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An inversion relation of multinomial type

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Abstract

Recently Heuvers, Cummings, and Rao proved that if Ψ and Φ are functions satisfying the relation

$$\Psi(x_1,\ldots,x_n)=\sum_{|s|=n}\binom{n}{s}\Phi(x_1^{s_1},\ldots,x_n^{s_n}),$$

then there exist unique numbers c_s such that

$$\Phi(x_1,\ldots,x_n)=\sum_{|s|=n}c_s\,\Psi(x_1^{s_1},\ldots,x_n^{s_n}).$$

In this paper, an explicit expression for c_s is established.

1. Introduction

The following inversion theorem of multinomial type was proved in 1988 by Heuvers, Cummings, and Bhaskara Rao [3, Theorem 3]. It was a major tool in the recent characterization of determinant and permanent functions by the Binet–Cauchy theorem [4–6].

Theorem 1.1. *Let K be a field of characteristic zero, let X be a nonempty set, and let V be a vector space over K. Let $\Phi,\Psi:X^n\to V$ be functions satisfying*

$$\Psi(x_1,\ldots,x_n)=\sum_{|s|=n}\binom{n}{s}\Phi(x_1^{s_1},\ldots,x_n^{s_n}) \tag{1}$$

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for all $(x_1, \ldots, x_n) \in X^n$, *where*

$$(x_1^{s_1}, \ldots, x_n^{s_n}) = (\underbrace{x_1, \ldots, x_1}_{s_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{s_n}).$$

Then

$$\Phi(x_1, \ldots, x_n) = \sum_{|s|=n} c_s \Psi(x_1^{s_1}, \ldots, x_n^{s_n}) \tag{2}$$

for fixed constants c_s *depending only on the* $\binom{n}{s}$.

An important example where this theorem applies is when $\Phi(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$ and $\Psi(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)^n$.

For the proof of the theorem the existence of unique coefficients c_s was established but their values were not determined. In an attempt to investigate them we used a computer algebra system to calculate the c_s for $2 \leqslant n \leqslant 6$. As a result we were led to the conjecture

$$c_s = \frac{(-1)^{n-d}}{s_{i_1} \cdots s_{i_d} n^{n-d} \langle n \rangle_d}, \tag{3}$$

where $s_{i_1}, \ldots, s_{i_d}$ are the nonzero components of s and $\langle n \rangle_d = n!/(n-d)!$. The goal of this paper is to prove (3) for $2 \leqslant n$.

2. Notation

Let $\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}_+^n$ Let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\alpha, \beta \in \boldsymbol{Z}_+^n$ let $\beta^\alpha = \beta_1^{\alpha_1} \cdots \beta_n^{\alpha_n}$. If $s = (s_1, \ldots, s_n) \in \boldsymbol{Z}_+^n$ and $|s| = n$ let $s! = s_1! \cdots s_n!$. Then $\binom{n}{s}$ is the multinomial coefficient

$$\frac{n!}{s_1! \cdots s_n!} = \frac{n!}{s!}.$$

If $x = (x_1, \ldots, x_n)$ is an n-tuple of indeterminates let $|x| = x_1 + \cdots + x_n$, and if we have $1 \leqslant p \leqslant q \leqslant n$ then let $x' = (x_1, \ldots, x_q)$ and $x'' = (x_1, \ldots, x_p)$. Also we let $\varepsilon = (1, \ldots, 1) \in \boldsymbol{Z}_+^n$ be the n-tuple all of whose components are 1.

An order will be introduced on $\boldsymbol{Z}_+^n$. We will say $\alpha \prec \beta$ if $\alpha_i < \beta_i$ for all $i = 1, \ldots, n$. The symbols $\preccurlyeq$, $\succ$, and $\succcurlyeq$ have the obvious interpretations. Let $\Omega_{q,n}$ be the set of multi-indices with q terms and sums equal to n, i.e. $\Omega_{q,n} = \{\alpha' \in \boldsymbol{Z}_+^q : |\alpha'| = n\}$. Also let $\Omega_{q,n}^+ = \{\alpha' \in \Omega_{q,n} : \alpha_i > 0, \text{ i} = 1, \ldots, q\}$.

