The Global Homological Dimensions of Trivial Extensions of Rings

CLAS LöFWALL

Department of Mathematics, University of Stockholm, S-113 85 Stockholm, Sweden

Communicated by I. N. Herstein

Received December 9, 1974

INTRODUCTION

Let $R$ be a ring, and $M$ a bimodule over $R$. The trivial extension $R \times M$ is as abelian group $R \oplus M$ and the multiplication is given by $(r, m)(r', m') = (rr', rm' + mr')$. Palmer and Roos give in [11] an almost general formula for the global homological dimension of $R \times M$. However, their proof is incomplete. They want to prove that a spectral sequence degenerates, but they only give a proof of the fact that the first differential is zero. In this note, we prove that also the higher differentials are zero. Moreover, our methods (very similar to those in [11]) give a completely general formula for the global homological dimension of $R \times M$ (and also for $\text{wgldim} R \times M$). It was an idea of Roos that "multiple-Tor" should be relevant to compute $\text{wgldim} R \times M$. He found the definition in [10] ("multiple-Tor" is defined there in a more complicated situation, which will not be studied in this note). Let $U$ be a left $R$-module, and $V$ a right $R$-module. The definition of $\text{Tor}(V, M, ..., M, U)$ is as follows. Choose a projective resolution $P_\ast$ of $U$, choose a projective resolution of the complex $M \oplus P_\ast$ (to do this, observe that the category of complexes is an abelian category with enough projectives or use the method given in the last chapter of [1]), take the associated simple complex of this resolution, apply the functor $M \otimes \cdot$, and so on. Apply at last the functor $V \otimes \cdot$ and take the homology of the corresponding complex. One can show that the definition is independent of the choice of projective resolutions. One of the main results in this note is: $\text{wgldim} R \times M \leq n \iff \text{Tor}_q(V, M, ..., M, U) = 0$ ($p$ copies of $M$) for all $p, q \geq 0$ such that $p + q = n + 1$ and all left $R$-modules $U$, and all right $R$-modules $V$.

Palmer and Roos make the following assumption on $M$ in [11] (which is not true in general): $M$ has a resolution $Q_\ast$ consisting of $R$-bimodules that are flat as left (or right) $R$-modules. (This is true for instance if $R$ is commutative and $M$ is symmetric.) In this case, we have $\text{Tor}(V, M, ..., M, U) =$
$H(P_\bullet(V) \otimes Q_\bullet \otimes \cdots \otimes Q_\bullet \otimes P_\bullet(U))$, where $P_\bullet(U)$ (resp. $P_\bullet(V)$) is a projective resolution of $U$ (resp. $V$). Hence, our result generalizes [11]. The formula for gldim $R \times M$ is also valid, with an obvious change of notations, for gldim $(\mathcal{A}, F)$ (cf. [4] or [5]) if $\mathcal{A}$ is an abelian category with enough projectives and injectives and $F$ is a right exact functor, which has a right adjoint.

**Summary**

In Section 1, we study the general concept of the “iterated homology of a composed functor”. We make frequent use of [2, Satz 3.1] and I wish to thank Gunnar Sjödin, who made the usefulness of the theorem clear to me. "Multiple-Ext" and "Multiple-Tor" are studied in Section 2. Section 3 is a self-contained development of the homology theory for trivial extensions (more details can be found in [4]).

In Section 4, we construct a spectral sequence, that converges to

$$\text{Ext}^n_{R \times M}((U, f), (V, 0)).$$

(A left $R \times M$-module is given by a left $R$-module $U$ and a map $f : M \otimes U \to U$ such that $f \circ (M \otimes f) = 0$.) There are no new ideas in Sections 3 and 4, all of them can be found essentially in [11]. In Section 5, we prove that all the differentials in the spectral sequence above are zero if $f = 0$.

Section 6 is an application to the theory of commutative local noetherian rings. (This application was shown to me by Jan-Eric Roos.) Let $(R, m)$ be such a ring, and $M$ a finitely generated $R$-module. The Poincaré-series $P_R^M$ is defined by $P_R^M(z) = \sum_{i \geq 0} \dim_k \text{Tor}_i^R(k, M) z^i (k = R/m)$. We prove the following: $P_{R \times M}^k(z) = P_R^k(z)(1 - zP_R^M(z))^{-1}$. Gulliksen's proof of this in [9] uses Massey-operations, and as he notes, the formula shows that if Serre’s hypothesis is true, that $P_R^k$ is rational for every ring $R$, then also, $P_R^M$ is rational for every finitely generated module $M$. (At this point, the reader may wish to refer to the Appendix for a summary of the notation used throughout the paper.

1. **On Iterated Homology**

Let $\mathcal{C}$ be an abelian category with enough injectives (projectives). Then, $\mathcal{K}^-, \mathcal{K}^+, \mathcal{K}^-$ are abelian categories with enough injectives (projectives). We only treat the case of $\mathcal{K}^+$ (the case $\mathcal{K}^-$ is dual to $\mathcal{K}^+$, because $\mathcal{K}^-(\mathcal{C}) \cong \mathcal{K}^+(\mathcal{C}^{op}))$. This can be proved directly (cf. [7, Lemma 11.5.2.1, p. 36]) but
also can be seen as a special case of the theory of trivial extensions (cf. [4, Example 4.15, p. 84]): $\mathcal{Y}^+$ is a trivial extension of the category $\prod N \mathcal{E}$. Indeed let $F: \prod N \mathcal{E} \to \prod N \mathcal{E}$ be defined by $F(K^0, K^1, ...) = (0, K^0, K^1, ...)$. A complex is an object $A \in \prod N \mathcal{E}$ and a map $d: FA \to A$ such that $d \cdot Fd = 0$. A map between two complexes $(A, d)$ and $(A', d')$ is a map $f: A \to A'$ such that

\[
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
\downarrow d & & \downarrow d' \\
A & \xrightarrow{f} & A'
\end{array}
\]

commutes.

We now know how the projectives and injectives in $\mathcal{X}^+$ look. (Observe that $F$ has a right adjoint $G$, $G(K^0, K^1, ...) = (K^1, K^2, ...)$. ) The projective objects in $\mathcal{X}^+$ are of the following form:

\[
\begin{array}{c}
FFP @ FP \\
\text{or written in a more familiar way:}
\end{array}
\]

\[
0 \longrightarrow P^0 \xrightarrow{(1 \ 0)} P^0 \oplus P^1 \xrightarrow{(0 \ 1)} P^1 \oplus P^2 \xrightarrow{(0 \ 1)} \cdots ,
\]

i.e., precisely the projective complexes, which are homotopic to zero. The injective objects in $\mathcal{X}^+$ are of the following form:

\[
\begin{array}{c}
GI @ I \\
\text{or,}
\end{array}
\]

\[
0 \longrightarrow I^0 \oplus I^1 \xrightarrow{(0 \ 1)} I^1 \oplus I^2 \xrightarrow{(0 \ 1)} \cdots ,
\]

i.e., injective resolutions of injective objects.

