On the separability problem for isometric shifts on $C(X)$

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Abstract

We provide examples of nonseparable spaces $X$ for which $C(X)$ admits an isometric shift, which solves in the negative a problem proposed by Gutek et al. [A. Gutek, D. Hart, J. Jamison, M. Rajagopalan, Shift operators on Banach spaces, J. Funct. Anal. 101 (1991) 97–119].

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1. Introduction

The usual concept of shift operator in the Hilbert space $\ell^2$ has been introduced in the more general context of Banach spaces in the following way (see [4,15]): Given a Banach space $E$ over $\mathbb{K}$ (the field of real or complex numbers), a linear operator $T : E \to E$ is said to be an isometric shift if

1. $T$ is an isometry,
2. the codimension of $T(E)$ in $E$ is 1,
3. $\bigcap_{n=1}^{\infty} T^n(E) = \{0\}$.

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One of the main settings where isometric shifts have been studied is $E = C(X)$, that is, the Banach space of all $\mathbb{K}$-valued continuous functions defined on a compact and Hausdorff space $X$, equipped with its usual supremum norm. In this setting, major breakthroughs were made in [9] and [11]. On the one hand, in [9], Gutek, Hart, Jamison, and Rajagopalan studied in depth these operators. In particular, using the well-known Holsztyński’s theorem [14], they classified them into two types, called type I and type II. On the other hand, in [11], Haydon showed a general method for providing isometric shifts of type II, as well as concrete examples.

However, a very basic question has remained open since the publication in 1991 of the seminal paper [9]: If $C(X)$ admits an isometric shift, must $X$ be separable? This question is only meaningful for type I isometric shifts since it was already proved in [9, Corollary 2.2] that type II isometric shifts yield the separability of $X$. Let us recall the definitions. If $T : C(X) \to C(X)$ is an isometric shift, then there exist a closed subset $Y \subset X$, a continuous and surjective map $\phi : Y \to X$, and a function $a \in C(Y)$, $|a| \equiv 1$, such that $(Tf)(x) = a(x) \cdot f(\phi(x))$ for all $x \in Y$ and all $f \in C(X)$. $T$ is said to be of type I if $Y$ can be taken to be equal to $X \setminus \{p\}$, where $p \in X$ is an isolated point, and is said to be of type II if $Y$ can be taken equal to $X$. Moreover, if $T$ is of type I, then the map $\phi : X \setminus \{p\} \to X$ is indeed a homeomorphism.

Not much is known about the possibility of finding a nonseparable space $X$ such that $C(X)$ admits an isometric shift since the problem was proposed. Interesting results in this direction say that such an $X$ must have the countable chain condition (see [10, Theorem 1.4] or [20, Lemma 5.6]). In [10, Theorem 1.9], it is even proved that $C_0(X \setminus \cl_X \{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\})$ must have cardinality at most equal to $\mathfrak{c}$, that is, the cardinality of $\mathbb{R}$ (where, as usual, $\cl_X A$ denotes the closure of $A$ in $X$ and $C_0(Z)$ is the space of $\mathbb{K}$-valued continuous functions on $Z$ vanishing at infinity).

From this fact, we can easily deduce that if $C(X)$ admits an isometric shift, then there exists a set $S$ of cardinality at most $\mathfrak{c}$ that is dense in $X$. To see it, we write $C_0(X \setminus \cl_X \{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}) = \{f_\alpha : \alpha \in I \subset \mathbb{R}\}$. For each $\alpha \in I$ such that $f_\alpha \neq 0$, we pick a point $x_\alpha \in X$ such that $f_\alpha(x_\alpha) \neq 0$. Obviously, given any (nonempty) open set $U \subset X \setminus \cl_X \{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}$, there exists $f_\alpha \neq 0$ whose support is contained in $U$. This implies that the set $S$ consisting of the union of all points $x_\alpha$ and $\{p, \phi^{-1}(p), \ldots, \phi^{-n}(p), \ldots\}$ is dense in $X$.

