R},\ (-1)^{k-1}y > 0,\ t \geqslant 0, \tag{2.78}$$

$$\lambda_1 \partial_y w_1 - \lambda_2 \partial_y w_2 = 0 \quad \text{on } y = 0, \tag{2.79}$$

$$\partial_t(w_1 - w_2) - \gamma \cdot \nabla w_1 = G(x,t) \quad \text{on } y = 0, \tag{2.80}$$

$$w_1(x,0,0) - w_2(x,0,0) = 0, \tag{2.81}$$

$$\lim_{y\to+\infty} w_1(x,y,t) \text{ is bounded}, \qquad \lim_{y\to-\infty} w_2(x,y,t) \text{ is bounded}. \tag{2.82}$$

Making Fourier transformation with respect to x in the system (2.78)–(2.82), we obtain that

$$-a_{11}^{(k)}|\xi|^2\widetilde{w}_k(\xi,y,t)+2a_{12}^{(k)}i\xi\frac{d}{dy}\widetilde{w}_k(\xi,y,t)+a_{22}^{(k)}\frac{d^2}{dy^2}\widetilde{w}_k(\xi,y,t)=0,$$
$$\xi\in\mathbb{R},\ (-1)^{k-1}y>0,\ t>0, \tag{2.83}$$

$$\lambda_1\frac{d}{dy}\widetilde{w}_1-\lambda_2\frac{d}{dy}\widetilde{w}_2=0\quad\text{on } y=0, \tag{2.84}$$

$$\partial_t(\widetilde{w}_1-\widetilde{w}_2)-\gamma_1 i\xi\widetilde{w}_1-\gamma_2\frac{d}{dy}\widetilde{w}_1=\widetilde{G}(\xi,t)\quad\text{on } y=0, \tag{2.85}$$

$$\widetilde{w}_1(\xi,0,0)-\widetilde{w}_2(\xi,0,0)=0, \tag{2.86}$$

$$\lim_{y\to+\infty}\widetilde{w}_1(\xi,y,t)\text{ is bounded},\qquad \lim_{y\to-\infty}\widetilde{w}_2(\xi,y,t)\text{ is bounded}. \tag{2.87}$$

From (2.83)–(2.87) we obtain

$$\widetilde{w}_1(\xi,0,t)=\frac{\lambda_2}{\lambda_1+\lambda_2}\int_0^t\exp\left\{-\frac{\lambda_2}{\lambda_1+\lambda_2}\left[\frac{\gamma_2}{a_{22}}\sqrt{a_{11}a_{22}-a_{12}^2}|\xi|\right.\right.$$
$$\left.\left.-\,i\left(\gamma_1+\frac{\gamma_2a_{12}}{a_{22}}\right)\xi\right](t-\tau)\right\}\widetilde{G}(\xi,\tau)\,d\tau. \tag{2.88}$$

In a derivation analogous to that in the proof of Lemma 2.2, we can get

$$\sum_{k=1}^2\left|w_k(x,0,t)\right|_{C([0,T];C^{2+\alpha}(\mathbb{R}))}\leqslant C|G|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}. \tag{2.89}$$

The only deference is that equality (2.57) now becomes

$$S_{ij}(x,t)=\int\limits_{|z-x-\mu|\leqslant 2^{-j}}+\int\limits_{|z-x-\mu|\geqslant 2^{-j}}=I_1+I_2,$$

where

$$\mu=\frac{\lambda_2}{\lambda_1+\lambda_2}\left(\nu_1+\frac{\nu_2a_{12}}{a_{22}}\right)(t-\tau).$$

This completes the proof of Lemma 2.3. □

Next we consider the problem (2.1)–(2.6). By employing the methods of localization, freezing coefficients and Lemma 2.3 we conclude:

Lemma 2.4. *If* $u_k\in C([0,T];C^{2+\alpha}(\overline{\Omega}_k))$, $\partial_t(u_1(x,0,t)-u_2(x,0,t))\in C([0,T];C^{1+\alpha}(\mathbb{R}))$ *and* (u_1,u_2) *is the solution of the problem* (2.1)–(2.6)*, then* (u_1,u_2) *admits a priori estimate*

$$\sum_{k=1}^2|u_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))}+\left|\partial_t\big(u_1(x,0,t)-u_2(x,0,t)\big)\right|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$

$$\leqslant C\left[\sum_{k=1}^{2}\big(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))}+|g_k|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}+|h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\big)\right.$$
$$\left.+|u_0|_{C^{2+\alpha}(\mathbb{R})}\right]. \tag{2.90}$$