To write $K^* \in \mathcal{X}^+$ as a subobject of an injective object, we only choose injectives $I^n$ and monomorphisms $f^n: K^n \to I^n$ and construct the following diagram:

\[
\begin{array}{c}
0 \longrightarrow K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} K^2 \xrightarrow{d^2} \cdots
\end{array}
\]

\[
\begin{array}{c}
\text{(f^0, f^0 d^0)} \downarrow \quad \text{(f^1, f^1 d^1)} \uparrow \quad \text{(f^2, f^2 d^2)} \uparrow \quad \cdots
\end{array}
\]

\[
\begin{array}{c}
0 \longrightarrow I^0 \oplus I^1 \longrightarrow I^1 \oplus I^2 \longrightarrow I^2 \oplus I^3 \longrightarrow \cdots
\end{array}
\]

\[
\begin{array}{c}
\text{(f^0, f^1)} \uparrow \quad \text{(f^1, f^2)} \uparrow \quad \text{(f^2, f^3)} \uparrow \quad \cdots
\end{array}
\]
There is a similar construction of projective resolutions, but the injective resolutions have nicer properties according to

**Lemma 1.1.** Let $K^* \in \mathcal{K}^+$. There is an injective resolution of $K^*$. (See Appendix for the definition).

**Proof.** Choose an injective resolution of $K^*$ in category sense and take the associated simple complex of the corresponding double complex. Q.E.D.

The following lemma is in Dold [2, Satz 3.10] (the premises there, however, are more general, and they may be still more general, according to Sjödin).

**Lemma 1.2.** Suppose given $X^*, Y^*, I^* \in \mathcal{K}^+$ with $I^n$ injective for each $n$ and a quism $f: X^* \Rightarrow I^*$. Then,

$$\Phi: [I^*, Y^*] \to [X^*, Y^*]$$

is an isomorphism, $\Phi$ is defined by $\Phi[g] = [g \circ f]$.

**Proof.** See [2].

**Corollary 1.1.** If $\phi: I_1^* \Rightarrow I_2^*$ is a quism and $I_i^n$ is injective for each $n$ ($i = 1, 2$), then $\phi$ is a homotopy equivalence.

**Proof.** See [2].

**Corollary 1.2.** Suppose the following diagram is given

$$\begin{array}{ccc}
I^* & \rightarrow & I_1^* \\
\downarrow f & & \downarrow f_1 \\
K^* & \rightarrow & K_1^*,
\end{array}$$

where the vertical maps are injective resolutions. Then, there exists a map $\phi: I^* \rightarrow I_1^*$, with $[\phi]$ uniquely defined, such that $\phi f$ and $f_1 \phi$ are homotopic. If $\phi$ is a quism, then $I^*$ and $I_1^*$ are homotopy equivalent.

**Proof.** Lemma 1.2 yields an isomorphism $[I^*, I_1^*] \to [K^*, I_1^*]$. Hence, there is $\phi: I^* \rightarrow I_1^*$ such that $\phi f$ and $f_1 \phi$ are homotopic, and $[\phi]$ is uniquely defined. If $\phi$ is a quism, then so is $\phi$, and hence, a homotopy equivalence according to Corollary 1.1. Q.E.D.

Let $\mathcal{C}_0, \mathcal{C}_1, ..., \mathcal{C}_n$ be abelian categories, $\mathcal{C}_0, ..., \mathcal{C}_n$ with enough injectives and $T_0, T_1, ..., T_n$ covariant additive functors, $T_i: \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$ for $i = 0, 1, ..., n$. Let $A \in \mathcal{C}_0$. Choose an injective resolution $I_0^*$ of $A$. Inductively, we choose an injective resolution $I_i^*$ of $T_{i-1}(I_{i-2}^*)$ for $i = 1, 2, ..., n$. 


Definition. \( H^*(T_n, T_{n-1}, \ldots, T_0)(A) = H^*(T_0(A^*)) \) is called the iterated (co)homology of \( T_n, \ldots, T_0 \) in \( A \).

The definition is independent of the choice of the complexes \( I_i^* \) \( i = 0, 1, \ldots, n \) according to Corollary 1.2.

Spectral Sequences

Let \( I_n^* \) be an injective resolution of \( T_{n-1}(I_{n-1}^*) \) in the sense of Cartan–Eilenberg (see [1]). We get the following two spectral sequences

\[
E_2^{pq} = H^p((R^qT_n \circ T_{n-1}), T_{n-1}, \ldots, T_0) \Rightarrow H^*(T_n, \ldots, T_0),
\]

\[
E_2^{pq} = R^qT_n(H^q(T_{n-1}, \ldots, T_0)) \Rightarrow H^*(T_n, \ldots, T_0).
\]

It is the second spectral sequence that requires resolutions in the sense of Cartan–Eilenberg. When the spectral sequences arise in practice, they are easy to derive directly (at least in this paper).

Remark. Some of the functors \( T_0, \ldots, T_n \) may be contravariant and we may also start with a projective resolution of \( A \in \mathcal{C}_0 \). But we must always choose injective resolutions of complexes in \( \mathcal{X}^+ \) and projective resolutions of complexes in \( \mathcal{X}^- \).

Proposition 1.1. \( \{H^i(T_n, \ldots, T_0)(-)\}_{i \geq 0} \) is an exact \( \partial \)-functor (in general not universal). (For the terminology see Grothendieck [8]).

Proof. Corollary 1.2 shows that \( H^i(T_n, \ldots, T_0)(-) \) is a functor. Suppose, that we have a commutative diagram of complexes in \( \mathcal{X}^+ \) with exact rows:

\[
0 \rightarrow A_1^* \rightarrow A_2^* \rightarrow A_3^* \rightarrow 0
\]

\[
0 \rightarrow A_1^* \rightarrow A_2^* \rightarrow A_3^* \rightarrow 0.
\]

By general theory for injective resolutions in category sense (e.g., regard the diagram as one exact sequence of double complexes) we may construct a commutative diagram of double complexes “lying above” the given one and consisting of injective resolutions of \( A_1^*, A_2^*, \ldots \) (in category sense). Take associated simple complexes. We get commutative diagrams with exact rows:

\[
0 \rightarrow I_1^* \rightarrow I_2^* \rightarrow I_3^* \rightarrow 0
\]

\[
0 \rightarrow A_1^* \rightarrow A_2^* \rightarrow A_3^* \rightarrow 0.
\]

(The vertical maps are injective resolutions.)
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a similar primed diagram, and

\[
\begin{array}{ccc}
0 & \rightarrow & I_1^* \\
\downarrow & & \downarrow \\
0 & \rightarrow & I_1'^* \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
\end{array}
\]

Apply the appropriate functor \( T_i \) and repeat the construction. At last we use that \( \{ H^k \}_{k \geq 0} \) is an exact \( \delta \)-functor on \( \mathcal{X}^+ \).

**Proposition 1.2.** Suppose \( T_0, \ldots, T_n \) are covariant functors. In computing \( H^s(T_n, \ldots, T_0)(A) \) one may replace the complex \( I_i^* \) with a complex \( Q_i^* \), where \( Q_i^* \) is \( T_i \)-exact for each \( n \) (i.e., \( R^q T_i(Q_i^*) = 0 \) for \( q \geq 1 \)) provided \( T_i \) is left exact.

**Proof.** Suppose that we have a quism \( q_i : T_{i-1}(I_{i-1}^*) \rightarrow Q_i^* \), where \( Q_i^* \) is \( T_i \)-exact for each \( n \). Choose an injective resolution \( \phi : Q_i^* \rightarrow I_i^* \) in category-sense. Then, \( \int \phi : Q_i^* \rightarrow \int I_i^* \) is a quism and \( T_i(\int I_i^*) \) is the "right" complex to continue the computation with. Furthermore, \( \int T_i(\phi) : T_i(Q_i^*) \rightarrow \int T_i(I_i^*) \) is a quism, because there is a spectral sequence with \( E_2(T_i(\phi)) \Rightarrow \int T_i(\phi) \) and \( E_2(T_i(\phi)) \) is an isomorphism, since \( R^q T_i(Q_i^*) = 0 \) for \( q \geq 1 \) and \( R^q T_i(Q_i^*) = T_i(Q_i^*) \).

