Base-normality and product spaces

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Dedicated to Professor Takao Hoshina on his 60th birthday

Abstract

We introduce the notion of base-normality, which is a natural generalization of base-paracompactness introduced by J.E. Porter. We prove the following: (1) For a base-normal space $X$ and a metrizable space $Y$, the product space $X \times Y$ is normal if and only if $X \times Y$ is base-normal. (2) For the countable product $X = \prod_{i \in \mathbb{N}} X_i$ of spaces $X_i$, such that finite subproducts $\prod_{i \leq n} X_i$, $n \in \mathbb{N}$, are base-normal, $X$ is normal if and only if $X$ is base-normal. (3) Every $\Sigma$-product of metric spaces is base-normal. Many applications for analogue of classical theorems on normality of products are also given.

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1. Introduction

Throughout this paper, all spaces are assumed to be $T_1$ topological spaces. The symbol $\mathbb{N}$ denotes the set of all natural numbers. Let $\kappa$ denote an infinite cardinal and $\omega$ the first infinite cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. The cardinality of a set $X$ is denoted by $|X|$. For a space $X$, $w(X)$ stands
for the weight of $X$, and $T(X)$ denotes the collection of all open subsets of $X$. For a space $X$ and a subspace $A$ of $X$, the closure of $A$ in $X$ is denoted by $\overline{A}$. For a collection $\mathcal{A}$ of subspaces of a space $X$, $\{\overline{A} : A \in \mathcal{A}\}$ is denoted by $\overline{\mathcal{A}}$.

Weakening total-paracompactness, in [15] J.E. Porter introduced the notion of base-paracompactness and proved some interesting results containing those for product spaces; a space $X$ is said to be base-paracompact [15] if there is a base $B$ for $X$ with $|B| = w(X)$ such that every open cover of $X$ has a locally finite refinement by members of $B$.

In this paper, we introduce a new notion, called base-normality, which is a natural generalization of base-paracompactness. A space $X$ is said to be base-normal if there is a base $B$ for $X$ with $|B| = w(X)$ such that every binary open cover $\{U_0, U_1\}$ of $X$ admits a locally finite cover $B'$ of $X$ by members of $B$ such that $\overline{B'}$ refines $\{U_0, U_1\}$. The definition is motivated by the well-known fact that $X$ is normal if and only if every binary open cover $\{U_0, U_1\}$ of $X$ admits a locally finite open cover $V$ of $X$ such that $\overline{V}$ refines $\{U_0, U_1\}$.

In this paper, we are principally concerned with the study of base-normality in product spaces. Namely, we prove the following three results, which are related to products with a metric factor, infinite products and $\Sigma$-products, respectively.

**Theorem 1.1.** Let $X$ be a base-normal space and $Y$ a metrizable space. Then, the product space $X \times Y$ is normal if and only if $X \times Y$ is base-normal.

Theorem 1.1 together with other results in this paper provide analogues of the Morita–Rudin–Starbird Theorem, the Dowker Theorem and the Morita Theorem related to normality of products (Theorems 6.1–6.3).

Next, we give the following result on infinite products, which is motivated by the Nagami–Zenor Theorem.

**Theorem 1.2.** Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product of spaces $X_i$ and assume finite subproducts $\prod_{i \in n} X_i$, $n \in \mathbb{N}$, are base-normal. Then, $X$ is normal if and only if $X$ is base-normal.

Theorem 1.2 will be applied to prove other results on infinite products. In particular, results for base-paracompact spaces will be given (Corollaries 6.9 and 6.10), which seem to be the first observations on base-paracompactness for infinite products.

A proper $\Sigma$-product is never paracompact [2], hence, is never base-paracompact. On the other hand, we have the following improvement of the Gul’ko–Rudin Theorem.

**Theorem 1.3.** Every $\Sigma$-product of metric spaces is base-normal.

For undefined terminology, see [4,16].
2. Preliminary facts on base-normal spaces

First, we list the following classical theorems for normality of product spaces, which will be frequently used in this paper. By $I$ we denote the closed unit interval $[0, 1]$.

**The Dowker Theorem** [3]. For a space $X$, the following statements are equivalent: (1) $X$ is normal and countably paracompact; (2) $X \times Y$ is normal for every compact metrizable space $Y$; (3) $X \times I$ is normal.

**The Morita Theorem** [8]. For a space $X$, the following statements are equivalent: (1) $X$ is normal and $\kappa$-paracompact; (2) $X \times Y$ is normal for every compact Hausdorff space $Y$ with $w(Y) \leq \kappa$; (3) $X \times I^\kappa$ is normal.

**The Morita–Rudin–Starbird Theorem** [9,18]. For a normal space $X$ and a non-discrete metrizable space $Y$, $X \times Y$ is normal if and only if $X \times Y$ is countably paracompact.

**The Nagami–Zenor Theorem** ([12,21], see also [4, 5.5.19]). Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product, where each $X_i$ contains at least two points. Then, $X$ is normal if and only if finite subproducts $\prod_{i \leq n} X_i$, $n \in \mathbb{N}$, are normal and $X$ is countably paracompact.

**The Gul'ko–Rudin Theorem** [6,17]. Every $\Sigma$-product of metric spaces is normal.

The proof of the following lemma is easy and omitted.

**Lemma 2.1.** For a space $X$, $X$ is base-normal if and only if $X$ is normal and satisfies the following condition ($*$):

\[(*)\] there is a base $B$ for $X$ with $|B| = w(X)$ such that every binary open cover of $X$ has a locally finite refinement by members of $B$.

It is not difficult to show, with no use of normality of $X$, that ‘binary’ in ($*$) on Lemma 2.1 can be replaced by ‘finite’.

We call a space $X$ base-$\kappa$-paracompact if there is a base $B$ for $X$ with $|B| = w(X)$ such that every open cover of $X$ of cardinality at most $\kappa$ has a locally finite refinement by members of $B$. In particular, a space $X$ is said to be base-countably paracompact if $X$ is base-$\omega$-paracompact. Clearly, base-paracompactness of $X$ implies base-$\kappa$-paracompactness of $X$, and this implication reverses if $\kappa \geq |T(X)|$ (in fact, if $\kappa \geq w(X)$). Note that $X$ is base-paracompact if and only if $X$ is base-$\kappa$-paracompact for every $\kappa$.

**Proposition 2.2.** For a space $X$, the following statements are equivalent:

(1) $X$ is base-normal and base-$\kappa$-paracompact;
(2) $X$ is base-normal and $\kappa$-paracompact;
(3) $X$ is normal and base-$\kappa$-paracompact;
(4) $X$ is normal and $\kappa$-paracompact, and $X$ satisfies ($*$) in Lemma 2.1;
(5) there is a base $B$ for $X$ with $|B| = w(X)$ satisfying the following condition: for every open cover $U$ of $X$ with $|U| \leqslant \kappa$, there is a locally finite cover $B'$ of $X$ by members of $B$ such that $\overline{B'}$ refines $U$.

**Proof.** (1) $\Rightarrow$ (3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (2): This follows from Lemma 2.1.

(2) $\Rightarrow$ (5): Assume $X$ is base-normal and $\kappa$-paracompact. Let $B$ be a base which witnesses base-normality for $X$. We shall show that $B$ is the base required in (5). To prove this, let $U = \{U_\alpha: \alpha \in \Omega\}$ be an open cover of $X$ with $|\Omega| \leqslant \kappa$. Since $X$ is normal $\kappa$-paracompact, we can take locally finite open covers $V = \{V_\alpha: \alpha \in \Omega\}$ and $W = \{W_\alpha: \alpha \in \Omega\}$ such that $V_\alpha \subset W_\alpha \subset U_\alpha$ for every $\alpha \in \Omega$. Since $X$ is base-normal, for every $\alpha \in \Omega$, we can take a locally finite collection $B_\alpha$ of $X$ by members of $B$ such that $V_\alpha \subset \bigcup B_\alpha$ and $\overline{B_\alpha} \subset W_\alpha$ for all $B \in B_\alpha$. Now, $\bigcup_{\alpha \in \Omega} B_\alpha$ is the required refinement. Hence, (5) holds.

(5) $\Rightarrow$ (1): Obvious. This completes the proof.

**Corollary 2.3.** For a Hausdorff space $X$, the following statements are equivalent:

(1) $X$ is base-paracompact;

(2) $X$ is paracompact and base-normal;

(3) $X$ is paracompact and base-countably paracompact;

(4) $X$ is paracompact, and $X$ satisfies $(*)$ in Lemma 2.1.

It is unknown whether the statement 'every paracompact space is base-paracompact' holds or not [15], and a list of some equivalent conditions is given in [15, Theorem 4.1]. By Corollary 2.3, among Hausdorff spaces, the statement 'every paracompact Hausdorff space is base-normal (or is base-countably paracompact, or satisfies $(*)$ in Lemma 2.1)' can be added equivalently into this list.

**Remark 2.4.** For a later use, let us note that Proposition 2.2 actually proves the following: Let $X$ be a base-normal and paracompact, and $B$ a base which witnesses base-normality for $X$. Then, $B$ is a base which witnesses the base-paracompactness for $X$.