In order to simplify our notation we will adopt a formal 'product' notation for repeated adjacent identical terms inside n-tuples. Thus

$$(\underbrace{x_1, \ldots, x_1}_{s_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{s_n})$$

will be denoted by $(x_1^{s_1}, \dots, x_n^{s_n})$ or (x^s), where s_i the number of times that x_i appears together inside the n-tuple. If $s_i = 0$ then x_i does not appear.

We define $H_m(n)$ by

$$H_m(n) = \sum_{s \in \Omega_{m,n}^+} \frac{1}{s_1 s_2 \cdots s_m}. \tag{4}$$

For $p \in \boldsymbol{Z}_+$ and one indeterminate x, the pth degree falling factorial polynomial $\langle x \rangle_p$ is defined by $\langle x \rangle_p = 1$ if $p = 0$ and $\langle x \rangle_p = \prod_{i=1}^{p} (x - i + 1)$ if $p \geqslant 1$. For $\alpha' \in \boldsymbol{Z}_+^q$ with $|\alpha'| = n$ and $x' = (x_1, \dots, x_q)$ the nth degree polynomial $\langle x' \rangle_{\alpha'}$ is defined by $\langle x' \rangle_{\alpha'} = \langle x_1 \rangle_{\alpha_1} \cdots \langle x_q \rangle_{\alpha_q}$ [11, p. 113; 4, pp. 200–203].

Remark. The notation for falling factorials is not fixed yet. Often the notation $(x)_p$ is used to denote falling factorials (e.g. see [10, p. 3; 1, p. 6]). However, the Pochammer symbols $(x)_p$, or rising factorials, are defined by $(x)_p = 1$ if $p = 0$ and $(x)_p = \prod_{i=1}^{p} (x + i - 1)$ if $p \geqslant 1$. This is the established notation in the study of hypergeometric functions (e.g. see [9, p. 22; 7, pp. 8, 9]). We use the notation $\langle x \rangle_p$ for falling factorials to avoid confusion with the Pochammer symbols.

3. Some preliminary results

If $\Phi(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$ and $\Psi(x_1, x_2, \dots, x_n) = (x_1 + x_2 + \cdots + x_n)^n$ then it immediately follows from (2) that the c_s are the unique coefficients satisfying

$$x_1 x_2 \cdots x_n = \sum_{|s| = n} c_s (s_1 x_1 + s_2 x_2 + \cdots + s_n x_n)^n. \tag{5}$$

Taking n partial derivatives on both sides and comparing coefficients we obtain the following result.

Proposition 3.1. *The c_s are the unique numbers satisfying*

$$1 = n! \sum_{|s| = n} c_s s_1 s_2 \cdots s_n \tag{6}$$

and

$$0 = \sum_{|s| = n} c_s s^t, \quad t \in \Omega_{n,n} - \{\varepsilon\}. \tag{7}$$

The left-hand side of (5) is symmetric in $x_1, \dots, x_n$. Consequently, we have the following corollary.

Corollary 3.2. *c_s is a symmetric function of $s_1, s_2, \dots, s_n$.*

In order to establish the main theorem we also need the following tool.

Proposition 3.3. *If n is a strictly positive integer and α is any complex number then*

$$\sum_{m=1}^{n} \frac{(-\alpha)^m}{m!} H_m(n) = (-1)^n \binom{\alpha}{n}. \tag{8}$$

Proof. Start with the identity $(1-x)^\alpha = e^{\alpha \ln(1-x)}$, then expand $(1-x)^\alpha$, $\ln(1-x)$, and e^x into their Maclaurin series. The result is

$$\sum_{i=0}^{\infty} \binom{\alpha}{i} (-1)^i x^i = \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\alpha \sum_{j=1}^{\infty} \frac{x^j}{j} \right)^m. \tag{9}$$

Expanding out the right side, we obtain

$$\sum_{i=0}^{\infty} \binom{\alpha}{i} (-1)^i x^i = \sum_{\substack{m \geqslant 0 \\ j_1 \geqslant 1 \cdots j_m \geqslant 1}} \frac{(-\alpha)^m x^{j_1 + \cdots + j_m}}{m! j_1 \cdots j_m}. \tag{10}$$

Now compare the coefficients of x^n on both sides of (10). The result is (8). □

Another generating function identity is also needed.