2. **Multiple-Ext and Multiple-Tor**

Let \( R_i \), \( i = 1, \ldots, n \) be rings, \( M_i \), \( i = 1, \ldots, n - 1 \) be \( R_{i+1} \rightarrow R_i \)-bimodules, \( V \) a right \( R_n \)-module, and \( U \) a left \( R_1 \)-module. Let

\[
T_i = M_i \otimes_{R_i} : R_i \rightarrow R_{i+1} \rightarrow \text{Mod}, \quad i = 1, \ldots, n - 1
\]

\[
T_n = V \otimes_{R_n} : R_n \rightarrow \text{Z-Mod}.
\]

**Definition.** \( Tor_*(V, M_{n-1}, \ldots, M_1, U) = H_*(T_n, \ldots, T_1)(U) \).

Let

\[
T_i' = \cdot \otimes_{R_{i+1}} R_i \rightarrow R_{n-i}, \quad i = 1, \ldots, n - 1
\]

and

\[
T_n' = \cdot \otimes_{R_1} U: \text{Mod} \rightarrow \text{Z-Mod}.
\]

**Proposition 2.1.** \( Tor_*(V, M_{n-1}, \ldots, M_1, U) = H_*(T_n', \ldots, T_1')(V) \).

**Proof.** Choose a projective resolution \( Q_* \rightarrow V \), and inductively projective resolutions \( Q_i^* \rightarrow Q_i^{i-1} \otimes M_{n-i}, \ i = 1, \ldots, n - 1 \). Choose in the same way a
projective resolution $P^0_* \to U$ and projective resolutions $P^i_\ast \to M_i \otimes P^{i-1}_\ast$
$i = 1, \ldots, n - 1$

$$\text{Tor}_\ast(V, M_{n-1}, \ldots, M_1, U)$$
$$= H_\ast(V \otimes P^{n-1}_\ast) = H_\ast \left( \int Q^0_* \otimes P^{n-1}_\ast \right)$$
$$= H_\ast \left( \int Q^0_* \otimes M_{n-1} \otimes P^{n-2}_\ast \right) = H_\ast \left( \int Q^1_* \otimes P^{n-2}_\ast \right) = \ldots$$
$$= H_\ast(Q^{n-1}_* \otimes U) = H_\ast(T'_n, \ldots, T'_1)(V).$$

Q.E.D.

Let $S_i = \text{Hom}_R(M_i, \cdot): R_{i+1}\text{-Mod} \to R_i\text{-Mod}$, $i = 1, \ldots, n - 1$. Let $V \in R_1\text{-Mod}$, $U \in R_n\text{-Mod}$, $S_0 = \text{Hom}_{R_1}(V, \cdot)$, and $S = \text{Hom}_{R_n}(\cdot, U)$.

**DEFINITION.** $\text{Ext}^\ast(V, M_1, \ldots, M_{n-1}, U) = H_\ast(S_0, \ldots, S_{n-1})(U)$.

**PROPOSITION 2.2.** $\text{Ext}^\ast(V, M_1, \ldots, M_{n-1}, U) = H_\ast(S, T_{n-1}, \ldots, T_1)(V)$.

**Proof.** Similar to the proof of Proposition 2.1. Q.E.D.

**Special cases.**

$n = 1$.

$\text{Tor}(V, U) = \text{usual Tor}$. $\text{Ext}(V, U) = \text{usual Ext}$.

$n = 2$.

$\text{Tor}_\ast(V, M, U) = L_\ast(\cdot \otimes_{R_2} M \otimes_{R_1} \cdot)(V, U)$.

$\text{Ext}^\ast(V, M, U) = R_\ast \text{Hom}_{R_2}(M \otimes_{R_1} \cdot, \cdot)(V, U)$.

**PROPOSITION 2.3.** Let $R$ be a ring, and $M$ an $R$-bi-module. Suppose $\text{Tor}_i^R(M, M^{\otimes r}) = 0$, $i \geq 1$, $r \geq 1$. Then, for all $r$

$\text{Ext}^\ast(V, M(r)M, U) = R_\ast \text{Hom}_R(M^{\otimes r} \otimes \cdot, \cdot)(V, U)$,

$\text{Tor}_\ast(V, M(r)M, U) = L_\ast(\cdot \otimes_R M^{\otimes r} \otimes \cdot)(V, U)$.

**Proof.** $\text{Tor}_i(M, M^{\otimes r} \otimes P) = \text{Tor}_i(M, M^{\otimes r}) \otimes P = 0$, $i \geq 1$, $r \geq 1$, if $P$ is projective. Hence, $M^{\otimes r} \otimes P$ is $(M \otimes_R \cdot)$-exact for $r \geq 1$. Now, apply Proposition 1.2.

**PROPOSITION 2.4.** Let $M$ be a bimodule over a ring $R$. The following are equivalent

(i) $\text{Tor}_i^R(M^{\otimes r}, M) = 0$, $i \geq 1$, $r \geq 1$,
(ii) $\text{Tor}_i^R(M, M^{\otimes r}) = 0$, $i \geq 1$, $r \geq 1$,
(iii) $\text{Tor}_i(M(r)M) = 0$, $i \geq 1$, $r \geq 2$. 
Proof. We prove (i) \(\Leftrightarrow\) (iii) ((ii) \(\Leftrightarrow\) (iii) is analogous). There is a spectral sequence:

\[
E_2^{p,q} = \text{Tor}_p(\text{Tor}_q(M(r)M), M) \twoheadrightarrow \text{Tor}_n(M(r+1)M)
\]

Suppose (iii) is true. Then, \(\text{Tor}_i^R(M, M) = 0\) for \(i \geq 1\), and for \(r \geq 2\) the spectral sequence degenerates and gives

\[
\text{Tor}_n(M^\otimes r, M) \simeq \text{Tor}_n(M(r+1)M) = 0, \quad (n \geq 1).
\]

Suppose (i) is true. We prove (iii) by induction over \(r\). For \(r = 2\) it is trivial. Suppose \(\text{Tor}_i^R(M(r)M) = 0\) for \(q \geq 1\). Then, the spectral sequence degenerates and we get \(\text{Tor}_n(M(r+1)M) \simeq \text{Tor}_n(M^\otimes r, M) = 0, (n \geq 1)\).

**Proposition 2.5.** Let \(R\) be a ring and \(M\) a bimodule over \(R\). Suppose \(M\) has a resolution \(Q_* \rightarrow M\) consisting of bimodules \(Q_i\) that are flat as right or left \(R\)-modules (cf. Palmér-Roos [11]). Let \(U, V \in \text{R-Mod}, V' \in \text{Mod-R}\). Let \(P_* \rightarrow V, P'_* \rightarrow V'\) be projective resolutions and \(U \rightarrow I^*\) an injective resolution. Then,

\[
\text{Ext}^*(V, M(r)M, U) = H_\ast \int \text{Hom}_R(Q^\otimes_r \otimes P_*, I^*)
\]

and

\[
\text{Tor}_*(V', M(r)M, U) = H_\ast \int (P'_* \otimes Q^\otimes_r \otimes P_*)
\]

Proof. If \(Q_i\) are right \(R\)-flat, the first formula is just the definition (use the adjoint formula \(\text{Hom}(Q \otimes P, I) = \text{Hom}(P, \text{Hom}(Q, I))\) and observe that \(\text{Hom}(Q, I)\) is injective). The second formula, and the first formula when \(Q_i\) are left \(R\)-flat, follows from the Propositions 1.2, 2.1, and 2.2.

