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The computable dimension of ordered abelian groups

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Abstract

Let G be a computable ordered abelian group. We show that the computable dimension of G is either 1 or ω, that G is computably categorical if and only if it has finite rank, and that if G has only finitely many Archimedean classes, then G has a computable presentation which admits a computable basis.

MSC: 03D; 06F

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1. Introduction

In this article, we examine countable ordered abelian groups from the perspective of computable algebra. We begin with the definition and some examples of ordered abelian groups.

Definition 1.1. An *ordered abelian group* is a pair $(G, \leqslant_G)$, where G is an abelian group and $\leqslant_G$ is a linear order on G such that if $a \leqslant_G b$, then $a + g \leqslant_G b + g$ for all $g \in G$.

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The simplest examples of ordered abelian groups are the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ with their usual orders. Another example is $\sum_\omega \mathbb{Z}$, the restricted sum of ω many copies of $\mathbb{Z}$. The elements of this group are functions $g : \mathbb{N} \to \mathbb{Z}$ with finite support. To compare two distinct elements g and h, find the least n such that $g(n) \neq h(n)$ and set $g < h$ if and only if $g(n) < h(n)$.

An abelian group is orderable if and only if it is torsion free. Therefore, all groups in this article are torsion free. Also, since we consider only computable groups (defined below), all groups in this article are countable.

One of the fundamental problems in computable algebra is to determine which classical theorems are effectively true. That is, we ask whether a classical theorem holds when all the algebraic objects are required to be computable. To illustrate this perspective, consider the following two classical theorems of field theory: every field has an algebraic closure, and a field is orderable if and only if it is formally real. Rabin [14] proved that the first theorem is effectively true, and Metakides and Nerode [12] proved that the second theorem is not effectively true. That is, every computable field has a computable algebraic closure, but there are computable formally real fields which do not have a computable order.

To apply the techniques of computability theory to a class of algebraic structures, we must first code these structures into the natural numbers. In the case of ordered abelian groups, this means that we choose a computable set $G \subset \mathbb{N}$ of group elements along with a computable function $+_G : G \times G \to G$ and a computable relation $\leqslant_G \subset G \times G$ which obey the axioms for an ordered abelian group. The triple $(G, +_G, \leqslant_G)$ is called a *computable ordered abelian group*. For simplicity, we often drop the subscripts on $+_G$ and $\leqslant_G$, and we abuse notation by referring to the computable ordered abelian group as G. If H is an abstract ordered abelian group and G is a computable ordered group such that $H \cong G$, then G is called a *computable presentation* of H. The intuition is that G is a coding of H into the natural numbers to which we can apply the techniques of computability theory.

For completeness, we give a more general definition of a computable structure, which agrees with the definition above for the class of ordered abelian groups. The most general definition, which allows the possibility of infinite languages, is not needed here.

Definition 1.2. An algebraic structure $\mathfrak{A}$ with finitely many functions and relations is *computable* if the domain of the structure and each of the functions and relations is computable. A *computable presentation* of a structure $\mathfrak{B}$ is a computable structure $\mathfrak{A}$ which is isomorphic to $\mathfrak{B}$.

In this article, we consider only abstract ordered abelian groups which have some computable presentation. Notice that this includes the examples given above, as well as most naturally occurring countable examples. That is, it takes some work to build a countable ordered group that has no computable presentation.

If an abstract ordered abelian group H has a computable presentation, then it will have many different computable presentations. One of the goals of computable algebra is to study how the effective properties of H depend upon the chosen presentation or coding. Consider the following example. Downey and Kurtz [2] proved that there is a computable torsion free abelian group which has no computable order and also no computable basis. Therefore, the theorem stating that every torsion free abelian group has both an order and a basis is not effectively true. In their proof, Downey and Kurtz gave a complicated coding of $\sum_{\omega} \mathbb{Z}$ which diagonalized against the existence of a computable order. However, it is clear that if the group $\sum_{\omega} \mathbb{Z}$ is coded in a "nice" way, then it will have a computable basis and the lexicographic order described above will be computable.

The next reasonable question to ask is if every torsion free abelian group which has a computable presentation also has one which admits a computable basis and a computable order. The answer turns out to be yes, as shown for a basis in [1] and for an order (which is a trivial consequence of Dobritsa's work) in [18]. Therefore, if a computable torsion free abelian group does not have a computable basis or a computable order, then it is a consequence of the coding as opposed to a fundamental property of the abstract isomorphism type of the group.

Unfortunately, Dobritsa's methods do not in general preserve orders. However, we will prove that an analogue of Dobritsa's result does hold for a wide class of computable ordered abelian groups. (The terms from ordered group theory are defined after the introduction.)

Theorem 1.3. *If G is a computable Archimedean ordered group, then G has a computable presentation which admits a computable basis.*

Theorem 1.4. *If G is a computable ordered abelian group with finitely many Archimedean classes, then G has a computable presentation which admits a computable nonshrinking basis.*

The computable ordered abelian groups which are the least affected by issues of coding are those for which there is a computable isomorphism between any two computable presentations. Such groups are called *computably categorical*. More generally, we look at computable structures up to computable isomorphism. That is, we regard two computable structures as equivalent if there is a computable isomorphism between them. This intuition motivates the following definition.

Definition 1.5. Let $\mathfrak{A}$ be a computable structure. The *computable dimension* of $\mathfrak{A}$ is the number of computable presentations of $\mathfrak{A}$ up to computable isomorphism. If the computable dimension of $\mathfrak{A}$ is 1, then $\mathfrak{A}$ is called *computably categorical* or *autostable*.

A considerable amount of work has been done on the question of which computable dimensions occur in various classes of algebraic structures.

Theorem 1.6 (Dzgoev and Goncharov [3]; Goncharov [6,7]; Larouche [11]; Metakides and Nerode [12]; Nurtazin [13]; Remmel [15]). *Every computable linear order, Boolean algebra, abelian group, algebraically closed field, and real closed field has computable dimension* 1 or ω.

For several of these classes of structures, there are algebraic conditions which separate the computably categorical structures from those which have computable dimension ω. For example, a computable linear order is computably categorical if and only if it has finitely many successive pairs of elements, and a computable Boolean algebra is computably categorical if and only if it has finitely many atoms.

These examples, unfortunately, give a picture that is too simple to hold in general. The following theorem shows that for other classes of algebraic structures, there exist computable structures which have finite computable dimensions other than 1.

Theorem 1.7 (Dzgoev and Goncharov [3]; Goncharov et al. [9]). *For each* $1 \leqslant n \leqslant \omega$, *the following classes of algebraic structures contain examples which have computable dimension exactly n: partially ordered sets, graphs, lattices, and nilpotent groups.*

The class of ordered abelian groups is interesting from the perspective of computable dimension because these groups have both an addition function and an ordering relation. Of the examples listed above, only Boolean algebras have both functions and an ordering, but for Boolean algebras, the order is definable from the meet and join. Furthermore, Goncharov has proved two general theorems, the Unbounded Models Theorem and the Branching Models Theorem (see [4]), stating conditions under which all computable structures from a particular class of structures must have dimension 1 or ω. For ordered abelian groups, neither of these theorems appears to apply. However, our main result, Theorem 1.8, shows that computable ordered abelian groups must have computable dimension 1 or ω. Theorems 1.3 and 1.4 will be established during the proof of Theorem 1.8.

Theorem 1.8. *Every computable ordered abelian group has computable dimension* 1 *or* ω. *Furthermore, such a group is computably categorical if and only if it has finite rank.*

If G has finite rank, then clearly G is computably categorical. In fact, not only are any two computable presentations of G computably isomorphic, every isomorphism between two computable presentations is computable. It remains to show that if G has infinite rank, then the computable dimension of G is ω. We use the following theorem from computable model theory to simplify our work.

Theorem 1.9 (Goncharov [8]). *If a countable model* $\mathscr{A}$ *has two computable presentations,* $\mathscr{A}_1$ *and* $\mathscr{A}_2$, *which are* Δ^0_2 *but not computably isomorphic, then* $\mathscr{A}$ *has computable dimension* ω.

We split the proof of Theorem 1.8 into three cases. Since the interplay between the group structure and the ordering can be quite complicated, we have to introduce new algebra in each case to handle the internal combinatorics.

Theorem 1.10. *If G is a computable ordered abelian group with infinitely many Archimedean classes, then G has computable dimension ω.*

Theorem 1.11. *If G is a computable Archimedean ordered group, then G has computable dimension* 1 *or ω. Furthermore, G is computably categorical if and only if G has finite rank.*

Theorem 1.12. *If G is a computable abelian ordered group with finitely many Archimedean classes, then G has computable dimension* 1 *or ω. Furthermore, G is computably categorical if and only if G has finite rank.*

In Section 2, we present some background material in ordered abelian group theory. In Section 3, we present the algebra necessary to prove Theorem 1.10, and we give the proof in Section 4. In Sections 5 and 6, we describe the computability theory and the algebra, respectively, used in the proofs of Theorems 1.11 and 1.3. We prove Theorems 1.11 and 1.3 in Section 7 and we prove Theorems 1.12 and 1.4 in Section 8.

The notation is standard and follows [16] for computability theory, and both [5,10] for ordered abelian groups. The term computable always means Turing computable and we use φ_e, $e \in \omega$, to denote an effective list of the partial computable functions. If we designate a number n as "large" during a construction, let n be the least number which is larger than any number used in the construction so far.

2. Ordered abelian groups

In this section, we introduce several useful concepts from the theory of ordered groups.

Definition 2.1. Let G be an ordered group. The *absolute value* of $g \in G$, denoted by $|g|$, is whichever of g or $-g$ is positive. For $g, h \in G$, we say g is *Archimedean equivalent to* h, denoted $g \approx h$, if there exist $n, m \in \mathbb{N}$ with $n, m > 0$, such that $|g| \leqslant_G |nh|$ and $|h| \leqslant_G |mg|$. If $g \not\approx h$ and $|g| < |h|$, g is *Archimedean less than* h, denoted $g \ll h$. G is an *Archimedean* group if $g \approx h$ for every $g, h \in G \setminus \{0_G\}$.

The Archimedean classes of G are the equivalence classes under $\approx$. Although technically 0_G forms its own Archimedean class, we typically ignore this class and consider only the nontrivial Archimedean classes.

In Section 5, we give a full discussion of Hölder's Theorem, but we state it here since it is used in the proof of Lemma 3.5.

Hölder's Theorem. *If G is an Archimedean ordered group, then G is isomorphic to a subgroup of the naturally ordered additive group $\mathbb{R}$.*

Definition 2.2. Let G be a torsion free abelian group. The elements $g_0, \dots, g_n \in G$ are *linearly independent* if, for all $c_0, \dots, c_n \in \mathbb{Z}$, the equality

$$c_0 g_0 + c_1 g_1 + \cdots + c_n g_n = 0$$

implies that $c_i = 0$ for all i. An infinite set is linearly independent if every finite subset is independent. A maximal linearly independent set is called a *basis*, and the cardinality of any basis is called the *rank* of G.

If a torsion free abelian group is divisible, then it forms a vector space over $\mathbb{Q}$. In this case, these definitions agree with the corresponding terms for a vector space. Notice that if g and h are in different Archimedean classes, then they are independent. Therefore, if G has infinitely many Archimedean classes, then G has infinite rank.

Definition 2.3. If $X = \{x_i | i \in \mathbb{N}\}$ is a basis for G, then each $g \in G$, $g \neq 0_G$, satisfies a *dependence relation* (*or equation*) of the form

$$\alpha g = c_0 x_0 + \cdots + c_n x_n,$$

where $\alpha \in \mathbb{N}$, $\alpha \neq 0$, and each $c_i \in \mathbb{Z}$. A dependence relation is called *reduced* if $\alpha > 0$ and the greatest common divisor of α and the nonzero c_i coefficients is 1.

Obviously, any dependence relation can be transformed into a reduced one by dividing. Suppose g and h both satisfy the equation $\alpha y = c_0 x_0 + \cdots + c_n x_n$. Then, $\alpha(g - h) = 0_G$, and since we consider only torsion free groups, $g = h$. Therefore, any dependence relation (regardless of whether $x_0, \dots, x_n$ are independent) has at most one solution. It will also be important that in any reduced equation, the coefficient α is required to be positive.

Definition 2.4. For any $X \subset G$, we define the *span* of X to be the set of solutions to the reduced equations $\alpha y = c_0 x_0 + c_1 x_1 + \cdots + c_k x_k$, where each $x_i \in X$. The span of X is denoted by $\mathrm{Span}(X)$.

The notion of t-independence will be used to approximate a basis during the constructions.

Definition 2.5. The elements $g_0, \dots, g_n$ are *t-independent* if for all $c_0, \dots, c_n \in \mathbb{Z}$ with $|c_i| \leqslant t$, $c_0 g_0 + \cdots c_n g_n = 0_G$ implies that each $c_i = 0$. The elements $g_0, \dots, g_n$ are *t-dependent* if they are not t-independent.

Definition 2.6. A subgroup H is *convex* if for all $x, y \in H$ and all $g \in G$, $x \leqslant g \leqslant y$ implies that $g \in H$.

If H is a convex subgroup of G, then there is a natural order on the quotient group G/H. The *induced ordered* on G/H is defined by $a+H \leqslant_{G/H} b+H$ if and only if $a+H=b+H$ or $a+H \neq b+H$ and $a<b$. In Section 8, we will use the fact that $a+H <_{G/H} b+H$ implies that $a <_G b$.

3. Algebra for Theorem 1.10

Throughout Sections 3 and 4, G denotes a computable ordered abelian group with infinitely many Archimedean classes.

Definition 3.1. $B \subset G$ has the *nonshrinking property* if for all $\{b_1, \dots, b_n\} \subset B$ with $b_1 \approx \cdots \approx b_n$, and for all $x \in \mathrm{Span}(b_1, \dots, b_n)$, if $x \neq 0_G$, then $x \approx b_1$. A basis with the nonshrinking property is called a *nonshrinking basis*.

We first establish, noneffectively, the existence of a nonshrinking basis.

Lemma 3.2. *For any (possibly finite) independent set $B = \{b_1, b_2, \dots\}$, there is an independent set with the nonshrinking property $B' = \{b'_1, b'_2, \dots\}$ such that for every i, $\mathrm{Span}(b_1, \dots, b_i) = \mathrm{Span}(b'_1, \dots, b'_i)$.*

Proof. Set $b'_1 = b_1$. For $n > 1$, consider all sums of the form $c_1 b'_1 + \cdots + c_{n-1} b'_{n-1} + c_n b_n$, where $c_i \in \mathbb{Z}$ and $c_n \neq 0$. These sums can lie in at most $n+1$ different Archimedean classes, so there is a least Archimedean class which contains one of these elements. Set b'_n to be any of these sums which lies in this least Archimedean class. Since $c_n \neq 0$, $b_n \in \mathrm{Span}(b'_1, \dots, b'_n)$.

To verify that B' has the nonshrinking property, assume that $b'_{i_1} \approx \cdots \approx b'_{i_n}$ with $i_1 < \cdots < i_n$. Suppose there is an $x \in \mathrm{Span}(b'_{i_1}, \dots, b'_{i_n})$ such that $x \neq 0_G$ and $x \ll b'_{i_1}$. Then, x satisfies a reduced equation of the form $\alpha x = c_{i_1} b'_{i_1} + \cdots + c_{i_n} b'_{i_n}$. Without loss of generality, assume that $c_{i_n} \neq 0$. By our construction of B', b'_{i_n} can be expressed as a sum of $b'_1, \dots, b'_{i_n - 1}, b_{i_n}$ in which the coefficient of b_{i_n} is not zero. Replace b'_{i_n} in the equation for x by this sum and notice that the coefficient of b_{i_n} is not zero. Therefore, when b'_{i_n} was chosen, αx was one of the other elements considered, contradicting our choice of b'_{i_n}. $\square$

The following two lemmas follow directly from Lemma 3.2 and Definition 3.1.

Lemma 3.3. *Any finite independent set with the nonshrinking property can be extended to a nonshrinking basis.*

Lemma 3.4. *If B is a nonshrinking basis and $\{b_1, \dots, b_n\} \subset B$ with $b_1 \lessapprox b_2 \lessapprox \cdots \lessapprox b_n$, then for all $x \in \mathrm{Span}(b_1, \dots, b_n)$, if $x \neq 0_G$, then $b_1 \lessapprox x$. We use $x \lessapprox y$ to denote that either $x \approx y$ or $x \ll y$.*

The reason for working with a nonshrinking bases is that there are no "large" elements which combine with other "large" elements to become "small". To be more specific, suppose B is a nonshrinking basis and $x \approx y$ are represented by the reduced equations $\alpha x = \sum_{i \in I} c_i b_i$ and $\beta y = \sum_{j \in J} d_j b_j$. Since $\alpha, \beta > 0$, $x \leqslant y$ if and only if $\alpha\beta x \leqslant \alpha\beta y$. To determine if $x \leqslant y$, it suffices to compare the sums from the expressions $\alpha\beta x = \sum_{i \in I} (\beta c_i) b_i$ and $\alpha\beta y = \sum_{j \in J} (\alpha d_j) b_j$. Let $X = \{b_k | k \in I \cup J\}$ and let Y be the set of all k such that $b_k \in X$ and b_k is an element of the largest Archimedean class occurring among the members of X. Define $x' = \sum_{i \in I \cap Y} (\beta c_i) b_i$ and $y' = \sum_{j \in J \cap Y} (\alpha d_j) b_j$. Because B is a nonshrinking basis, $x' \approx b_k$ and $y' \approx b_k$ for all $k \in Y$. Therefore, $x' < y'$ implies that $x < y$. On the other hand, if $x' = y'$, then we can compare the parts of the sums for βx and αy generated by the basis elements in the second greatest Archimedean class in X. Assuming that $x \neq y$, we must eventually find a largest Archimedean class within X for which the sums for $\alpha\beta x$ and $\alpha\beta y$ restricted to the basis elements in X in this class disagree. Then $x < y$ if and only if the restricted sum for $\alpha\beta x$ is less than the restricted sum for $\alpha\beta y$.