In this paper, we will give an answer in the negative to the separability question: There are indeed examples of isometric shifts on $C(X)$, with $X$ not separable, and even having $2^\mathfrak{c}$ infinite components (see Sections 3 and 4). The latter example can be connected with the question addressed in [15], where it was conjectured that the space $X$ cannot have an infinite connected component (the only examples which appeared so far in the literature for type I isometric shifts, both of spaces containing exactly one infinite component, can be found in [9, Corollary 2.1] and [2]; for the case of type II isometric shifts in the complex setting, see [11]). Related to this, one of the main results in [9] states that $C(X)$ does not admit any isometric shifts, whenever $X$ has a countably infinite number of components, all of whom are infinite.

Some other papers have recently studied questions related to isometric shifts (also defined on other spaces of functions). Among them, we will mention for instance [1,3,5,6,16,17,19–22] (see also references therein).
2. Preliminaries and notation

The unit circle in \( \mathbb{C} \) will be denoted by \( \mathbb{T} \). \( L^\infty(\mathbb{T}) \) will be the space of all Lebesgue-measurable essentially bounded complex-valued functions on \( \mathbb{T} \), and \( \mathcal{M} \) will be its maximal ideal space. \( m \) will denote the Lebesgue measure on \( \mathbb{T} \).

It is well known that if \( \rho \) is an irrational number, then the rotation map \([\rho]: \mathbb{T} \to \mathbb{T}\) sending each \( z \in \mathbb{T} \) to \( ze^{2\pi \rho i} \) satisfies that \( \{[\rho]^n(z): n \in \mathbb{N}\} \) is dense in \( \mathbb{T} \) for every \( z \in \mathbb{T} \) (see [23, Proposition III.1.4]). Indeed, it is easy to see that this fact can be generalized to separable powers of \( \mathbb{T} \), that is, those of the form \( \mathbb{T}^\kappa \) for \( \kappa \leq \aleph \) (similarly as it is mentioned for finite powers in [23, III.1.14]): Let \( \Lambda := \{\rho_\alpha: \alpha \in \mathbb{R}\} \) be a set of irrational numbers linearly independent over \( \mathbb{Q} \); if \( P \) is any nonempty subset of \( \mathbb{R} \) and \( [\rho_\alpha]_{\alpha \in P}: \mathbb{T}^P \to \mathbb{T}^P \) is defined as \( [\rho_\alpha]_{\alpha \in P}((z_\alpha)_{\alpha \in P}) := (z_\alpha e^{2\pi \rho_\alpha i})_{\alpha \in P} \), then the set \( \{[\rho_\alpha]^n_{\alpha \in P}((z_\alpha)_{\alpha \in P}): n \in \mathbb{N}\} \) is dense in \( \mathbb{T}^P \) for every \( (z_\alpha)_{\alpha \in P} \in \mathbb{T}^P \).

Given two topological spaces \( Z \) and \( W \), we denote by \( Z + F \) their topological sum, that is, the union \( Z \cup W \) endowed with the topology consisting of unions of open subsets of these spaces (see [24, p. 65]).

\( \mathcal{N} := \{1, 2, \ldots, n, \ldots\} \) will be a discrete infinite countable space, and \( \mathcal{N} \cup \{\infty\} \) will denote its one-point compactification. In our examples, the point \( 1 \) will play the same role as \( p \) in the definition of isometric shift of type I.

Throughout “homeomorphism” will be synonymous with “surjective homeomorphism.”