3. **On \(\text{Ext}_{R \times M}((U, f), (V, 0))\)**

Let \(R\) be a ring, and \(M\) a bimodule over \(R\). The trivial extension \(R \times M\) is as abelian group \(R \oplus M\) and the multiplication is given by \((r, m)(r', m') = (rr', rm' + mr')\). A left \(R \times M\)-module is given by a left \(R\)-module \(U\) and a map \(f: M \otimes U \rightarrow U\) such that \(f \circ (M \otimes f) = 0\). If \(u \in U\) and \((r, m) \in R \times M\), then \((r, m)u = ru + f(m \otimes u)\). A map between two modules \((U, f)\) and \((V, g)\) is a map \(\phi: U \rightarrow V\) (\(R\)-homomorphism) such that

\[
\begin{align*}
M \otimes U & \xrightarrow{M \otimes f} M \otimes V \\
\downarrow f & \downarrow g \\
U & \xrightarrow{\phi} V,
\end{align*}
\]

commutes.
(0, M) is an ideal in R x M, (0, M)^2 = 0, and we have a ring epimorphism
R x M → R (≃ R x M/(0, M)) given by (r, m) → r. This gives an
"adjoint-situation":

\[ R \times M \text{-Mod} \xrightarrow{C} R \text{-Mod}, \]

where C is the tensor product functor and Z is the forgetful functor.
\( C(U, f) \simeq (U, f)/(0, M)(U, f) \approx U/\text{im } f - \text{coker } f, \ Z(V) = (V, 0). \)

Hence, we get the following adjoint-formula:

\[ \text{Hom}_{R \times M}(U, f), (V, 0)) = \text{Hom}_R(\text{coker } f, V). \]

**Standard resolutions in R x M-Mod**

There is another ring-homomorphism relating R and R x M, namely,
\( R \to R \times M \) given by \( r \mapsto (r, 0) \). This gives the adjoint-situation:

\[ R \text{-Mod} \xrightarrow{T} R \times M \text{-Mod}, \]

where \( TU = (R \times M) \otimes_R U = ((M \otimes U) \oplus U, (0, 0)) \) and \( D(V, g) = V \).

Hence, \( \text{Hom}_{R \times M}(TU, (V, g)) = \text{Hom}_R(U, V) \).

Let \((U, f) \in R \times M\text{-Mod}. \) Write \( U \) as a quotient of a projective \( R\) module \( P_0 \). By \( \text{Hom}_R(P_0, U) = \text{Hom}_{R \times M}(TP_0, (U, f)) \), we get a map \( TP_0 \to (U, f) \) and this map is surjective. Let \((K, h)\) be the kernel of \( TP_0 \to (U, f) \). Write \( K \) as a quotient of a projective \( R\) module \( P_1 \). We get an exact sequence \( TP_1 \to TP_0 \to (U, f) \to 0 \). Repeating the construction, we get a projective resolution \((K_*, g_*) \to (U, f)\), where for each \( n, (K_n, g_n) = TP_n \) and \( P_n \) is a projective \( R\)-module (observe that it would be misleading to write \((K_*, g_*)\) as \( TP_*, \) cf. Section 4). The following formula is fundamental \((V \in \text{Mod}-R)\):

\[ \text{Ext}^n_{R \times M}((U, f), (V, 0)) = H^n \text{Hom}_{R \times M}((K_*, g_*), (V, 0)) \]
\[ = H^n \text{Hom}_R(\text{coker } g_*, V) \]
\[ = H^n \text{Hom}_R(P_*, V). \] (1)

Similarly, we get a formula for \( \text{Tor}^n_{R \times M}(V, 0), (U, f)) \), \( V \in \text{Mod}-R\):

\[ (V, 0) \otimes_{R \times M} (U, f) = (V \otimes_R R) \otimes_{R \times M} (U, f) = V \otimes_R (R \otimes_{R \times M} (U, f)) \]
\[ - \ V \otimes_R \text{coker } f. \]

Hence,

\[ \text{Tor}^n_{R \times M}(V, 0), (U, f)) = H_n(V \otimes_R P_*) \] (2)

**Remark.** \( 0 \to (\text{im } g, 0) \to (V, g) \to (\text{coker } g, 0) \to 0 \) is an exact sequence, whence, \( \text{hd}(U, f) \leq n \iff \text{Ext}_{R \times M}^{n+1}((U, f), (V, 0)) = 0 \) for all \( V \in \text{R-Mod}, \) and \( \text{lgldim } R \times M \leq n \iff \text{Ext}_{R \times M}^{n+1}((U, 0), (V, 0)) = 0 \) for all \( U, V \in \text{R-Mod}. \)
4. CONSTRUCTION OF A SPECTRAL SEQUENCE CONVERGING TO \( \text{Ext}^n_{R \otimes M}((U, f), (V, 0)) \)

Notations are as above. \((K_*, g_*) \to (U, f)\) is described by the following diagram:

\[
\begin{array}{c}
K_*: \cdots \longrightarrow M \otimes P_1 \oplus P_1 \longrightarrow M \otimes P_0 \oplus P_0 \longrightarrow U \longrightarrow 0 \\
\end{array}
\]

It follows that the differential on \(K_*\) is of the form \((M \otimes d, h)\), where \(d\) is the differential on \(P_*\) and \(h: P_* \to S(M \otimes P_*)\) is some map of complexes. Hence, we get an exact sequence of complexes: \(0 \to M \otimes P_* \to K_* \to P_* \to 0\). Moreover, \(K_* \to U\) is a resolution of \(U\), but not projective in general. Therefore, we want to replace \(M \otimes P_*\) by a projective complex.

**Lemma 4.1.** Suppose \(0 \to A_* \to B_* \to C_* \to 0\) is a semisplit exact sequence of complexes \((A^*, B^*, C^* \in \mathcal{K}^-)\), i.e., \(B_*\) is the “mapping cone” of a map \(\phi: C_* \to SA_*\). Suppose \(p^A: P_*^A \to A_*\) and \(p^C: P_*^C \to C_*\) are projective resolutions. Then, there is a projective resolution \(p^B: P_*^B \to B_*\) such that

\[
\begin{array}{c}
0 \longrightarrow P_*^A \longrightarrow P_*^B \longrightarrow P_*^C \longrightarrow 0 \\
\end{array}
\]

commutes with exact rows.

Further, suppose (*) is given, with \(P_*^B = \text{the mapping cone of } \phi\), then

\[
\begin{array}{c}
P_*^C \xrightarrow{\delta} SP_*^A \\
\downarrow \phi \quad \downarrow \phi \\
C_* \xrightarrow{\phi} SA_* \\
\end{array}
\]

commutes modulo homotopy.