Theorem 2.5. *Under the assumptions of* (2.7)–(2.13)*, the problem* (2.1)–(2.6) *has a unique solution* $(u_1,u_2)\in C([0,T];C^{2+\alpha}(\overline{\Omega}_1))\times C([0,T];C^{2+\alpha}(\overline{\Omega}_2))$ *with* $\partial_t(u_1(x,0,t)-u_2(x,0,t))\in C([0,T];C^{1+\alpha}(\mathbb{R}))$. *Moreover,*

$$\sum_{k=1}^{2}|u_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))}+\big|\partial_t\big(u_1(x,0,t)-u_2(x,0,t)\big)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}$$
$$\leqslant C\left[\sum_{k=1}^{2}\big(|f_k|_{C([0,T];C^{\alpha}(\overline{\Omega}_k))}+|g_k|_{C([0,T];C^{1+\alpha}(\mathbb{R}))}+|h_k|_{C([0,T];C^{k+\alpha}(\mathbb{R}))}\big)\right.$$
$$\left.+|u_0|_{C^{2+\alpha}(\mathbb{R})}\right], \tag{2.91}$$

where C is a positive constant.

Proof. The proof is based on Theorem 2.1 (the result of existence for Poisson's equation) and a priori estimate (2.90). Suppose $\lambda\in[0,1]$, and consider problem

$$(1-\lambda)\Delta u_k+\lambda\big(a_{11}^{(k)}\partial_x^2u_k+2a_{12}^{(k)}\partial_x\partial_yu_k+a_{22}^{(k)}\partial_y^2u_k+b_1^{(k)}\partial_xu_k+b_2^{(k)}\partial_yu_k+c_ku_k\big)$$
$$=f_k,\quad (x,y)\in\Omega_k,\ t\geqslant 0,$$
$$\partial_yu_1=h_1(x,t)\quad\text{on } y=a,$$
$$u_2=h_2(x,t)\quad\text{on } y=-a,$$
$$(1-\lambda)(\lambda_1\partial_yu_1-\lambda_2\partial_yu_2)+\lambda\beta\cdot(\lambda_1\nabla u_1-\lambda_2\nabla u_2)=g_1(x,t)\quad\text{on } y=0,$$
$$\partial_t(u_1-u_2)-(1-\lambda)\partial_yu_1-\lambda(\gamma\cdot\nabla u_1+d_1u_1+d_2u_2)=g_2(x,t)\quad\text{on } y=0,$$
$$u_1(x,0,0)-u_2(x,0,0)=u_0(x).$$

From Theorem 2.1, (2.90) and applying a standard method of parameter extension (see [13, Theorem 5.2]) we can obtain that the problem (2.1)–(2.6) has a solution which satisfies the estimate (2.91).

From (2.90) we also get uniqueness. We complete the proof of Theorem 2.5. □

Remark on Theorem 2.5. Theorem 2.5 remains valid in any multidimensional cases. The reason is that the key part, Lemma 2.2, can be established in any multidimensional cases. The method of the proof is similar to that of this paper.

3. Global existence of (1.21)–(1.27)

In this section we prove global existence of the problem (1.21)–(1.27). It is convenient to straighten the free boundary. Make a transformation of variables

$$x = x, \qquad z = y - \phi(y)\rho(x,t), \qquad t = t,$$

where $\phi(y)$ is a cutoff function such that

$$\phi(y) = \begin{cases} 1, & \text{when } |y| < a/3, \\ 0, & \text{when } |y| > 2a/3, \end{cases}$$

and $|\phi'(y)| \leqslant 4/a$. Note that $\partial z/\partial y = 1 - \phi'(y)\rho(x,t)$. Hence if

$$|\rho| \leqslant a/6, \tag{3.1}$$

then $\partial z/\partial y \geqslant 1/3$; in this case the transformation is invertible. Let

$$w_k(x,z,t) = v_k(x,y,t), \quad k = 1, 2.$$

Then

$$\partial_x v_k = \partial_x w_k - \phi \partial_x \rho \partial_z w_k,$$
$$\partial_y v_k = (1 - \phi'\rho)\partial_z w_k.$$

It means

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_x - \phi\partial_x\rho\partial_z \\ (1-\phi'\rho)\partial_z \end{pmatrix} := \nabla_\rho$$

and

$$\begin{aligned} \nabla_\rho^2 = \partial_x^2 &+ \big[\phi^2(\partial_x\rho)^2 + (1-\phi'\rho)^2\big]\partial_z^2 - 2\phi\partial_x\rho\partial_{xz} \\ &+ \left\{\phi\phi'\frac{\partial y}{\partial z}(\partial_x\rho)^2 - \left[\phi'\left(\frac{\partial y}{\partial x} - \phi\partial_x\rho\frac{\partial y}{\partial z}\right)\partial_x\rho + \phi\partial_x^2\rho\right]\right\}\partial_z \\ &- (1-\phi'\rho)\phi''\frac{\partial y}{\partial z}\rho\partial_z. \end{aligned}$$