Proposition 3.4. *Let q be a strictly positive integer, n be a nonnegative integer, and $a' = (a_1, \ldots, a_q) \in \mathbf{Z}_+^q$. Then*

$$\sum_{s' \in \Omega_{q,n}} \langle s' \rangle_{a'} = a'! \binom{n+q-1}{|a'|+q-1}. \tag{11}$$

Proof. Consider the identity

$$\frac{x^{a_1} a_1!}{(1-x)^{a_1+1}} \frac{x^{a_2} a_2!}{(1-x)^{a_2+1}} \cdots \frac{x^{a_q} a_q!}{(1-x)^{a_q+1}} = a'! \frac{x^{|a'|}}{(1-x)^{|a'|+q}}. \tag{12}$$

Represent each fraction in (12) with its series expansion. The result is

$$\left(\sum_{s_1=0}^{\infty} \langle s_1 \rangle_{a_1} x^{s_1} \right) \left(\sum_{s_2=0}^{\infty} \langle s_2 \rangle_{a_2} x^{s_2} \right) \cdots \left(\sum_{s_q=0}^{\infty} \langle s_q \rangle_{a_q} x^{s_q} \right)$$
$$= \frac{a'!}{(|a'|+q-1)!} x^{1-q} \sum_{k=0}^{\infty} \langle k \rangle_{|a'|+q-1} x^k. \tag{13}$$

Now compare the coefficients of x^n on both sides of (13). The result is (11). □

We also need the following change of basis result.

Proposition 3.5. *Let $\alpha' \in \Omega_{q,n}$ and let $x_1, x_2, \ldots, x_q$ be q indeterminates. Then there exist coefficients $\mathscr{S}(\beta', \alpha')$, $\beta' \preccurlyeq \alpha'$, such that*

$$x'^{\alpha'} = \sum_{\beta' \preccurlyeq \alpha'} \mathscr{S}(\beta', \alpha') \langle x' \rangle_{\beta'}. \tag{14}$$

Moreover, if $\alpha' \in \Omega^+_{q,n}$ then

$$x'^{\alpha'} = \sum_{e' \preccurlyeq \beta' \preccurlyeq \alpha'} \mathcal{S}(\beta', \alpha') \langle x' \rangle_{\beta'}. \tag{15}$$

Proof. A proof of (14) can be found in [8, Lemma 2]. Also, we have from [8],

$$\mathcal{S}(\beta', \alpha') = \prod_{i=1}^{q} S(\beta_i, \alpha_i), \quad \alpha', \beta' \in \Omega_{q,n}, \tag{16}$$

where $S(\beta_i, \alpha_i)$ are the ordinary Stirling numbers of the second kind defined by

$$x^m = \sum_{k=0}^{m} S(k, m) \langle x \rangle_k. \tag{17}$$

Upon setting $x=0$ it becomes clear that $S(0,m)=0$ for $m>0$. If $\alpha' \in \Omega^+_{q,n}$, then it follows from (16) that $\mathcal{S}(\beta', \alpha')=0$ if any component of β' is zero. This proves (15). □

Finally two binomial coefficient identities will be needed:

$$\sum_{k=0}^{n-m} \binom{x+k}{k} \binom{n-k}{m} = \binom{x+n+1}{n-m}, \quad n \geqslant m, \tag{18}$$

where n, m are nonnegative integers and x is any complex number. This follows from Gould [2, (3.2)] together with the symmetry relation $\binom{n}{k} = \binom{n}{n-k}$.