We prove a sequence of lemmas, culminating in the main combinatorial lemma needed for the proof of Theorem 1.10. Our eventual goal is to show that if we have a finite set $G_s \subset G$ with subsets $C, P \subset G_s$ satisfying particular conditions, then there is a map $\delta : G_s \to G$ which preserves $+$ and $<$, which is the identity on P, and which collapses the elements of C to a single Archimedean class. This property will allow us to diagonalize against computable isomorphisms.

Lemma 3.5. *Let $g_1, \ldots, g_k$ be elements in the least nontrivial Archimedean class of G such that $g_i - g_j \approx g_i$ for all $1 \leqslant i \neq j \leqslant k$. There is a map $\varphi : \{g_1, \ldots, g_k\} \to \mathbb{Z}$ such that for all $1 \leqslant x, y, z \leqslant k$, $g_x + g_y = g_z$ if and only if $\varphi(g_x) + \varphi(g_y) = \varphi(g_z)$ and $g_x < g_y$ if and only if $\varphi(g_x) \leqslant \varphi(g_y)$. Furthermore, if $g_x > 0_G$, then $\varphi(g_x) > 0$.*

Proof. Consider the Archimedean subgroup $H = \{g \in G | g \approx g_1 \vee g = 0_G\}$, let $b_1, \ldots, b_n \in H$ be independent positive elements such that each g_i is dependent on $\{b_1, \ldots, b_n\}$, and let t be such that each g_i is actually t-dependent on $\{b_1, \ldots, b_n\}$. Each g_i satisfies a unique reduced equation $\alpha g_i = \alpha_1 b_1 + \cdots + \alpha_n b_n$ in which $0 < \alpha \leqslant t$ and each $|\alpha_i| \leqslant t$. Applying Hölder's Theorem, fix an embedding $\psi : H \to \mathbb{R}$ such that $\psi(b_1) = 1$ and assume $\psi(b_i) = r_i$ for $1 < i \leqslant n$.

Look at all sums of the form $\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n$ in which each $\beta_i \in \mathbb{Z}$ and $|\beta_i| \leqslant 2t^3$. Because $r_1, \ldots, r_n$ are independent, the sums corresponding to different choices of coefficients are different. Let $q \in \mathbb{Q}$, $q > 0$, be strictly less than the difference between any two distinct sums of this form, let $q' \in \mathbb{Q}$ be such that $0 < q' < q/9nt^3$, and pick $q_2, \ldots, q_n \in \mathbb{Q}$ such that $|r_i - q_i| \leqslant q'$.

Next, we prove four claims about sums involving the numbers r_i and q_i. Fix arbitrary distinct sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$, $\langle \beta_1, \ldots, \beta_n \rangle$, and $\langle \gamma_1, \ldots, \gamma_n \rangle$ such that each $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$ and $|\alpha_i|, |\beta_i|, |\gamma_i| \leqslant t^3$.

Our first claim is that for such sequences,

$$\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n < \beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n$$
$$\Leftrightarrow \alpha_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n < \beta_1 + \beta_2 q_2 + \cdots + \beta_n q_n.$$

This claim follows because

$$|(\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) - (\alpha_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)| \leqslant nt^3 q' \leqslant q/9,$$

$$|(\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n) - (\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n)| \leqslant nt^3 q' \leqslant q/9,$$

and

$$|(\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) - (\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n)| > q.$$

Our second claim is that for all sequences as above, we have

$$(\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) + (\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n) = (\gamma_1 + \gamma_2 r_2 + \cdots + \gamma_n r_n)$$
$$\Leftrightarrow (\alpha_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n) + (\beta_1 + \beta_2 q_2 + \cdots + \beta_n q_n) = (\gamma_1 + \gamma_2 q_2 + \cdots + \gamma_n q_n).$$

Since $1, r_2, \dots, r_n$ are independent, we have that the top equality holds if and only if $\gamma_i = \alpha_i + \beta_i$ for each i. Therefore, the $(\Rightarrow)$ direction is clear. To establish the $(\Leftarrow)$ direction, assume that the bottom equality holds but the top does not. We get a contradiction by considering the inequalities used to prove the first claim, together with the following inequalities:

$$|(\gamma_1 + \gamma_2 r_2 + \cdots + \gamma r_n) - (\gamma_1 + \gamma_2 q_2 + \cdots + \gamma q_n)| \leqslant q/9,$$

and

$$|[(\alpha_1 + \beta_1) + (\alpha_2 + \beta_2) r_2 + \cdots + (\alpha_n + \beta_n) r_n] - (\gamma_1 + \gamma_2 r_2 + \cdots + \gamma r_n)| > q.$$

To verify the last inequality, notice that $|\alpha_i + \beta_i| \leqslant 2t^3$.

Let m be the least common multiple of the denominators of the reduced fractions $q_2, \dots, q_n$. Let $m' = m \cdot t!$, and define $p_1 = m'$, $p_2 = m' q_2, \dots, p_n = m' q_n$. Notice that $p_i \in \mathbb{Z}$ and $t!$ divides p_i for each i.

Our third claim is that

$$\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n < \beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n$$
$$\Leftrightarrow \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n < \beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n.$$

This claim follows from the first claim because

$$\alpha_1 p_1 + \cdots + \alpha_n p_n = m'(\alpha_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)$$

and

$$\beta_1 p_1 + \cdots + \beta_n p_n = m'(\beta_1 + \beta_2 q_2 + \cdots + \beta_n q_n).$$

Our fourth (and final) claim is that

$$(\alpha_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) + (\beta_1 + \beta_2 r_2 + \cdots + \beta_n r_n) = (\gamma_1 + \gamma_2 r_2 + \cdots + \gamma_n r_n)$$

$$\Leftrightarrow (\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) + (\beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n)$$

$$= (\gamma_1 p_1 + \gamma_2 p_2 + \cdots + \gamma_n p_n).$$

This claim follows from the second claim just as the third claim follows from the first claim.

For each g_i, consider the unique reduced equation $\alpha g_i = \alpha_1 b_1 + \cdots + \alpha_n b_n$. Since ψ is a homomorphism, the equation $\alpha x = \alpha_1 + \alpha_2 r_2 \cdots + \alpha_n r_n$ has the unique solution $x = \psi(g_i)$ in $\mathbb{R}$. Because $t!$ divides each p_i and $0 < \alpha \leqslant t$, we have that

$$u_i = \alpha_1 \frac{p_1}{\alpha} + \cdots + \alpha_n \frac{p_n}{\alpha} \in \mathbb{Z}.$$

Define φ by $\varphi(g_i) = u_i$.

To verify that φ has the appropriate properties, fix x, y, z between 1 and k. There are positive numbers α, β, and γ, and integer sequences $\langle \alpha_1, \ldots, \alpha_n \rangle$, $\langle \beta_1, \ldots, \beta_n \rangle$, and $\langle \gamma_1, \ldots, \gamma_n \rangle$ with the absolute value of all numbers bounded by t such that

$$\alpha g_x = \alpha_1 b_1 + \cdots + \alpha_n b_n, \quad \beta g_y = \beta_1 b_1 + \cdots + \beta_n b_n, \quad \text{and} \quad \gamma g_z = \gamma_1 b_1 + \cdots + \gamma_n b_n.$$

Because G is torsion free, $g_x + g_y = g_z$ if and only if $\alpha\beta\gamma g_x + \alpha\beta\gamma g_y = \alpha\beta\gamma g_z$. Since the coefficients in the sums for $\alpha\beta\gamma g_x$, $\alpha\beta\gamma g_y$, and $\alpha\beta\gamma g_z$ are all bounded by t^3, all four claims apply to these sums. The following calculation proves that addition is preserved under φ:

$$g_x +_G g_y = g_z \Leftrightarrow \alpha\beta\gamma g_x +_G \alpha\beta\gamma g_y = \alpha\beta\gamma g_z$$

$$\Leftrightarrow \beta\gamma(\alpha_1 p_1 + \cdots + \alpha_n p_n) +_{\mathbb{Z}} \alpha\gamma(\beta_1 p_1 + \cdots + \beta_n p_n) = \alpha\beta(\gamma_1 p_1 + \cdots + \gamma_n p_n)$$

$$\Leftrightarrow 1/\alpha(\alpha_1 p_1 + \cdots \alpha_n p_n) +_{\mathbb{Z}} 1/\beta(\beta_1 p_1 + \cdots + \beta_n p_n) = 1/\gamma(\gamma_1 p_1 + \cdots + \gamma_n p_n)$$

$$\Leftrightarrow u_x +_{\mathbb{Z}} u_y = u_z \Leftrightarrow \varphi(g_x) +_{\mathbb{Z}} \varphi(g_y) = \varphi(g_z).$$

The following equivalences prove that $<$ is preserved under φ:

$$
\begin{aligned}
g_x < g_y &\Leftrightarrow \alpha\beta g_x < \alpha\beta g_y \\
&\Leftrightarrow \beta(\alpha_1 p_1 + \cdots + \alpha_n p_n) < \alpha(\beta_1 p_1 + \cdots + \beta_n p_n) \\
&\Leftrightarrow 1/\alpha(\alpha_1 p_1 + \cdots + \alpha_n p_n) < 1/\beta(\beta_1 p_1 + \cdots + \beta_n p_n) \\
&\Leftrightarrow u_x < u_y \Leftrightarrow \varphi(g_x) < \varphi(g_y).
\end{aligned}
$$

Finally, the fact that $g_x > 0_G$ if and only if $\varphi(g_x) > 0$ is similar. $\square$

Lemma 3.6. *Let $g_1, \ldots, g_k$ be nonidentity elements such that $g_i \approx g_j$ and $g_i - g_j \approx g_i$ for all $1 \leqslant i \neq j \leqslant k$. There is a map $\varphi : \{g_1, \ldots, g_k\} \to \mathbb{Z}$ such that for all $1 \leqslant x, y, z \leqslant k$, $g_x + g_y = g_z$ implies that $\varphi(g_x) + \varphi(g_y) = \varphi(g_z)$, and $g_x < g_y$ implies that $\varphi(g_x) < \varphi(g_y)$. Furthermore, $g_x > 0_G$ if and only if $\varphi(g_x) > 0$.*

Proof. If $\{g_1, \ldots, g_k\}$ are in the least nontrivial Archimedean class, then we have the stronger result of Lemma 3.5. Otherwise, let $N = \{g \in G | g \ll g_1\}$ be the subgroup of elements Archimedean less than g_1. The elements $g_1 + N, \ldots, g_k + N$ are in the least nontrivial Archimedean class of G/N. Also, if $g_x \neq g_y$, then $g_x - g_y \approx g_x$ and so $g_x - g_y \notin N$. Therefore if $x \neq y$, then $g_x + N \neq g_y + N$, so Lemma 3.5 applies to the elements $g_1 + N, \ldots, g_k + N$ in G/N. The lemma now follows from the fact that $g_x < g_y$ implies $g_x + N < g_y + N$ and that $g_x + g_y = g_z$ implies $g_x + N + g_y + N = g_z + N$. $\square$

Lemma 3.7. *Let $C = \{g_1, \ldots, g_m\}$ be such that $g_1 \lessapprox g_i \lessapprox g_m$ for each i. There is a map $\delta : C \to G$ such that for all $u, v, w \in C$, we have*

1. $\delta(u) \approx g_m$,
2. $u + v = w$ *implies* $\delta(u) + \delta(v) = \delta(w)$, *and*
3. $u < v$ *implies* $\delta(u) < \delta(v)$.

Proof. First, fix a nonshrinking basis B for G and let $\{b_1, \ldots, b_k\} \subset B$ be such that $C \subset \mathrm{Span}(b_1, \ldots, b_k)$ and $b_i \lessapprox g_m$ for each i. Let t be such that $|c| < t$ for all coefficients c used in the reduced equations for elements of C relative to $\{b_1, \ldots, b_k\}$. Thus, every element of C satisfies a unique reduced equation of the form $\alpha x = c_1 b_1 + \cdots + c_k b_k$, with $\alpha < t$ and each $|c_i| < t$.

Second, divide $\{b_1, \ldots, b_k\}$ (by possibly renumbering the indices) into $\{b_1, \ldots, b_j\} \cup \{b_{j+1}, \ldots, b_k\}$ where $g_1 \lessapprox b_i \lessapprox g_m$ for all $i \leqslant j$ and $b_i \ll g_1$ for all $i > j$. Let $A = \{b_1, \ldots, b_j\}$. Without loss of generality, assume that $A \subset C$ (by expanding C if necessary). Let C' be the set of elements of G corresponding to the sums $\sum_{i=1}^{j} c_i b_i$ for every choice of coefficients with $|c_i| \leqslant t^3$.

Since C is finite, it intersects a finite number r of Archimedean classes. Further partition A (again renumbering the indices if necessary) into

$$b_1 \approx \cdots \approx b_{d_1} \ll b_{d_1+1} \approx \cdots \approx b_{d_2} \ll b_{d_2+1} \cdots \ll b_{d_{r-1}+1} \approx \cdots \approx b_j.$$

For notational convenience, let $d_0 = 0$, $d_r = j$. Therefore, each Archimedean class within C is generated by $b_{d_{y-1}+1}, \ldots, b_{d_y}$ for some $0 < y \leqslant r$. Let $A_y = \{b_{d_{y-1}+1}, \ldots, b_{d_y}\}$ and $D_y = \mathrm{Span}(A_y) \cap (C \cup C')$. When we have to verify statements for each D_y, we will typically verify it for D_1 and note that the proofs for the other D_y are the same up to a change in subscripts.

The point of this notation is to think of dividing $C \cup C'$ into various categories. Each D_y has the property that all of its elements are Archimedean equivalent and, because our basis is nonshrinking, the difference between any two distinct elements still lies in the same Archimedean class. Therefore, Lemma 3.6 can be applied to each D_y. We will fix the images of these elements under δ first.

There are also elements $x \in \mathrm{Span}(A)$ such that $x \notin D_y$ for any y. Each $b_i \in A$ is in some D_y set, so $\delta(b_i)$ is already defined. Therefore, we can use the fact that the elements in $\mathrm{Span}(A)$ are all solutions of equations over A to define the images of the elements of $\mathrm{Span}(A) - \cup D_y$. Finally, there are the elements that involve the basis elements $\{b_{j+1}, \ldots, b_k\}$, and we fix the images of these elements last.

We begin by applying Lemma 3.6 to each D_y to define maps $\varphi_y : D_y \to \mathbb{Z}$ such that for all $u, v, w \in D_y$

$$u + v = w \Rightarrow \varphi_y(u) + \varphi_y(v) = \varphi_y(w),$$

$$u < v \Rightarrow \varphi_y(u) < \varphi_y(v), \quad \text{and} \quad u > 0_G \Leftrightarrow \varphi_y(u) > 0. \tag{1}$$

Next, we define a map $\varphi : \bigcup D_y \to \mathbb{Z}$ such that for all $u, v, w \in \cup D_y$,

$$u + v = w \Rightarrow \varphi(u) + \varphi(v) = \varphi(w),$$

$$u \leqslant v \Rightarrow \varphi(u) \leqslant \varphi(v), \quad \text{and} \quad u > 0_G \Leftrightarrow \varphi(u) > 0. \tag{2}$$

We define φ on each D_y by induction on y, verifying at each step that Eq. (2) holds. For $x \in D_1$, set $\varphi(x) = t!\varphi_1(x)$. It is clear from Eq. (1) that Eq. (2) holds for all $u, v, w \in D_1$. Let M_1 be such that $M_1 > |\varphi(x)|$ for all $x \in D_1$.

For $x \in D_2$, set $\varphi(x) = M_1 t! \varphi_2(x)$. Define M_2 such that $M_2 > |\varphi(x_1)| + |\varphi(x_2)|$ for all $x_1 \in D_1$ and $x_2 \in D_2$. To see that φ satisfies Eq. (2), let $u, v, w \in D_1 \cup D_2$. If $u + v = w$, then either $u, v, w \in D_1$ or $u, v, w \in D_2$, so Eq. (1) implies that $+$ is preserved. Similarly, if $u, v \in D_1$ or $u, v \in D_2$, then it is clear that $<$ is preserved. Consider $u \in D_1$ and $v \in D_2$. Then, $u < v$ implies that either u, v are both positive or else u is negative and v is positive. In the first case, $\varphi_1(u)$ and $\varphi_2(v)$ are both positive, so $\varphi(u) < \varphi(v)$ follows from the fact that $\varphi(u) < M_1$. In the second case, $\varphi_1(u)$ is negative and $\varphi_2(v)$ is positive, so $\varphi(u) < \varphi(v)$. The cases for $u \in D_2$ and $v \in D_1$ are similar.

We proceed by induction. For all $x \in D_y$, set $\varphi(x) = M_{y-1}t!\varphi_y(x)$ and define M_y such that $M_y > |\varphi(x_1)| + \cdots + |\varphi(x_y)|$ for all choices of $x_i \in D_i$. The verification that Eq. (2) holds is similar to the case of $y = 2$ done above. Also, the fact that for all $x \in \bigcup D_y$, $x > 0_G$ if and only if $\varphi(x) > 0$ follows from the fact that this holds for each φ_y.

Fix $h \in G$ such that $h \approx g_m$ and h is positive. We begin to define the map δ by setting $\delta(x) = \varphi(x)h + x$ for all $x \in \cup D_y$. In particular, $\delta(b_i)$ is now defined for all $b_i \in A$.

