The AOR iterative method for new preconditioned linear systems

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Abstract

In this paper we apply the AOR method to preconditioned linear systems different from those considered in Evans and Martins (Internat. J. Comput. Math. 5 (1995) 69–76), Gunawardena et al. (Linear Algebra Appl. 154–156 (1991) 123–143) and Li and Evans (Technical Report No. 901, Department of Computer Studies, University of Loughborough, 1994). Our results show that some improvements in the convergence rate of this iterative method can be obtained. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

For several years many researchers have been studying iterative methods which approximate the solution of the linear system

\[ A\bar{x} = \bar{b}, \]

where \( A \in \mathbb{C}^n \times n \), \( b \in \mathbb{C}^n \) are given and \( x \in \mathbb{C}^n \) is unknown. Some techniques of preconditioning which improve the rate of convergence of these iterative methods have been developed.

In this paper we show that, under certain assumptions, some iterative methods applied to some preconditioned systems are faster than when we apply them to the original system (1.1). Thus, let us consider the preconditioned linear system

\[ \tilde{A}\bar{x} = \tilde{b}, \]

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where
\[ \tilde{A} = (I + S)A \quad \text{and} \quad \tilde{b} = (I + S)b \] (1.3)
with
\[
S = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & 0 & \cdots & 0
\end{bmatrix}
\] (1.4)
and the preconditioned linear system
\[ A'\tilde{x} = b', \] (1.5)
where
\[ A' = (I + S')A \quad \text{and} \quad b' = (I + S')b \] (1.6)
with
\[
S' = \begin{bmatrix}
0 & 0 & \cdots & -a_{1n} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
\] (1.7)
Without loss of generality, let the matrix \( A \) of (1.1) be
\[ A = I - L - U, \] (1.8)
where \( I \) is the identity matrix, \( -L \) and \( -U \) are strictly lower and upper triangular matrices obtained from \( A \), respectively.
The accelerated overrelaxation (AOR) iterative method [4] is given by
\[ \tilde{x}^{(i+1)} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU]\tilde{x}^{(i)} + (I - rL)^{-1}wb, \quad i = 0, 1, \ldots \] (1.9)
whose iteration matrix is
\[ L_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU], \] (1.10)
where \( w \) and \( r \) are real parameters with \( w \neq 0 \).
It is well known [1,4] that, for certain values of the parameters \( w \) and \( r \), we obtain the Jacobi (J), the Gauss–Seidel (GS) and the successive overrelaxation (SOR) methods.
In the next section we prove that, under certain assumptions, the rate of convergence of the AOR iterative method can be enlarged if we apply this method to the preconditioned linear systems (1.2) and (1.5). In the following we are going to consider the preconditioned linear system (1.2), where
\[ \tilde{A} = (I + S)A = (I + S - L - SU) - U = \tilde{D} - \tilde{L} - \tilde{U} \] (1.11)
with
\[ \tilde{D} = \text{diag}(\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_n) \quad \text{where} \quad \tilde{d}_i = 1, \ i = 1, \ldots, n - 1 \text{ and } \tilde{d}_n = 1 - a_{1n}a_{1n}; \]
and the preconditioned linear system (1.5), where
\[ A' = (I + S')A = (I + S' - U - S'L) - L = D' - L' - U' \] (1.12)
with
\[ D' = \text{diag}(d'_1, d'_2, \ldots, d'_n) \] where \( d'_i = 1, \ i = 2, \ldots, n \) and \( d'_1 = 1 - a_{1n}a_{n1} \).

In [2,3,5] a different preconditioned linear system was considered and some results, on several iterative methods applied to it were obtained.

In the sequel, we need the following definitions and results.

**Definition 1.1** (Young [7]). A matrix \( A \) is an \( L \)-matrix if \( a_{ii} > 0, \ i = 1, \ldots, n \) and \( a_{ij} \leq 0, \) for all \( i, j = 1, \ldots, n \) such that \( i \neq j \).

**Definition 1.2** (Varga [6]). A matrix \( A \) is irreducible if the directed graph associated to \( A \) is strongly connected.

**Theorem 1.3** (Varga [6]). Let \( A \geq 0 \) be an irreducible \( n \times n \) matrix. Then

1. A has a positive real eigenvalue equal to its spectral radius.
2. To \( \rho(A) \) there corresponds an eigenvector \( x > 0 \).
3. \( \rho(A) \) is a simple eigenvalue of \( A \).