**Proof.** Corollary 1.2 gives us the map \(\phi\) and a homotopy \(j: \phi p^C \to Sp^A\phi\). Define \((P_*^B, d_n)\) as

\[
\begin{bmatrix}
P_*^A \oplus P_*^C, & \begin{pmatrix} d_n' \\ 0 & d_n \end{pmatrix} \\
\end{bmatrix}
\]

\((d')\) is the differential of \(P_*^A\), \(d''\) of \(P_*^C\), and

\[
p^B = \begin{pmatrix} p^A & j \\ 0 & p^C \end{pmatrix}.
\]
A simple computation shows that, "$p^B$ is a chain map" is equivalent to "$j$ is a homotopy from $\phi p^C$ to $Sp^A \phi$". $p^B$ is a quism according to the 5-lemma. Conversely, suppose $(*)$ is given. From the commutativity of $(*)$ it follows that

$$p^B = \begin{pmatrix} p^A & j \\ 0 & p^C \end{pmatrix}$$

for some $j$, and from above, $j$ is a homotopy: $\phi p^C \rightarrow Sp^A \phi$. Q.E.D.

Choose a projective resolution of $M \otimes P_\ast$, $p^0: X_\ast^0 \simeq M \otimes P_\ast$ and inductively projective resolutions, $p^n: X_\ast^n \simeq M \otimes X_\ast^{n-1}$. According to Lemma 4.1, we may construct $K_\ast^\ast$ such that

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_\ast^0 & \longrightarrow & K_0^\ast & \longrightarrow & P_\ast & \longrightarrow & 0 \\
\downarrow p^0 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M \otimes P_\ast & \longrightarrow & K_\ast & \longrightarrow & P_\ast & \longrightarrow & 0.
\end{array}$$

commutes with exact rows. Since $K_\ast \rightarrow U$ is a resolution, $K_0^\ast$ is a projective resolution of $U$. Inductively, we construct $K_n^\ast$ for $n \geq 1$, such that for $n \geq 1$ ($X_\ast^{-1} = P_\ast$):

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_\ast^n & \longrightarrow & K_n^\ast & \longrightarrow & X_\ast^{n-1} & \longrightarrow & 0 \\
\downarrow p^n & & \downarrow & & \downarrow & & \downarrow p^{n-1} & \\
0 & \longrightarrow & M \otimes X_\ast^{n-1} & \longrightarrow & M \otimes K_{n-1}^\ast & \longrightarrow & M \otimes X_\ast^{n-2} & \longrightarrow & 0,
\end{array}$$

commutes with exact rows.

$K_\ast^\ast$ becomes a double complex, by $K_\ast^\ast \rightarrow X_\ast^{n-1} \rightarrow K_{n-1}^\ast$. We have the following picture:
The diagram is commutative to the right, since
\[
\begin{array}{c}
\hat{K}_{1*} \rightarrow X_{*}^0 \rightarrow \hat{K}_{0*} \\
\downarrow \downarrow \downarrow \\
M \otimes \hat{K}_{0*} \rightarrow M \otimes P_{*} \rightarrow K_{*} \\
\downarrow \downarrow \\
M \otimes K_{*} \rightarrow M \otimes P_{*} \rightarrow K_{*} \\
\downarrow \\
M \otimes U \rightarrow U \\
\end{array}
\]
is commutative, and suppose
\[
\begin{array}{c}
\hat{K}_{n*} \rightarrow X_{*}^{n-1} \rightarrow \hat{K}_{n-1*} \\
\downarrow \\
M \otimes U \rightarrow M \otimes X_{*}^{n-1} \rightarrow M \otimes K_{n-1*} \\
\downarrow \\
M \otimes U \rightarrow M \otimes U \\
\end{array}
\]
is commutative, then
\[
\begin{array}{c}
\hat{K}_{n+1*} \rightarrow X_{*}^{n} \rightarrow \hat{K}_{n*} \\
\downarrow \\
M \otimes \hat{K}_{n*} \rightarrow M \otimes X_{*}^{n-1} \rightarrow M \otimes K_{n-1*} \\
\downarrow \\
M \otimes \hat{K}_{n*} \rightarrow M \otimes U \\
\end{array}
\]
is commutative.

The augmentation $\hat{K}_{**} \rightarrow P_{*}$ gives rise to quisms:
\[
\int \text{Hom}(\hat{K}_{**}, V) \simeq \text{Hom}(P_{*}, V), \quad V \in R-\text{Mod},
\]
and
\[
\int V \otimes \hat{K}_{**} \simeq V \otimes P_{*}, \quad V \in \text{Mod} - R.
\]

The construction of $\hat{K}_{p*}$ and the formulae (1) and (2) will now give us two spectral sequences:
\[
\begin{aligned}
\tilde{E}_{1}^{pq} &= \text{Ext}^q(U, M(p)M, V) \Rightarrow \text{Ext}^n_{R \otimes M}((U, f), (V, 0)), \quad U, V \in R-\text{Mod}, \\
\hat{E}_{1}^{pq} &= \text{Tor}_q(V, M(p)M, U) \Rightarrow \text{Tor}^n_{R \otimes M}((V, 0), (U, f)), \quad U \in R-\text{Mod}, \\
& \quad V \in \text{Mod} - R.
\end{aligned}
\]
Remark 1. We may regard a module over $R \times M$ as a pair $[V, g]$, where $V \in R$-Mod and $g : V \to \text{Hom}_R(M, V)$ satisfying $\text{Hom}(M, g) \otimes = 0$. If we do so, we can dualize the construction of standard resolutions and the complex $\mathcal{K}_{n*}$. Starting with the adjoint formula $\text{Hom}_{R \times M}((U, 0), (V, g)) = \text{Hom}_R(U, \ker g)$, we get the spectral sequence:

$$E_{pq}^q = \text{Ext}^q(U, M(p)M, V) \Rightarrow \text{Ext}^n_{R \times M}((U, 0), (V, g)).$$

Remark 2. It is easy to see (by using Corollary 1.2), that a map $f : M \otimes U \to U$ induces maps $\text{Tor}_i(V, M(p)M, U) \to \text{Tor}_i(V, M(p-1)M, U)$ for every $p \geq 1, q \geq 0$. This is the first differential $E_{pq}^1 \Rightarrow E_{p-1q}^1$ . (There is of course a similar interpretation of the first differential of (3).)

5. DEGENERATION OF THE SPECTRAL SEQUENCES (3) AND (4) IF $f = 0$

When we have proved that all differentials are zero if $f = 0$, the main result of this paper will follow (cf. remark at the end of Section 3).

Theorem. $\text{lglldim } R \times M \leq n \iff \text{Ext}^q(U, M(p)M, V) = 0$, for all $p, q \geq 0$ such that $p + q = n + 1$, and for all $U, V \in R$-Mod.

$\text{wglldim } R \times M \leq n \iff \text{Tor}_i(V, M(p)M, U) = 0$, for all $p, q \geq 0$ such that $p + q = n + 1$, all $U \in R$-Mod and all $V \in \text{Mod}-R$.

Corollary 5.1. $\text{lglldim } R \times M < \infty \iff \text{lglldim } R < \infty$, and $\exists N$: $\text{Tor}_i(M(N)M) = 0$, for all $i \geq 0$.

$\text{wglldim } R \times M < \infty \iff \text{wglldim } R < \infty$, and $\exists N$: $\text{Tor}_i(M(N)M) = 0$, for all $i \geq 0$.

Proof of Corollary 5.1. We prove the first assertion (the other is similar).