To give an equivalent definition for $\delta(x)$, assume $x \in D_1$ and x satisfies the reduced equation $\alpha x = \alpha_1 b_1 + \cdots + \alpha_{d_1} b_{d_1}$. By the proof of Lemma 3.5 and the fact that $b_i \in D_1$ for $1 \leqslant i \leqslant d_1$, we have $\alpha\varphi_1(x) = \alpha_1\varphi_1(b_1) + \cdots + \alpha_{d_1}\varphi_1(b_{d_1})$. Multiplying by $t!$ shows $\alpha\varphi(x) = \alpha_1\varphi(b_1) + \cdots + \alpha_{d_1}\varphi(b_{d_1})$, which gives us

$$\begin{aligned}\alpha\delta(x) &= \alpha\varphi(x)h + \alpha x \\ &= (\alpha_1\varphi(b_1) + \cdots + \alpha_{d_1}\varphi(b_{d_1}))h + (\alpha_1 b_1 + \cdots + \alpha_{d_1} b_{d_1}) \\ &= \alpha_1\delta(b_1) + \cdots + \alpha_{d_1}\delta(b_{d_1}).\end{aligned}$$

Therefore, once we have defined $\delta(b_i) = \varphi(b_i)h + b_i$, we can define $\delta(x)$ to be the unique solution to

$$\alpha x = \alpha_1\delta(b_1) + \cdots + \alpha_{d_1}\delta(b_{d_1}).$$

(By the calculations above, this equation does have a solution.) The same calculations with different subscripts give analogous results for each D_y.

Before continuing with the definition of δ, we verify that for all $u, v, w \in (\bigcup D_y) \cap C'$,

$$u + v = w \Rightarrow \delta(u) + \delta(v) = \delta(w) \quad \text{and} \quad u < v \Rightarrow \delta(u) < \delta(v).$$

To see that $<$ is preserved, notice that $u < v$ implies that $\varphi(u) < \varphi(v)$, which in turn implies that $\delta(u) = \varphi(u)h + u < \varphi(v)h + v = \delta(v)$. To see that $+$ is preserved, it is easiest to use the definition of δ in terms of solutions of equations. Without loss of generality assume that $u, v, w \in D_1$. Since they are also in C', they satisfy equations $u = \alpha_1 b_1 + \cdots + \alpha_{d_1} b_{d_1}$, $v = \beta_1 b_1 + \cdots + \beta_{d_1} b_{d_1}$, and $w = \gamma_1 b_1 + \cdots + \gamma_{d_1} b_{d_1}$. If $u + v = w$, then $\alpha_i + \beta_i = \gamma_i$ for each $i \leqslant d_1$. Therefore,

$$\alpha_1\delta(b_1) + \cdots + \alpha_{d_1}\delta(b_{d_1}) + \beta_1\delta(b_1) + \cdots + \beta_{d_1}\delta(b_{d_1}) = \gamma_1\delta(b_1) + \cdots + \gamma_{d_1}\delta(b_{d_1}),$$

and hence $\delta(u) + \delta(v) = \delta(w)$. The same argument works for any D_y with the appropriate index substitutions.

Next, consider $x \in \mathrm{Span}(A)$, write $\alpha x = \alpha_1 b_1 + \cdots + \alpha_j b_j$ as a reduced equation, and recall that $0 < \alpha < t$. Define $\varphi(x)$ as the solution to $\alpha x = \alpha_1\varphi(b_1) + \cdots + \alpha_j\varphi(b_j)$. The fact that $t!$ divides each $\varphi(b_i)$ guarantees that $\varphi(x) \in \mathbb{Z}$. If $x \in D_y$, this definition agrees with value of $\varphi(x)$ we have already assigned. Set $\delta(x) = \varphi(x)h + x$, and as above, notice that this definition is equivalent to defining $\delta(x)$ as the solution to

$\alpha z = \alpha_1 \delta(b_1) + \cdots + \alpha_j \delta(b_j)$. Because this equation is equivalent to

$$\alpha z = (\alpha_1 \varphi(b_1) + \cdots + \alpha_j \varphi(b_j))h + (\alpha_1 b_1 + \cdots + \alpha_j b_j),$$

and because α divides each $\varphi(b_i)$ as well as $\alpha_1 b_1 + \cdots + \alpha_j b_j$, this equation does have a solution.

Again, we verify some properties before finishing the definition of δ. We have now defined δ for all elements of C'. The argument that for all $u, v, w \in C'$,

$$u + v = w \Rightarrow \delta(u) + \delta(v) = \delta(w) \quad \text{and} \quad u < v \Rightarrow \delta(u) < \delta(v)$$

is essentially the same as for $(\bigcup D_y) \cap C'$. Also, we verify that for all $x \in \mathrm{Span}(A)$, $x > 0_G$ if and only if $\varphi(x) > 0$. Fix x and suppose it satisfies the reduced equation $\alpha x = \alpha_1 b_1 + \cdots + \alpha_j b_j$. Consider the largest Archimedean class with nonzero terms in $\alpha_1 b_1 + \cdots + \alpha_j b_j$. Let z be the element of C' which is the restriction of the sum $\alpha_1 b_1 + \cdots + \alpha_j b_j$ to the terms from this largest Archimedean class. Because our basis is nonshrinking, z lies in this largest Archimedean class, and hence it determines whether x is positive or not. Therefore, $x > 0_G$ if and only if $z > 0_G$. Since $z \in D_y$ for some y, we have already verified that $z > 0_G$ if and only if $\varphi(z) > 0$. Finally, since $\varphi(z)$ is a multiple of M_{y-1} and M_{y-1} is larger than any sum of images of elements of smaller Archimedean classes under φ, we have that $\varphi(z)$ determines the sign of $\varphi(x)$. Altogether, these equivalences imply that $x > 0_G$ if and only if $\varphi(x) > 0$.

To finish the definition of δ, consider a remaining element g_i and assume g_i is a solution to the reduced equation $\alpha z = c_1 b_1 + \cdots + c_j b_j + c_{j+1} b_{j+1} + \cdots + c_k b_k$. Since $g_i \notin \mathrm{Span}(A)$, there must be at least one $c_i \neq 0$ for $i > j$. Define $\delta(g_i)$ to be the solution to

$$\alpha z = c_1 \delta(b_1) + \cdots + c_j \delta(b_j) + c_{j+1} b_{j+1} + \cdots + c_k b_k.$$

As above, this equation does have a solution. Also, this definition for δ agrees with our earlier definitions in the case that $g_i \in \cup D_y$ or $g_i \in \mathrm{Span}(A)$. Therefore, it can be taken as the final definition covering all cases.

It remains to verify the properties of δ. First, we show that for all $g_i \in C$, $\delta(g_i) \approx h$ and hence $\delta(g_i) \approx g_m$. Suppose $g_i > 0_G$ satisfies $\alpha g_i = \alpha_1 b_1 + \cdots + \alpha_k b_k$, and consider $z = \alpha_1 b_1 + \cdots + \alpha_j b_j \in C'$. If $g_i > 0_G$, then $z > 0_G$, and hence $\varphi(z) > 0$. Since $\delta(z) = \varphi(z)h + z$, we have $\delta(z) > \varphi(z)h$, and since $z \lessapprox g_m$, it follows that $\delta(z) \approx h$. Because $\alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k \ll g_1$, we get $\delta(z) + \alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k \approx h$. Dividing by α cannot change the Archimedean class, so $\delta(g_i) \approx h$. The argument for $g_i < 0_G$ is similar.

Second, we check that $<$ is preserved. Assume g_i satisfies the equation above and g_j satisfies $\beta g_j = \beta_1 b_1 + \cdots + \beta_k b_k$. If $g_i < g_j$, then $\alpha\beta g_i < \alpha\beta g_j$ since α and β are positive. We therefore have

$$\begin{aligned} &\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) + \beta(\alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k) \\ &\quad < \alpha(\beta_1 b_1 + \cdots + \beta_j b_j) + \alpha(\beta_{j+1} b_{j+1} + \cdots + \beta_k b_k). \end{aligned}$$

We claim that this implies that $\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) \leqslant \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$. If not, then $\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) > \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$. Since our basis is nonshrinking, both of these sums are Archimedean greater than the parts involving $b_{j+1}, \ldots, b_k$. Therefore, $\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) > \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$ implies that $\alpha\beta g_i > \alpha\beta g_j$, which is a contradiction.

There are now two cases to consider. If $\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) = \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$, then $\alpha\beta g_i < \alpha\beta g_j$ implies that $\beta(\alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k) < \alpha(\beta_{j+1} b_{j+1} + \cdots + \beta_k b_k)$. Also, since the elements $x = \beta(\alpha_1 b_1 + \cdots + \alpha_j b_j)$ and $y = \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$ are in C', we have that $x = y$ implies $\delta(x) = \delta(y)$. However, $\alpha\beta\delta(g_i) = \delta(x) + \beta(\alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k)$ and $\alpha\beta\delta(g_j) = \delta(y) + \alpha(\beta_{j+1} b_{j+1} + \cdots + \beta_k b_k)$. Therefore, $\alpha\beta\delta(g_i) < \alpha\beta\delta(g_j)$ and hence $\delta(g_i) < \delta(g_j)$.

The second case is when $\beta(\alpha_1 b_1 + \cdots + \alpha_j b_j) < \alpha(\beta_1 b_1 + \cdots + \beta_j b_j)$. In this case, with x and y as above, $x < y$ and so $\delta(x) < \delta(y)$. However, $\delta(x), \delta(y) \approx h$ and so are Archimedean greater than $b_{j+1}, \ldots, b_k$. Therefore, $\alpha\beta\delta(g_i) < \alpha\beta\delta(g_j)$ and $\delta(g_i) < \delta(g_j)$.

Last, we check that $+$ is preserved. Let g_i and g_j satisfy reduced sums as above and let g_l satisfy $\gamma g_l = \gamma_1 b_1 + \cdots + \gamma_k b_k$. If $g_i + g_j = g_l$, then $\alpha\beta\gamma g_i + \alpha\beta\gamma g_j = \alpha\beta\gamma g_l$. Since our basis is nonshrinking,

$$\beta\gamma(\alpha_1 b_1 + \cdots + \alpha_j b_j) + \alpha\gamma(\beta_1 b_1 + \cdots + \beta_j b_j) = \alpha\beta(\gamma_1 b_1 + \cdots + \gamma_j b_j)$$

and

$$\beta\gamma(\alpha_{j+1} b_{j+1} + \cdots + \alpha_k b_k) + \alpha\gamma(\beta_{j+1} b_{j+1} + \cdots + \beta_k b_k) = \alpha\beta(\gamma_{j+1} b_{j+1} + \cdots + \gamma_k b_k).$$

The terms in the top equation are in C', so the addition is preserved by δ. The terms in the bottom sum are not moved by δ. Therefore, $\alpha\beta\gamma\delta(g_i) + \alpha\beta\gamma\delta(g_j) = \alpha\beta\gamma\delta(g_l)$ and so $\delta(g_i) + \delta(g_j) = \delta(g_l)$. $\square$

The following lemma expresses the main combinatorial fact needed to do the diagonalization in the proof of Theorem 1.10.

Lemma 3.8. *Let $G_s \subset G$ be a finite set with two subsets $P = \{p_1, \ldots, p_n\} \subset G_s$ (called the protected elements) and $C = \{g_1, \ldots, g_m\} \subset G_s$ (called the collapsing elements). Assume that the elements of C satisfy $g_1 \lessapprox g_i \lessapprox g_m$ for each i. Let $G' = \{g \in G | g_1 \lessapprox g \lessapprox g_m\}$. Assume that $G_s \cap G' = C$ and $\mathrm{Span}(P) \cap G' = \emptyset$. Then, there is a map $\delta : G_s \to G$ such that the following conditions hold:*

1. *For all $x \in \mathrm{Span}(P) \cap G_s$, $\delta(x) = x$.*
2. *For all $1 \leqslant i \leqslant m$, $\delta(g_i) \approx g_m$.*
3. *For all $x, y, z \in G_s$, $x + y = z$ implies $\delta(x) + \delta(y) = \delta(z)$ and $x < y$ implies $\delta(x) < \delta(y)$.*

Proof. Apply Lemma 3.2 to get $P' = \{p'_1, \dots, p'_n\}$ such that P is independent, has the nonshrinking property, and satisfies $\mathrm{Span}(p_1, \dots, p_n) = \mathrm{Span}(p'_1, \dots, p'_n)$. Let $B = \{b_i | i \in \omega\}$ be a nonshrinking basis for G that extends P'.

Run the construction of Lemma 3.7 using the basis B to obtain $\delta : C \to G$. We use the same notation as in the proof of Lemma 3.7. That is, by possibly renumbering the indices in B, we assume that $j < k$ are such that $C \subset \mathrm{Span}(b_1, \dots, b_k)$, $g_1 \lessapprox b_i \lessapprox g_m$ for all $i \leqslant j$, and $b_i \ll g_1$ for all $j < i \leqslant k$. Furthermore, let $l > k$ be such that $G_s \subset \mathrm{Span}(b_1, \dots, b_l)$.

To extend δ to G_s, write $x \in G_s$ as the solution to the reduced equation $\alpha x = c_1 b_1 + \cdots + c_l b_l$ and define $\delta(x)$ to be the solution to

$$\alpha z = c_1 \delta(b_1) + \cdots + c_j \delta(b_j) + c_{j+1} b_{j+1} + \cdots + c_l b_l.$$

The verification that this equation has a solution and that $+$ and $<$ are preserved under δ is essentially the same as in Lemma 3.7. Therefore, we restrict ourselves to showing that $<$ is preserved. By possibly increasing k and renumbering indices, we can assume that $b_{k+1}, \dots, b_l \gg g_m$. Suppose $u, v \in G_s$ satisfy the reduced equations $\alpha u = \alpha_1 b_1 + \cdots + \alpha_l b_l$ and $\beta v = \beta_1 b_1 + \cdots + \beta_l b_l$. If $u < v$, then $\alpha\beta u < \alpha\beta v$, and so $\beta(\alpha_1 b_1 + \cdots + \alpha_l b_l) < \alpha(\beta_1 b_1 + \cdots + \beta_l b_l)$.

We now split into cases. Let $x = \beta(\alpha_{k+1} b_{k+1} + \cdots + \alpha_l b_l)$ and $y = \alpha(\beta_{k+1} b_{k+1} + \cdots + \beta_l b_l)$. Notice that δ does not move x or y and also, since our basis is nonshrinking, that $g_m \ll x, y$. Therefore, if $x < y$, then $\alpha\beta\delta(u) < \alpha\beta\delta(v)$ since the parts of the sums for $\delta(u)$ and $\delta(v)$ which are distinct from x and y generate elements which are $\lessapprox g_m$. Similarly, if $y < x$, then $\alpha\beta u > \alpha\beta v$, which is a contradiction. If $x = y$, then to determine which of $\alpha\beta\delta(u)$ and $\alpha\beta\delta(v)$ is larger, we examine $\alpha\beta\delta(u) - x$ and $\alpha\beta\delta(v) - y$. In this case, we are back within the realm of Lemma 3.7 and the argument there applies.

It remains to check that $\delta(x) = x$ for all $x \in \mathrm{Span}(P) \cap G_s$. Let $x \in \mathrm{Span}(P)$. Because $\mathrm{Span}(P') \cup G' = \emptyset$. We can assume without loss of generality that the elements of P' are among the basis elements $b_{j+1}, \dots, b_l$. Therefore, x can be written in the form

$$\alpha x = c_{j+1} b_{j+1} + \cdots + c_l b_l$$

since the other basis elements are not needed to generate x. The definition of δ shows that $\delta(x) = x$ as required. □

4. Proof of Theorem 1.10

This section is devoted to a proof of Theorem 1.10. Fix a computable ordered abelian group G which has infinitely many Archimedean classes. By Theorem 1.9, it suffices to build a computable ordered abelian group H with a Δ^0_2 isomorphism $f : H \to G$, and to meet the requirements

$$R_e : \varphi_e : G \to H \text{ is not an isomorphism.}$$

In this context, an isomorphism must preserve order as well as addition.

We use ω for the elements of H. At stage s of the construction, we have a finite initial segment of ω, denoted H_s, and a map $f_s : H_s \to G$, with range G_s. We define the operations on H by $x + y = z$ if and only if there is an s such that $f_s(x) + f_s(y) = f_s(z)$ and $x \leqslant y$ if and only if there is an s such that $f_s(x) \leqslant f_s(y)$. To insure that these operations are well defined and computable, we require that for all s

$$f_s(x) + f_s(y) = f_s(z) \Rightarrow \forall t \geqslant s(f_t(x) + f_t(y) = f_t(z))$$

and

$$f_s(x) \leqslant f_s(y) \Rightarrow \forall t \geqslant s(f_t(x) \leqslant f_t(y)).$$

We let $f = \lim_s f_s$. To insure that f is well defined and Δ^0_2, we also meet the requirements

$$S_e : \lim_s f_s(e) \text{ exists.}$$

The priority on these requirements is $R_0 < S_0 < R_1 < S_1 < \cdots$.

The strategy for S_e is to make $f_{s+1}(e) = f_s(e)$. The strategy for R_e is to pick witnesses $w_{e,0}$ and $w_{e,1}$ from G_s which currently look like $w_{e,0} \not\approx w_{e,1}$. R_e then waits for $\varphi_e(w_{e,0})\downarrow$ and $\varphi_e(w_{e,1})\downarrow$. If it looks like $\varphi_e(w_{e,0}) \not\approx \varphi_e(w_{e,1})$ (which we measure by looking at the elements $f_s(\varphi_e(w_{e,0}))$ and $f_s(\varphi_e(w_{e,1})))$, then we apply Lemma 3.8 to change the map f_s to a map f_{s+1} which forces $f_{s+1}(\varphi_e(w_{e,0})) \approx f_{s+1}(\varphi_e(w_{e,1}))$. This action may move the images of all the elements in H_s which are between the Archimedean classes for $\varphi_e(w_{e,0})$ and $\varphi_e(w_{e,1})$. R_e then wants to restrict any other R_i requirement from changing $f_t(\varphi_e(w_{e,0}))$ or $f_t(\varphi_e(w_{e,1}))$ at a later stage.