We will usually write \( T = T[a, \phi, \Delta] \) to describe a codimension 1 linear isometry \( T: C(X) \to C(X) \), where \( X \) is compact and contains \( \mathcal{N} \). It means that \( \phi: X \setminus \{1\} \to X \) is a homeomorphism, satisfying in particular \( \phi(n + 1) = n \) for all \( n \in \mathbb{N} \). It also means that \( a \in C(X \setminus \{1\}) \), \( |a| \equiv 1 \), and that \( \Delta \) is a continuous linear functional on \( C(X) \) with \( ||\Delta|| \leq 1 \). Finally, the description of \( T \) we have is \( (Tf)(x) = a(x)f(\phi(x)) \), when \( x \neq 1 \), and \( (Tf)(1) = \Delta(f) \), for every \( f \in C(X) \).

In general, given a continuous map \( f \) defined on a space \( X \), we also denote by \( f \) its restrictions to subspaces of \( X \) and its extensions to other spaces containing \( X \).

All our results will be valid in the real and complex settings, unless otherwise stated. The only exceptions are the following: Results exclusively given for \( \mathbb{K} = \mathbb{C} \) appear just in Section 5. The only result valid just for the case \( \mathbb{K} = \mathbb{R} \) is given in Example 5.3. \( C_C(X) \) and \( C_R(X) \) will denote the Banach spaces of continuous functions on \( X \) taking complex and real values, respectively.

3. Nonseparable examples

**Theorem 3.1.** \( C(\mathcal{M} + \mathcal{N} \cup \{\infty\}) \) admits an isometric shift.

Once we have a first example, we can get more. For instance, the next result is essentially different in that it provides examples with \( 2^\kappa \) infinite connected components.

**Theorem 3.2.** Let \( \kappa \) be any cardinal such that \( 1 \leq \kappa \leq \aleph \). Then \( C(\mathcal{M} \times \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\}) \) admits an isometric shift.

Finally, we can also give examples with just one infinite component.

**Theorem 3.3.** Let \( \kappa \) be any cardinal such that \( 1 \leq \kappa \leq \aleph \). Then \( C(\mathcal{M} + \mathbb{T}^\kappa + \mathcal{N} \cup \{\infty\}) \) admits an isometric shift.
Remark 3.1. Even if our space $\mathcal{M}$ is based on an algebra of complex-valued functions, Theorems 3.1–3.3 are valid both if $K = \mathbb{R}$ or $\mathbb{C}$. Nevertheless, there are examples that can be constructed just in the complex setting (see Theorem 5.2 and Example 5.3).

4. The proofs

It is well known that $\mathcal{M}$ is extremally disconnected, that is, the closure of each open subset is also open. In fact, each measurable subset $A$ of $\mathbb{T}$ determines via the Gelfand transform an open and closed subset $G(A)$ of $\mathcal{M}$, and the sets obtained in this way form a basis for its topology (see [13, p. 170]). Now it is straightforward to see that $\mathcal{M}$ is not separable: Let $(x_n)$ be a sequence in $\mathcal{M}$, and consider a partition $(a.e)$ of $\mathbb{T}$ by $k$ arcs of equal length, $k \geq 3$. This determines a partition of $\mathcal{M}$ into $k$ closed and open subsets of $\mathbb{T}$. Select the arc $A_1$ such that $G(A_1)$ contains $x_1$. Next do the same process with $k^2$ arcs of equal length, and pick $A_2$ with $x_2 \in G(A_2)$. Repeat the process infinitely many times, in such a way that each time we take $A_n$ of length $1/k^n$ such that $x_n \in G(A_n)$. It is clear that if $A := \bigcup_{n=1}^{\infty} A_n$, then $m(A) < 2\pi$, so $G(\mathbb{T} \setminus A)$ is a nonempty closed and open subset of $\mathcal{M}$ containing no point $x_n$.

Notice that, since $\mathcal{M}$ is not separable, every isometric shift on $C(\mathcal{M})$ must be of type I. But there are none because $\mathcal{M}$ has no isolated points. Even more, in [9, Corollary 2.5], it is proved that no space $L^\infty(\mathbb{Z}, \Sigma, \mu)$ admits an isometric shift if $\mu$ is non-atomic.