$\Rightarrow$ Suppose $\text{lglldim } R \times M = n < \infty$. The theorem implies that $\text{Ext}^{n+1}(U, V) = 0$, for all $U, V \in R$-Mod, hence, $\text{lglldim } R \leq n$. Put $U = R$ and for $i \geq 0$ choose an injective $V_i$ such that $\text{Tor}_i(M(n+1)M) \subseteq V_i$. The theorem implies that

$$\text{Ext}^i(R, M(n+1)M, V_i) = 0, \quad \text{for all } i \geq 0 \Rightarrow$$
$$\text{Hom}(\text{Tor}_i(M(n+1)M), V_i) = 0, \quad \text{for all } i \geq 0 \Rightarrow$$
$$\text{Tor}_i(M(n+1)M) = 0, \quad \text{for all } i \geq 0.$$

$\Leftarrow$ Suppose $\text{lglldim } R = t < \infty$ and $\text{Tor}_i(M(N)M) = 0$ for all $i \geq 0$.

Claim. $\text{lglldim } R \times M \leq Nt + N - 1$.
Proof. \( \text{Tor}_p(\text{Tor}_q(M(N)M), M) \Rightarrow \text{Tor}_n(M(N + 1)M) \), hence,
\( \text{Tor}_i(M(j)M) = 0, \) for all \( i \geq 0, \ j \geq N. \)
\( \text{Tor}_p(M, \text{Tor}_q(M(j)M, U)) \Rightarrow \text{Tor}_n(M(j + 1)M, U), \ U \in R-\text{Mod}. \)

With induction, it follows that, for \( j \geq 1, \)
\( \text{Tor}_q(M(j)M, U) = 0, \) for all \( q > j, \) and all \( U \in R-\text{Mod}. \)
\( \text{Ext}^p(\text{Tor}_q(M(r)M, U), V) \Rightarrow \text{Ext}^{p+q}(U, M(r)M, V), \ U, V \in R-\text{Mod}, \)
hence, \( p + q + r > Nt + N - 1 \)
\( \Rightarrow r > N - 1, \) or \( p > t, \) or \( (q > (N - 1)t, \) and \( r \leq N - 1). \)

In all these three cases, we conclude from above that
\( \text{Ext}^{p+q}(U, M(r)M, V) = 0. \)

From the theorem we get \( \lgldim R \times M \leq Nt + N - 1. \)

**Corollary 5.2** (cf. [11, Theorem 4]). Suppose \( \text{Tor}_i(M\otimes r, M) = 0, \) for all \( i \geq 1, \ r \geq 1. \) Then, \( \lgldim R \times M \leq n \Rightarrow \text{Ext}^p(M\otimes r \otimes \cdot, -) = 0, \) for all \( p, q \geq 0, \) such that \( p + q = n + 1. \)

\( \text{wlgldim} R \times M \leq n \Rightarrow \text{Ext}^p(\cdot \otimes M\otimes r \otimes -) = 0, \) for all \( p, q \geq 0 \) such that \( p + q = n + 1. \)

**Proof.** Propositions 2.3 and 2.4.

**Corollary 5.3.** \( \lgldim R \times M \leq 1 \iff \)
(a) \( \lgldim R \leq 1, \) (b) \( M_R \text{ flat}, \)
(c) \( M \otimes M = 0, \) (d) \( M \otimes U \text{ projective for all} U \in R-\text{Mod}. \)

(This was first proved by Reiten [13, Proposition 2.3.3].)

**Proof.** \( \lgldim R \times M \leq 1 \iff \)
\( \text{Ext}^q(\cdot, M(p)M, -) = 0, \) for all \( p, q \geq 0, \) such that \( p + q = 2. \)
\( p = 0, q = 2: \text{Ext}^2(\cdot, -) = 0 \Rightarrow \lgldim R \leq 1, \)
\( p = 1, q = 1: \text{Ext}^1(M \otimes \cdot, -) = 0, \)
\( E_2^{pq} = \text{Ext}^p(\text{Tor}_q(M, \cdot), -) \Rightarrow R^n(M \otimes \cdot, -), \)
\( E_2^{10} = E_2^{10}, \quad E_2^{10} = 0 \Rightarrow E_2^{20} = 0 \Rightarrow E_2^{01} = E_2^{01}. \)
Hence, \( R^i(M \otimes \cdot, -) = 0 \Leftrightarrow \text{Ext}^i(M \otimes \cdot, -) = 0 \), and
\[
\text{Hom}(\text{Tor}_i(M \otimes \cdot), -) = 0 \Leftrightarrow M \otimes U, \text{projective for all } U \in R\text{-Mod and } M_R \text{ flat.}
\]

\( p = 2, q = 0: M \otimes M = 0 \). Q.E.D.

It is also easy to derive equivalent conditions to \( \text{lgldim } R \times M < 2 \).

It turns out to be eight conditions, and these will be found in [12, Theorem 41], where the proof, however, is quite different.

To prove the theorem, we start with a general lemma concerning spectral sequences arising from double complexes. This is just another formulation of the definitions in Godement [6], more suitable for the case of double complexes. It also can be seen as a direct translation of the definitions in [3]. Let \( (K^{p,q}, d', d'')_{p,q \in \mathbb{Z}} \) be a double complex with \( d' d'' + d'' d' = 0 \) and \( (K^n, d')_{n\in \mathbb{Z}} \) the associated simple complex. Consider the filtration
\[
F^pK = \bigsqcup_{r \geq p} K^{rs} \quad (K = \bigsqcup_{n \in \mathbb{Z}} K^n).
\]

Godement makes the following definitions:

\[(r \geq 1) \quad Z^{pq}_r = \{x \in F^pK^{p+q}; dx \in F^{p+r}K^{p+q+1}\},
\]
\[
E^p_r = Z^{pq}_r / Z^{pq}_{r-1} + dZ^{r+1, q-r-2},
\]

\( d^p_r \) induced by \( d: Z^{pq}_r \to Z^{p+r, q-r+1} \). Now, we define \( \bar{Z}^{pq}_r, \bar{B}^{pq}_r, \bar{E}^{pq}_r, \bar{d}^{pq}_r \) as follows:

\[
\bar{Z}^{pq}_r = \{x_0 \in K^{pq}; \exists x_i \in K^{p+i}, 1 \leq i \leq r - 1, \text{such that } d \left( \sum_{i=0}^{r-1} x_i \right) \in F^{p+r}K\},
\]

\[
\bar{B}^{pq}_r = \{x_0 \in K^{pq}; \exists y_i \in K^{p-i}, 0 \leq i \leq r - 1, \text{such that } d \left( \sum_{i=0}^{r-1} y_i \right) - x_0 \in F^{p+1}K\},
\]

\[
\bar{E}^{pq}_r = \bar{Z}^{pq}_r / \bar{B}^{pq}_r,
\]

\[
\bar{d}^{pq}_r(x_0) = d'x_{r-1} \text{ mod } \bar{B}^{p+r, q-r+1}_r.
\]

**Lemma 5.1.** \((\bar{E}^{pq}_r, \bar{d}^{pq}_r)\) and \((\bar{E}^{pq}_r, \bar{d}^{pq}_r)\) are isomorphic as complexes.

\[
\ker \bar{d}^{pq}_r = \bar{Z}^{pq}_{r+1} / \bar{B}^{pq}_r, \quad \text{im} \bar{d}^{p-r, q+r-1}_r = \bar{B}^{pq}_{r+1} / \bar{B}^{pq}_r.
\]
Proof. There is a natural map $f: Z^p_r \to \bar{E}^p_r$, indeed, $f(\sum_{i<0} x_i) = \bar{x}_0$.

$$f(Z^{p+1}_{r-1} q-1 + dZ^{p-r+1}_{r-1} q+r-1) = 0$$

implies that $f$ induces a map $\bar{f}: E^p_r \to \bar{E}^p_r$. It is evident that $\bar{f}$ is surjective. Suppose $\sum_{i<0} x_i \in Z^p_r$ and $f(\sum_{i<0} x_i) = 0$. Then, there exists $y_i$ such that $\sum y_i \in F^{p-r+1}K$ and $d(\sum y_i) = -x_0 \in F^{p+1}K$. $\sum x_i = (\sum x_i - d(\sum y_i)) + d(\sum y_i)$. The first term is in $Z^{p+1}_{r-1} q-1$, and the second is in $dZ^{p-r+1}_{r-1} q+r-2$. It is evident that $\bar{f}$ commutes with the differentials. The rest of the lemma is a simple verification. Q.E.D.