**Example 2.5.** Some typical examples are base-paracompact or base-normal. The Michael line and the Sorgenfrey line are base-paracompact (see [14,15]). S. Kawaguchi [7] communicated to the author that $\omega_1$ is base-normal, which is, of course, countably paracompact (hence, base-countably paracompact) but not paracompact. Now, we use the method of [15] to provide various examples. Let $Y$ be any normal, $\kappa$-paracompact but non-paracompact space, for example, $\kappa^+$ with the usual order topology is such one space. The direct sum $Y \oplus (|\mathcal{T}(Y)| + 1)$, where $|\mathcal{T}(Y)| + 1$ has the usual order topology, is base-normal and base-$\kappa$-paracompact but not paracompact. If we construct $Y \oplus (|\mathcal{T}(Y)| + 1)$ by using any non-normal $\kappa$-paracompact space $Y$ or any Dowker space $Y$, respectively, we can get a non-normal base-$\kappa$-paracompact space or a base-normal Dowker space, respectively.
In the first version of the paper, we commented that we did not know whether every normal space is base-normal or not. The referee kindly informed us that certain (necessarily consistent) examples of well-known separable normal non-metrizable Moore spaces are not base-normal. To see this, let $X$ be a normal separable Moore space of weight $\omega_1$ which is not collectionwise-Hausdorff (e.g., take $X$ to be the tangent disk space over a $Q$-set). Then, there is a closed set $D$ of cardinality $\omega_1$ which cannot be separated. Suppose $\mathcal{B}$ is a base witnessing base-normality of $X$, with $|\mathcal{B}| = \omega_1$. It is easy to construct by induction disjoint subsets $H$ and $K$ of $D$ such that, if $B \in \mathcal{B}$ and $|B \cap D| = \omega_1$, then $B \cap H \neq \emptyset$ and $B \cap K \neq \emptyset$. But then $\mathcal{B}$ has no locally finite subcover $\mathcal{B}'$ refining $\{X - H, X - K\}$, for suppose it did. By separability, $\mathcal{B}'$ is countable, so some member of $\mathcal{B}'$ meets $D$ in an uncountable set, hence meets both $H$ and $K$, contradiction.

Let us note that the above example implies that it is consistent with ZFC that following facts hold: Base-normal spaces are not preserved under open perfect mappings. Base-normal spaces are not hereditarily to clopen subsets. Base-normality of the product space $X \times Y$ need not imply base-normality of $X$. To prove these, let a space $X$ be normal but not base-normal, $D(|T(X)|)$ the discrete space of cardinality $|T(X)|$, and $p$ a point. Consider base-normal spaces $X \oplus (|T(X)| + 1) \oplus \{p\}$ and $X \times (D(|T(X)|))$, and the map $f : X \oplus (D(|T(X)|) + 1) \oplus \{p\} \to X \oplus \{p\}$ defined by $f(x) = x$ if $x \in X$, $f(x) = p$ if $x \in (D(|T(X)|) + 1) \oplus \{p\}$, and apply a method similar to that of J.E. Porter [15].

On a study of base-paracompactness of product spaces, difficulties lie on the unknown fact whether ‘locally finite cover $\mathcal{B}'$’ in the definition can be replaced by ‘$\sigma$-locally finite cover $\mathcal{B}'$’ or not. Indeed, on proofs of [15, Theorem 3.15] or [20, Corollaries 4.3 and 4.4], it takes some effort to avoid this problem. For a study of base-normality of product spaces, there are difficulties similar to the above.

3. Proof of Theorem 1.1

Before the proof of Theorem 1.1, let us note that for a base-normal space (even for a base-paracompact space) $X$ and a metrizable space $Y \times Y$ is not necessarily normal (Example 2.5).

We first give the following key lemma, which is used not only for the proof of Theorem 1.1 but also for that of Theorem 1.2. For a collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of a space $X$ and a map $f : Z \to X$, let us denote $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $f^{-1}(A) = \{f^{-1}(A) : A \in \mathcal{A}\}$. Also, for collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of spaces $X$ and $Y$, respectively, we express $\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

**Lemma 3.1.** Let $X$ be a base-normal space, and $\mathcal{B}_X$ a base which witnesses base-normality for $X$. Let $\mathcal{U}$ be a locally finite open cover of $X$, and $R^0$, $R^1$ and $K$ closed subsets of $X$ such that $R \cap K = \emptyset$, where $R = R^0 \cap R^1$. Then, there is a locally finite cover $\mathcal{B}$ of $X$ by members of $\mathcal{B}_X \wedge \mathcal{U}$ satisfying the following conditions: for every $B \in \mathcal{B}$,

(a) $B \cap R = \emptyset \implies B \cap R^0 = \emptyset \lor B \cap R^1 = \emptyset$;
(b) $B \cap R \neq \emptyset \implies B \cap K = \emptyset$. 

Proof. Since $R \cap K = \emptyset$, there is a locally finite collection $V_1$ of $X$ by members of $B_X$ such that $R \subset \bigcup V_1$, and that $V \cap K = \emptyset$ for every $V \in V_1$. Set $B_1 = \{ V \cap U : V \in V_1, U \in \mathcal{U}, V \cap U \cap R = \emptyset \}$. Then, we have:

1. $B_1$ is a locally finite collection of $X$ by members of $B_X \cup \mathcal{U}$;
2. $R \subset \bigcup B_1$;
3. $\overline{B} \cap K = \emptyset$ for every $B \in B_1$;
4. $B \cap R \neq \emptyset$ for every $B \in B_1$.

By (12), $R^0 - \bigcup B_1$ and $R^1$ are closed and disjoint. Hence, there is a locally finite collection $V_2$ of $X$ by members of $B_X$ such that $R^0 - \bigcup B_1 \subset \bigcup V_2$ and $V \cap R^1 = \emptyset$ for every $V \in V_2$. Set $B_2 = \{ V \cap U : V \in V_2, U \in \mathcal{U} \}$. Then, we have:

1. $B_2$ is a locally finite collection of $X$ by members of $B_X \cup \mathcal{U}$;
2. $R^0 - \bigcup B_1 \subset \bigcup B_2$;
3. $\overline{B} \cap R^1 = \emptyset$ for every $B \in B_2$.

In a parallel construction with $B_2$, we have $B_3$ satisfying that:

1. $B_3$ is a locally finite collection of $X$ by members of $B_X \cup \mathcal{U}$;
2. $R^1 - \bigcup B_1 \subset \bigcup B_3$;
3. $\overline{B} \cap R^0 = \emptyset$ for every $B \in B_3$.

By (22) and (32), we have $R^0 \cup R^1 \subset \bigcup (B_1 \cup B_2 \cup B_3)$. Hence, there is a locally finite collection $V_4$ of $X$ by members of $B_X$ such that $X - \bigcup (B_1 \cup B_2 \cup B_3) \subset \bigcup V_4$, and $V \cap (R^0 \cup R^1) = \emptyset$ for every $V \in V_4$. Set $B_4 = \{ V \cap U : V \in V_4, U \in \mathcal{U} \}$. Then, we have:

1. $B_4$ is a locally finite collection of $X$ by members of $B_X \cup \mathcal{U}$;
2. $X - \bigcup (B_1 \cup B_2 \cup B_3) \subset \bigcup B_4$;
3. $\overline{B} \cap (R^0 \cup R^1) = \emptyset$ for every $B \in B_4$.

Set $B = B_1 \cup B_2 \cup B_3 \cup B_4$. Then, it follows from (11), (21), (31), (41) and (42) that $B$ is a locally finite cover of $X$ by members of $B_X \cup \mathcal{U}$.

To prove (a), let $B \in B$ satisfying the condition $B \cap R = \emptyset$. Because of (14), we have $B \notin B_1$, that is, $B \in B_2 \cup B_3 \cup B_4$. So, it follows from (21), (31) and (41) that either $\overline{B} \cap R^0 = \emptyset$ or $\overline{B} \cap R^1 = \emptyset$ holds. Hence (a) holds.

To prove (b), assume $B \in B$ and $\overline{B} \cap R \neq \emptyset$. By (23), (33) and (43), we have $B \notin B_2 \cup B_3 \cup B_4$, that is, $B \in B_1$. It follows from (13) that $\overline{B} \cap K = \emptyset$. Hence (b) holds.

This completes the proof. □

Proof of Theorem 1.1. Let $X$ be a base-normal space and $Y$ a metrizable space, and assume $X \times Y$ is normal. In case $Y$ is discrete, $X \times Y$ is clearly base-normal. Hence, we may assume $Y$ is non-discrete, and therefore, $X \times Y$ is countably paracompact by the Morita–Rudin–Starbird Theorem.
Let $\mathcal{B}_X$ be a base which witnesses base-normality for $X$. We may assume $\mathcal{B}_X$ is closed under finite intersections. We can take a base $\mathcal{G}$ of $Y$ so that the following conditions are satisfied:

(i) $\mathcal{G}$ is written as $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$, where each $\mathcal{G}_n$ is a locally finite open cover of $Y$;
(ii) for a suitable index set $\Omega$ with $|\Omega| = w(Y)$, each $\mathcal{G}_n$ is expressed as $\mathcal{G}_n = \{G(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega\}$;
(iii) for $\alpha_1, \ldots, \alpha_n \in \Omega$ and $n \in \mathbb{N}$, $G(\alpha_1, \ldots, \alpha_n) = \bigcup_{i \in \mathbb{N}} G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$;
(iv) for $\alpha_1, \ldots, \alpha_n \in \Omega$, $\forall n \in \mathbb{N}$, the diameter of $G(\alpha_1, \ldots, \alpha_n) < 1/2^n$.