As usual, we consider $\mathbb{T}$ oriented counterclockwise, and denote by $A(\alpha, \beta)$ the (open) arc of $\mathbb{T}$ beginning at $e^{i\alpha}$ and ending at $e^{i\beta}$.

Proof of Theorem 3.1. We start by defining a linear and surjective isometry on $L^\infty(\mathbb{T})$. We first consider the rotation $\psi(z) := ze^{iz}$ for every $z \in \mathbb{T}$, and then define the isometry $S : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ as $Sf := -f \circ \psi$ for every $f \in L^\infty(\mathbb{T})$. On the other hand, using the Gelfand transform we have that the Banach algebra $L^\infty(\mathbb{T})$ is isometrically isomorphic to $C(\mathcal{M})$, so $S$ determines a linear and surjective isometry $T_S : C(\mathcal{M}) \to C(\mathcal{M})$. Also, by the Banach-Stone theorem, there exists a homeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ such that $T_S f = -f \circ \phi$ for every $f \in C(\mathcal{M})$. Notice that this is valid both in the real and complex cases (see for instance [7, p. 187]).

Let $X := \mathcal{M} + \mathcal{N} \cup \{\infty\}$. The definition of $T_S$ can be extended to a new isometry $T : C(X) \to C(X)$ in three steps. First, for each $f \in C(X)$, we put $(T f)(x) := (T_S f)(x)$ if $x \in \mathcal{M}$. In the same way $(T f)(n) := (f \circ \phi)(n)$ if $n \in \mathcal{N} \cup \{\infty\} \setminus \{1\}$ (where $\phi : \mathcal{N} \setminus \{1\} \to \mathcal{N}$ is the canonical map sending each $n$ into $n - 1$, which obviously can be extended as $\phi(\infty) := \infty$). Finally, we put

$$(T f)(1) := \frac{1}{2\pi} \int_{A(0, 2\pi \phi)} f \, dm,$$

where $\Phi := (\sqrt{5} - 1)/2$ is the golden ratio conjugate. It is easy to verify that $T$ is a codimension one linear isometry, so we just need to prove that $\bigcap_{i=1}^{\infty} T^i(C(X)) = \{0\}$.

Suppose then that $f \in \bigcap_{i=1}^{\infty} T^i(C(X))$. It is easy to check that

$$f(n) = (T^{-n+1} f)(1) = (T(T^{-n} f))(1)$$
\[
\frac{(-1)^n}{2\pi} \int_{A(\alpha, 2\pi \Phi)} f \circ \psi^{-n} \, dm = \frac{(-1)^n}{2\pi} \int_{A(n, n+2\pi \Phi)} f \, dm.
\]

On the other hand, if we fix any \( \alpha \in \mathbb{T} \), then there exist two increasing sequences \((n_k)\) and \((m_k)\) in \(2\mathbb{N}\) and \(2\mathbb{N} + 1\), respectively, converging to \( \alpha \mod 2\pi \). An easy application of the Dominated Convergence Theorem proves that
\[
\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm = 2\pi \lim_{k \to \infty} f(n_k),
\]
and
\[
\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm = -2\pi \lim_{k \to \infty} f(m_k).
\]
By continuity, we deduce that
\[
\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm = 2\pi f(\infty) = -\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm.
\]
Obviously, this implies that
\[
\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm = 0
\]
for every \( \alpha \in \mathbb{T} \), and \( f(\infty) = 0 \). In particular this proves that \( f(n) = 0 \) for every \( n \in \mathbb{N} \). As a consequence we can identify \( f \) with an element \( f \in L^\infty(\mathbb{T}) \) satisfying
\[
\int_{A(\alpha, \alpha+2\pi \Phi)} f \, dm = 0
\]
for every \( \alpha \in \mathbb{T} \). On the other hand, it is clear that we may assume that \( f \) takes values just in \( \mathbb{R} \).