We now construct a very special type of double complexes, which will be of interest for our purposes.

Let $\{Y_i\}_{i \in \mathbb{Z}}$ be a family of complexes, and for each $i$, $\phi_i: Y_i \to SY_{i+1}$ a map of complexes. We get a double complex $L^{**}$:

$$L^{**} = \left[ Y^*_p \oplus Y^*_{p+1}, \left( \begin{array}{cc} d_p & \phi_p \\ 0 & d_{p+1} \end{array} \right) \right]$$

(i.e. $L^{p*}$ is the mapping cone of $\phi_p$) and $L^{p*} \to L^{p+1*}$ is given by the natural composition:

$$Y^*_p \oplus Y^*_{p+1} \to Y^*_{p+1} \to Y^*_p \oplus Y^*_{p+2}.$$

Lemma 5.2. Consider the $p$-filtration for $L^{**}$ and suppose it is regular (e.g., $L^{pq} = 0$ for $q < 0$), then, $d^{pq}_r = 0$, for all $r \geq 1$, and all $p, q$ if $H^0(\phi_p)$, mono for all $p, q$.

Proof. Without any loss of generality, we put $p = q = 0$ and derive an equivalent condition to $d^{pq}_r = 0$ for all $r \geq 1$. By Lemma 5.1, this is equivalent to $Z_1^{00} = Z_2^{00}$ for all $r \geq 1$.

\[
\begin{array}{c}
\left( \begin{array}{cc} d_0 & \phi_0 \\ 0 & d_1 \end{array} \right)
\end{array}
\]

\[
\begin{array}{c}
Y^0_0 \oplus Y^0_1 \xrightarrow{(0 \ 1)} Y^0_1 \oplus Y^0_2
\end{array}
\]

\[
\begin{array}{c}
Y^{-1}_1 \oplus Y^{-1}_2 \xrightarrow{(0 \ 1)} Y^{-1}_2 \oplus Y^{-1}_3
\end{array}
\]
Suppose \((x_0, y_0) \in \mathbb{Z}^{20}\). This means that
\[
\begin{align*}
d_0x_0 + \phi_0^1 y_0 &= 0 \\
d_1 y_0 &= 0
\end{align*}
\]
i.e., \(H(\phi^1)(\bar{y}_0) = 0\).

Suppose \((x_0, y_0) \in \mathbb{Z}^{20}\). Then, there exists \((x_1, y_1) \in Y_{1}^{-1} \oplus Y_2^{-1}\) such that
\[
\begin{pmatrix}
d_1 \\
\phi_1 y_1 \\
d_2
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
y_1 \\
0
\end{pmatrix},
i.e., \phi_1 y_1 = y_0 - d_1 x_1.
\]
i.e., there is \(\bar{y}_1 \in H(Y_2^{-1})\) such that \(H(\phi^1)(\bar{y}_1) = \bar{y}_0\).

The generalization is obvious: \((x_0, y_0) \in \mathbb{Z}^{20}\) \(\Rightarrow\) there is \(\bar{y}_{r-1} \in H(Y_{r-r+1})\) such that \(H(\phi^1) \cdots H(\phi^{r-1})(\bar{y}_{r-1}) = \bar{y}_0\) (and \((x_0, y_0) \in \mathbb{Z}^{20}_1\)). But the filtration is regular, hence, \(Y_{r-r+1} = 0\) for \(r\) large enough. Hence, \(Z^{20} = \mathbb{Z}^{20}\) for all \(r \Rightarrow H(\phi^1)\) mono. Q.E.D.

The complex \(\text{Hom}(\mathcal{K}, V)\) that gives rise to the spectral sequence (3) is of the type, that is treated in Lemma 5.2.

Let \(\phi^n: X^{n-1}_* \rightarrow SX_*^{n} \ n \geq 0 \ (X_*^0 = P_*\) be the map that defines \(\mathcal{K}\) (as the mapping cone of \(\phi^n\)). We must prove that \(H(\phi^n, V)\) is mono for every \(n\). \(\mathcal{K}_0^* \rightarrow U \rightarrow 0\) is a projective resolution. Hence, \(H_i(\mathcal{K}_0^*) = 0\) for \(i \geq 1\). \(0 \rightarrow X_*^0 \rightarrow \mathcal{K}_0^* \rightarrow P_* \rightarrow 0\) is exact. This gives a long exact sequence:

\[
\begin{array}{c}
\rightarrow H_i(\mathcal{K}_0^*) \rightarrow H_i(P_*^0) \xrightarrow{H_i(\phi^n)} H_{i-1}(X_*^0) \rightarrow H_{i-1}(\mathcal{K}_0^*) \rightarrow \cdots
\end{array}
\]

Hence, \(H_i(\phi^n)\) iso for \(i \geq 2\).

\[
\begin{array}{c}
0 \rightarrow H_i(P_*^0) \xrightarrow{H_i(\phi^n)} H_0(X_*^0) \rightarrow H_0(\mathcal{K}_0^*) \rightarrow H_0(P_*^0)
\end{array}
\]
is exact, and

\[
\begin{array}{c}
\mathcal{K}_1^* \rightarrow X_*^0 \rightarrow \mathcal{K}_0^*
\end{array}
\]

commutes, so the composition \(H_0(\mathcal{K}_1^*) \rightarrow H_0(X_*^0) \rightarrow H_0(\mathcal{K}_0^*)\) is equal to \(f\). But \(H_0(\mathcal{K}_1^*) \rightarrow H_0(X_*^0)\) is epi, and therefore, \(f = 0\) implies that \(H_0(X_*^0) \rightarrow H_0(\mathcal{K}_0^*)\) is zero. From the exactness of the sequence above we get \(H_1(\phi^n)\) iso.

To summarize, we have for \(\phi^n: \phi^n = 0\) and \(H_i(\phi^n)\) iso for \(i \geq 1\).

\(\text{Hom}(\phi^n, V): \text{Hom}(X_*^{n-1}, V) \rightarrow \text{Hom}(SX_*^n, V)\).