There are some obvious conflicts between the requirements. R_e needs to change the images of certain elements, but it does not know which elements until the witnesses $w_{e,i}$ stabilize and the functions $\varphi_e(w_{e,i})$ converge. Both R_e and S_e want to restrain other requirements from moving particular elements. To see how to resolve these conflicts consider R_0, S_0, and R_1. R_0 can act whenever it wants to, and once R_0 has acted, S_0 is can prevent $f_s(0)$ from changing ever again. R_1 cannot change $f_s(0)$, $f_s(\varphi_0(w_{0,0}))$, or $f_s(\varphi_0(w_{0,1}))$. The span of these three elements, however, can intersect at most three Archimedean classes. Therefore, we give R_1 8 witnesses, $w_{1,i}$ for $i \leqslant 7$. If $\varphi_1(w_{1,i})\downarrow$ for all $i \leqslant 7$, and $f_s(\varphi_1(w_{1,i})) \not\approx f_s(\varphi_1(w_{1,j}))$ for $i \neq j$, then by the Pigeonhole Principle there must be two witnesses $w_{1,i}$ and $w_{1,j}$ for which $f_s(\varphi_1(w_{1,i})) \ll f_s(\varphi_1(w_{1,j}))$ and

$$\mathrm{Span}(f_s(0), f_s(\varphi_0(w_{0,0})), f_s(\varphi_0(w_{0,1}))) \cap \{g \in G_s \mid f_s(\varphi_1(w_{1,i}) \lessapprox g \lessapprox f_s(\varphi_1(w_{1,j}))\} = \emptyset.$$

Thus, by Lemma 3.8, there is a way to protect $0, f_s(\varphi_0(w_{0,0}))$, and $f_s(\varphi_0(w_{0,1}))$ while forcing $f_{s+1}(\varphi_1(w_{1,i})) \approx f_{s+1}(\varphi_1(w_{1,j}))$.

In general, we define a function $\tau(e)$ and let R_e have $\tau(e)$ many witnesses. Let $\tau(0) = 2$ and $\tau(e+1) = 2(e+1+\sum_{i \leqslant e} \tau(i)) + 2$. There are $e+1$ S_i requirements (each with one number to protect) of higher priority than R_{e+1}, and each R_i with $i \leqslant e$ has $\tau(i)$ witnesses to protect. Therefore, there are $e+1+\sum_{i \leqslant e} \tau(i)$ many numbers protected by requirements of higher priority than R_{e+1} and the span of these numbers intersects at most $e+1+\sum_{i\leqslant e}\tau(i)$ many Archimedean classes. $\tau(e)$ is defined to be the smallest number of witnesses that will guarantee R_{e+1} has some pair that can be collapsed to the same Archimedean class without moving the elements protected by the higher priority requirements.

Definition 4.1. Let $F\subset G$ be a finite set. For $x,y\in F$, we define

$$x\approx_s y\Leftrightarrow\exists u,\ v\leqslant s\,(u,v>0\wedge u|x|\geqslant|y|\wedge v|y|\geqslant|x|).$$

If $x\not\approx_s y$ and $|x|\leqslant|y|$, then $x\ll_s y$.

The following lemma follows immediately from this definition.

Lemma 4.2. *For all* $x,y\in G$, $x\approx y\Leftrightarrow\exists s(x\approx_s y)$, $x\approx_s y\Rightarrow\forall t\geqslant s(x\approx_t y)$, *and* $x\ll y\Leftrightarrow\forall s(x\ll_s y)$.

Construction

Stage 0: Let $H_0=\{0\}$, $G_0=\{0_G\}$, and $f_0(0)=0_G$.

Stage $s+1$: The first step is to define what appear to be the ω-least representatives for the Archimedean classes. Define $a_i^s\in G_s$ by induction on i until every $x\in G_s$, $x\neq 0_G$, satisfies $x\approx_s a_i^s$ for some a_i^s. Let a_0^s be the ω-least strictly positive element in G_s. Let a_{i+1}^s be the ω-least element of G_s such that $a_{i+1}^s\not\approx_s a_j^s$ for all $j\leqslant i$. Let A_s be the set of the a_i^s.

The second step is to assign witnesses to the R_e requirements by induction on e. We continue to assign witnesses until the elements of A_s are all taken. By induction on e we assign R_e $\tau(e)$ many witnesses, $w_{e,i}^s$ for $i<\tau(e)$, which are chosen from A_s in increasing ω-order and which are removed from A_s once they are chosen. For each R_e which has a full set of witnesses, R_e is *active* if either R_e did not have a full set of witnesses at the previous stage, or one of R_e's witnesses has changed, or R_e has the same witnesses and was active at the end of the previous stage. Otherwise, R_e is not active.

We say that R_e *needs attention* if R_e is active, $\varphi_{e,s}(w_{e,i}^s)\downarrow$ for all $i<\tau(e)$, and $f_s(\varphi_{e,s}(w_{e,i}^s))\not\approx_s f_s(\varphi_{e,s}(w_{e,j}^s))$ for all $i\neq j$. Consider the least e such that R_e needs attention. (If no R_e needs attention, then proceed as if the search procedure below ended because of option (1).) Run the following two search procedures concurrently.

1. Search for some $i\neq j$ for which $f_s(\varphi_{e,s}(w_{e,i}^s))\approx f_s(\varphi_{e,s}(w_{e,j}^s))$.
2. Search for some $i\neq j$ and a map $\delta: G_s\to G$ such that
   (a) $\delta(x)=x$ for all $x=f_s(k)$ with $k<e$ and all $x=f_s(\varphi_{k,s}(w_{k,l}^s))$ with $k<e$, $l<\tau(k)$, and $\varphi_{k,s}(w_{k,l}^s)\downarrow$.
   (b) For all $x,y,z\in G_s$, $x+y=z$ implies $\delta(x)+\delta(y)=\delta(z)$, and $x<y$ implies $\delta(x)<\delta(y)$.
   (c) $\delta(f_s(\varphi_{e,s}(w_{e,i}^s)))\approx\delta(f_s(\varphi_{e,s}(w_{e,j}^s)))$.

At least one of these search procedures must terminate (see the verification below).

If the search in (1) terminates first, then let n_G be the ω-least element of $G - G_s$ and let n_H be the ω-least number not in H_s. Define $G_{s+1} = G_s \cup \{n_G\}$, $H_{s+1} = H_s \cup \{n_H\}$, $f_{s+1}(x) = f_s(x)$ for all $x \in H_s$, and $f_{s+1}(n_H) = n_G$.

If the search in (2) terminates first, then let $\{g_1, \dots, g_m\} = G_s - \mathrm{range}(\delta)$, let n_G be the ω-least element in $G - (G_s \cup \mathrm{range}(\delta))$, and let $r_1, \dots, r_{m+1}$ be the $m+1$ ω-least numbers not in H_s. Define $H_{s+1} = H_s \cup \{r_1, \dots, r_{m+1}\}$, $G_{s+1} = G_s \cup \mathrm{range}(\delta) \cup \{n_G\}$, $f_{s+1}(x) = \delta(x)$ for all $x \in H_s$, $f_{s+1}(r_i) = g_i$ for $i \leqslant m$, and $f_{s+1}(r_{m+1}) = n_G$. Declare R_e to be not active, and for all R_i with $i > e$, if R_i is not active, declare it to be active. We say that R_e acted at stage $s+1$.

End of construction

Lemma 4.3. *The following properties hold of this construction*:

1. $\bigcup_s G_s = G$.
2. *For all s and all $x, y, z \in H_s$, if $f_s(x) + f_s(y) = f_s(z)$, then $f_{s+1}(x) + f_{s+1}(y) = f_{s+1}(z)$, and if $f_s(x) < f_s(y)$, then $f_{s+1}(x) < f_{s+1}(y)$.*
3. *If $g_1, \dots, g_s$ are the ω-least elements of G, then $\{g_1, \dots, g_s\} \subset G_{s+1}$.*

Lemma 4.4. *For each i, $\lim_s a_i^s = a_i$ exists and for all $i \neq j$, $a_i \not\approx a_j$.*

Proof. Let s be such that there are $i+1$ distinct Archimedean classes represented among the first s (in terms of $\mathbb{N}$) elements of G. These elements are all in G_{s+1}, and so $a_0^s, \dots, a_i^s$ are all permanently defined and have reached limits at stage $s+1$. To see that $a_i \not\approx a_j$, suppose $a_i \approx a_j$ and $i < j$. Then, there is an s such that $a_i \approx_s a_j$ and so $\forall t \geqslant s\ (a_i \approx_t a_j)$. Without loss of generality, $a_i^s = a_i$ has already reached its limit. Therefore, for every $t \geqslant s$, $a_j^t \neq a_j$, which is a contradiction. $\square$

Lemma 4.5. *For each $e \in \omega$ and $i < \tau(e)$, $\lim_s w_{e,i}^s = w_{e,i}$ exists, and for all $\langle e, i \rangle \neq \langle e', i' \rangle$, $w_{e,i} \not\approx w_{e',i'}$.*

Proof. Immediate from Lemma 4.4. $\square$

Lemma 4.6. *One of the two concurrent search procedures must terminate.*

Proof. Assume that the search in (1) never terminates. Then, $f_s(\varphi_e(w_{e,i}^s)) \not\approx f_s(\varphi_e(w_{e,j}^s))$ for $i \neq j$. Let P be the set consisting of $f_s(k)$ for $k < e$ and all $f_s(\varphi_{k,s}(w_{k,l}^s))$ for $k < e$, $l < \tau(k)$, and for which $\varphi_{k,s}(w_{k,l}^s)\downarrow$. Notice that $\mathrm{Span}(P)$ intersects at most $e + 1 + \sum_{k<e} \tau(k)$ many Archimedean classes. Therefore, by the Pigeonhole Principle, there must be $i \neq j$ such that $f_s(\varphi_e(w_{e,i}^s)) \ll f_s(\varphi_e(w_{e,i}^s))$ and for all $x \in \mathrm{Span}(P)$, either $x \ll f_s(\varphi_e(w_{e,i}^s))$ or $f_s(\varphi_e(w_{e,j}^s)) \ll x$. Let $C = \{g \in G_s \mid f_s(\varphi_e(w_{e,i}^s)) \lessapprox g \lessapprox f_s(\varphi_e(w_{e,j}^s))\}$ and apply Lemma 3.8 to see the existence of a map δ with the required properties. $\square$

Lemma 4.7. *Each R_e requirement acts at most finitely often and is eventually satisfied.*

Proof. The proof proceeds by induction on e. Let s be a stage such that all R_i with $i<e$ have ceased to act and $w_{e,i}^t = w_{e,i}$ for all $t \geqslant s$ and $i < \tau(e)$. The lemma is trivial if $\varphi_e(w_{e,i})\uparrow$ for some i. Therefore, assume $\varphi_{e,s}(w_{e,i})\downarrow$ for all i. Suppose $f_s(\varphi_e(w_{e,i})) \approx_s f_s(\varphi_e(w_{e,j}))$ for some $i \neq j$. Then, since R_e does not act, since no requirement of higher priority acts and since no requirement of lower priority can change either $f_s(\varphi_e(w_{e,i}))$ or $f_s(\varphi_e(w_{e,j}))$, we have that for all $t \geqslant s$, $f_t(\varphi_e(w_{e,i})) = f_s(\varphi_e(w_{e,i}))$ and $f_t(\varphi_e(w_{e,j})) = f_s(\varphi_e(w_{e,j}))$. Therefore, $f(\varphi_e(w_{e,i})) = f_s(\varphi_e(w_{e,i}))$, and $f(\varphi_e(w_{e,j})) = f_s(\varphi_e(w_{e,j}))$. It follows that $\varphi_e(w_{e,i}) \approx \varphi_e(w_{e,j})$ in H, but $w_{e,i} \not\approx w_{e,j}$ in G, so R_e is satisfied.

If $f_s(\varphi_e(w_{e,i})) \not\approx_s f_s(\varphi_e(w_{e,j}))$ for all $i \neq j$, then R_e acts at stage $s+1$. Either R_e discovers that $f_s(\varphi_e(w_{e,i})) \approx f_s(\varphi_e(w_{e,j}))$ for some $i \neq j$, in which case R_e does not act and is satisfied as above, or else R_e finds an appropriate δ. In that case, $f_{s+1}(\varphi_e(w_{e,i})) \approx f_{s+1}(\varphi_e(w_{e,j}))$ and R_e is declared not active. Since no requirement of higher priority ever acts again and no witness $w_{e,i}$ changes again, we have that R_e never acts again. Therefore, R_e is satisfied as above. $\square$

Lemma 4.8. *Each S_e requirement is satisfied.*

Proof. Let s be a stage such that all requirements R_i with $i \leqslant e$ have stopped acting. No requirement is allowed to change $f_s(e)$ after this stage, and hence S_e is satisfied. $\square$

5. Effective Hölder's Theorem

In this section, we turn to the effective algebra we need to prove Theorems 1.11 and 1.3. In Sections 5, 6 and 7, G denotes a computable Archimedean ordered group with infinite rank. Hölder's Theorem characterizes the Archimedean ordered groups.

Hölder's Theorem. *If G is an Archimedean ordered group, then G is isomorphic to a subgroup of the naturally ordered additive group $\mathbb{R}$.*

Notice that Hölder's Theorem implies that every Archimedean ordered group is abelian. It is possible to give an effective proof of Hölder's Theorem (see [17] for the details of such a proof). To describe the effective version of Hölder's Theorem formally, we need the following definitions. The first definition says that a computable real number is one which has a computable dyadic expansion.

Definition 5.1. A *computable real* is a computable sequence of rationals $x = \langle q_k | k \in \mathbb{N} \rangle$ such that $\forall k \forall i (|q_k - q_{k+i}| \leqslant 2^{-k})$. Let $y = \langle q'_k | k \in \mathbb{N} \rangle$ be another real. We say $x = y$ if $|q_k - q'_k| \leqslant 2^{-k+1}$ for all k. Similarly, $x < y$ if there is a k such that $q_k + 2^{-k+1} < q'_k$. (Notice that the latter condition is Σ^0_1.)

The next definition formalizes the notion of a computable ordered subgroup of the reals. Since reals are second-order objects (that is, they are infinite sequences of rationals), we specify a computable subgroup by uniformly coding a countable sequence of reals such that we can compute the sum and the order relation of two reals in the sequence effectively in the indices of these elements.

Definition 5.2. A *computable ordered subgroup of* $\mathbb{R}$ (indexed by a computable set X) is a computable sequence of computable reals $A = \langle r_n | n \in X \rangle$ together with a partial computable function $+_A : X \times X \to X$, a partial computable binary relation $\leqslant_A$ on X, and a distinguished number $i \in X$ such that

1. $r_i = 0_{\mathbb{R}}$.
2. $n +_A m = p$ if and only if $r_n +_{\mathbb{R}} r_m = r_p$.
3. $n \leqslant_A m$ if and only if $r_n \leqslant_{\mathbb{R}} r_m$.
4. $(X, +_A, \leqslant_A)$ satisfies the ordered group axioms with i as the identity element.

Effective Hölder's Theorem. *If G is a computable Archimedean ordered abelian group, then G is isomorphic to a computable ordered subgroup of $\mathbb{R}$, indexed by G, for which $+_A$ and $\leqslant_A$ are exactly $+_G$ and $\leqslant_G$.*

To prove this version of Hölder's Theorem, one builds a uniform sequence of computable reals r_g, for $g \in G$, such that $r_g +_{\mathbb{R}} r_h = r_{g+h}$ and $r_g \leqslant_{\mathbb{R}} r_h$ if and only if $g \leqslant_G h$. We will use this correspondence to give us a measure of distance in G. Notice that while the computable ordered subgroup of the reals here is not a computable group in the ordinary sense (since the elements are second-order objects), there still is a sense in which the isomorphism is computable. For each $g \in G$, we can uniformly compute the corresponding real r_g. Therefore, we can think of the isomorphism as effectively giving us an index for the Turing machine computing the dyadic expansion of the corresponding real in such a way that both the addition function and the order relation are effective in these indices.

The proof of Proposition 5.3 can be found in [10].

Proposition 5.3. *If* $\operatorname{rank}(G) > 1$ *and G is Archimedean, then G is dense in the sense that for every $g < h$, there is an x such that $g < x < h$.*

If $\{a, b\}$ is independent, then the element x from Proposition 5.3 can be taken to be a linear combination $c_1 a + c_2 b$ in which both c_1 and c_2 are nonzero.

Proposition 5.4. *Let G be a subgroup of $(\mathbb{R}, +)$ with rank $\geqslant 2$. For every $r \in \mathbb{R}$ with $r > 0$, there is an $h \in G$ with $h \in (0, r)$. Notice, $r \in \mathbb{R}$, but it need not be in G.*

Proof. Let $g \in G$ be such that $g > 0$. By Proposition 5.3, there is an $x \in G$ such that $0 < x < g$, and hence, either $x \in (0, g/2)$ or $g - x \in (0, g/2)$. Thus, there is an $h \in G$ such that $h \in (0, g/2)$. Repeat this argument to get elements in $(0, g/4)$, $(0, g/8)$, and so on, until an element appears in $(0, r)$. $\square$

Proposition 5.5. *Let G be a subgroup of $(\mathbb{R},+)$ with rank $\geqslant 2$. For every $r_1 <_{\mathbb{R}} r_2$, there is an $h \in G$ with $h \in (r_1, r_2)$. Notice, $r_1, r_2 \in \mathbb{R}$, but they need not be in G.*

Proof. Let $d = r_2 - r_1$ and let $g \in G$ be such that $g \in (0, d)$. Then, since $\mathbb{R}$ is Archimedean ordered, there is an $m \in \mathbb{N}$ such that $r_1 < mg < r_2$. Setting $h = mg$ proves the theorem. $\square$

If $\{a, b\}$ is independent, then by the comments following Proposition 5.3, we can assume that the h in Propositions 5.4 and 5.5 has the form $h = c_1 a + c_2 b$ with $c_1, c_2 \neq 0$.