We also need the following identity from Gould [2, (3.49)]:

$$\sum_{k=0}^{n} \binom{n}{k} \binom{x-k}{m} (-1)^k = \binom{x-n}{m-n}, \tag{19}$$

where n, m are nonnegative integers and x is any complex number.

4. The main result

Theorem 4.1. *Let the coefficients c_s be defined by* (5). *Then*

$$c_s = \frac{1}{s_{i_1} s_{i_2} \cdots s_{i_d} \langle n \rangle_d (-n)^{n-d}}, \tag{20}$$

where $s_{i_1}, s_{i_2}, \ldots, s_{i_d}$ are the nonzero entries of s.

Proof. By Proposition 3.1 it suffices to show that if c_s is given by (20) then (6) and (7) both hold. The only nonzero term in (6) is the term where $s = \varepsilon = (1, 1, \ldots, 1)$, so (6) clearly holds for the c_s given by (20).

We now need to prove (7), so we assume that $t \in \Omega_{n,n}$ and that $t \neq \varepsilon$. Due to the symmetry of the c_s, it suffices to show that (7) holds if $t = (t_1, \ldots, t_n)$, where $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_n$. Clearly $t_1 > 1$ and $t_n = 0$. Accordingly, we let $t_1, t_2, \ldots, t_q$ be the nonzero components of t, and we also assume that $t_1>1, t_2>1, \ldots, t_p>1$. It is clear that the nonzero terms in the sum (7) occur where the first q components of s are strictly positive. Now let $s'=(s_1,\ldots,s_q)$, $t'=(t_1,\ldots,t_q)$, $\varepsilon'=(1^q)$, and $\tau'=(t_1-1,\ldots,t_q-1)=t'-\varepsilon'$. Also let $s''=(s_1,\ldots,s_p)$, $t''=(t_1,\ldots,t_p)$, $\varepsilon''=(1^p)$, and $\tau''=t''-\varepsilon''=(t_1-1,\ldots,t_p-1)$. We now have

$$\sum_{|s|=n} c_s s^t = \sum_{j=q}^{n} \sum_{\substack{s\in\Omega_{n,n}\\ s'\in\Omega^+_{q,j}}} c_s s'^{t'}. \tag{21}$$

Our goal is to show that the sum on the right is zero. To simplify the notation in what follows, we will always assume that $s\in\Omega_{n,n}$ without explicitly stating it all the time.

Now let $s_{i_1}, s_{i_2}, \ldots, s_{i_c}$ be the nonzero entries among the last $n-q$ entries of s. Then we obtain the following for the right-hand side of (21):

$$\sum_{s'\in\Omega^+_{q,n}} s'^{\tau'} \frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} + \sum_{j=q}^{n-1} \sum_{s'\in\Omega^+_{q,j}} \sum_{c=1}^{n-j} s'^{\tau'} \frac{(-1)^{n-c-q}}{n^{n-c-q}\langle n\rangle_{c+q}} \sum_{\substack{\{i_1,\ldots,i_c\}\subseteq\{q+1,\ldots,n\}\\ (s_{i_1},\ldots,s_{i_c})\in\Omega^+_{c,n-j}}} \frac{1}{s_{i_1}\cdots s_{i_c}}. \tag{22}$$

For each choice $i_1, i_2, \ldots, i_c$ out of the set $\{q+1,\ldots,n\}$ we have

$$\sum_{(s_{i_1},\ldots,s_{i_c})\in\Omega^+_{c,n-j}} \frac{1}{s_{i_1}\cdots s_{i_c}} = H_c(n-j).$$

Therefore (22) becomes

$$\sum_{s'\in\Omega^+_{q,n}} s'^{\tau'} \frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} + \sum_{j=q}^{n-1} \sum_{s'\in\Omega^+_{q,j}} \sum_{c=1}^{n-j} s'^{\tau'} \frac{(-1)^{n-c-q}}{n^{n-c-q}\langle n\rangle_{c+q}} \binom{n-q}{c} H_c(n-j). \tag{23}$$