Then the problem (1.21)–(1.27) becomes

$$\nabla_\rho^2 w_1 = 0, \quad x \in \mathbb{R},\ 0 < z < a,\ t \geqslant 0, \tag{3.2}$$
$$\nabla_\rho^2 w_2 = 0, \quad x \in \mathbb{R},\ -a < z < 0,\ t \geqslant 0, \tag{3.3}$$
$$\partial_z w_1 = f_1(x,t), \quad x \in \mathbb{R},\ z = a,\ t \geqslant 0, \tag{3.4}$$
$$w_2 = f_2(x,t), \quad x \in \mathbb{R},\ z = -a,\ t \geqslant 0, \tag{3.5}$$
$$\left(\frac{k_1}{k_2} - 1\right)g_1(t)\sigma = w_1 - w_2, \quad x \in \mathbb{R},\ z = 0,\ t \geqslant 0, \tag{3.6}$$
$$\begin{aligned} &k_1\big\{\big[1 + (\varepsilon\partial_x\sigma)^2\big]\partial_z w_1 - \varepsilon\partial_x\sigma\partial_x w_1\big\} \\ &\quad = k_2\big\{\big[1 + (\varepsilon\partial_x\sigma)^2\big]\partial_z w_2 - \varepsilon\partial_x\sigma\partial_x w_2\big\}, \quad x \in \mathbb{R},\ z = 0,\ t \geqslant 0, \end{aligned} \tag{3.7}$$
$$\partial_t\sigma = -k_1\big\{\big[1 + (\varepsilon\partial_x\sigma)^2\big]\partial_z w_1 - \varepsilon\partial_x\sigma\partial_x w_1\big\}, \quad x \in \mathbb{R},\ z = 0,\ t \geqslant 0, \tag{3.8}$$
$$\sigma(x,0) = \sigma_0(x). \tag{3.9}$$

From (3.6) we have

$$\partial_t\sigma = \frac{k_2}{(k_1-k_2)g_1(t)}\partial_t(w_1-w_2) - \frac{k_2 g_1'}{(k_1-k_2)g_1^2}(w_1-w_2).$$

Substituting it into (3.8), we obtain that

$$\partial_t(w_1-w_2) = \frac{k_1}{k_2}(k_2-k_1)g_1(t)\big\{\big[1+(\varepsilon\partial_x\sigma)^2\big]\partial_z w_1 - \varepsilon\partial_x\sigma\,\partial_x w_1\big\} + \frac{g_1'}{g_1}(w_1-w_2). \tag{3.10}$$

Combining (3.6), (3.9) and (3.10) we get

$$(w_1-w_2)(x,0,0) = \left(\frac{k_1}{k_2}-1\right)g_1(0)\sigma_0(x), \quad x\in\mathbb{R}. \tag{3.11}$$

Finally, we prove that the system (3.2)–(3.7), (3.10) and (3.11) has a classical solution globally in time provided ε is small enough.

Suppose that

$$f_1\in C\big([0,T];C^{1+\alpha}(\mathbb{R})\big), \qquad f_2\in C\big([0,T];C^{2+\alpha}(\mathbb{R})\big), \tag{3.12}$$

k_1, k_2 are positive constants with $k_1<k_2$, (3.13)

$$g_1(t)\in C^1[0,T], \qquad g(t)<0, \qquad \left|\int_0^T k_1 g_1(t)\,dt\right| \leqslant a/12, \tag{3.14}$$

$$\sigma_0(x)\in C^{2+\alpha}(\mathbb{R}). \tag{3.15}$$

We have

Theorem 3.1. *Under the assumptions of* (3.12)–(3.15)*, if ε is small enough which is independent of T, the problem* (3.2)–(3.7), (3.10), (3.11) *has a unique solution* $(w_1(x,z,t), w_2(x,z,t), \sigma(x,t)) \in C([0,T];C^{2+\alpha}(\overline{\Omega}_1)) \times C([0,T];C^{2+\alpha}(\overline{\Omega}_2)) \times C([0,T]; C^{2+\alpha}(\mathbb{R}))$ *with* $\partial_t(w_1-w_2)(x,0,t), \partial_t\sigma \in C([0,T];C^{1+\alpha}(\mathbb{R}))$. *Moreover,*

$$\sum_{k=1}^{2}|w_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))} + \big|\partial_t(w_1-w_2)(x,0,t)\big|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |\sigma|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\partial_t\sigma|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} \leqslant C\big(|f_1|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |f_2|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\sigma_0|_{C^{2+\alpha}(\mathbb{R})}\big), \tag{3.16}$$

where C depends on $|g_1(t)|_{C^1[0,T]}$ and is independent of ε.