**2. On preconditioned AOR iterations**

In this section we will show that the rate of convergence of the standard AOR method can be improved if we precondition the linear system (1.1). Thus, if we apply the AOR method to (1.2) we have the preconditioned AOR iterative method, whose iteration matrix is

\[
\tilde{L}_{r,w} = (\tilde{D} - r\tilde{L})^{-1}[(1 - w)\tilde{D} + (w - r)\tilde{L} + w\tilde{U}].
\] (2.1)

If we apply the same iterative method to (1.5) we get the corresponding preconditioned iterative method whose iteration matrix \( L'_{r,w} \) has the same expression of (2.1), with \( \tilde{D}, \tilde{L} \) and \( \tilde{U} \), replaced by \( D', L' \) and \( U' \), respectively.

**Lemma 2.1.** Let \( A = (a_{ij}) \), \( \tilde{A} \) and \( A' \) be the coefficient matrices of the linear systems (1.1), (1.2) and (1.5). If \( 0 \leq r \leq w \leq 1 \) \( (w \neq 0) \ (r \neq 1) \) and \( A \) is an \( L \)-matrix with \( 0 < a_{i+1}a_{i+1} \), \( i = 1, \ldots, n - 1 \) and \( 0 < a_{1n}a_{n1} < 1 \) then the iteration matrices \( L_{r,w}, \tilde{L}_{r,w} \) and \( L'_{r,w} \) associated to the AOR method applied to the linear systems (1.1), (1.2) and (1.5), respectively, are nonnegative and irreducible.

**Proof.** From (1.10) we have

\[
L_{r,w} = (1 - w)I + w(1 - r)L + wU + T
\]

with \( T \geq 0 \).

Then, from Lemma 1 of [5], we can conclude that \( L_{r,w}, \tilde{L}_{r,w} \) and \( L'_{r,w} \) are nonnegative and irreducible. □
Theorem 2.2. Let \( L_{r,w} \) and \( \tilde{L}_{r,w} \) be the iteration matrices of the AOR method given by (1.10) and (2.1), respectively. If the matrix \( A \) of (1.1) is an \( L \)-matrix with \( 0 < a_{ij}a_{i+1,j+1} \), \( i = 1, \ldots, n-1 \) and \( 0 < a_{ii}a_{i+1} < 1 \), \( \rho(L_{r,w}) < 1 \) and \( 0 \leq r \leq w \leq 1 \) \( (w \neq 0) \) \( (r \neq 1) \) we have
\[
\rho(\tilde{L}_{r,w}) < \rho(L_{r,w}) < 1.
\]

Proof. From Lemma 2.1 it is clear that \( L_{r,w} \) and \( \tilde{L}_{r,w} \) are nonnegative and irreducible matrices. Thus, from Theorem 1.3 there is a positive vector \( x \), such that
\[
L_{r,w}x = \lambda x,
\]
where \( \lambda = \rho(L_{r,w}) \) or, equivalently,
\[
[(1 - w)I + (w - r)L + wU]x = \lambda (I - rL)x.
\]
Therefore,
\[
\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}[(1 - w)\tilde{D} + (w - r)\tilde{L} + w\tilde{U} - \lambda (\tilde{D} - r\tilde{L})]x.
\]
From (1.11) and (2.3) we have
\[
 w\tilde{U}x = wUx = (\lambda - 1 + w)x + (r - w - \lambda r)Lx
\]
and
\[
\lambda(\tilde{D} - r\tilde{L})x = \lambda (1 - r)\tilde{D}x + \lambda r(\tilde{D} - \tilde{L})x
\]
\[
= \lambda (1 - r)\tilde{D}x + \lambda r(I + S - L - SU)x.
\]
Then
\[
\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}[(1 - w - \lambda + \lambda r)\tilde{D} + \lambda (I - rL) - (1 - w)I
\]
\[
+ (w - r)(\tilde{L} - L) - \lambda r(I + S - L - SU)]x
\]
or, equivalently,
\[
\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}[(1 - \lambda)(1 - r)(\tilde{D} - I) + (w - r + \lambda r)(SU - S)]x.
\]
Using (2.3), we can still write
\[
\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}\{(1 - \lambda)(1 - r)(\tilde{D} - I) - (w - r + \lambda r)S + r(\lambda - 1)SU
\]
\[
+ S[\lambda(I - rL) - (1 - w)I - (w - r)L]\}x
\]
or, equivalently,
\[
\tilde{L}_{r,w}x - \lambda x = (\tilde{D} - r\tilde{L})^{-1}\{(1 - \lambda)(1 - r)(\tilde{D} - I) - (1 - r)(1 - \lambda)S + r(\lambda - 1)SU
\]
\[
+ (r - w - \lambda r)SL\}x.
\]
If \( Z = \tilde{L}_{r,w}x - \lambda x \) and \( \lambda < 1 \), from the previous equality we have \( Z \leq 0 \).
Thus, from Theorem 2 of [5] we get the required result. \( \square \)