Let \(V \rightarrow I_0^*\) be an injective resolution of \(V\), and define inductively \(I_i^*\)
as an injective resolution of $\text{Hom}(M, I^*_{i-1})$ $i \geq 1$. By Lemma 4.1 we have a diagram, commutative modulo homotopy:

\[
\begin{array}{ccc}
X^{n-1}_{*} & \xrightarrow{\phi^n} & SX^n_{*} \\
\downarrow & & \downarrow \\
M \otimes X^{n-2}_{*} & \xrightarrow{M \otimes d^{n-1}} & M \otimes SX^{n-1}_{*}, \quad n \geq 1.
\end{array}
\]

This yields the following diagrams, all commutative modulo homotopy:

\[
\begin{array}{ccc}
\text{Hom}(X^{n-1}_{*}, V) & \xleftarrow{\text{Hom}(\phi^n, V)} & \text{Hom}(SX^n_{*}, V) \\
\uparrow & & \uparrow \\
\int \text{Hom}(X^{n-1}_{*}, I^*_0) & \xleftarrow{} & \int \text{Hom}(SX^n_{*}, I^*_0) \\
\uparrow & & \uparrow \\
\int \text{Hom}(M \otimes X^{n-2}_{*}, I^*_0) & \xleftarrow{} & \int \text{Hom}(M \otimes SX^{n-1}_{*}, I^*_0) \\
\uparrow & & \uparrow \\
\int \text{Hom}(X^{n-2}_{*}, \text{Hom}(M, I^*_0)) & \xleftarrow{\text{Hom}(\phi^n, \text{Hom}(M, I^*_0))} & \int \text{Hom}(SX^{n-1}_{*}, \text{Hom}(M, I^*_0)) \\
\uparrow & & \uparrow \\
\int \text{Hom}(X^{n-2}_{*}, I^*_1) & \xleftarrow{} & \int \text{Hom}(SX^{n-1}_{*}, I^*_1) \\
\uparrow & & \uparrow \\
& \vdots & \\
\int \text{Hom}(P^*_*, I^*_n) & \xleftarrow{\text{Hom}(\phi^0, I^*_n)} & \int \text{Hom}(SX^{n}_*, I^*_n).
\end{array}
\]

It follows that it is enough to prove that $H \int \text{Hom}(\phi^0, I^*_n)$ is mono, and this will follow from

**Lemma 5.3.** Let $\phi: K^{**} \to L^{**}$ be a map of double complexes in the first quadrant. Consider the spectral sequence arising from the $q$-filtration. Suppose $E^0_q(\phi)$ is mono, and $E^p_q(\phi)$ is an isomorphism for $p > 0$ and every $q$. Then, $H(\int \phi): H \int K^{**} \to H \int L^{**}$ is mono.

**Proof.** We use induction over $r$ to prove that

\[
\begin{cases}
E^q_r(\phi) \text{ mono} & \text{all } q, \\
E^p_r(\phi) \text{ iso } p > 0
\end{cases}
\]
for $r \geq 2$. For every $r \geq 2$, we have the commutative diagram:

\[
\begin{array}{ccc}
E_r^{p+q-1} \cong (K) & \longrightarrow & E_r^{pq}(K) \\
\downarrow \text{iso} & & \downarrow \text{mono} \\
E_r^{p+q-1} \cong (L) & \longrightarrow & E_r^{pq}(L)
\end{array}
\]

By diagram chasing, we get:

\[
E_r^{pq}(\phi) \text{ mono } \Rightarrow E_{r+1}^{pq}(\phi) \text{ mono}.
\]

\[
E_r^{pq}(\phi) \text{ iso } \Rightarrow E_{r+1}^{pq}(\phi) \text{ iso}.
\]

Hence, $E_r^{pq}(\phi)$ is mono for $p = 0$, and iso for $p > 0$. Since the filtration is regular (finite) it follows that $H(J^\infty \phi)$ is mono.

This ends the proof of the degeneration of (3). The proof of the same for (4) is dual. In that case, we consider the double complex $V \otimes \mathcal{K}_{**}$ and use the dual forms of Lemmas 5.2 and 5.3.

6. An Application

Let $(R, \mathfrak{m})$ be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$, and residue-field $k = R/\mathfrak{m}$. The Poincaré-series for $R$ is defined by $P_R(z) = \sum_{i=0}^{\infty} B_i z^i$, where $B_i = \dim_k \text{Tor}_i^R(k, k)$. For a finitely generated $R$-module $M$, we define $P_R^M(z) = \sum_{i=0}^{\infty} B_i^M z^i$, where $B_i^M = \dim_k \text{Tor}_i^R(k, M)$. Observe that $R \times M$ is again a commutative noetherian local ring with the same residue-field $k$, and $M$ acts trivially on $k$ as $R \times M$-module. Thus, $k$ is of the form $(k, 0)$ so the spectral sequence (4) degenerates (with $U = V = k$ and $f = 0$). Hence,

\[
\dim_k \text{Tor}_n^{R\times M}(k, k) = \dim_k \text{GrTor}_n^{R\times M}(k, k) = \sum_{p+q=n} \dim_k \text{Tor}_q(k, M(p)M, k).
\]

Choose minimal projective resolutions $P_*$ of $k$, and $P_*^M$ of $M$.

\[
\text{Tor}_q(k, M(p)M, k) = H_q(k \otimes P_*^M \otimes P_+, k) = (k \otimes P_*^M \otimes P_+)_{q}.
\]

Thus, $\dim_k \text{Tor}_n^R(k, M(p)M, k) = [(P_R^M(z))^p \cdot P_R(z)]_n$, where $[\cdot]_q$ means the $q$th coefficient in the corresponding series.

\[
\dim_k \text{Tor}_n^{R\times M}(k, k) = \sum_{p=0}^{n} [z^p(P_R^M(z))^p P_R(z)]_n.
\]
Hence,
\[ P_{R X M}(x) = P_R(x)(1 + x P_R^M(x) + (x P_R^M(x))^2 + \cdots) = P_R(x)(1 - z P_R^M(x))^{-1}. \]

**APPENDIX: NOMENCLATURE**

- \( R \) = a ring with identity.
- \( R\text{-Mod} \) = the category of left \( R \)-modules.
- \( \text{Mod-}R \) = the category of right \( R \)-modules.
- \( \mathcal{C} \) = an abelian category.

\( \mathcal{K} = \mathcal{K}(\mathcal{C}) \) = the category of complexes over \( \mathcal{C} \) with differential of degree +1.

\( \mathcal{K}^+(\mathcal{K}^-) \) = the full subcategory of \( \mathcal{K} \) consisting of complexes \( K^* \) with \( K^n = 0 \) for \( n < 0 \) \((n > 0)\).

Let \( X^*, Y^*, Z^* \) be objects in \( \mathcal{K} \). \( 0 \rightarrow X^* \rightarrow Y^* \rightarrow Z^* \rightarrow 0 \) is called *split exact* if \( 0 \rightarrow X^n \rightarrow Y^n \rightarrow Z^n \rightarrow 0 \) is split exact for every \( n \).

\( SX^* \) = the "suspension" of \( X^* \), i.e., \( (SX^*)^n = X^{n-1} \), \( d_{SX^*} = -d_{X^*} \).

A map between complexes \( f \), is called a quasi-isomorphism or *quism* if \( H(f) \) is iso.

\( f : X^* \cong Y^* \) = the notation for a quism.

\( f : X^* \hookrightarrow Y^* \) = called an *injective resolution* if \( Y^n \) is injective for every \( n \).

An injective resolution in \( \mathcal{K}^+ \) is called an injective resolution in *category-sense*.

\( [X^*, Y^*] = \) the set of homotopy classes of chain maps of degree zero.

\( [f] = \) the homotopy class of \( f : X^* \rightarrow Y^* \).

\( \int K^{**} = \) the associated simple complex to a double complex \( K^{**} \).

\( M(\cdot)p\) = short for \( M, M, \ldots, M \) with \( p \) copies of \( M \).

\( M \otimes^r \) = short for \( M \otimes \cdots \otimes M \), \( r \) factors.

\( \mathbb{N} = \) the set of natural numbers

\( \mathbb{Z} = \) the ring of integers.

\( = \) the identity map.

**REFERENCES**

GLOBAL HOMOLOGICAL DIMENSIONS