This can be rewritten as

$$\sum_{s'\in\Omega^+_{q,n}} s'^{\tau'} \frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} + \sum_{j=q}^{n-1} \sum_{s'\in\Omega^+_{q,j}} s'^{\tau'} \frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} \sum_{c=1}^{n-j} \frac{(-n)^c}{c!} H_c(n-j). \tag{24}$$

Now we use Proposition 3.3 to obtain

$$\sum_{s'\in\Omega^+_{q,n}} s'^{\tau'} \frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} + \sum_{j=q}^{n-1} \sum_{s'\in\Omega^+_{q,j}} s'^{\tau'} \frac{(-1)^{2n-q-j}}{n^{n-q}\langle n\rangle_q} \binom{n}{n-j}. \tag{25}$$

Now the right-hand side of (21) becomes

$$\frac{(-1)^{n-q}}{n^{n-q}\langle n\rangle_q} \sum_{i=0}^{n-q} \binom{n}{i} (-1)^i \sum_{s'\in\Omega^+_{q,n-i}} s'^{\tau'}. \tag{26}$$

Clearly it suffices to show that

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{s'\in\Omega^+_{q,n-i}} s'^{\tau'}=0. \tag{27}$$

At this point we consider two cases.

4.1. *Case $q>p$*

Here we have $t_1>1,\ t_2>1,\dots,t_p>1,\ t_{p+1}=\cdots=t_q=1$, and $t_{q+1}=\cdots=t_n=0$. Let $\tilde{s}=(s_{p+1},\dots,s_q)$. Then we can write the sum in (27) as

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=p}^{n+p-i-q}\left(\sum_{\tilde{s}\in\Omega^+_{p,k}} s''^{\tau''}\right)\left(\sum_{\tilde{s}''\in\Omega^+_{q-p,n-i-k}} 1\right). \tag{28}$$

So (28) becomes

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=p}^{n+p-i-q}\binom{n-i-k-1}{q-p-1}\sum_{s''\in\Omega^+_{p,k}} s''^{\tau''}. \tag{29}$$

Since $\tau''\succcurlyeq e''$, $s''^{\tau''}=0$ whenever one of the components of s'' is zero. Hence

$$\sum_{s''\in\Omega^+_{p,k}} s''^{\tau''}=\sum_{s''\in\Omega_{p,k}} s''^{\tau''}.$$

Moreover, if $k<p$ and $s''\in\Omega_{p,k}$ then at least one component of s'' is zero which implies that $s''^{\tau''}=0$. So the k sum can start at 0 rather than at p. Hence (29) can be rewritten as

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=0}^{n+p-i-q}\binom{n-i-k-1}{q-p-1}\sum_{s''\in\Omega_{p,k}} s''^{\tau''}. \tag{30}$$

Then we can use Proposition 3.5 on $s''^{\tau''}$ to express (30) as

$$\sum_{e''\preccurlyeq\alpha''\preccurlyeq\tau''}\mathcal{S}(\alpha'',\tau'')\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=0}^{n+p-i-q}\binom{n-i-k-1}{q-p-1}\sum_{s''\in\Omega_{p,k}}\langle s''\rangle_{\alpha''}. \tag{31}$$

Our goal is to show that the above sum is zero. It suffices to show that

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=0}^{n+p-i-q}\binom{n-i-k-1}{q-p-1}\sum_{s''\in\Omega_{p,k}}\langle s''\rangle_{\alpha''}=0 \quad \text{for } e''\preccurlyeq\alpha''\preccurlyeq\tau''. \tag{32}$$

Apply Proposition 3.4 on the sum in (32) and the result is

$$\alpha''!\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i \sum_{k=0}^{n+p-i-q}\binom{n-i-k-1}{q-p-1}\binom{k+p-1}{|\alpha''|+p-1}. \tag{33}$$

Next we will show that the sum in (33) is zero. Setting $m=n+p-q-i-k$ it becomes

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i\sum_{m=0}^{n+p-i-q}\binom{q-p+m-1}{q-p-1}\binom{n+2p-q-i-m-1}{|\alpha''|+p-1}$$
$$=\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i\sum_{m=0}^{n+p-i-q}\binom{q-p+m-1}{m}\binom{n+2p-q-i-m-1}{|\alpha''|+p-1}. \quad (34)$$