Since $|\mathcal{G}| = w(Y)$, we have $|\mathcal{B}_X \times \mathcal{G}| = w(X \times Y)$. We shall show that $\mathcal{B}_X \times \mathcal{G}$ witnesses base-normality for $X \times Y$. As in the proof of the Morita–Rudin–Starbird Theorem, for every $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and every $i = 0, 1$, set

$$R^i(\alpha_1, \ldots, \alpha_n) = \pi_X(F_i \cap \{X \times G(\alpha_1, \ldots, \alpha_n)\}),$$

where $\pi_X : X \times Y \to X$ is the natural projection, and define $R(\alpha_1, \ldots, \alpha_n) = R^0(\alpha_1, \ldots, \alpha_n) \cap R^1(\alpha_1, \ldots, \alpha_n)$. Note that, for every $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and every $\alpha_{n+1} \in \Omega$,

$$R(\alpha_1, \ldots, \alpha_n) \supseteq R(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}).$$

(1)

For every $n \in \mathbb{N}$, define

$$R_n = \bigcup\{R(\alpha_1, \ldots, \alpha_n) \times G(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega\}.$$

Then, $\{R_n : n \in \mathbb{N}\}$ is a decreasing collection of closed subsets of $X \times Y$ with $\bigcap_{n \in \mathbb{N}} R_n = \emptyset$.

Since $X \times Y$ is countably paracompact, take a decreasing open collection $\{Q_n : n \in \mathbb{N}\}$ of $X \times Y$ such that $R_n \subseteq Q_n$, $n \in \mathbb{N}$, and $\bigcap_{n \in \mathbb{N}} Q_n = \emptyset$.

For every $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, set

$$K(\alpha_1, \ldots, \alpha_n) = \{x \in X : (\{x\} \times G(\alpha_1, \ldots, \alpha_n)) \cap Q_n = \emptyset\}.$$

Note that $K(\alpha_1, \ldots, \alpha_n)$ is a closed subset of $X$, and that $K(\alpha_1, \ldots, \alpha_n) \cap R(\alpha_1, \ldots, \alpha_n) = \emptyset$ for every $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$.

**Claim 1.** There is a collection $\{\mathcal{B}(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega, n \in \mathbb{N}\}$ of locally finite open covers of $X$, where each $\mathcal{B}(\alpha_1, \ldots, \alpha_n)$ consists of members of $\mathcal{B}_X$, such that the following conditions are satisfied:

(a) for $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and $\alpha_{n+1} \in \Omega$,

$$\mathcal{B}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$$

refines $\mathcal{B}(\alpha_1, \ldots, \alpha_n)$;

(b) for $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and $B \in \mathcal{B}(\alpha_1, \ldots, \alpha_n)$,

$$B \cap R(\alpha_1, \ldots, \alpha_n) = \emptyset$$

$\Rightarrow$ $\overline{B} \cap R^0(\alpha_1, \ldots, \alpha_n) = \emptyset$ or $\overline{B} \cap R^1(\alpha_1, \ldots, \alpha_n) = \emptyset$;

(c) for $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and $B \in \mathcal{B}(\alpha_1, \ldots, \alpha_n)$,

$$B \cap R(\alpha_1, \ldots, \alpha_n) \neq \emptyset$$

$\Rightarrow$ $\overline{B} \cap K(\alpha_1, \ldots, \alpha_n) = \emptyset$.

**Proof.** Fix $\alpha_1 \in \Omega$. We put $\mathcal{U} = \{X\}$, $R^0 = R^0(\alpha_1)$, $R^1 = R^1(\alpha_1)$ (therefore $R = R(\alpha_1)$), $K = K(\alpha_1)$, apply Lemma $3.1$, and denote the resulting cover $\mathcal{B}$ by $\mathcal{B}(\alpha_1)$. It follows from $\mathcal{B}(\alpha_1) \subset \mathcal{B}_X \cup \{X\} = \mathcal{B}_X$ that $\mathcal{B}(\alpha_1)$ is as required.
Next, fix $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$. Assume $B(\alpha_1, \ldots, \alpha_n)$ has been constructed so as to satisfy conditions (a), (b) and (c) above. Now, fix $\alpha_{n+1} \in \Omega$. To apply Lemma 3.1 again, we put $U = B(\alpha_1, \ldots, \alpha_n)$, $R^0 = R^0(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, $R^1 = R^1(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ (hence, $R = R(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$) and put $K = K(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$, and define the resulting cover $B$ by $B(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$. Since $B_X$ is closed under finite intersections, we have $B(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \subset B_X \cap B(\alpha_1, \ldots, \alpha_n) \subset B_X$. By the construction, $B(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ satisfies the conditions (a), (b) and (c). This completes the proof of Claim 1. □

For all $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, set $A(\alpha_1, \ldots, \alpha_n) = \{ B \in B(\alpha_1, \ldots, \alpha_n): B \cap R(\alpha_1, \ldots, \alpha_n) = \emptyset \}$ and $A'(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_n) - A(\alpha_1, \ldots, \alpha_n)$. Then, for every $\alpha_1, \ldots, \alpha_n \in \Omega$, $n \in \mathbb{N}$, and $\alpha_{n+1} \in \Omega$, we have

$$\bigcup \alpha(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \subset \bigcup \alpha'(\alpha_1, \ldots, \alpha_n).$$

(2)

To prove (2), let $x \in \bigcup \alpha'(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$. Take $B \in \alpha'(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ such that $x \in B$. By the condition (a) of Claim 1, there is $B' \in B(\alpha_1, \ldots, \alpha_n)$ such that $B \subset B'$. By the definition of $\alpha'(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ and (1), we have $\emptyset \neq B \cap R(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \subset B' \cap R(\alpha_1, \ldots, \alpha_n)$. Hence, $B' \in \alpha'(\alpha_1, \ldots, \alpha_n)$. Thus, $x \in B \subset B' \subset \bigcup \alpha'(\alpha_1, \ldots, \alpha_n)$, which completes the proof of (2).

Define $L_0 = \{ B \times G(\alpha_1): \alpha_1 \in \Omega, B \in A(\alpha_1) \}$. Moreover, for $n \in \mathbb{N}$, define

$$L_n = \{ B' \cap B \times G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}): B' \in \alpha'(\alpha_1, \ldots, \alpha_n), B \in A(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega \}.$$

Set $L = \bigcup_{n \geq 0} L_n$. Since each member of $L$ is in $B_X \times G$, to complete the proof, it suffices to show the following claims.

Claim 2. For every $L \in L$, either $\bar{L} \cap F_0 = \emptyset$ or $\bar{L} \cap F_1 = \emptyset$ holds.

Proof. Let $L \in L$. In both cases $L \in L_0$ and $L \in \bigcup_{i \in \mathbb{N}} L_i$, there are $\alpha_1, \ldots, \alpha_n \in \Omega$ and $B \in A(\alpha_1, \ldots, \alpha_n)$ such that $L \subset B \times G(\alpha_1, \ldots, \alpha_n)$. It follows from the condition (b) of Claim 1 that $\bar{B} \cap R^0(\alpha_1, \ldots, \alpha_n) = \emptyset$ or $\bar{B} \cap R^1(\alpha_1, \ldots, \alpha_n) = \emptyset$. By the definition of $R^i(\alpha_1, \ldots, \alpha_n)$, we have

$$\bar{B} \times G(\alpha_1, \ldots, \alpha_n) \cap F_i = \emptyset \quad \text{for } i = 0 \text{ or } 1.$$

Hence, $\bar{L} \cap F_i = \emptyset$ for $i = 0$ or 1, which completes the proof of Claim 2. □

Claim 3. $L$ is a cover of $X \times Y$.

Proof. Let $(x, y) \in X \times Y$. Since $\bigcap_{n \in \mathbb{N}} \overline{Q_n} = \emptyset$ and $\{ Q_n: n \in \mathbb{N} \}$ is decreasing, there are $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \Omega$ such that $\overline{(x \times G(\alpha_1, \ldots, \alpha_n))} \cap \overline{Q_n} = \emptyset$ and $y \in G(\alpha_1, \ldots, \alpha_n)$. Then, we have $x \in K(\alpha_1, \ldots, \alpha_n)$. Since $B(\alpha_1, \ldots, \alpha_n)$ is a cover of $X$, take $B \in B(\alpha_1, \ldots, \alpha_n)$ such that $x \in B$. Since $B \cap K(\alpha_1, \ldots, \alpha_n) \neq \emptyset$, it follows from the condition (c) of Claim 1 that $\bar{B} \cap R(\alpha_1, \ldots, \alpha_n) = \emptyset$. Hence we have
$B \in \mathcal{A}(\alpha_1, \ldots, \alpha_n)$, which shows that $x \in \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_n)$. Now, let $m$ be the minimum $m$ ($\leq n$) such that $x \in \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_m)$. Then, either $x \in \bigcup \mathcal{A}(\alpha_1)$ or $x \in \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_m) - \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_{m-1})$ holds.