Claim. \( \int_{A(\alpha, \alpha+2\pi \Phi^n)} f \, dm = (-1)^n F(n-1) \int_{\mathbb{T}} f \, dm \) for every \( \alpha \in \mathbb{T} \) and \( n \in \mathbb{N} \), where \( F(n) \) denotes the \( n \)th Fibonacci number.

Let us prove the claim inductively on \( n \). We know that it holds for \( n = 1 \). Also notice that \( \Phi + \Phi^2 = 1 \), so \( \Phi^n + \Phi^{n+1} = \Phi^{n-1} \) for every \( n \in \mathbb{N} \).

The case \( n = 2 \) is immediate because, since
\[
\mathbb{T} = A(\alpha, \alpha + 2\pi \Phi^2) \cup A(\alpha + 2\pi \Phi, \alpha + 2\pi(\Phi + \Phi^2))
\]
a.e., then we have \( \int_{\mathbb{T}} f \, dm = \int_{A(\alpha, \alpha + 2\pi \Phi^2)} f \, dm \) for every \( \alpha \in \mathbb{T} \).

Now assume that, given \( k \geq 2 \), the claim is true for every \( n \leq k \). Then we see that, for any \( \alpha \in \mathbb{T} \),
\[
A(\alpha, \alpha + 2\pi \Phi^k) = A(\alpha, \alpha + 2\pi \Phi^{k+1}) \cup A(\alpha + 2\pi \Phi^{k+1}, \alpha + 2\pi(\Phi^k + \Phi^{k+1}))
\]
a.e., so
\[
(-1)^{k-1} F(k-2) \int_{\mathbb{T}} f \, dm = \int_{A(\alpha, \alpha + 2\pi \Phi^{k+1})} f \, dm + (-1)^k F(k-1) \int_{\mathbb{T}} f \, dm,
\]
and the conclusion proves the claim.

The claim, combined with the fact that \( f \) is essentially bounded, implies that \( \int_{\mathbb{T}} f \, dm = 0 \), and consequently \( \int_{A(\alpha, \alpha + 2\pi \Phi^n)} f \, dm = 0 \) for every \( \alpha \in \mathbb{T} \) and every \( n \in \mathbb{N} \).

Now, it is easy to see that if \( U \) is an open subset of \( \mathbb{T} \), then \( U \) is the union of countably many pairwise disjoint arcs whose lengths belong to the set \( \{2\pi \Phi^n : n \in \mathbb{N}\} \). Now, applying again the Dominated Convergence Theorem, we see that \( \int_{U} f \, dm = 0 \). Obviously, this implies that \( \int_{K} f \, dm = 0 \) whenever \( K \subset \mathbb{T} \) is compact.
Finally take \( C^+ := \{ z \in \mathbb{T} : f(z) > 0 \} \). We know that there exists a sequence of compact subsets \( K_n \) of \( C^+ \), with \( K_n \subset K_{n+1} \) for every \( n \in \mathbb{N} \), and such that \( \lim_{n \to \infty} m(C^+ \setminus K_n) = 0 \). Clearly, the above fact and the Monotone Convergence Theorem imply that \( \int_{C^+} f \, dm = 0 \), and then \( m(C^+) = 0 \). Now we can easily conclude that \( f \equiv 0 \) a.e., and consequently \( T \) is a shift. \( \square \)

Next we prove Theorem 3.2. It provides nonseparable examples with \( 2^\kappa \) infinite connected components, each homeomorphic to a (finite or infinite dimensional) torus: It follows from the fact that \( \mathcal{M} \) is homeomorphic to an infinite closed subset of \( \beta \mathbb{N} \setminus \mathbb{N} \) that its cardinality must be \( 2^\kappa \) (see [18] and [8, Corollary 9.2]).