Notice that if $m>n+p-q-i-|\alpha''|$ then $\binom{n+2p-q-i-m-1}{|\alpha''|+p-1}=0$. Hence we can adjust the upper index of the inner sum to obtain

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i\sum_{m=0}^{n+p-i-q-|\alpha''|}\binom{q-p+m-1}{m}\binom{n+2p-q-i-m-1}{|\alpha''|+p-1}. \quad (35)$$

Next use the binomial coefficient identity (18) to obtain

$$\sum_{i=0}^{n-q}\binom{n}{i}\binom{n+p-i-1}{n+p-i-q-|\alpha''|}(-1)^i=\sum_{i=0}^{n-q}\binom{n}{i}\binom{n+p-i-1}{q+|\alpha''|-1}(-1)^i. \quad (36)$$

Since $\alpha''\geqslant e''$, $|\alpha''|\geqslant p$, thus $\binom{n+p-i-1}{q+|\alpha''|-1}=0$ for $i>n-q$. Hence we can extend the upper limit of the sum to n, so (36) becomes

$$\sum_{i=0}^{n}\binom{n}{i}\binom{n+p-i-1}{q+|\alpha''|-1}(-1)^i. \quad (37)$$

Use the binomial coefficient identity (19) on (37) to obtain

$$\binom{p-1}{q+|\alpha''|-n-1}. \quad (38)$$

Here $|\alpha''|\leqslant|\tau''|=|t''|-|e''|=|t'|-(q-p)-p=n-q$. So $q+|\alpha''|-n-1\leqslant-1$ and hence the above expression is zero.

4.2. Case $q=p$

Here the sum in (27) becomes

$$\sum_{i=0}^{n-q}\binom{n}{i}(-1)^i\sum_{s'\in\Omega^+_{q,n-i}}s'^{\tau'}. \quad (39)$$

Here $q=p$ implies that all the components of t' are greater than one. Thus all components of τ' are strictly positive. So if any component of s' is zero, then $s'^{\tau'}=0$. Thus

$$\sum_{s'\in\Omega^+_{q,n-i}}s'^{\tau'}=\sum_{s'\in\Omega_{q,n-i}}s'^{\tau'},\quad i=0,1,\ldots,n-q.$$

Also if $i>n-q$ and $s'\in\Omega_{q,n-i}$ then at least one component of s' is zero which gives $s'^{\tau'}=0$. Hence the upper index of the i sum can be extended to n. Hence (39) becomes

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{s'\in\Omega_{q,n-i}} s'^{\tau'}. \tag{40}$$

Use Proposition 3.5 on (40) to obtain

$$\sum_{\alpha'\leqslant\tau'}\mathscr{S}(\alpha',\tau')\sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{s'\in\Omega_{q,n-i}}\langle s'\rangle_{\alpha'}. \tag{41}$$

We want to show that the above sum is zero and it suffices to show that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{s'\in\Omega_{q,n-i}}\langle s'\rangle_{\alpha'}=0 \quad \text{for } \alpha'\leqslant\tau'. \tag{42}$$

Using Proposition 3.4 the sum in (42) becomes

$$\alpha'!\sum_{i=0}^{n}\binom{n}{i}\binom{n+q-i-1}{|\alpha'|+q-1}(-1)^i. \tag{43}$$

We will now show that the sum in (43) is zero. When the binomial coefficient identity (19) is used on the above sum we obtain

$$\binom{q-1}{|\alpha'|+q-n-1}. \tag{44}$$

Here $\alpha'\leqslant\tau'$ implies that $|\alpha'|\leqslant|\tau'|=|t'|-|e'|=n-q$. Hence $|\alpha'|+q-n-1\leqslant-1$, so the above expression is zero. Therefore

$$\sum_{|s|=n} c_s s^t=0 \tag{45}$$

in the case $q=p$ as well. □

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