**Case 1.** $x \in \bigcup \mathcal{A}(\alpha_1)$. Note that $y \in G(\alpha_1, \ldots, \alpha_m) \subset G(\alpha_1)$ by (iii). Take $B \in \mathcal{A}(\alpha_1)$ such that $x \in B$. Now, we have $(x, y) \in B \times G(\alpha_1) \subset L_0$.

**Case 2.** $x \in \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_m) - \bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_{m-1})$. By the reason that $x \in (\bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_m)) \cap (\bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_{m-1}))$, take $B \in \mathcal{A}(\alpha_1, \ldots, \alpha_m)$ and $B' \in \mathcal{A}(\alpha_1, \ldots, \alpha_{m-1})$ such that $x \in B \cap B'$. By (iii), we have $y \in G(\alpha_1, \ldots, \alpha_m) \subset G(\alpha_1, \ldots, \alpha_m)$. Thus, it follows that $(x, y) \in (B' \cap B) \times G(\alpha_1, \ldots, \alpha_m) \in L_{m-1}$. This completes the proof of Claim 3. □

**Claim 4.** $L$ is locally finite in $X \times Y$.

**Proof.** Let $(x, y) \in X \times Y$. Since $\bigcap_{n \in \mathbb{N}} Q_n = \emptyset$ and $\{Q_n: n \in \mathbb{N}\}$ is decreasing, take a neighborhood $U_x$ of $x$ in $X$ and $n \in \mathbb{N}$ such that $(U_x \times B(y; 1/2^{n+1})) \cap Q_n = \emptyset$, where $B(y; 1/2^{n+1})$ is the open $(1/2^{n+1})$-ball of $y$ in $Y$. Since each $L_i$ is locally finite, it suffices to show that $\bigcup_{m \geq n} L_m$ is locally finite at $(x, y)$.

Since $G_n$ is locally finite, there is a neighborhood $V_y$ of $y$ in $Y$ such that $V_y \subset B(y; 1/2^n)$ and $|\{(\alpha_1, \ldots, \alpha_n) \in \Omega^n: G(\alpha_1, \ldots, \alpha_n) \cap V_y \neq \emptyset\}| < \omega$. Set $\delta = \{(\alpha_1, \ldots, \alpha_n) \in \Omega^n: G(\alpha_1, \ldots, \alpha_n) \cap V_y \neq \emptyset\}$. Now, we have

$$U_x \cap \left(\bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_n)\right) = \emptyset \quad \text{for every } (\alpha_1, \ldots, \alpha_n) \in \delta. \quad (3)$$

To prove (3), let $(\alpha_1, \ldots, \alpha_n) \in \delta$. Then, by (iv), we have that $G(\alpha_1, \ldots, \alpha_n) \subset B(y; 1/2^{n+1})$. Hence, $(U_x \times G(\alpha_1, \ldots, \alpha_n)) \cap Q_n = \emptyset$, which implies $U_x \subset K(\alpha_1, \ldots, \alpha_n)$. On the other hand, by the condition (c) of the Claim 1, $K(\alpha_1, \ldots, \alpha_n) \cap (\bigcup \mathcal{A}(\alpha_1, \ldots, \alpha_n)) = \emptyset$. This completes the proof of (3).

To complete the proof of Claim 4, it suffices to show that

$$(U_x \times V_y) \cap L = \emptyset \quad \text{for every } L \in \bigcup_{m \geq n} L_m. \quad (4)$$

To prove (4), assume on the contrary that $(U_x \times V_y) \cap L \neq \emptyset$ for some $L \in L_m$ with $m \geq n$. Then, $L$ is expressed as $L = (B' \cap B) \times G(\alpha_1', \ldots, \alpha_m', \alpha_{m+1}')$ for some $\alpha_1', \ldots, \alpha_m', \alpha_{m+1}' \in \Omega$, $B' \in \mathcal{A}(\alpha_1', \ldots, \alpha_m')$ and $B \in \mathcal{A}(\alpha_1', \ldots, \alpha_m', \alpha_{m+1}')$. It follows from (2) that

$$\emptyset 
\neq U_x \cap B' \cap B \subset U_x \cap \left(\bigcup \mathcal{A}(\alpha_1', \ldots, \alpha_m')\right) \subset U_x \cap \left(\bigcup \mathcal{A}(\alpha_1', \ldots, \alpha_{m}')\right).$$

Hence, it follows from (3) that $(\alpha_1', \ldots, \alpha_{m}') \notin \delta$. On the other hand, by (iii), we have $\emptyset \neq V_y \cap G(\alpha_1', \ldots, \alpha_{m}', \alpha_{m+1}') \subset V_y \cap G(\alpha_1', \ldots, \alpha_{m}')$. Thus, we have $(\alpha_1', \ldots, \alpha_{m}') \in \delta$, which induces a contradiction. Hence, (4) holds. This completes the proof of Claim 4. □

**Remark 3.2.** By a similar method as above, we can prove a finer result: Assume $X$ satisfies $(\ast)$ in Lemma 2.1 and $Y$ a metric space. If the product space $X \times Y$ is countably paracompact, then $X \times Y$ satisfies $(\ast)$ in Lemma 2.1.
4. Proof of Theorem 1.2

First, note that the assumption of base-normality of \( \prod_{i \leq n} X_i \) for all \( n \in \mathbb{N} \) does not necessarily imply the normality of \( \prod_{i \in \mathbb{N}} X_i \). For, let \( X_1 \) be the Michael line and \( X_i = \mathbb{N} \) for every \( i \geq 2 \). Then, \( \prod_{i \leq n} X_i \) is base-normal (in fact, base-paracompact) for every \( n \in \mathbb{N} \) (Example 2.5), but \( \prod_{i \in \mathbb{N}} X_i \) is not normal.

For a collection \( \mathcal{A} \) of subsets of a space \( X \), \( \bigwedge \mathcal{A} \) stands for all finite intersections of elements of \( \mathcal{A} \), that is, \( \bigwedge \mathcal{A} = \{ \bigcap A' : A' \in [\mathcal{A}]^{<\omega} \} \).

Theorem 1.2 immediately follows from Theorem 4.1 below.

**Theorem 4.1.** Let \( X = \operatorname*{lim} \{ X_i, \pi_i^j \} \) be the limit of an inverse sequence of base-normal spaces \( X_i \) with open continuous onto mappings \( \pi_i^j : X_j \to X_i \), where \( i \leq j \). If \( X \) is countably paracompact, then \( X \) is base-normal.

First, we apply Theorem 4.1 to prove Theorem 1.2.

**Proof of Theorem 1.2.** Assume \( X \) is normal. We may assume \( |X_i| \geq 2 \) for every \( i \in \mathbb{N} \). By the Nagami–Zenor Theorem, \( X \) is countably paracompact. Hence, it follows from Theorem 4.1 that \( X \) is base-normal, which completes the proof of Theorem 1.2. \( \square \)

**Proof of Theorem 4.1.** Assume \( X \) is countably paracompact. First note that the natural projection \( \pi_n : X \to X_n \) is also an open continuous onto map for every \( n \in \mathbb{N} \) (see [4, 2.5.B.(a)] and [1, Corollary 1]). We may assume \( w(X) \geq \omega \).

For every \( n \in \mathbb{N} \), let \( \mathcal{G}_n \) be a base which witnesses base-normality for \( X_n \). For the later use, for \( n \in \mathbb{N} \), we set
\[
\mathcal{G}_n^\ast = \mathcal{G}_n \cup (\pi_{n-1})^{-1}(\mathcal{G}_{n-1}) \cup (\pi_{n-2})^{-1}(\mathcal{G}_{n-2}) \cup \cdots \cup (\pi_1)^{-1}(\mathcal{G}_1),
\]
and \( \mathcal{G}_n^\ast = \bigwedge \mathcal{G}_n^\ast \). Define \( \mathcal{G} = \bigcup_{n \in \mathbb{N}} \pi_n^{\ast -1}(\mathcal{G}_n^\ast) \). Then, \( \mathcal{G} \) is closed under finite intersections. Since \( |\mathcal{G}| \leq \sup_{n \in \mathbb{N}} w(X_n) \cdot \omega = w(X) \) and \( \mathcal{G} \) is a base for \( X \) [4, 2.5.5], we shall show that \( \mathcal{G} \) witnesses base-normality for \( X \).

To prove this, let \( F_0 \) and \( F_1 \) be disjoint closed subsets of \( X \). As in the proof of the Nagami–Zenor Theorem, for every \( n \in \mathbb{N} \), and every \( i = 0, 1 \), set \( R^i = \pi_n(F_i)^{X_n} \) and \( R(n) = R^0(n) \cap R^1(n) \). Then, \( R(n) \) is closed in \( X_n \) for every \( n \in \mathbb{N} \). By using the fact that \( \pi_n \) are open, we can show that for every \( n \in \mathbb{N} \),
\[
\pi_n^{-1}(R(n)) \supset \pi_{n+1}^{-1}(R(n+1)) \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(R(n)) = \emptyset. \tag{5}
\]
Since \( X \) is countably paracompact, take a decreasing open collection \( \{ Q(n) : n \in \mathbb{N} \} \) of \( X \) such that \( \pi_n^{-1}(R(n)) \subset Q(n) \), \( n \in \mathbb{N} \), and \( \bigcap_{n \in \mathbb{N}} Q(n)^X = \emptyset \).