Proposition 5.6. *Let G be a subgroup of $(\mathbb{R},+)$ with infinite rank, $B = \{b_0, \dots, b_m\} \subset G$ be a linearly independent set, $X = \{x_0, \dots, x_n\} \subset G$ be any set of nonidentity elements, and $d \in \mathbb{R}$ with $d > 0$. Then there are elements $a_i \in G$, for $0 \leqslant i \leqslant n$, such that $\{b_0, \dots, b_m, (x_0 + a_0), \dots, (x_n + a_n)\}$ is linearly independent and for each i, $|a_i| < d$. Furthermore, we can require that for any fixed $p \in \mathbb{N}$, $p \neq 0$, each a_i is divisible by p in G.*

Proof. It is enough to consider a single element $x_0 \in G$, and proceed by induction. If x_0 is independent from B, then let $a_0 = 0_G$. Otherwise, let $b \in G$ be such that $\{b_0, \dots, b_m, b\}$ is linearly independent. By Proposition 5.4, there are coefficients $c_1, c_2 \in \mathbb{Z}$ (which we can assume are both nonzero) such that $c_1 b + c_2 b_0 \in (0, d/p)$. Let $a_0 = c_1 pb + c_2 pb_0$. Clearly, a_0 is divisible by p in G, $|a_0| < d$, and $\{b_0, \dots, b_m, (x_0 + a_0)\}$ is linearly independent (since we assumed that $c_1 \neq 0$). $\square$

To prove Theorem 1.11, it suffices, by Theorem 1.9, to build a computable ordered group H which is Δ^0_2 isomorphic but not computably isomorphic to G. We build H in stages so that at each stage we have a finite set H_s and a map $f_s : H_s \to G$ with range G_s. Assuming that $\lim_s f_s(x)$ converges for each x, the Limit Lemma shows that $f = \lim_s f_s$ is Δ^0_2. During the construction, we meet the requirements

$$R_e : \varphi_e : H \to G \text{ is not an isomorphism.}$$

Notice that we are treating φ_e as a map from H to G.

We define $+_H$ and $\leqslant_H$ as before: $a +_H b = c$ if and only if $\exists s\ (f_s(a) +_G f_s(b) = f_s(c))$, and $a <_H b$ if and only if $\exists s (f_s(a) \leqslant_G f_s(b))$. To insure that these operations are well-defined and computable, we guarantee that

$$f_s(a) + f_s(b) = f_s(c) \Rightarrow \forall t \geqslant s (f_t(a) + f_t(b) = f_t(c)) \tag{3}$$

and

$$f_s(a) \leqslant_G f_s(b) \Rightarrow \forall t \geqslant s (f_t(a) \leqslant_G f_t(b)). \tag{4}$$

To defeat a single requirement R_e, our strategy is to guess a basis for G. The inverse image under f of such a basis will be a basis for H. The strategy for R_e

proceeds as follows:

1. Pick two elements a_e^s and b_e^s from our guess at the basis for H. We will settle on longer and longer initial segments of a basis, so eventually, R_e will choose two linearly independent elements. Without loss of generality, we assume $a_e^s <_H b_e^s$.
2. Do nothing until a stage $t \geqslant s$ occurs for which $\varphi_{e,t}(a_e^t)\downarrow$, $\varphi_{e,t}(b_e^t)\downarrow$, and $\varphi_e(a_e^t) <_G \varphi_e(b_e^t)$. If these calculations do not appear, then φ_e is not an isomorphism from H to G, so R_e is satisfied.
3. Define $f_{t+1}(b_e) \neq f_t(b_e)$ such that for some large $n, m \in \mathbb{N}$, we have $n\varphi_e(a_e^t) <_G m\varphi_e(b_e^t)$ and $mf_{t+1}(b_e^t) <_G nf_{t+1}(a_e^t)$. In this case, we have also satisfied R_e. The algebra behind the definition of f_{t+1} is discussed in Section 6.

The general idea for Step 3 is to fix an effective map $\psi : G \to \mathbb{R}$, which we use to measure distances in G. We want to move the image of b_e^t just enough to make the diagonalization possible, but not so far as to upset the order or addition relations defined to far. Propositions 5.4 and 5.6 will allow us to diagonalize as long as $f_s(a_e^t)$ and $f_s(b_e^t)$ really are independent. Therefore, we initiate a search process for an appropriate new image of b_e^t, which, to keep the requirements R_e and R_i from interfering with each other, we require to be in the span of $f_s(a_e^t)$ and $f_s(b_e^t)$. Either we find an appropriate image, or we find a dependence relation between a_e^t and b_e^t. In the latter case, we know that the witnesses for R_e are bound to change.

The injury in this construction is finite. Once the higher priority requirements have ceased to act, R_e can use the next two linearly independent elements to diagonalize.

6. Algebra for Theorems 1.11 and 1.3

From the description in the previous section, it should be clear that when we change the image of a basis element b_e^t, we need to make sure that we preserve both the addition and ordering facts specified so far in H. To preserve the addition facts, we use the notion of an approximate basis for a finite subset G' of G.

Before giving the formal definition of an approximate basis, we give some motivation for the conditions which occur in the definition. Suppose G' is a finite subset of G and $B = \{b_0, \dots, b_k\}$ is an independent set which spans G'. Then, each $g \in G'$ satisfies a unique reduced relation of the form $\alpha y = c_0 b_0 + \cdots + c_k b_k$. Furthermore, if g, h, and $g + h \in G'$, then the reduced relation satisfied by $g + h$ can be found by adding the relations for g and h, and dividing by the greatest common divisor of the nonzero coefficients. That is, if $\alpha g = c_0 b_0 + \cdots + c_k b_k$ and $\beta h = d_0 b_0 + \cdots + d_k b_k$, then $g + h$ is the solution to the reduced version of

$$\alpha\beta y = (\beta c_0 + \alpha d_0) b_0 + \cdots + (\beta c_k + \alpha d_k) b_k.$$

At each stage of the construction, we will guess at an independent subset of G, and our guess at each stage will be an approximate basis. We want our guesses to have these two properties of an actual independent set. Therefore, assume that G' is the finite subset of G which is the range of the partial isomorphism f_s we have defined at stage s.

To imitate the first property, we want our approximate basis $X_s = \{x_0^s, \ldots, x_k^s\}$ at stage s to be t-independent, where t is large enough that each element $g \in G'$ is the solution to a unique reduced dependence relation of the form

$$\alpha y = c_0 x_0^s + c_1 x_1^s + \cdots + c_k x_k^s,$$

where each coefficient has absolute value $\leqslant t$. Notice that if g is the solution to more than one relation of this form, then we know X_s is not independent. Since there is some independent set which spans G', there must be a set which is t'-independent (for some t') and which does have this uniqueness property.

As new elements enter H during the construction, they will be assigned reduced dependence relations. If h enters H at stage s and is assigned the reduced relation $\alpha y = c_0 x_0 + \cdots + c_k x_k$, then for every stage $t \geqslant s$, we will define $f_t(h)$ to be the unique solution to $\alpha y = c_0 x_0^t + \cdots + c_k x_k^t$ (where $x_0^t, \ldots, x_k^t$ is an initial segment of our approximate basis at stage t). Therefore, the second property we want X_s to have is that if g, h, and $g+h$ are all in G', then the dependence relation for $g+h$ relative to the approximate basis is the sum of the dependence relations for g and h, as described above. This property will guarantee that Eq. (3) holds. The key point is that if $g+h$ satisfies some other reduced dependence relation, then, as above, we know that X_s is not independent, and therefore, there must be another set with the required properties.

By the comments above, if X_s is independent and spans G', then it will have both of these properties. It follows that every finite G' has an approximate basis and that during the construction we can add additional requirements on the level of independence of an approximate basis, such as requiring that it be at least s-independent at stage s.

Definition 6.1. Let G' be a finite subset of G. An *approximate basis* for G' with weight $t > 0$ is a finite sequence $X = \langle x_0, \ldots, x_k \rangle$ such that

1. $\{x_0, \ldots, x_k\}$ is t-independent,
2. every $g \in G' \cup X$ satisfies a unique reduced dependence relation of the form $\alpha y = c_0 x_0 + \cdots + c_k x_k$ with $0 < \alpha \leqslant t$ and $|c_i| \leqslant t$, and
3. for every $g, h \in G' \cup X$ with $g + h \in G' \cup X$, if g and h satisfy the reduced dependence relations $\alpha g = c_0 x_0 + \cdots + c_k x_k$ and $\beta h = d_0 x_0 + \cdots + d_k x_k$ with α, β, $|c_i|$, and $|d_i| \leqslant t$, then the reduced coefficients in

$$\alpha\beta(g+h) = (\beta c_0 + \alpha d_0)x_0 + \cdots + (\beta c_k + \alpha d_k)x_k$$

have absolute value less that t.

We use sequences to represent approximate bases to emphasize the fact that their elements are ordered. We will abuse notation, however, and simply treat them as sets, with the understanding that the set $\{x_0, \ldots, x_k\}$ is really the ordered sequence $\langle x_0, \ldots, x_k \rangle$. Also, whenever we refer to $g \in G'$ satisfying a reduced equation of an approximate basis of weight t, we assume that the absolute value of all the coefficients is bounded by t.

Returning to the description of the construction, at stage s we have an approximate basis $X_s = \{x_0^s, \ldots, x_{k_s}^s\}$ for G_s which is t_s-independent. Each h which enters H at stage s is assigned a reduced dependence relation $\alpha y = c_0 x_0 + \cdots + c_{k_s} x_{k_s}$ with $\alpha, |c_i| \leqslant t_s$. For every $t \geqslant s$, we define $f_t(h)$ so that

$$\alpha f_t(h) = c_0 x_0^t + \cdots + c_{k_s} x_{k_s}^t.$$

The properties of an approximate basis guarantee that Eq. (3) holds.

However, it is not clear that Eq. (4) will hold or that the relation $\alpha y = c_0 x_0^t + \cdots + c_{k_s} x_{k_s}^t$ will have a solution unless we do something to insure that our choices for approximate bases at stages s and $t \geqslant s$ fit together in a nice way. Therefore, we introduce the notion of coherence between approximate bases.

Definition 6.2. Let $G_0 \subset G_1$ be finite subsets of G, with approximate bases $X_0 = \{x_0^0, \ldots, x_{k_0}^0\}$ of weight t_0 and $X_1 = \{x_0^1, \ldots, x_{k_1}^1\}$ with weight t_1, respectively. We say that X_1 *coheres* with X_0 if the following conditions are met:

1. $k_0 \leqslant k_1$ and $t_0 \leqslant t_1$.
2. For each $i \leqslant k_0$, if $\{x_0^0, \ldots, x_i^0\}$ is linearly independent, then $x_j^1 = x_j^0$ for every $j \leqslant i$.
3. If $g \in G_0$ satisfies the reduced equation $\alpha y = c_0 x_0^0 + \cdots + c_{k_0} x_{k_0}^0$, then there is a solution to $\alpha y = c_0 x_0^1 + \cdots + c_{k_0} x_{k_0}^1$ in G.
4. If $g <_G h \in G_0$ satisfy the reduced sums $\alpha y = c_0 x_0^0 + \cdots + c_{k_0} x_{k_0}^0$ and $\beta z = d_0 x_0^0 + \cdots + d_{k_0} x_{k_0}^0$, respectively, then the solutions $g', h' \in G$, respectively, to the equations $\alpha y = c_0 x_0^1 + \cdots + c_{k_0} x_{k_0}^1$ and $\beta z = d_0 x_0^1 + \cdots + d_{k_0} x_{k_0}^1$ satisfy $g' <_G h'$.

Lemma 6.3. *Let $G_0 \subset G_1$ be finite subsets of G, and let X_0 be an approximate basis for G_0. There exists an approximate basis X_1 for G_1 which coheres with X_0.*

Proof. Since we are not yet worried about effectiveness issues, we can assume by Hölder's Theorem that $G \subset \mathbb{R}$. If X_0 is linearly independent, then we can extend it to a set X_1 which is linearly independent and spans G_1. Such a set X_1 satisfies the conditions in Definition 6.2.

Therefore, assume that X_0 is not linearly independent and that $i < k_0$ is such that $\{x_0^0, \ldots, x_i^0\}$ is linearly independent, but $\{x_0^0, \ldots, x_{i+1}^0\}$ is not. Let d' be the minimum distance between any pair $g \neq h \in G_0$, and let $d = d'/(3t_0 k_0)$.

Apply Proposition 5.6 with $B = \{x_0^0, \ldots, x_i^0\}$, $X = \{x_{i+1}^0, \ldots, x_{k_0}^0\}$, d as above, and $p = t_0!$. We obtain $a_{i+1}, \ldots, a_{k_0}$ such that $\{x_0^0, \ldots, x_i^0, (a_{i+1} + x_{i+1}^0), \ldots, (a_{k_0} + x_{k_0}^0)\}$ is linearly independent, and, for each j with $i + 1 \leqslant j \leqslant k_0$, $t_0!$ divides a_j and $|a_j| < d$.

For $0 \leqslant j \leqslant i$, set $x_j^1 = x_j^0$, and for $i+1 \leqslant j \leqslant k_0$, set $x_j^1 = a_j + x_j^0$. Since $Y = \{x_0^1, \dots, x_{k_0}^1\}$ is linearly independent, we let X_1 be a finite linearly independent set that extends Y and that spans G_1. Clearly, X_1 is an approximate basis for G_1 and satisfies conditions 1 and 2 of Definition 6.2.

To see that X_1 satisfies condition 3, fix an arbitrary $g \in G_0$, and suppose $\alpha g = c_0 x_0^0 + \cdots + c_{k_0} x_{k_0}^0$ is a reduced dependence relation with $\alpha, |c_i| \leqslant t_0$. Then,

$$\alpha y = c_0 x_0^1 + \cdots + c_{k_0} x_{k_0}^1 = (c_0 x_0^0 + \cdots + c_{k_0} x_{k_0}^0) + (c_{i+1} a_{i+1} + \cdots + c_{k_0} a_{k_0}).$$

Since $\alpha \leqslant t_0$ and $t_0!$ divides each of the a_j in G, the equation $\alpha y = c_0 x_0^1 + \cdots + c_{k_0} x_{k_0}^1$ has a solution $g' \in G$.

To see that X_1 satisfies condition 4, we consider the distance between the solutions $g \in G_0$ and $g' \in G_1$ to the dependence relation above. Since each $|c_j| \leqslant t_0$, $|a_j| < d$, and there are at most k_0 of the a_j's, we have

$$|\alpha g - \alpha g'| \leqslant |c_{i+1} a_{i+1} + \cdots + c_k a_k| \leqslant k_0 t_0 d \leqslant d'/3.$$

Furthermore, since $\alpha > 0 \in \mathbb{N}$, $|g - g'| \leqslant |\alpha g - \alpha g'| \leqslant d'/3$. Suppose $h \in G_0$ with $h \neq g$ satisfies $\beta h = d_0 x_0^0 + \cdots + d_{k_0} x_{k_0}^0$ and $h' \in G_1$ is the solution to $\beta y = d_0 x_0^1 + \cdots + d_{k_0} x_{k_0}^1$. An identical argument shows that $|h - h'| \leqslant d'/3$. Combining the facts that $|g - h| \geqslant d'$, $|g - g'| \leqslant d'/3$, and $|h - h'| \leqslant d'/3$, it is clear that $g <_G h$ implies $g' <_G h'$. $\square$

It remains to fix an effective method for finding bases which cohere. The algorithm below is not the most obvious one, but it has properties which will be important in our proof.

Suppose $G_0 \subset G_1$ are finite subsets of G. Let $X_0 = \{x_0, x_1, \dots, x_{k_0}\}$ be an approximate basis for G_0 which is t_0-independent. We find an approximate basis X_1 for G_1 which coheres with X_0 in three phases.

In the first phase, we guess (until we find evidence to the contrary) that X_0 is linearly independent. We perform the following two tasks concurrently:

1. Search for a dependence relation among the elements of X_0.
2. Search for a Y such that $X_0 \cup Y$ is an approximate basis for G_1 which coheres with X_0 as follows. Begin with $n = t_0 + 1$ and $i = 0$.
   (a) Let y_i be the $\mathbb{N}$-least element of G such that $X \cup \{y_0, \dots, y_i\}$ is n-independent. Check if this set spans G_1 using coefficients with absolute value $\leqslant n$. If so, then proceed to (b), and if not, repeat (a) with i set to $i+1$.
   (b) Check if $X_0 \cup \{y_0, \dots, y_i\}$ satisfies condition 3 from Definition 6.1. If it does, then it coheres with X_0 and we end the algorithm. If it is not an approximate basis, then return to (a) with n set to $n+1$ and $i = 0$.

This phase will terminate, since if X_0 is linearly independent, then, at worst, we repeat (a) and (b) until we pick elements $y_0 <_{\mathbb{N}} \cdots <_{\mathbb{N}} y_i$ which are the $\mathbb{N}$-least such that $\{x_0, \dots, x_{k_0}, y_0, \dots, y_i\}$ is a linearly independent and spans G_1. This set coheres with X_0. If this phase ends because we find an approximate basis in Step 2, then the algorithm terminates. However, if this phase ends because we find a dependence relation in Step 1, then we proceed to the second phase with the knowledge that X_0 is not linearly independent.

For the second phase, assume that we know $\{x_0, \dots, x_{i+1}\}$ is n-dependent, but $\{x_0, \dots, x_i\}$ is n-independent. We search for elements y_{i+1} through y_{k_0} from which to construct elements which play the role of the a_j's in the proof of Lemma 6.3. Before starting this phase, fix a computable embedding $\psi : G \to \mathbb{R}$, let d' be any positive real less than the minimum of $|\psi(g - h)|$, where $g \neq h$ range over G_0, and set $d = d'/3k_0t_0$.