Theorem 2.3. Let \( L_{r,w} \) and \( L'_{r,w} \) be the iteration matrices associated to the AOR method applied to the linear systems (1.1) and (1.5), respectively. Under the assumptions formulated in Theorem 2.2, we have
\[
\rho(L'_{r,w}) < \rho(L_{r,w})
\]
if \( \rho(L_{r,w}) < 1 \).
Proof. Following the proof of Theorem 2.2, let us consider

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)D' + (w - r)L' + wU' - \lambda(D' - rL')]x \]

or, equivalently,

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)D' - r(1 - \lambda)L' - w(D' - U') + wL']x. \]

From (1.12) and (2.3), we have

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)D' - r(1 - \lambda)L - w(I + S' - U - S'L) + wL]x \]

or, equivalently,

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)(D' - I) - w(S' - S'L)]x. \]

From (2.3) we can still write

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)(D' - I) - wS' + S'[\lambda(I - rL) - wU - (1 - w)I + rL]]x \]

or

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1}[(1 - \lambda)(D' - I) - wS'U + (\lambda - 1)S' + r(1 - \lambda)S'L]x \]

Using, again, (2.3) we have

\[ L_{r,w}^i x - \lambda x = (D' - rL')^{-1} \left\{ (1 - \lambda)(D' - I) - wS'U + \left(\frac{\lambda - 1}{\lambda}\right) S'[(1 - w)I + (w - r)L + wU] \right\} x. \]

Thus, as \( 0 < \lambda < 1 \), \( t = L_{r,w}^i x - \lambda x \) is a vector less than or equal to zero and from [5, Theorem 2] we obtain the required result. \( \Box \)

Corollary 2.4. Let \( L_w, \hat{L}_w \) and \( L'_w \) be the iteration matrices of the successive overrelaxation (SOR) iterative method associated to (1.1), (1.2) and (1.5), respectively. If the matrix \( A \) of (1.1) is an L-matrix with \( 0 < a_{i,i+1}a_{i+1,i}, i = 1, \ldots, n - 1 \) and \( 0 < a_{n1}a_{1n} < 1 \) and \( 0 < w < 1 \), we have

\[ \rho(\hat{L}_w) < \rho(L_w) \quad \text{if} \quad \rho(L_w) < 1 \]

and

\[ \rho(L'_w) < \rho(L_w) \quad \text{if} \quad \rho(L_w) < 1. \]

If in (1.9) and (2.1) we consider \( w = 1 \) and \( r = 0 \) we obtain the iteration matrices of the Jacobi method, associated to (1.1) and (1.2). Therefore, we also have the following result.

Corollary 2.5. Let \( B, B' \) and \( B' \) be the iteration matrices of the Jacobi method associated to (1.1), (1.2) and (1.5), respectively. If the matrix \( A \) of (1.1) is an L-matrix with \( 0 < a_{i,i+1}a_{i+1,i}, i = 1, \ldots, n - 1 \) and \( 0 < a_{n1}a_{1n} < 1 \), we have:

\[ \rho(B) < \rho(B) \quad \text{if} \quad \rho(B) < 1 \]

and

\[ \rho(B') < \rho(B) \quad \text{if} \quad \rho(B) < 1. \]
Remark. From the previous results we can conclude that the rate of convergence of the Jacobi, SOR and AOR iterative methods can be enlarged if we apply these iterative methods to the linear system (1.2) and (1.5).

3. Numerical example

In this section we give a numerical example to illustrate the results obtained in Section 2. Thus, let us consider the matrix $A$ of (1.1), similar to the one suggested in [3], and given by

$$A = \begin{bmatrix}
1 & -0.1 & -0.2 & 0 & -0.3 & -0.5 \\
-0.2 & 1 & -0.3 & 0 & -0.4 & -0.1 \\
0 & -0.3 & 1 & -0.6 & -0.2 & 0 \\
-0.2 & -0.3 & -0.1 & 1 & -0.1 & -0.3 \\
0 & -0.3 & -0.2 & -0.1 & 1 & -0.2 \\
-0.2 & -0.3 & 0 & -0.3 & -0.1 & 1
\end{bmatrix}$$

and the parameters $w$ and $r$ according to the conditions imposed in Theorems 2.2 and 2.3.

Thus, we have $\rho(B) = 0.971121$, $\rho(\bar{B}) = 0.958673$ and $\rho(B') = 0.956038$.

Analogously, for the SOR method, with $w = 0.9$, we obtain $\rho(L_w) = 0.9545$, $\rho(\tilde{L}_w) = 0.9376$ and $\rho(L'_w) = 0.9304$.

For the AOR method, with $w = 0.9$ and $r = 0.85$ we get $\rho(L_{r,w}) = 0.9564$, $\rho(\tilde{L}_{r,w}) = 0.9399$ and $\rho(L'_{r,w}) = 0.9332$.

References