For every \( n \in \mathbb{N} \), define \( K(n) = X_n - \pi_n(Q(n)) \). Note that, for every \( n \in \mathbb{N} \), \( K(n) \) is closed in \( X_n \) and \( K(n) \cap R(n) = \emptyset \).

**Claim 1.** There are locally finite covers \( \mathcal{B}(n) \) of \( X_n \), \( n \in \mathbb{N} \), where each \( \mathcal{B}(n) \) is consisting of members of \( \mathcal{G}_n^\ast \), such that the following conditions are satisfied:

1. \( \mathcal{B}(n) \) is locally finite for every \( n \in \mathbb{N} \).
2. \( \mathcal{B}(n) \) is a base for \( X_n \) for every \( n \in \mathbb{N} \).
3. \( \mathcal{B}(n) \) is normal.
4. \( \mathcal{B}(n) \) is closed under finite intersections.
Claim 4. \( \mathcal{L} \) is locally finite in \( X \).

Proof. Let \( x \in X \). Since \( \bigcap_{n \in \mathbb{N}} \overline{Q(n)}^X = \emptyset \) and \( \{ Q(n) : n \in \mathbb{N} \} \) is decreasing, there is \( n \in \mathbb{N} \) such that \( \pi_n^{-1}(x) \cap \overline{Q(n)}^X = \emptyset \). Then, by using (c) of Claim 1 and the similar technique of the proof of Claim 3 of Theorem 1.1, we can prove \( \pi_n(x) \in \bigcup A(n) \). Now, let \( m \) be the minimum \( m \leq n \) such that \( \pi_m(x) \in \bigcup A(m) \). Then, either \( x \in \bigcup \pi_m^{-1}(A(1)) \) or \( x \in \bigcup \pi_m^{-1}(A(m)) - \bigcup \pi_{m-1}^{-1}(A(m-1)) \) holds. If \( x \in \bigcup \pi_m^{-1}(A(1)) \), we clearly have \( x \in \bigcup \mathcal{L} \). So, we may assume \( x \in \bigcup \pi_m^{-1}(A(m)) - \bigcup \pi_{m-1}^{-1}(A(m-1)) \). It follows from \( X = (\bigcup \pi_{m-1}^{-1}(A(m-1))) \cup (\bigcup \pi_m^{-1}(A(m))) \) that \( x \in \bigcup \pi_{m-1}^{-1}(A(m-1)) \). Now, take \( B' \in A'(m-1) \) and \( B \in A(m) \) such that \( x \in \pi_{m-1}^{-1}(B') \cap \pi_m^{-1}(B) \). The, we have \( x \in \bigcup \mathcal{L}_{m-1} \). This completes the proof of Claim 3. \( \square \)
Proof. Let \( x \in X \). Since \( \bigcap_{n \in \mathbb{N}} Q(n) \times = \emptyset \) and \( \{ Q(n) \colon n \in \mathbb{N} \} \) is decreasing, take \( n \in \mathbb{N} \) and a neighborhood \( U \) of \( \pi_n(x) \) in \( X_n \) such that \( \pi_n^{-1}(U) \cap Q(n) = \emptyset \). Since each \( L_i \) is locally finite in \( X \), it suffices to show that \( \bigcup_{m \geq n} L_m \) is locally finite at \( x \).

Now, we have
\[
U \cap \left( \bigcup A(n) \right) = \emptyset.
\] (7)

To prove (7), first notice that \( U \subset K(n) \). On the other hand, it follows from the condition (c) of Claim 1 that \( K(n) \cap (\bigcup A(n)) = \emptyset \). Thus, we have \( U \cap (\bigcup A(n)) = \emptyset \), which completes the proof of (7).

To finish the proof of Claim 4, it suffices to show that
\[
\pi_n^{-1}(U) \cap L = \emptyset \quad \text{for every} \quad L \in \bigcup_{m \geq n} L_m.
\] (8)

To prove (8), assume on the contrary that \( \pi_n^{-1}(U) \cap L \neq \emptyset \) for some \( L \in L_m \) with \( m \geq n \). Then, \( L \) is expressed as \( L = \pi_{m-1}(B') \cap \pi_{m+1}(B) \) for some \( B' \in A(m) \) and \( B \in A(m+1) \). It follows from (6) that \( \emptyset \neq \pi_n^{-1}(U) \cap \pi_{m-1}(B') \cap \pi_{m+1}(B) \subset \pi_n^{-1}(U) \cap \pi_{m-1}(A(m)) \subset \pi_n^{-1}(U) \cap \pi_n^{-1}(\bigcup A(n)) \). Hence, \( U \cap (\bigcup A(n)) \neq \emptyset \), which contradicts (7). Thus, (8) holds. This completes the proof of Claim 4. \( \square \)

Claims 2, 3 and 4 complete the proof of Theorem 4.1. \( \square \)

Remark 4.2. For a later use, note that the proof of Theorem 4.1 actually shows: Let \( X_i \), \( i \in \mathbb{N} \), be spaces with \( |X_i| \geq 2 \) such that finite subproducts \( \prod_{i \in \mathbb{N}} X_i \), \( n \in \mathbb{N} \), are base-normal. For every \( n \in \mathbb{N} \), let \( B_n \) be a base which witnesses base-normality for \( \prod_{i \leq n} X_i \). Assume that \( \prod_{i \in \mathbb{N}} X_i \) is normal. Then, \( \bigwedge \{ \bigcup_{m \in \mathbb{N}} \pi_n^{-1}(B_m) \} \) is a base which witnesses base-normality for \( \prod_{i \in \mathbb{N}} X_i \), where \( \pi_{[1,\ldots,n]} \colon \prod_{i \in \mathbb{N}} X_i \to \prod_{i \leq n} X_i \) is the natural projection.

Remark 4.3. By a similar method, we can prove a finer result: Let \( X = \lim_{\leftarrow} \{ X_i, \pi^j_i \} \) be the limit of an inverse sequence of spaces \( X_i \) which satisfy \( (\ast) \) with \( \pi^j_i : X_j \to X_i \), where \( i \leq j \), open continuous onto mappings. If \( X \) is countably paracompact, then \( X \) satisfies \( (\ast) \).

5. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 applying the fact mentioned in Remark 4.2. Let \( X = \prod_{A \subseteq \Omega} X_A \) be a product space and let \( p = (p_A) \) be a fixed point of \( X \). The subspace \( \Sigma = \{ x = (x_A) \in X : \{ \alpha \in \Omega : x_A \neq p_A \} \leq \omega \} \) of \( X \) is called a \( \Sigma \)-product of spaces \( X_{A}, \alpha \in \Omega \) (about \( p \)). A \( \Sigma \)-product \( \Sigma \) of spaces \( X_{A}, \alpha \in \Omega \), is called proper if uncountably many spaces \( X_{A} \) contain at least two elements.

Let \( \Sigma \) be the \( \Sigma \)-product of spaces \( X_{A}, \alpha \in \Omega \), about \( p = (p_A) \). For \( A \subseteq \Omega \), the map \( r_A : \Sigma \to \Sigma \) is defined by \( r_A(x) = x_A \) if \( \alpha \in A \), and \((r_A(x))_\alpha = p_\alpha \) otherwise,
where \( x = (x_a)_{a \in \Omega} \). For \( A \subseteq \Omega \), the natural projection is denoted by \( \pi_A : \prod_{a \in \Omega} X_a \to \prod_{a \in A} X_a \). Sometimes, \( \pi_A \) itself stands for the restricted map of \( \pi \) to \( \Sigma \), that is, \( \pi_A|_{\Sigma} : \Sigma \to \prod_{a \in A} X_a \). Also, for \( B \subseteq A \subseteq \Omega \), \( \pi_A^B : \prod_{a \in A} X_a \to \prod_{a \in B} X_a \) denotes the natural projection. Note that \( r_A(\Sigma) \) and \( \prod_{a \in A} X_a \) naturally corresponds with each other for every \( A \subseteq [\Omega]^{\leq \omega} \). The \( \Sigma \) is said to be \( r_A \)-distinguished if \( r_A^{-1}(r_A(\Sigma)) = U \) [2].

**Proof.** For every \( A \subseteq \Omega \), the \( \Sigma \) witnesses the base-paracompactness for \( \prod_{a \in A} X_a \) if \( \Sigma \) is a base which witnesses base-normality for \( \prod_{a \in A} X_a \). By Claim 1, \( \Sigma \) is base-paracompact.

**Claim 1.** \( w(\Sigma) = |\Sigma| \cdot \sup_{a \in \Omega} w(X_a) \).

**Proof.** By usual calculations, we have \(|\Sigma| \cdot \sup_{a \in \Omega} w(X_a) \leq w(\Sigma) \leq w(\prod_{a \in \Omega} X_a) \leq |\Omega| \cdot \sup_{a \in \Omega} w(X_a). \) \( \Box \)

For every \( \delta \in [\Omega]^{\leq \omega} \), since \( \prod_{a \in \delta} X_a \) is base-paracompact, take a base \( B_\delta \) which witnesses base-paracompactness for \( \prod_{a \in \delta} X_a \). Set \( B' = \bigcup_{\delta \in [\Omega]^{\leq \omega}} \pi_\delta^{-1}(B_\delta) \) and define \( B = \bigwedge B' \). Since \(|B| = |B'| = |\Omega| \cdot \sup_{a \in \Omega} w(X_a)\) and \( B \) is a base for \( \Sigma \), by Claim 1, we shall show that \( B \) witnesses base-normality for \( \Sigma \).