1. For $i + 1 \leqslant j \leqslant k_0$, pick $y_j \in G$ to be the $\mathbb{N}$-least such that $\{x_0, \dots, x_i, y_{i+1}, \dots, y_j\}$ is n-independent.
2. Check the following Σ^0_1 conditions concurrently:
   (a) Search for a dependence relation among $\{x_0, \dots, x_i, y_{i+1}, \dots, y_{k_0}\}$. If we discover that $\{x_0, \dots, x_{j+1}\}$ is dependent, then restart Phase 2 with $\{x_0, \dots, x_j\}$. If we discover that $\{x_0, \dots, x_i, y_{i+1}, \dots, y_j\}$ is dependent for some j, then we return to Step 1 of this phase, set n to be large, and repick y_{i+1} through y_{k_0}.
   (b) For each $i + 1 \leqslant j \leqslant k_0$, search for coefficients $b_j, d_j \neq 0$ such that, for $a_{i+1} = b_{i+1}t_0!x_i + d_{i+1}t_0!y_{i+1}$ and $a_j = b_jt_0!y_{j-1} + d_jt_0!y_j$ (for $j > i + 1$), we have $\psi(a_j) \in (0, d)$. If we find such a_j, then end Phase 2.

Determining if $\psi(a_j) \in (0, d)$ is a Σ^0_1 fact, so by dove-tailing our computations, we can effectively perform the search in (b). This phase will terminate, since once $\{x_0, \dots, x_i\}$ has shrunk to a linearly independent set (by finitely many discoveries of dependence relations in (a)), we know that there are linearly independent y_j's and coefficients b_j, d_j, with the required properties. By continually choosing the $\mathbb{N}$-least elements which look independent, we eventually find such elements.

We verify two properties of $X' = \{x_0, \dots, x_i, x_{i+1} + a_{i+1}, \dots, x_{k_0} + a_{k_0}\}$. First, as in Lemma 6.3, if $\alpha y = c_0x_0 + \cdots + c_{k_0}x_{k_0}$, with $\alpha, |c_i| \leqslant t_0$, has a solution $g \in G_0$, then

$$\alpha y = c_0x_0 + \cdots + c_ix_i + c_{i+1}(x_{i+1} + a_{i+1}) + \cdots + c_{k_0}(x_{k_0} + a_{k_0})$$

has a solution in $g' \in G$. Second, $|\psi(g) - \psi(g')| \leqslant d'/3$, also as in Lemma 6.3. Therefore, if $g < h \in G_0$ and g', h' are the solutions to the dependence relations for g and h, respectively, with $x_{i+1}, \dots, x_{k_0}$ replaced by $x_{i+1} + a_{i+1}, \dots, x_{k_0} + a_{k_0}$, then $g' < h'$. Therefore, any extension of X' which is an approximate basis for G_1 will cohere with X_0.

To find such an extension, we use a search similar to Phase 1. Perform the following two tasks concurrently:

1. Search for a dependence relation among the elements of X'. If we find such a relation, then either $\{x_0, \dots, x_i\}$ is dependent, in which case we return to the beginning of Phase 2 with a shorter initial segment of X_0, or else $\{x_0, \dots, x_i, y_{i+1}, \dots, y_j\}$ is dependent for some $j \leqslant k_0$. In this case, we return to

Step 1 of Phase 2 with $\{x_0, \ldots, x_i\}$ and repick y_{i+1} through y_{k_0} with n chosen to be large.

2. Search for a Y such that $X' \cup Y$ is an approximate basis for G_1 which coheres with X_0 as follows. Set m to be large and $i = 0$.
   (a) Let w_i be the $\mathbb{N}$-least element of G such that $X' \cup \{w_0, \ldots, w_i\}$ is m-independent. Check if this set spans G_1 using coefficients with absolute value $\leqslant m$. If so, then proceed to (b), and if not, repeat (a) with i set to $i+1$.
   (b) Check if $X' \cup \{w_0, \ldots, w_i\}$ satisfies condition 3 from Definition 6.1. If it does, then, by the comments above, it coheres with X_0, and we end the algorithm. If it is not an approximate basis, then return to (a) with m set to $m+1$.

This phase must terminate since we can return to Phase 2 only finitely often without picking a linearly independent set $\{x_0, \ldots, x_i, y_{i+1}, \ldots, y_{k_0}\}$. From here, it is clear that we will eventually pick a spanning set for G_1 with the correct level of independence.

We could easily have added requirements that the approximate basis X_1 has a specified higher level of independence or a larger size. We summarize this discussion with the following lemma.

Lemma 6.4. *Let G be a computable Archimedean ordered group with infinite rank, $G_0 \subset G_1$ be finite subsets of G, and X_0 be a t_0-independent approximate basis for G_0 of size k_0. For any m, n with $t_0 < m$ and $k_0 < n$, we can effectively find an approximate basis X_1 for G_1 which coheres with X_0, which is at least m-independent, and which has size at least n.*

It remains to discuss the diagonalization process for an R_e requirement. Recall that R_e has two witnesses, a_e and $b_e \in H_s$ such that $f_s(a_e)$ and $f_s(b_e)$ are elements of our approximate basis X_s (of weight t_s) for G_s, where G_s is the image of H_s under f_s. Also, we have a fixed map $\psi : G \to \mathbb{R}$. If we want to diagonalize for R_e at stage s, then we search for an element x in the subgroup generated by $t_s!f_s(a_e)$ and $t_s!f_s(b_e)$ such that $\psi(x)$ is sufficiently close to 0 in $\mathbb{R}$ and x meets the diagonalization strategy discussed at the end of Section 5. (We will provide the exact bounds for $\psi(x)$ and the exact diagonalization properties during the construction when we have established the necessary notation.) If we find an appropriate x, then we replace $f_s(b_e)$ in our approximate basis by $f_s(b_e) - x$. As above, the fact that $t_s!$ divides x allows us to solve the necessary equations in G to preserve addition and the fact that $\psi(x)$ is sufficiently close to 0 guarantees that the new solutions have the same ordering relations as ones from G_s. However, since we have introduced large multiples of $f_s(a_e)$ and $f_s(b_e)$, it need not be the case that $X' = (X_s - \{f_s(b_e)\}) \cup \{f_s(b_e) - x\}$ is still t_s-independent.

We handle this situation as follows. If we are diagonalizing for R_e, assume that

$$X_s = \{f_s(a_0), f_s(b_0), \ldots, f_s(a_e), f_s(b_e), f_s(y_1), \ldots, f_s(y_k)\}.$$

Every element $g \in G_s$ is the solution to a unique reduced dependence relation over X_s with coefficients whose absolute value is bounded by t_s. We want to find a new approximate basis X_s' (of weight $\geqslant t_s$) for some subset G_s' of G such that we have met our diagonalization requirements and such that all the equations which were satisfied by some $g \in G_s$ over X_s are also satisfied by some $g' \in G_s'$ over X_s'. Notice that addition is automatically preserved because $g_1 + g_2 = g_3$ in G_s if and only if the defining equations for g_1, g_2, and g_3 satisfy this additive relationship. Therefore, if g_1', g_2', and g_3' are the solutions in G_s' to the equations for g_1, g_2, and g_3 over X_s', we must have $g_1' + g_2' = g_3'$. Lastly, we want that $<$ is preserved in the sense that if $g < h$ in G_s, then $g' < h'$ holds in G_s'.

Therefore, we perform two searches concurrently. First, we search for a dependence relation among $\{f_s(a_0), f_s(b_0), \ldots, f_s(a_e), f_s(b_e)\}$. If we find such a dependence relation, we know that the witnesses a_e and b_e are going to change, so there is no need to diagonalize at this point. Second, we search for nonzero coefficients c_1 and c_2 and for elements $u_1, \ldots, u_k$ of G such that

1. $\psi(c_1 t_s! f_s(a_e) + c_2 t_s! f_s(b_e))$ is as close to 0 as we want it to be and meets our diagonalization strategy (and we set $x = c_1 t_s! f_s(a_e) + c_2 t_s! f_s(b_e)$), and
2. $t_s!$ divides each u_i and $\psi(u_i)$ is as close to 0 as we want it to be, and
3. $X_s' = \{f_s(a_0), f_s(b_0), \ldots, f_s(a_e), f_s(b_e) - x, f_s(y_1) + u_1, \ldots, f_s(y_k) + u_k\}$ is t_s' independent for some $t_s' \geqslant 2(t_s)^3$, and
4. for every $g \in G_s$, the equation satisfied by g over X_s has a solution g' over X_s' (and we let G_s' be the set of solutions to these equations), and
5. $<$ is preserved in the sense mentioned above.

Assuming that $\{f_s(a_0), f_s(b_0), \ldots, f_s(a_e), f_s(b_e)\}$ is independent, Propositions 5.4 and 5.5 will tell us that we can find an appropriate x and Proposition 5.6 will tell us that we can find appropriate u_i elements.

Now, we define $f_s' : H_s \to G_s'$ on the approximate basis X_s' by $f_s'(a_i) = f_s(a_i)$ for $i \leqslant e$, $f_s'(b_i) = f_s(b_i)$ for $i < e$, $f_s'(b_e) = f_s(b_e) - x$, and $f_s'(y_i) = f_s(y_i) + u_i$. We can extend this map across H_s by mapping $h \in H_s$ to the solution over X_s' for the equation defining $f_s(h)$ over X_s. The map f_s' preserves all the ordering and addition facts about H_s.

To see that X_s' is an approximate basis for G_s', we need to check condition (3) of Definition 6.1. Therefore, assume that $g, h \in G_s$ satisfy the equations

$$\alpha g = c_0 f_s(a_0) + c_1 f_s(b_0) + \cdots + c_{2e} f_s(a_e) + c_{2e+1} f_s(b_e)$$
$$+ \; c_{2e+2} f_s(y_1) + \cdots + c_{2e+1+k} f_s(y_k),$$

$$\beta h = d_0 f_s(a_0) + d_1 f_s(b_0) + \cdots + d_{2e} f_s(a_e) + d_{2e+1} f_s(b_e)$$
$$+ \; d_{2e+2} f_s(y_1) + \cdots + d_{2e+1+k} f_s(y_k)$$

over X_s with $|c_i|, |d_i| \leqslant t_s$ and $0 < \alpha, \beta \leqslant t_s$. Let g', h' be the solutions to these equations over X_s', and suppose $g' + h' \in G_s'$. Then, since G_s' is exactly the set of solutions to the equations (over X_s') for the elements $g \in G_s$, we know that $g' + h'$ satisfies an equation of the form

$$\gamma(g' + h') = l_0 f'_s(a_0) + \cdots + l_{2e+1} f'_s(b_e) + l_{2e+2} f'_s(y_1) + \cdots + l_{2e+1+k} f'_s(y_k) \qquad (5)$$

with $|l_i| \leqslant t_s$ and $0 < \gamma \leqslant t_s$. However, summing the equations for g and h, we see that $g' + h'$ also satisfies

$$\alpha\beta(g' + h') = (\beta c_0 + \alpha d_0) f_s(a_0) + \cdots + (\beta c_{2e+1+k} + \alpha d_{2e+1+k}) f_s(y_k). \qquad (6)$$

We need to show that Eqs. (6) and (5) are equivalent when reduced. If we multiply Eq. (5) by $\beta\alpha$ and Eq. (6) by γ, we obtain two equations for $\alpha\beta\gamma(g' + h')$. Each of these equations has its coefficients bounded by $2(t_s)^3$, and since X'_s is $2(t_s)^3$ independent, these equations must have equal coefficients. Therefore, they remain the same when reduced. This completes the proof that X'_s is an approximate basis for G'_s.

To finish the stage, we let X_{s+1} be an approximate basis for $G_s \cup G'_s$ which coheres with the basis $\{f'_s(a_0), f'_s(b_0), \ldots, f'_s(a_e), f'_s(b_e), f'_s(y_1), \ldots, f'_s(y_k)\}$ for G'_s. We can assume that $f'_s(a_0), f'_s(b_0), \ldots, f'_s(a_e), f'_s(b_e)$ forms an initial segments of X_{s+1}, since otherwise there must be a dependence relation between $f_s(a_0), f_s(b_0), \ldots, f_s(a_e), f_s(b_e)$.

For each $h \in H_s$, the dependence relation defining $f_s(h)$ over X_s has a solution over X'_s, and hence it has a solution in G over X_{s+1}. Let G''_s be the set of solutions to the equations for $h \in H_s$ over X_{s+1}.

We let $G_{s+1} = G_s \cup G'_s \cup G''_s \cup X_{s+1}$ and we expand H_s to H_{s+1} by adding $|G_{s+1} \setminus G_s|$ many new elements. To define the map f_{s+1} on H_{s+1}, we first consider $f_{s+1}(h)$ for $h \in H_s$. We know that $f_s(h)$ satisfies a reduced equation over X_s and that this equation has a solution in G_{s+1} over X_{s+1}. Therefore, we defined $f_{s+1}(h)$ to be the solution to this equation in G_{s+1}. For the new elements in H_{s+1}, we map these elements to the elements of G_{s+1} which are not hit by elements of H_s under f_{s+1}. Each of the new elements in H_{s+1} is assigned the dependence relation satisfied by $f_{s+1}(h)$ over X_{s+1}.

The final thing to notice is that since $f'_s(a_0), f'_s(b_0), \ldots, f'_s(a_e), f'_s(b_e)$ forms an initial segments of X_{s+1}, we have that $f_{s+1}(a_e) = f_s(a_e)$ and $f_{s+1}(b_e) = f_s(b_e) - x$. Hence, we have diagonalized as we wanted.

7. Proof of Theorems 1.11 and 1.3

At stage s of the construction, we will have an approximate basis $X_s = \{x^s_0, \ldots, x^s_{k_s}\} \subset G$, with $k_s \geqslant 2s$, which is t_s-independent, with $t_s > s$. If h enters H at stage $s+1$, then h is assigned a reduced dependence relation of the form $\alpha y = c_0 x_0 + \cdots + c_{k_{s+1}} x_{k_{s+1}}$. We say that $g \in G$ *satisfies* this relation relative to the approximate basis X_t, with $t \geqslant s+1$, if $\alpha g = c_0 x^t_0 + \cdots + c_{k_{s+1}} x^t_{k_{s+1}}$. Each requirement R_e, with $e \leqslant s$, has two distinct witnesses, a^s_e and b^s_e, such that $f_s(a^s_e) \in X_s$

and $f_s(b_e^s) \in X_s$. R_e *does not need attention* at stage s if any of the following conditions hold:

1. $\varphi_{e,s}(a_e^s)\uparrow$ or $\varphi_{e,s}(b_e^s)\uparrow$, or
2. for some $0<m,n<s$, $m\varphi_{e,s}(b_e^s)\downarrow \leqslant_G n\varphi_{e,s}(a_e^s)\downarrow$ and $nf_s(a_e^s) <_G mf_s(b_e^s)$, or
3. for some $0<m,n<s$, $m\varphi_{e,s}(a_e^s)\downarrow \leqslant_G n\varphi_{e,s}(b_e^s)\downarrow$ and $nf_s(b_e^s) <_G mf_s(a_e^s)$, or
4. R_e was declared satisfied at some stage $t<s$ and both a_e^s and b_e^s are the same as a_e^t and b_e^t.

R_e *requires attention* at stage s if none of these conditions hold.

Construction

Stage 0: Fix a computable embedding $\psi : G \to \mathbb{R}$. Set $H_0 = \{0\}$, $f_0(0) = 0_G$, and $X_0 = \emptyset$. Assign $0 \in H$ the empty reduced dependence relation.

Stage $s+1$: Assume we have defined H_s and $f_s : H_s \to G$, with $G_s = \mathrm{range}(f_s)$. We have a set $X_s \subset G_s$ which is an approximate basis for G_s, which is t_s-independent and which has size $k_s \geqslant 2s$. Each element $h \in H_s$ has been assigned a reduced dependence relation of the form $\alpha y = c_0x_0 + \cdots + c_ix_i$ for some $i \leqslant k_s$. We split the stage into four steps.

Step 1: Let g be the $\mathbb{N}$-least element of G not in G_s. Let $X_s' = \{x_{0,s}', \ldots x_{k_s',s}'\}$ be an approximate basis for $G_s \cup \{g\}$ which coheres with X_s, which has size $k_s' \geqslant 2(s+1)$, and which is t_s'-independent, for some $t_s' > (s+1)$. Because X_s' coheres with X_s, every dependence relation assigned to an element $h \in H_s$ has a solution over X_s'. Let G_s' contain G_s, $\{g\}$, X_s', and the solution to the dependence relation for each $h \in H_s$ over X_s'. Let $n = |G_s' \setminus G_s|$, let $h_1, \ldots h_n$ be the n least elements of $\mathbb{N}$ not in H_s, and let $H_s' = H_s \cup \{h_1, \ldots, h_n\}$. Define $f_s' : H_s' \to G_s'$ as follows. For $h \in H_s$, $f_s'(h)$ is the solution to the dependence for h over X_s'. For h_i, $1 \leqslant i \leqslant n$, let $f_s'(h_i)$ map to the elements of G_s' not in the image of H_s under f_s'. Each new $h_i \in H_s'$ is assigned the reduced dependence relation $\alpha y = c_0x_0 + \cdots + c_{k_s'}x_{k_s'}$ with $\alpha, |c_j| \leqslant t_s'$ such that

$$\alpha f_s'(h_i) = c_0x_{0,s}' + c_1x_{1,s}' + \cdots + c_{k_s'}x_{k_s',s}'.$$

Step 2: Define the witnesses for R_e with $e \leqslant s$ by setting a_e^{s+1} and b_e^{s+1} to be the elements of H_s' such that $f_s'(a_e^{s+1}) = x_{2e,s}'$ and $f_s'(b_e^{s+1}) = x_{2e+1,s}'$. Check if any R_e requires attention. If so, let R_e be the least such requirement and go to Step 3. Otherwise, proceed to Step 4.