Proof. We use Schauder fixed point theorem. Define

$$\mathcal{D} = \big\{\sigma(x,t)\in C\big([0,T];C^{2+\alpha}(\mathbb{R})\big),\ \partial_t\sigma\in C\big([0,T];C^{1+\alpha}(\mathbb{R})\big),\ \sigma(x,0)=\sigma_0(x)\big\},$$
$$\mathcal{D}_K = \big\{\sigma\in\mathcal{D},\ |\sigma|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\partial_t\sigma|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} \leqslant K\big\},$$

where K is a positive constant which is independent of ε and is determined later. It is clear that $\mathcal{D}_K$ is a compact convex set in Banach space $C(\mathbb{R} \times [0, T])$.

For given $\sigma \in \mathcal{D}_K$, recalling (1.20) we let $\rho(x, t) = \rho^*(t) + \varepsilon\sigma(x, t)$ and consider the problem (3.2)–(3.5), (3.7), (3.10), (3.11) for unknown $(w_1(x, z, t), w_2(x, z, t))$. We observe that if $\varepsilon = 0$, then $\partial_x \rho = 0$, $\partial y/\partial x = 0$ and

$$\nabla_\rho^2 = \partial_x^2 + (1 - \phi'\rho)^2 \partial_z^2 - (1 - \phi'\rho)\phi'' \frac{\partial y}{\partial z} \rho \partial_z,$$

which is an elliptic operator, so there exists a $\varepsilon_0 > 0$, such that if $0 \leqslant \varepsilon \leqslant \varepsilon_0$, ∇_ρ^2 is an elliptic operator too. We may suppose $\varepsilon_0 \leqslant a/12K$. In this way

$$|\varepsilon\sigma|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} \leqslant a/12$$

for $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $\sigma \in \mathcal{D}_K$. In this case

$$\bigl|\rho(x, t)\bigr| \leqslant \bigl|\rho^*(t)\bigr| + |\varepsilon\sigma| \leqslant a/12 + a/12 = a/6,$$

which satisfies condition (3.1).

From Theorem 2.5 we know that the problem (3.2)–(3.5), (3.7), (3.10), (3.11) has a unique solution $(w_1(x, z, t), w_2(x, z, t)) \in C([0, T]; C^{2+\alpha}(\overline{\Omega}_1)) \times C([0, T]; C^{2+\alpha}(\overline{\Omega}_2))$ with $\partial_t(w_1 - w_2)(x, 0, t) \in C([0, T]; C^{1+\alpha}(\mathbb{R}))$. Moreover,

$$\begin{aligned} &\sum_{k=1}^{2} |w_k|_{C([0,T];C^{2+\alpha}(\overline{\Omega}_k))} + \bigl|\partial_t(w_1 - w_2)(x, 0, t)\bigr|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} \\ &\quad \leqslant C\bigl(|f_1|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |f_2|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\sigma_0|_{C^{2+\alpha}(\mathbb{R})}\bigr), \end{aligned} \tag{3.17}$$

where C depends on $|g_1(t)|_{C^1[0,T]}$ and is independent of K and ε.

In view of (3.6) we define

$$\overline{\sigma}(x, t) = \frac{k_2}{(k_1 - k_2)g_1(t)}\bigl(w_1(x, 0, t) - w_2(x, 0, t)\bigr)$$

and, by (3.17),

$$\begin{aligned} &|\overline{\sigma}|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\partial_t\overline{\sigma}|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} \\ &\quad \leqslant C\bigl(|f_1|_{C([0,T];C^{1+\alpha}(\mathbb{R}))} + |f_2|_{C([0,T];C^{2+\alpha}(\mathbb{R}))} + |\sigma_0|_{C^{2+\alpha}(\mathbb{R})}\bigr) := K. \end{aligned}$$

Define a mapping $\mathcal{F} : \mathcal{D}_K \to \mathcal{D}$ by

$$\mathcal{F}(\sigma) = \overline{\sigma}.$$

Previously we have proved that $\mathcal{F}$ maps $\mathcal{D}_K$ into itself. From the uniqueness of solution to the problem (3.2)–(3.5), (3.7), (3.10), (3.11) and the compactness, it follows that $\mathcal{F}$ is a continuous mapping. Hence, according to the Schauder fixed point theorem, $\mathcal{F}$ has a fixed point $\mathcal{F}(\sigma) = \sigma$ which together with the corresponding (w_1, w_2) provides a solution to the problem (3.2)–(3.7), (3.10), (3.11). The existence is proved.

The uniqueness had been proved in [6]. $\square$

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