**Claim 2.** Let \( A \subseteq [\Omega]^{\leq \omega} \). Then, every open cover of \( r_A(\Sigma) \) has a locally finite refinement by members of \( \bigwedge (\bigcup_{a \in [\Omega]^{\leq \omega}}(\pi_A^a)^{-1}(B_\delta)) \).

**Proof.** On the case \( A \) is finite, Claim 2 is obvious. Assume \( A \) is infinite countable. Since \( w(\prod_{a \in A} X_a) = |\prod_{a \in [\Omega]^{\leq \omega}}(\pi_A^a)^{-1}(B_\delta)| \), by Remark 4.2, it follows that \( \bigwedge (\bigcup_{a \in [\Omega]^{\leq \omega}}(\pi_A^a)^{-1}(B_\delta)) \) is a base which witnesses base-normality for \( \prod_{a \in A} X_a \). Since \( \prod_{a \in A} X_a \) is paracompact, it follows from Remark 2.4 that \( \bigwedge (\bigcup_{a \in [\Omega]^{\leq \omega}}(\pi_A^a)^{-1}(B_\delta)) \) witnesses the base-paracompactness for \( \prod_{a \in A} X_a \). \( \Box \)

Let \( F_0 \) and \( F_1 \) be disjoint non-empty closed subsets of \( \Sigma \). For \( A \subseteq [\Omega]^{\leq \omega} \), \( r_A(\Sigma) \) is metrizable by a metric \( \rho_A \), and denote by \( d_A \) a continuous pseudo-metric on \( \Sigma \) which extends \( \rho_A \) so as to satisfy \( d_A((x, y)) = \rho_A(r_A(x), r_A(y)) \).

**Claim 3.** By induction on \( n \), we construct a sequence \( \{ U_n : n \in \mathbb{N} \} \) of locally finite covers of \( \Sigma \), each \( U_n \) consisting by members of \( B \), maps \( p_n : U_n \to U_{n-1} \), \( n \in \mathbb{N} \), and for every \( U \in U_n \), \( n \in \mathbb{N} \), take \( A(U) \subseteq [\Omega]^{\leq \omega} \), a continuous pseudo-metric \( \mu_U \) on \( \Sigma \) which metrizes \( r_A(U)(\Sigma) \), and \( x_{f_j}^U \in U \cap F_j \) (if exists) for \( j = 0, 1 \) so that the following conditions are satisfied:

For \( W \in U_{n-1} \),

(a) \( W = \bigcup \{ U \in U_n : p_n(U) = W \} \).
For $U \in \mathcal{U}_n$, 

(b) $|\{W \in \mathcal{U}_{n-1}: W \cap U \neq \emptyset\}| < \omega$.

(c) $U$ is $r_{A(p_n(U))}$-distinguished,

(d) $A(W) \subset A(U)$ for every $W \in \mathcal{U}_{n-1}$ with $W \cap U \neq \emptyset$,

(e) $\text{supp}(x_U^j) \subset A(U), \ j = 0, 1$.

(f) $\mu_U(x, y) = \mu_U(r_{A(U)}(x), r_{A(U)}(y))$.

(g) $\mu_{r_{A(p_n(U))}}(\text{diam}(U)) < 1/2^{n-1}$.

(h) $\mu_U = \max[\mu_{W^1}, \max\{\mu_U: W \in \mathcal{U}_{n-1}, W \cap U \neq \emptyset\}]$.

**Proof.** First, let $\mathcal{U}_0 = \{\Sigma\}$, take $A(\Sigma) \in [\Omega]^{<\omega} \setminus \{\emptyset\}$ and $x^0_\Sigma, x^1_\Sigma \in \Sigma$ arbitrarily, and set $\mu_\Sigma = d_{A(\Sigma)}$. Assume $\mathcal{U}_k$ and $p_k, k \leq n - 1$, are constructed so as to satisfy conditions from (a) to (h) above.

Fix $U \in \mathcal{U}_{n-1}$. Now, we have:

$$r_{A(U)}(W): W \in \mathcal{U}_{n-1}, W \cap U \neq \emptyset$$ is locally finite in $r_{A(U)}(\Sigma)$.

To prove (9), it suffices to show that $W$ is $r_{A(U)}$-distinguished for every $W \in \mathcal{U}_{n-1}$ with $W \cap U \neq \emptyset$.

It follows from (a) that $W \subset p_{n-1}(W)$, hence we have $p_{n-1}(W) \cap U \neq \emptyset$. By (d), we have $A(p_{n-1}(W)) \subset A(U)$. It follows from (c) that $W$ is $r_{A(p_{n-1}(W))}$-distinguished, hence, $W$ is $r_{A(U)}$-distinguished. So, (9) is proved.

By (h), notice that $\mu_U$ metrizes $r_{A(U)}(\Sigma)$. Hence, for every $x \in r_{A(U)}(\Sigma)$, take a neighborhood $O_x$ of $x$ in $r_{A(U)}(\Sigma)$ such that $|\{W \in \mathcal{U}_{n-1}: r_{A(U)}(W) \cap O_x \neq \emptyset, W \cap U \neq \emptyset\}| < \omega$ and that $\mu_U(\text{diam}(O_x)) < 1/2^{n-1}$. For every $\delta \in [A(U)]^{<\omega}$, let us consider the natural projection $r^{A(U)}_{\delta}: \prod_{\alpha \in A(U)} X_\alpha(= r_{A(U)}(\Sigma)) \to \prod_{\alpha \in \delta} X_\alpha$. By Claim 2, there exists a locally finite cover $\mathcal{A}_\delta$ of $r_{A(U)}(\Sigma)$ by members of $\bigwedge\{\bigcup_{x \in \delta(r_{A(U)}(\Sigma))}\mu(x)\}$ such that $\mathcal{A}_\delta$ refines $\{O_x: x \in r_{A(U)}(\Sigma)\}$. Define $\mathcal{V}_\delta = \{r^{A(U)}_{\delta}(B) \cap U: B \in \mathcal{A}_\delta\}$. Then, $U = \bigcup_{\delta} \mathcal{V}_\delta$, and $\mathcal{V}_\delta$ is a locally finite collection of $\Sigma$ by members of $\mathcal{B}$. From assumptions (a), (d) and (e), we can show that $U$ is $r_{A(U)}$-distinguished. Hence, by the definition of $\mathcal{V}_U$, we have:

$$V \text{ is } r_{A(U)} \text{-distinguished for every } V \in \mathcal{V}_U.$$ (10)

Moreover, we have:

$$\mu_U(\text{diam}(V)) < 1/2^{n-1} \text{ for every } V \in \mathcal{V}_U.$$ (11)

$$|\{W \in \mathcal{U}_{n-1}: W \cap V \neq \emptyset\}| < \omega \text{ for every } V \in \mathcal{V}_U.$$ (12)

To prove (11) and (12), let $V \in \mathcal{V}_U$. Then, $V$ is expressed as $V = r_{A(U)}^{-1}(B) \cap U$ for some $B \in \mathcal{A}_\delta$. Take $O_x$ such that $B \subset O_x$. By (f), $\mu_U(\text{diam}(r_{A(U)}(B))) = \mu_U(\text{diam}(B)) < 1/2^{n-1}$. This completes the proof of (11). On the other hand, we can show that $r_{A(U)}(W) \cap B \neq \emptyset$ and $W \cap U \neq \emptyset$ for every $W \in \mathcal{U}_{n-1}$ with $W \cap V \neq \emptyset$. From the definition of $O_x$, we have $|\{W \in \mathcal{U}_{n-1}: W \cap V \neq \emptyset\}| \leq |\{W \in \mathcal{U}_{n-1}: r_{A(U)}(W) \cap B \neq \emptyset, W \cap U \neq \emptyset\}| < \omega$. Since $B \subset O_x$, the proof of (12) is completed.
Set \( \mathcal{U}_n = \bigcup_{U \in \mathcal{U}_{n-1}} \mathcal{V}_U \) and define \( p_n: \mathcal{U}_n \to \mathcal{U}_{n-1} \) by \( p_n(V) = U \) so as to satisfy \( V \in \mathcal{V}_U \). Take \( x_i^j \in \overline{V} \cap F_j \) for \( j = 0, 1 \) and \( V \in \mathcal{U}_n \) if exists. By (12), for \( V \in \mathcal{U}_n \), we can put \( A(V) = \bigcup \{ A(W): W \in \mathcal{U}_{n-1}, W \cap V \neq \emptyset \} \cup \text{supp}(x_i^0) \cup \text{supp}(x_i^1) \) (where \( \text{supp}(x_i^j) = \emptyset \) if \( \overline{V} \cap F_j = \emptyset \)), and define \( \mu_V = \max\{d_{A(V)}: W \in \mathcal{U}_{n-1}, W \cap V \neq \emptyset \} \). Then, \( \mu_V \) is continuous pseudometric on \( \Sigma \) which metrizes \( r_{A(V)}(\Sigma) \).