Step 3: In this step we do the actual diagonalization. First, calculate a safe distance to move the image of b_e^{s+1}. Set $d' \in \mathbb{R}$ to be such that $d' > 0$ and

$$d' \leqslant \min\{|\psi(f_s'(h)) - \psi(f_s'(g))| \mid h \neq g \in H_s'\}.$$

We can find such a d' effectively since H_s' is finite. Set $d = d'/(3t_s'(1 + k_s'))$.

Second, we search for an appropriate $x \in G$ to set $f_{s+1}(b_e^{s+1}) = f_s'(b_e^{s+1}) - x$. We say that x *diagonalizes* for R_e if there are $n, m > 0$ such that either

$$nf_s'(a_e^{s+1}) <_G m(f_s'(b_e^{s+1}) - x) \quad \text{and} \quad n\varphi_e^s(a_e^{s+1}) \geqslant_G m\varphi_e^s(b_e^{s+1})$$

or

$$nf'_s(a_e^{s+1}) >_G m(f'_s(b_e^{s+1}) - x) \quad \text{and} \quad n\varphi_e^s(a_e^{s+1}) \leqslant_G m\varphi_e^s(b_e^{s+1}).$$

We search concurrently for

1. elements $x, u_{2e+2}, \dots, u_{k'_s}$ in G such that
   (a) x has the form $c_1 t'_s! f'_s(a_e^{s+1}) + c_2 t'_s! f'_s(b_e^{s+1})$ with $c_1, c_2 \neq 0$ such that $\psi(x) \in (0, d)$, and x diagonalizes for R_e, and
   (b) $t'_s!$ divides each u_i in G and $|\psi(u_i)| < d$, and
   (c) $X''_s = \{f'_s(a_0^{s+1}), f'_s(b_0^{s+1}), \dots, f'_s(a_e^{s+1}), f'_s(b_e^{s+1}) - x, f'_s(x'_{2e+2}) + u_{2e+2}, \dots, f'_s(x'_{k'_s}) + u_{k'_s}\}$ is at least $2(t'_s)^3$ independent, or
2. $n, m \in \mathbb{N}$ such that $nf'_s(a_e^{s+1}) <_G mf'_s(b_e^{s+1})$ and $n\varphi_{e,s}(a_e^{s+1}) \geqslant_G m\varphi_{e,s}(b_e^{s+1})$, or
3. $n, m \in \mathbb{N}$ such that $nf'_s(b_e^{s+1}) <_G mf'_s(a_e^{s+1})$ and $n\varphi_{e,s}(b_e^{s+1}) \geqslant_G m\varphi_{e,s}(a_e^{s+1})$, or
4. a dependence relation among $\{f'_s(a_0^{s+1}), f'_s(b_0^{s+1}), \dots, f'_s(a_e^{s+1}), f'_s(b_e^{s+1})\}$ in G.

This process terminates (see Lemma 7.1). Furthermore, if we found X''_s, then because $t'_s!$ divides all the elements we are adding to the approximate basis elements, this set has the property that each dependence relation assigned to an $h \in H'_s$ has a solution over X''_s. Also, because $|\psi(u_i)| < d$ and $\psi(x) < d$, these solutions preserve $<$ in the sense described at the end of Section 6 (see Lemma 7.3).

If the process terminates with conditions 2, 3 or 4, then skip to Step 4. Otherwise, we define f_{s+1} using x and the u_i. For every $h \in H'_s$, there is a solution to the dependence relation for h over X''_s. Therefore, we can define G''_s as the set of solutions to the dependence relations assigned to $h \in H'_s$. As explained at the end of Section 6, because X''_s is $2(t'_s)^3$ independent, it is an approximate basis for G''_s. Let X_{s+1} be an approximate basis for $G'_s \cup G''_s$ which coheres with the approximate basis X''_s for G'_s. Let $G_{s+1} = G'_s \cup G''_s \cup X_{s+1}$ and let H_{s+1} contain H'_s plus $|G_{s+1} \setminus G'_s|$ many new elements. Define $f_{s+1} : H_{s+1} \to G_{s+1}$ as follows. For $h \in H'_s$, set $f_{s+1}(h)$ to be the solution to the dependence relation for h over X_{s+1}. Map the elements $h \in H_{s+1} \setminus H'_s$ to the elements of G_{s+1} which are not in the image of H'_s under f_{s+1} and assign to each such h the reduced dependence relation satisfied by $f_{s+1}(h)$ over X_{s+1}. Proceed to stage $s+2$.

Step 4: If we arrived at this step, then there is no diagonalization to be done. Define $f_{s+1} = f'_s$, $X_{s+1} = X'_s$, $H_{s+1} = H'_s$, $G_{s+1} = G'_s$, $k_{s+1} = k'_s$, and $t_{s+1} = t'_s$. If we arrived at Step 4 because condition 2 or 3 was satisfied in the search procedure in Step 3, then declare R_e satisfied. Proceed to stage $s+2$.

End of construction.

To prove the construction works, we verify the following lemmas.

Lemma 7.1. *The search procedure in Step* 3 *of stage* $s+1$ *terminates.*

Proof. Each condition in the search procedure is Σ^0_1. Therefore, it suffices to show that if conditions 2–4 do not hold, then condition 1 does hold.

Suppose conditions 2–4 are not true. Because condition 4 does not hold, $f_s'(a_e^{s+1})$ and $f_s'(b_e^{s+1})$ are linearly independent. Therefore, by Proposition 5.4, there are $n, m \in \mathbb{N}$ such that $|m\psi(f_s'(b_e^{s+1})) - n\psi(f_s'(a_e^{s+1}))| <_{\mathbb{R}} d$. Fix such n, m, and without loss of generality, assume that $n\psi(f_s'(a_e^{s+1})) <_{\mathbb{R}} m\psi(f_s'(b_e^{s+1}))$ (the case for the reverse inequality follows by a similar argument). Because ψ is an embedding, $nf_s'(a_e^{s+1}) <_G mf_s'(b_e^{s+1})$, and because condition 2 does not hold, $n\varphi_{e,s}(a_e^{s+1}) <_G m\varphi_{e,s}(b_e^{s+1})$.

Since $t_s'!f_s'(a_e^{s+1})$ and $t_s'!f_s'(b_e^{s+1})$ are linearly independent, we use Proposition 5.5 to conclude that there are nonzero c_1 and c_2 such that

$$\frac{m\psi(f_s'(b_e^{s+1})) - n\psi(f_s'(a_e^{s+1}))}{m} <_{\mathbb{R}} c_1 t_s' \psi(f_s'(a_e^{s+1})) + c_2 t_s' \psi(f_s'(b_e^{s+1})) <_{\mathbb{R}} \frac{d}{m}.$$

We set $x = c_1 t_s' f_s'(a_e^{s+1}) + c_2 t_s' f_s'(b_e^{s+1})$, and note that $mf_s'(b_e^{s+1}) - nf_s'(a_e^{s+1}) <_G mx$, $\psi(x) \in (0, d)$, and $t_s'!$ divides x in G. Furthermore, since $c_2 \neq 0$, $f_s'(b_e^{s+1}) - x$ is independent from $f_s'(a_e^{s+1})$. Finally, to see that x diagonalizes for R_e:

$$0 <_{\mathbb{R}} m\psi(f_s'(b_e^{s+1})) - n\psi(f_s'(a_e^{s+1})) <_{\mathbb{R}} m\psi(x)$$

$$\Rightarrow \quad mf_s'(b_e^{s+1}) - mx <_G nf_s'(a_e^{s+1}),$$

which implies that $m(f_s(b_e^{s+1}) - x) <_G nf_s(a_e^{s+1})$ as required.

Finally, Proposition 5.6 implies that elements $u_{2e+2}, \ldots, u_{k_s'}$ exist with the required level of independence. $\square$

Lemma 7.2. *Each $h \in H$ is assigned a unique reduced dependence relation of the form $\alpha y = c_0 x_0 + \cdots + c_n x_n$. Furthermore, if $h \in H_s$ and $x_0^s, \ldots, x_n^s$ are the initial elements of X_s, then this relation has a solution in G.*

Proof. The first time an approximate basis is chosen after h enters H, h is assigned a unique reduced dependence relation. If $h \in H_s \setminus H_{s-1}$, then $f_s(h)$ satisfies a dependence relation of the form $\alpha y = c_0 x_0^s + c_1 x_1^s + \cdots + c_{k_s} x_{k_s}^s$ with α, $|c_i| \leqslant t_s$. We show by induction that for all $u \geqslant s$ this equation has a solution in G. Notice that if $u \geqslant s$, then $|X_u| \geqslant |X_s|$, so there are enough approximate basis elements in X_u for this equation to make sense. Assume that the equation has a solution at stage u, and we consider it at stage $u+1$.

X_u' coheres with X_u, so $\alpha y = c_0 x_{0,u}' + c_1 x_{1,u}' + \cdots + c_{k_s} x_{k_s,u}'$ has a solution. If we do not diagonalize at stage $u+1$, then $X_{u+1} = X_u'$, and we are done. If we do diagonalize at stage $u+1$, then our conditions on X_u'' guarantee that the equation has a solution over X_u''. We then choose X_{u+1} so that it coheres with X_u'', and hence the equation has a solution over X_{u+1}. $\square$

Lemma 7.3. *Suppose $s+1$ is a stage at which we diagonalize, and $a <_G b \in G_s'$ satisfy the dependence relations $\alpha y = c_0 x_{0,s}' + \cdots + c_k x_{k,s}'$ and $\beta y = d_0 x_{0,s}' + \cdots + d_k x_{k,s}'$. If*

$a'', b'' \in G$ are the solutions to $\alpha y = c_0 x_0^{s+1} + \cdots + c_k x_k^{s+1}$ and $\beta y = d_0 x_0^{s+1} + \cdots + d_k x_k^{s+1}$, then $a'' <_G b''$.

Proof. At stage $s+1$, we set d' to be $<_{\mathbb{R}}$ the least distance between any pair $\psi(h)$ and $\psi(g)$, with $h \neq g \in G'_s$, and we set $d = d'/(3t_{s+1}(1+k'_s))$. Let a' and b' be the solutions to the equations for a and b over X''_s. We first show that $a' < b'$.

If $X''_s = \{x''_{0,s}, \ldots, x''_{k'_s,s}\}$, then by our restrictions on $\psi(x)$ and $\psi(u_i)$, we have that $|\psi(x'_{i,s}) - \psi(x''_{i,s})| \leqslant d$ for each i. Therefore, since $\alpha > 0$,

$$|\psi(a) - \psi(a')| \leqslant |\psi(\alpha a) - \psi(\alpha a')| \leqslant (k'_s + 1)t'_s d = d'/3.$$

Similarly, $|\psi(b) - \psi(b')| \leqslant d'/3$. However, since $|\psi(a) - \psi(b)| \geqslant d'$, we have that $\psi(a') < \psi(b')$ and hence, $a' < b'$.

Finally, since X_{s+1} coheres with X''_s, we know that $a' < b'$ implies that $a'' < b''$, as required. □

Lemma 7.4. *For each $s \in \mathbb{N}$, each $a, b, c \in H_s$ and each $t \geqslant s$, we have*

$$f_s(a) + f_s(b) = f_s(c) \Rightarrow f_t(a) + f_t(b) = f_t(c)$$

and

$$f_s(a) \leqslant f_s(b) \Rightarrow f_t(a) \leqslant f_t(b).$$

Proof. First, we check that addition is preserved. If a and b are assigned the dependence relations $\alpha y = c_0 x_0 + \cdots + c_k x_k$ and $\beta y = d_0 x_0 + \cdots + d_k x_k$, respectively, then by the definition of an approximate basis, c is assigned the reduced version of

$$\alpha\beta(g + h) = (\beta c_0 + \alpha d_0)x_0 + \cdots + (\beta c_k + \alpha d_k)x_k.$$

At every stage t after the assignment of these dependence relations, $f_t(a)$, $f_t(b)$, and $f_t(c)$ are defined to be the solutions of these relations relative to X_t. Therefore, $f_t(a) +_G f_t(b) = f_t(c)$.

Second, we check that the ordering is preserved. Assume that $a, b \in G_s$ are such that $f_s(a) <_G f_s(b)$. We show by induction on $t \geqslant s$ that $f_t(a) <_G f_t(b)$. Since X'_{t+1} coheres with X_t, we know that if $a', b' \in G$ are the solutions to the dependence relations assigned to a and b, respectively, relative to the basis X'_{t+1}, then $a' <_G b'$. If we do not diagonalize at stage $t+1$, then $X_{t+1} = X'_{t+1}$, so we are done. If we do diagonalize, then we apply Lemma 7.3. □

Lemma 7.5. *Each approximate basis element x_i^s reaches a limit, and the set of these limits forms a basis for G. Furthermore, each witness a_e^s and b_e^s reaches a limit and each requirement R_e is eventually satisfied.*

Proof. It is clear that if the elements x_i^s reach limits, then they will form a basis for G. Therefore, since each x_i^s is eventually chosen to be an a_e^s or a b_e^s, it suffices to show by induction on e that a_e^s and b_e^s reach limits and that each R_e requirement is eventually satisfied. Since $a_0^s = x_0^s$ is always defined to be the first nonidentity element in G, this element never changes, and hence reaches a limit a_0.

Consider $b_0^s = x_1^s$. Let y be the $\mathbb{N}$ least element of G such that $\{a_0, z\}$ is independent. Since our algorithm for choosing a coherent basis always chooses the $\mathbb{N}$ least elements it can, we eventually find a stage when we recognize that $\{a_0^s, b_0^s\}$ is dependent and we pick y_1 to be z in Phase 2 of the coherent basis algorithm. From this stage on, whenever we run this algorithm, we choose y_1 to be z, so eventually by Proposition 5.6 we will find an appropriate linear combination of b_0^s and z and set b_0^{s+1} to be this linear combination. Since b_0^{s+1} is now independent from a_0, it will not change again unless R_0 diagonalizes.

Once b_0^s has reached a limit, R_0 is guaranteed to win if it ever chooses to diagonalize. This is because once $\{a_0^s, b_0^s\}$ is independent, the search procedure in Step 3 cannot end in condition 4. If R_0 never wants to diagonalize, then R_0 is satisfied for trivial reasons. If R_0 does diagonalize, then b_0^s changes one last time, but it remains independent of a_0 and hence will never change again.

We can now consider the case for $e+1$. Assume we have passed a stage such that $a_0^s, b_0^s, \ldots, a_e^s, b_e^s$ have all reached their limits and no requirement R_i, with $i \leqslant e$, ever wants to act again. As above, let z_1 and z_2 be the $\mathbb{N}$ least elements of G such that $\{a_0^s, b_0^s, \ldots, a_e^s, b_e^s, z_1, z_2\}$ is independent. It is possible that the action of diagonalization for a higher priority requirement will have made a_{e+1}^s and b_{e+1}^s independent from the elements above. If not, then the algorithm for picking a coherent basis eventually finds that they are dependent and redefines a_{e+1}^s to be a linear combination with z_1 and redefines b_{e+1}^s to be a linear combination with z_2. After this point, a_{e+1}^s will never change again, and b_{e+1}^s will only change if R_{e+1} wants to act. As above, if R_{e+1} ever wants to act, then it is guaranteed to win because the search in Step 3 cannot end in condition 4. Therefore, R_{e+1} is eventually satisfied and b_{e+1}^s reaches a limit. □

Lemma 7.6. *For each s and each $h \in H_s$, the sequence $f_t(h)$ for $t \geqslant s$ reaches a limit.*

Proof. Suppose $h \in H_s$ and h is assigned the relation $\alpha y = c_0 x_0 + \cdots + c_k x_k$. For $t \geqslant s$, $f_t(h)$ is the solution to this equation over X_t. Therefore, once each x_i^t reaches a limit, so does $f_t(h)$. □

This ends the proof of Theorem 1.11. To finish the proof of Theorem 1.3, we need one more lemma.

Lemma 7.7. *H admits a computable basis.*

Proof. For $i \in \mathbb{N}$, define d_i to be the element assigned the reduced equation $y = x_i^s$ and let f be the pointwise limit of f_s. Then, $f(d_i) = \lim_s x_i^s = x_i$. Since $\{x_i | i \in \mathbb{N}\}$ is a basis for G, $\{d_i | i \in \mathbb{N}\}$ is a basis for H. □

8. Proofs of Theorems 1.12 and 1.4

For this section, we fix a computable ordered abelian group G with infinite rank which is not Archimedean, but has only finitely many Archimedean classes. Assume G has r nontrivial Archimedean classes and fix positive representatives $\Gamma_1, \dots, \Gamma_r$ for these classes. Since every nonidentity element $g \in G$ satisfies $g \approx \Gamma_i$ for a unique i, we can effectively determine the Archimedean class of each g.

For each $1 \leqslant i \leqslant r$, let L_i be the computable subgroup $\{g \in G | g \ll \Gamma_i\}$. Also, let E_i be the least nontrivial Archimedean class of the quotient group G/L_i. Since E_i is an Archimedean ordered group (with the induced order), we can fix maps $\psi_i : E_i \to \mathbb{R}$ by Hölder's Theorem. Since G has infinite rank, at least one of the E_i groups must have infinite rank. We will say that Γ_i *represents an infinite rank Archimedean class* if E_i has infinite rank. Otherwise, Γ_i *represents a finite rank Archimedean class*.

The key to proving Theorems 1.12 and 1.4 is to find the correct analogues of Propositions 5.4 and 5.6 and Lemma 6.3. Once we have these results, the arguments presented in Sections 6 and 7 can be used with minor changes.

Definition 8.1. A finite subset $G_0 \subset G$ is *closed under Archimedean differences* if for all $g, h \in G_0$ such that $g \approx h$ but $g - h \not\approx g$, we have $g - h \in G_0$.

Lemma 8.2. *If $G_0 \subset G$ is finite, then there is a finite set G_0' such that $G_0 \subset G_0'$ and G_0' is closed under Archimedean differences.*

Proof. Start with the largest Archimedean class occurring in G_0 and compare all pairs of elements in this class. For each pair such that $g \approx h$ and $g - h \not\approx g$, add $g - h$ to G_0. Considering each Archimedean class in G_0 in decreasing order, we close G_0 under Archimedean differences by adding only finitely many elements. □

Definition 8.3. $X \subset G$ is *nonshrinking* if for all $x_0 \approx \cdots \approx x_n \in X$ and coefficients $c_0, \dots, c_n$ with at least one $c_i \neq 0$, we have $c_0 x_0 + \cdots + c_n x_n \approx x_0$. X is *t-nonshrinking* if this property holds with the absolute values of the coefficients bounded by t.

Theorem 8.4. *There is a nonshrinking basis for G.*

Proof. For each $1 \leqslant i \leqslant r$, fix a set B_i of elements b_j^i such that each $b_j^i \approx \Gamma_i$ and the set of elements $b_j^i + L_i$ is a basis for E_i. The fact that the b_j^i elements are independent modulo L_i means that for any coefficients $c_1, \dots, c_k$ with at least one $c_j \neq 0$, we have

$$(c_1 b_1^i + \cdots + c_k b_k^i) + L_i \neq L_i$$

and hence $c_1 b_1^i + \cdots + c_k b_k^i \notin L_i$. Since each $b_j^i \approx \Gamma_i$, this implies that $c_1 b_1^i + \cdots + c_k b_k^i \approx \Gamma_i$. Therefore, B_i is nonshrinking.

It remains to show that $B = \bigcup_{1 \leqslant i \leqslant r} B_i$ is a basis for G. First, B is independent since each B_i is independent and nonshrinking. Second, to see that B spans G, let $g \in G$ be such that $g \approx \Gamma_i$. For some choice of coefficients $\alpha, c_1, \dots, c_k$ and elements $b_1^i, \dots, b_k^i$, we can write

$$\alpha g + L_i = (c_1 b_1^i + \cdots + c_k b_k^i) + L_i.$$

Therefore, $c_1 b_1^i + \cdots + c_k b_k^i - \alpha g \ll \Gamma_i$. If this element is equal to 0_G, then we are done. Otherwise, we can repeat this process with $c_1 b_1^i + \cdots + c_k b_k^i - \alpha g$. Since there are only finitely many Archimedean classes, this process must stop and show that some multiple of g is a linear combination of elements of B. $\square$

Definition 8.5. Let $G_0 \subset G$ be finite. X_0 is *an approximate nonshrinking basis* for G_0 with weight t_0 if X_0 is an approximate basis for G_0 with weight t_0 and X_0 is t_0-nonshrinking.

As before, an approximate nonshrinking basis is a sequence, but we abuse notation and treat it as a set. Furthermore, we think of X_0 as broken down into Archimedean classes, and we treat X_0 as a sequence of sequences, $\langle X_0^1, \dots, X_0^r \rangle$, where X_0^i is the sequence of elements $x \in X_0$ for which $x \approx \Gamma_i$.

Definition 8.6. If $G_0 \subset G_1$ are finite subsets and $X_0 = \{x_0^0, \dots, x_{k_0}^0\}$ is an approximate nonshrinking basis for G_0 of weight t_0, then the approximate nonshrinking basis $X_1 = \{x_0^1, \dots, x_{k_1}^1\}$ for G_1 of weight t_1 *coheres* with X_0 if

1. Conditions 1,3, and 4 from Definition 6.2 hold, and
2. for each Archimedean class X_0^i inside X_0, if $X_0^i = \{x_{i_0}^0, \dots, x_{i_l}^0\}$ and j is such that $\{x_{i_0}^0, \dots, x_{i_j}^0\}$ is independent, but $\{x_{i_0}^0, \dots, x_{i_{j+1}}^0\}$ is not, then $\{x_{i_0}^0, \dots, x_{i_j}^0\} \subset X_1$.

We can now give the analogues of Propositions 5.4 and 5.6 and of Lemma 6.3.

Proposition 8.7. *Let $\{b_0, b_1\} \subset G$ be independent and nonshrinking such that $b_0 \approx b_1 \approx \Gamma_i$. Then, for any $d > 0 \in \mathbb{R}$, there are nonzero coefficients c_0, c_1 such that*

$$|\psi_i((c_0 b_0 + L_i) + (c_1 b_1 + L_i))| < d.$$

Proof. This lemma is a direct consequence of Proposition 5.4. $\square$

Proposition 8.8. *Let $B = \{b_0, \dots, b_m\} \subset G$ be independent and nonshrinking such that $b_j \approx \Gamma_i$ for each j, and assume that Γ_i represents an infinite rank Archimedean class. Let $X = \{x_0, \dots, x_n\} \subset G$ be such that $x_j \approx \Gamma_i$ for each j, and fix $d > 0 \in \mathbb{R}$ and $p > 0 \in \mathbb{N}$. There are elements $a_0, \dots, a_n \in G$ such that*

1. *$\{b_0, \dots, b_m, x_0 + a_0, \dots, x_n + a_n\}$ is independent and nonshrinking, and*
2. *for each j, (p divides a_j), ($x_j + a_j \approx \Gamma_i$), and $|\psi_i(a_j + L_i)| < d$.*

Proof. As in Proposition 5.6, we prove this lemma for x_0 and then proceed by induction. If $B \cup \{x_0\}$ is independent and nonshrinking, then let $a_0 = 0_G$. Otherwise, there are coefficients $c_0, \dots, c_m, \alpha$ with $\alpha > 0$ such that $c_0 b_0 + \cdots + c_m b_m + \alpha x_0 = y \ll \Gamma_i$. Solving for αx_0 gives $\alpha x_0 = y - c_0 b_0 - \cdots - c_m b_m$.

Since Γ_i represents an infinite rank Archimedean class, we can fix a $b \approx \Gamma_i$ such that $\{b_0 + L_i, \dots, b_m + L_i, b + L_i\}$ is independent in G/L_i. Clearly, $B \cup \{b\}$ is independent, but by the argument in Theorem 8.4, it is also nonshrinking. Next, we apply Proposition 8.7 to get nonzero coefficients c_0, c_1 such that $|\psi_i((c_0 b_0 + L_i) + (c_1 b + L_i))| < d/p$ and we let $a_0 = pc_0 b_0 + pc_1 b$.

To see that $B \cup \{x_0 + a_0\}$ is independent and nonshrinking, suppose there are coefficients $d_0, \dots, d_m, \beta$ such that $d_0 b_0 + \cdots + d_m b_m + \beta(x_0 + a_0) = z \ll \Gamma_i$. Since B is independent and nonshrinking, we know $\beta \neq 0$. Multiplying by α and performing several substitutions, we get

$$\alpha d_0 b_0 + \cdots + \alpha d_m b_m + \beta \alpha x_0 + \beta \alpha a_0 = \alpha z \ll \Gamma_i,$$

$$\alpha d_0 b_0 + \cdots + \alpha d_m b_m + \beta(y - c_0 b_0 - \cdots - c_m b_m) + \beta\alpha(pc_0 b_0 + pc_1 b) = \alpha z$$

and

$$\begin{aligned}(\alpha d_0 - \beta c_0 + \beta\alpha p c_1) b_0 + (\alpha d_1 + \beta c_1) b_1 + \cdots \\ + (\alpha d_m - \beta c_m) b_m + \beta \alpha p c_1 b = \alpha z - \beta y \ll \Gamma_i.\end{aligned}$$

Since $\beta \alpha p c_1 \neq 0$, the bottom equation contradicts the fact that $B \cup \{b\}$ is independent and nonshrinking. $\square$

Lemma 8.9. *Let $G_0 \subset G_1$ be finite sets and assume that G_0 is closed under Archimedean differences. Let X_0 be an approximate nonshrinking basis for G_0 with weight t_0. There exists an approximate nonshrinking basis X_1 for G_1 which coheres with X_0.*

Proof. If X_0 is independent and nonshrinking, then let X_1 be any independent nonshrinking extension of X_0 which spans G_1. If X_0 is either not independent or not nonshrinking, then we begin by partitioning X_0 into Archimedean classes. For simplicity of notation, assume that there are only two Archimedean classes in X_0. The general case follows by a similar argument, which is sketched after the case of two Archimedean classes. Let $X_0 = \{b_1, \dots, b_l\} \cup \{e_1, \dots, e_m\}$, where $b_i \approx \Gamma_b$ and $e_i \approx \Gamma_e$ for Archimedean representatives $\Gamma_b \ll \Gamma_e$.

Second, consider each Archimedean class in X_0 and separate out the initial segment which is independent and nonshrinking. That is,

$$X_0 = \{b_1, \dots, b_i\} \cup \{b_{i+1}, \dots, b_l\} \cup \{e_1, \dots, e_j\} \cup \{e_{j+1}, \dots, e_m\},$$

where $\{b_0, \ldots, b_i\}$ is independent and nonshrinking, but $\{b_0, \ldots, b_{i+1}\}$ is not (and similarly for e_j). Let $d > 0 \in \mathbb{R}$ be less than the minimum of all the following conditions:

1. $|\psi_b(x + L_b)|$ for $x \in G_0$, $x \approx \Gamma_b$, and
2. $|\psi_b(x + L_b) - \psi_b(y + L_b)|$ for $x, y \in G_0$ with $x \approx y \approx \Gamma_b$ and $x + L_b \neq y + L_b$, and
3. $|\psi_e(x + L_e)|$ for $x \in G_0$ with $x \approx \Gamma_e$, and
4. $|\psi_e(x + L_e) - \psi_e(y + L_e)|$ for $x, y \in G_0$ with $x \approx y \approx \Gamma_e$ and $x + L_e \neq y + L_e$.

Let $d' = d/(3t_0(l + m))$.

Apply Proposition 8.8 with $B = \{b_1, \ldots, b_i\}$, ψ_b, d', $t_0!$, and $X = \{b_{i+1}, \ldots, b_l\}$ to get $\{a_{i+1}, \ldots, a_l\}$. Also, apply Proposition 8.8 with $B = \{e_1, \ldots, e_j\}$, ψ_e, d', $t_0!$, and $X = \{e_{j+1}, \ldots, e_m\}$ to get $\{a'_{j+1}, \ldots, a'_m\}$. Let

$$Y = \{b_1, \ldots, b_i\} \cup \{b_{i+1} + a_{i+1}, \ldots, b_l + a_l\} \cup \{e_1, \ldots, e_j\} \cup \{e_{j+1} + a'_{j+1}, \ldots, e_m + a'_m\}.$$

Since Y is independent and nonshrinking, we can extend it to X_1 which is independent, nonshrinking, spans G_1, and for which $|X_1| \geqslant |X_0|$. Clearly, X_1 is an approximate nonshrinking basis for G_1. To see that X_1 coheres with X_0, notice that X_1 is t_1-independent and t_1-nonshrinking for arbitrarily large t_1. Also, the fact that $t_0!$ divides each a_k and a'_k shows that every equation over X_0 which defines an element of G_0 has a solution over X_1.

To check the last condition, suppose $g < h \in G_0$ satisfy the reduced equations

$$\alpha y = c_1 b_1 + \cdots + c_l b_l + c_{l+1} e_1 + \cdots + c_{l+m} e_m$$

and

$$\beta y = d_1 b_1 + \cdots + d_l b_l + d_{l+1} e_1 + \cdots + d_{l+m} e_m,$$

respectively. Let $g', h' \in G$ be the solutions to these equations over X_1, that is, with $(b_{i+1} + a_{i+1})$ through $(b_l + a_l)$ in place of b_{i+1} through b_1, and $(e_{j+1} + a'_{j+1})$ through $(e_m + a'_m)$ in place of e_{j+1} through e_m.

We need to show that $g' < h'$. There are several cases to consider. First, suppose $g \approx \Gamma_b$ and $h \approx \Gamma_e$. $g < h$ implies that $h > 0_G$, and $g \approx \Gamma_b$ implies that the coefficients $c_{l+1}, \ldots, c_{l+m}$ are all 0. Therefore, $g' \approx \Gamma_b$, and similarly, $h' \approx \Gamma_e$. We claim that $h > 0_G$ implies that $h' > 0_G$. To see this fact, notice

$$\beta h - \beta h' = d_{i+1} a_{i+1} + \cdots + d_l a_l + d_{l+j+1} a'_{j+1} + \cdots + d_{l+m} a'_m.$$

Therefore, $|\psi_e((\beta h - \beta h') + L_e)| \leqslant (l + m) t_0 d' \leqslant d/3$. Since $|\psi_e(h + L_e)| > d$ and $|\psi_e((h - h') + L_e)| \leqslant |\psi_e((\beta h - \beta h') + L_e)|$, we have that $\psi_e(h' + L_e) > 0$. The definition of the quotient order and the fact that ψ_e is an embedding imply that $h' > 0_G$. Putting these facts together, we have that $g' \approx \Gamma_b \ll \Gamma_e \approx h'$ and $h' > 0_G$, and therefore $g' < h'$. A similar analysis applies when $g \approx \Gamma_e$ and $h \approx \Gamma_b$.

It remains to consider the case when $g \approx h$. Assume $g \approx h \approx \Gamma_e$ and consider the case when $g - h \approx \Gamma_e$. In this case, $g + L_e \neq h + L_e$, so $g + L_e < h + L_e$, and hence

$\psi_e(g + L_e) < \psi_e(h + L_e)$. By calculations similar to those above and those in Lemma 6.3 involving our choice of d', we have that $\psi_e(g' + L_e) < \psi_e(h' + L_e)$. Therefore $g' + L_e < h' + L_e$, which implies $g' < h'$.

If $g \approx h \approx \Gamma_e$, but $h - g \ll \Gamma_e$, then since G_0 is closed under Archimedean differences, we know that $h - g \in G_0$ and since $g < h$, we have $0_G < h - g$. Again, by our choice of d, this means that $0_G < h' - g'$, and so $g' < h'$.

Finally, if $g \approx h \approx \Gamma_b$, then since G_0 is closed under Archimedean differences and Γ_b represents the smallest Archimedean class in G_0, we know $g - h \approx \Gamma_b$. The analysis for the case when $g \approx h \approx \Gamma_e$ applies in this case as well.

We now sketch the general case. Suppose there are k Archimedean classes in X_0. We partition X_0 into Archimedean classes, $X_0 = \{b_1^1, \dots, b_{n_1}^1\} \cup \dots \cup \{b_1^k, \dots, b_{n_k}^k\}$ corresponding to the representatives $\Gamma_{b_1}, \dots, \Gamma_{b_k}$. Next, for each $j \leqslant k$, we separate out the initial segment of $\{b_1^j, \dots, b_{n_j}^j\}$ which is independent and nonshrinking, $\{b_1^j, \dots, b_{m_j}^j\} \cup \{b_{m_j+1}^j, \dots, b_{n_j}^j\}$. We fix $d > 0 \in \mathbb{R}$, as above, which is less that the minimum for all $j \leqslant k$ of

1. $|\psi_{b_j}(x + L_{b_j})|$ for $x \in G_0$, $x \approx \Gamma_{b_j}$, and
2. $|\psi_{b_j}(x + L_{b_j}) - \psi_{b_j}(y + L_{b_j})|$ for each $x, y \in G_0$ with $x \approx y \approx \Gamma_{b_j}$ and $x + L_{b_j} \neq y + L_{b_j}$.

Let $d' = d/3t_0(n_1 + \dots + n_k)$. For each $j \leqslant k$, we apply Proposition 8.8 to get $\{a_{m_j+1}^j, \dots, a_{n_j}^j\}$. Let $Y = \bigcup_{j \leqslant k}\{b_1^j, \dots, b_{m_j}^j, b_{m_j+1}^j + a_{m_j+1}^j, \dots, b_{n_j}^j + a_{n_j}^j\}$. Since Y is independent and nonshrinking, we can extend it to X_1 which is independent, nonshrinking, and spans G_1. As above, our definition of d' implies that if $h \in G_0$ is positive and satisfies a reduced equation over X_0, then the solution h' to the same equation over X_1 is also positive. The proof that X_1 coheres with X_0 now breaks into cases exactly as above. $\square$

Now that we have the appropriate replacements for Propositions 5.4 and 5.6 and Lemma 6.3, we sketch the remainder of the argument. There is a search procedure to make Lemma 8.9 effective just as in Section 6, except when we search for dependence relations, we also search for sums which shrink in terms of the Archimedean classes.

For our given group G, we build H and a Δ_2^0 isomorphism $f : H \to G$ in stages as before. We again meet the requirements

$$R_e\text{: } \varphi_e : H \to G \text{ is not an isomorphism}$$

by diagonalization. The first change in the construction is to use approximate nonshrinking bases instead of just approximate bases. These insure that our basis at the end of the construction is nonshrinking.

The second change is to fix the part of the basis for the finite rank Archimedean classes at stage 0. For each Γ_i which represents a finite rank Archimedean class, let $n_i = \operatorname{rank}(E_i)$, where E_i is the subgroup of G/L_i consisting of the least nontrivial Archimedean class. Pick a set B_i which is independent, nonshrinking, has size n_i, and such that for all $x \in B_i$, $x \approx \Gamma_i$. Place these elements in the approximate nonshrinking basis at stage 0. Since these elements are in fact independent and nonshrinking, they will remain in all approximate nonshrinking bases chosen later in the construction.

The third change is to fix the least i such that Γ_i represents an infinite rank Archimedean class. We perform the diagonalization to meet R_e using approximate basis elements which are $\approx \Gamma_i$. Just as Proposition 5.4 is used in Lemma 7.1 to perform the diagonalization, Proposition 8.7 is used here.

With these changes, the proofs for Theorems 1.12 and 1.14 proceed just as those for Theorems 1.11 and 1.3.

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