It remains to show conditions from (a) to (h) are satisfied. We only check (f). Let \( V \in \mathcal{U}_n \). By the definition, we have \( d_{A(V)}(x, y) = d_{A(V)}(r_{A(V)}(x), r_{A(V)}(y)) \). On the other hand, by (f) on the assumption of induction and (d), it follows that \( \mu_W(x, y) = \mu_W(r_{A(W)}(x), r_{A(W)}(y)) = \mu_W(r_{A(W)}(x), r_{A(W)}(y)) = \mu_W(r_{A(V)}(x), r_{A(V)}(y)) \) for every \( W \in \mathcal{U}_{n-1} \) with \( W \cap V \neq \emptyset \). Hence, (f) holds. This completes the proof of Claim 3.

Set \( \mathcal{U}_n^+ = \{ V \in \mathcal{U}_n: \overline{V} \cap F_j = \emptyset \) for \( j = 0 \) or \( 1 \} \) and \( \mathcal{U}_n^- = \mathcal{U}_n - \mathcal{U}_n^+ \). Next, we have:

Claim 4. \( \bigcup_{n \in \mathbb{N}} \mathcal{U}_n^+ \) is a cover of \( \Sigma \).

Proof. Since this proof is similar to that of Gulko [6], we leave it to the reader. □

Set \( \mathcal{L}_1 = \mathcal{U}_1^+ \) and \( \mathcal{L}_n = \mathcal{U}_n^+ \cap \mathcal{U}_{n-1}^- \) for every \( n \geq 2 \). Define \( \mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n \). Since \( \mathcal{L} \subset \mathcal{B} \), to complete the proof, it suffices to show the following Claims 5, 6 and 7.

Claim 5. For every \( L \in \mathcal{L} \), either \( \overline{L} \cap F_0 = \emptyset \) or \( \overline{L} \cap F_1 = \emptyset \) holds.

Proof. This follows from the fact that \( L \subset U \) for some \( U \in \mathcal{U}_n^+ \), \( n \in \mathbb{N} \). □

Claim 6. \( \mathcal{L} \) is a cover of \( \Sigma \).

Proof. Let \( x \in \Sigma \). By using Claim 4, we can take the minimum \( n \) satisfying that \( x \in \bigcup \mathcal{U}_n^+ \). Then, notice that \( x \in \bigcup \mathcal{U}_n^- \) if \( n \geq 2 \). □

Claim 7. \( \mathcal{L} \) is locally finite in \( \Sigma \).

Proof. Fix \( x \in \Sigma \). By Claim 4, there exist \( n \in \mathbb{N} \) and \( U \in \mathcal{U}_n^+ \) such that \( x \in U \). We may assume \( \overline{U} \cap F_0 = \emptyset \). By (a), (d) and (c) of Claim 3, \( U \) is \( r_{A(U)} \)-distinguished. Moreover, by (h) of Claim 3, \( \mu_U \) metrizes \( r_{A(U)}(\Sigma) \). Hence, we can take \( \varepsilon > 0 \) such that \( B_{\mu_U}(x; \varepsilon) \subset U \). Moreover, take \( m \in \mathbb{N} \) such that \( m \geq n \) and \( 1/2^m < \varepsilon \), and consider \( O = B_{\mu_U}(x; 1/2^{m+1}) \), which is an open neighborhood of \( x \) in \( \Sigma \). Now, we have:

\[
O \cap W \neq \emptyset, \ W \in \mathcal{U}_{m+2} \implies W \in \mathcal{U}_{m+2}^+.
\] (13)

To prove (13), let \( W \in \mathcal{U}_{m+2} \) with \( O \cap W \neq \emptyset \), and take \( y \in O \cap W \). Pick \( w \in W \) and fix it. Put \( W_{m+1} = p_{m+2}(W) \). Since \( y, w \in W \in \mathcal{U}_{m+2} \), by (g) of Claim 3, we have \( \mu_{W_{m+1}}(y, w) < 1/2^{m+1} \). Put \( W_m = p_{m+1}(W_{m+1}) \) and \( W_m = p_m(W_m) \), and continue this process until we have \( W \subset W_{m+1} \subset W_m \subset \cdots \subset W_{n+2} \subset W_{n+1} \) satisfying that
\( p_{k+1}(W_{k+1}) = W_k \in U_k, \ m \geq k \geq n + 1. \) Hence, by (h) of Claim 3, we have \( \mu_{W_{n+1}} \leq \mu_{W_{n+2}} \leq \cdots \leq \mu_{W_{m+1}}. \) Since \( U \cap W_{n+1} \supset O \cap W \neq \emptyset, \) it follows from (h) of Claim 3 again that \( \mu_U \leq \mu_{W_{n+1}}. \) So, we have \( \mu_U \leq \mu_{W_{m+1}}. \) Thus, \( \mu_U(x, w) \leq \mu_U(x, y) + \mu_U(y, w) \leq \mu_U(x, y) + \mu_{W_{m+1}}(y, w) < 1/2^{m+1} + 1/2^{m+1} < \varepsilon. \) This shows that \( w \in B_{\mu_U}(x; \varepsilon) \subset U. \)

Thus, we have \( W \subset U. \) Hence, \( \overline{W} \cap F_0 = \emptyset, \) which shows \( W \in U_{m+2}^+. \) This completes the proof of (13).

By using (13), we can show that:

\[
O \cap W \neq \emptyset, \ W \in U_j, \ j \geq m + 2 \implies W \in U_j^+. \tag{14}
\]

Finally, by using (14), we have:

\[
O \cap L \neq \emptyset, \ L \in \mathcal{L} \implies L \in \bigcup_{i \leq m+2} \mathcal{L}_i. \tag{15}
\]

Since each \( \mathcal{L}_i \) is locally finite at \( x, \) from (15), we have that \( \mathcal{L} \) is locally finite at \( x. \) This completes the proof of Claim 7. \( \square \)

This completes the proof of Theorem 1.3. \( \square \)

6. Applications for analogues of classical theorems

In this section, we apply Theorems 1.1, 1.2 and 1.3 and other results obtained in Section 2 to give analogue of classical theorems on normality of products.

A version of the Morita–Rudin–Starbird Theorem is given as follows:

**Theorem 6.1.** For a base-normal space \( X \) and a non-discrete metrizable space \( Y, \) the following statements are equivalent:

1. \( X \times Y \) is normal (or equivalently, countably paracompact);
2. \( X \times Y \) is base-normal;
3. \( X \times Y \) is base-countably paracompact.

**Proof.** This follows from Theorem 1.1, Proposition 2.2 and the Morita–Rudin–Starbird Theorem. \( \square \)

Next, we consider the base-normality of products with a compact factor. The following is a version of the Dowker Theorem.

**Theorem 6.2.** For a space \( X, \) the following statements are equivalent:

1. \( X \) is base-normal and base-countably paracompact;
2. \( X \times C \) is base-normal for every compact metrizable space \( C; \)
3. \( X \times I \) is base-normal.
A version of the Morita Theorem is also given as follows. Note that it is unknown whether base-normality of \( X \times I \) (or, of \( X \times C \)) implies base-normality of \( X \) or not.

**Theorem 6.3.** For a base-normal space \( X \), the following statements are equivalent:

1. \( X \) is base-\( \kappa \)-paracompact;
2. \( X \times C \) is base-normal for every compact Hausdorff space \( C \) with \( w(C) \leq \kappa \);
3. \( X \times I \) is base-normal.

For the proofs of Theorems 6.2 and 6.3, we need the following basic result.

**Proposition 6.4.** Let \( X \) be a base-\( \kappa \)-paracompact space and \( Y \) a compact space with \( w(Y) \leq \kappa \). Then, \( X \times Y \) is base-\( \kappa \)-paracompact.

**Proof.** Let \( B_X \) be a base which witnesses base-\( \kappa \)-paracompactness for \( X \), and \( B_Y \) a base for \( Y \) with \( w(Y) = |B_Y| \leq \kappa \). We express \( B_Y = \{ B_\beta : \beta \in \Omega \} \) with \( |\Omega| \leq \kappa \).

We shall show that \( B_X \times B_Y \) is a base which witnesses base-\( \kappa \)-paracompactness for \( X \times Y \). To do this, let \( \{ U_\alpha : \alpha \in A \} \) be an open cover of \( X \times Y \) with \( |A| \leq \kappa \). For every \( \{ (\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n) \} \in [A \times [\kappa]^\omega]^{< \omega} \) satisfying that \( \bigcup \{ B_\beta : \beta \in \bigcup_{i=1}^n \delta_i \} = Y \), we set \( G((\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n)) = \bigcup \{ O \in T(X) : O \times B_\beta \subseteq U_\alpha \text{ for every } \beta \in \delta_i \text{ and every } i = 1, \ldots, n \} \). Define \( W = \{ G((\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n)) : \{ (\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n) \} \in [A \times [\kappa]^\omega]^{< \omega} \text{ with } \bigcup_{\beta \in \bigcup_{i=1}^n \delta_i} B_\beta = Y \} \). Since \( W \) is an open cover of \( X \) with \( |W| \leq \kappa \), \( W \) has a locally finite refinement \( B' \) by members of \( B_X \). For each \( B' \in B' \), fix an \( \{ (\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n) \} \in [A \times [\kappa]^\omega]^{< \omega} \text{ such that } B' \subseteq G((\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n)) \), and put \( \delta_{B'} = \bigcup_{i=1}^n \delta_{i} \). Now, \( \{ B' \in B : \beta \in \delta_{B'}, B' \in B' \} \) is the required refinement by members of \( B_X \times B_Y \). Thus, \( X \times Y \) is base-\( \kappa \)-paracompact, which completes the proof. \( \square \)

Proposition 6.4 is a slight improvement of [15, Corollary 3.9]. On the other hand, it was proved that base-paracompactness is inverse invariant of perfect mappings [15, Theorem 3.6]. We do not know whether a similar result holds or not for base-\( \kappa \)-paracompact spaces.

**Proof of Theorem 6.2.** (1) \( \Rightarrow \) (2): Use the Dowker Theorem and Propositions 2.2 and 6.4 (or Theorem 1.1).

(2) \( \Rightarrow \) (3): Obvious.

(3) \( \Rightarrow \) (1): We may assume \( w(X) \geq \omega \). Since \( w(X \times I) = w(X) \), from the assumption (3), we have \( X \) is base-normal. Apply the Dowker Theorem and Proposition 2.2. \( \square \)

**Proof of Theorem 6.3.** (1) \( \Rightarrow \) (2): Use the Morita Theorem, Propositions 2.2 and 6.4.

(2) \( \Rightarrow \) (3): Obvious.

(3) \( \Rightarrow \) (1): Use the Morita Theorem, the assumption of the base-normality of \( X \) and Proposition 2.2. \( \square \)
Analogous to the work of M.E. Rudin and M. Starbird in [18], we further have the following: the proof is obtained by Theorem 1.1 together with [18, Theorem 2] and Proposition 2.2.

**Corollary 6.5.** For a base-normal and base-\(\kappa\)-paracompact space \(X\) and a metrizable space \(Y\), the product space \(X \times Y\) is normal (or equivalently, countably paracompact) if and only if \(X \times Y\) is base-normal and base-\(\kappa\)-paracompact.

By Corollary 6.5, we immediately have:

**Corollary 6.6.** For a base-paracompact Hausdorff space \(X\) and a metrizable space \(Y\), the product space \(X \times Y\) is normal if and only if \(X \times Y\) is base-paracompact.

Corollary 6.6 extends the following result obtained by J.E. Porter [15, Theorem 3.15]:

For a hereditarily Lindelöf space \(X\) and a metrizable space \(Y\), the product space \(X \times Y\) is base-paracompact. It should be noted that Y. Yajima [20] proved that for a base-paracompact Hausdorff space \(X\) and a base-paracompact stratifiable space \(Y\), the product space \(X \times Y\) is paracompact if and only if \(X \times Y\) is base-paracompact. This result and [18, Theorem 2] also imply Corollary 6.6. See [5] for basic facts of generalized metric spaces.

Moreover, by using [18,10,11,13], we have the following further results analogous to the studies of M.E. Rudin and M. Starbird, K. Morita and A. Okuyama.

Let \(X\) be a space, \(C\) a compact Hausdorff space, and \(M\) a metrizable space. If \(X \times C\) is base-normal and \(X \times M\) is normal, then \(X \times C \times M\) is base-normal.

Let \(X\) be a normal and \(\kappa\)-paracompact space, \(C\) a compact Hausdorff space, and \(M\) a metrizable space. If \(X \times C\) is base-normal and \(X \times M\) is normal, then \(X \times C \times M\) is base-normal and base-\(\kappa\)-paracompact.

A space \(X\) is a base-normal \(P(\kappa)\)-space if and only if \(X \times Y\) is base-normal for every metrizable space \(Y\) with \(\omega(Y) \leq \kappa\).

A space \(X\) is a base-normal weak \(P(\kappa)\)-space if and only if \(X \times Y\) is base-normal for every completely metrizable space \(Y\) with \(\omega(Y) \leq \kappa\).

See [10,11,13] for definitions of \(P(\kappa)\)-spaces and weak \(P(\kappa)\)-spaces.

Next, let us consider the case of infinite products. A version of the Nagami–Zenor Theorem is given as follows.

**Theorem 6.7.** Let \(X = \prod_{i \in \mathbb{N}} X_i\) be the countable product of spaces \(X_i\), and assume finite subproducts \(\prod_{i \leq n} X_i\), \(n \in \mathbb{N}\), are base-normal and base-countably paracompact. Then, the following statements are equivalent:

1. \(X\) is normal (or equivalently, countably paracompact);
2. \(X\) is base-normal;
3. \(X\) is base-countably paracompact.
In the above theorem, if each $X_j$ contains at least two points, this remains true without base-countable paracompactness of $\prod_{i \in \kappa} X_i$, $n \in \mathbb{N}$.

Recall another theorem obtained by K. Nagami [12] and P. Zenor (see [4, 5.5.19(c)]) that: Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product, where each $X_i$ is a Hausdorff space containing at least two points. Then, $X$ is normal $\kappa$-paracompact if and only if finite subproducts $\prod_{i \in \kappa} X_i$, $n \in \mathbb{N}$, are normal $\kappa$-paracompact and $X$ is countably paracompact.

By this result together with Theorem 6.7 and Proposition 2.2, we have:

**Corollary 6.8.** Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product of spaces $X_i$, and assume finite subproducts $\prod_{i \in \kappa} X_i$, $n \in \mathbb{N}$, are base-normal base-$\kappa$-paracompact. Then, $X$ is normal (or equivalently, countably paracompact) if and only if $X$ is base-normal and base-$\kappa$-paracompact.

By Corollary 6.8, we immediately have:

**Corollary 6.9.** Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product of Hausdorff spaces $X_i$, and assume finite subproducts $\prod_{i \in \kappa} X_i$, $n \in \mathbb{N}$, are base-paracompact. Then, $X$ is normal (or equivalently, countably paracompact) if and only if $X$ is base-paracompact.

Furthermore, by Corollary 6.9, we have the following results for uncountable products.

**Corollary 6.10.** Let $X = \prod_{\alpha \in \Omega} X_{\alpha}$ be the product of Hausdorff spaces $X_{\alpha}$ and assume all finite subproducts $\prod_{\alpha \in \delta} X_{\alpha}$, $\delta \in [\Omega]^{<\omega}$, are base-paracompact. Then, $X$ is normal (or equivalently, countably paracompact) if and only if $X$ is base-paracompact.

**Proof.** Assume $X = \prod_{\alpha \in \Omega} X_{\alpha}$ is normal. Now, we may assume all $X_{\alpha}$, $\alpha \in \Omega$, contain at least two points. For, if $X_{\alpha}$ contains at least two points for only countably many $\alpha \in \Omega$, the proof follows from Corollary 6.9. By A.H. Stone [19], there is $\Omega' \subset \Omega$ with $|\Omega'| \leq \omega_0$ such that $X_{\alpha}$ is countably compact for every $\alpha \in \Omega - \Omega'$. Hence, $\prod_{\alpha \in \Omega - \Omega'} X_{\alpha}$ is compact. On the other hand, it follows from Corollary 6.9 that $\prod_{\alpha \in \Omega} X_{\alpha}$ is base-paracompact. It follows from [15, Corollary 3.9] (or Proposition 6.4 above) that $X = (\prod_{\alpha \in \Omega - \Omega'} X_{\alpha}) \times (\prod_{\alpha \in \Omega - \Omega'} X_{\alpha})$ is base-paracompact. This completes the proof. $\Box$

In [12] (see also [4, 5.5.19(b)]), K. Nagami also proved a similar result for collectionwise normality. Here, we define a space $X$ base-$\kappa$-collectionwise normal if there is a base $B$ for $X$ with $|B| = w(X)$ such that for every discrete closed collection $\{F_{\alpha} : \alpha \in \Omega\}$ of $X$ with $|\Omega| \leq \kappa$, there is a locally finite cover $B'$ of $X$ by members of $B$ with, for every $B' \in B'$, $|\alpha \in \Omega : \overline{B} \cap F_{\alpha} \neq \emptyset| \leq 1$. We say a space $X$ base-collectionwise normal if $X$ is base-$\kappa$-collectionwise normal for every $\kappa$. By the same argument as in Section 2, we have the fact that: $X$ is base-$\kappa$-collectionwise normal if and only if $X$ is $\kappa$-collectionwise normal and base-normal.

We conclude this paper by giving two results. One is an analogue of the Nagami–Zenor Theorem for base-collectionwise normality: the proof is obtained from Theorem 1.2 and the Nagami Theorem [12] (see [4, 5.5.19(b)]). Another is a straightforward generalization of Theorem 1.3: the proof is obtained from Theorem 1.3 and [6, Remark, p. 1439].
Corollary 6.11. Let $X = \prod_{i \in \mathbb{N}} X_i$ be the countable product, where each $X_i$ contains at least two points, and assume finite subproducts $\prod_{i \leq n} X_i$, $n \in \mathbb{N}$, are base-collectionwise normal. Then, $X$ is normal (or equivalently, countably paracompact) if and only if and $X$ is base-collectionwise normal.

Corollary 6.12. Every $\Sigma$-product of metric spaces is base-collectionwise normal.

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References