**Proof of Theorem 3.2.** Write the isometric shift \( T : C(\mathcal{M} + \mathcal{N} \cup \{ \infty \}) \to C(\mathcal{M} + \mathcal{N} \cup \{ \infty \}) \) given in the proof of Theorem 3.1 as \( T = T[a, \phi, \Delta] \). Obviously, \( \Delta \equiv 0 \) on \( C(\mathcal{N} \cup \{ \infty \}) \), and it can be considered as an element of \( C(\mathcal{M}') \).

Consider a subset \( P \) of \( \mathbb{R} \) with cardinal equal to \( \kappa \), and suppose that \( \{ 1/2\pi \} \cup \{ \rho_\alpha : \alpha \in P \} \) is a family of real numbers linearly independent over \( \mathbb{Q} \). Then put \( \rho_\kappa := \{ \rho_\alpha \}_{\alpha \in P} \).

Define \( \phi_\kappa : \mathcal{M} \times \mathbb{T}^\kappa \to \mathcal{M} \times \mathbb{T}^\kappa \) as \( \phi_\kappa(x, z) := (\phi(x), \rho_\kappa(z)) \) for every \( x \in \mathcal{M} \), and \( z \in \mathbb{T}^\kappa \). Select now a point \( v_\kappa \in \mathbb{T}^\kappa = \mathbb{T}^\kappa \), and consider the evaluation map \( \Gamma_\kappa \in C(\mathbb{T}^\kappa)' \). Both \( \Delta \) and \( \Gamma_\kappa \) are positive linear functionals, and so is the product \( \Delta \times \Gamma_\kappa \in C(\mathcal{M} \times \mathbb{T}^\kappa)' \), which also satisfies \( \| \Delta \times \Gamma_\kappa \| \leq 1 \) (see [12, §13] for details).

Given \( f \in C(\mathcal{M} \times \mathbb{T}^\kappa) \) and \( z \in \mathbb{T}^\kappa \), we write \( f_\kappa : \mathcal{M} \to \mathbb{K} \) meaning \( f_\kappa(x) := f(x, z) \) for every \( x \in \mathcal{M} \). Obviously \( f_\kappa \) belongs to \( C(\mathcal{M}) \), and \( (\Delta \times \Gamma_\kappa)(f) = \Gamma_\kappa(\Delta(f_\kappa)) = \Delta(f_\kappa) \).

Now, for \( X_\kappa := \mathcal{M} \times \mathbb{T}^\kappa + \mathcal{N} \cup \{ \infty \} \), define \( a_\kappa \in C(X_\kappa \setminus \{ 1 \}) \) as \( a_\kappa \equiv -1 \) on \( \mathcal{M} \times \mathbb{T}^\kappa \), and \( a_\kappa \equiv 1 \) everywhere else, and put \( T_\kappa := T[a_\kappa, \phi_\kappa, \Delta \times \Gamma_\kappa] \).

Let \( \psi : \mathcal{T} \to \mathbb{T} \) and \( \Phi \) be as in the proof of Theorem 3.1. Given \( f \in \bigcap_{i=1}^\infty T_\kappa^i(X_\kappa) \), we have that for every \( k \in \mathbb{N} \),

\[
\begin{align*}
f(k) &= (T_\kappa^{-k+1} f)(1) \\
&= (\Delta \times \Gamma_\kappa)(T_\kappa^{-k} f) \\
&= (-1)^k (\Delta \times \Gamma_\kappa)(f \circ \phi_\kappa^{-k}) \\
&= (-1)^k \Delta(f \circ \phi_\kappa^{-k})_{\kappa} \\
&= (-1)^k \Delta(f_{\rho_\kappa^{-k}(v_\kappa)} \circ \phi_\kappa^{-k}) \\
&= \frac{(-1)^k}{2\pi} \int_{A(0,2\pi \Phi)} f_{\rho_\kappa^{-k}(v_\kappa)} \circ \psi^{-k} \, dm \\
&= \frac{(-1)^k}{2\pi} \int_{A(k,k+2\pi \Phi)} f_{\rho_\kappa^{-k}(v_\kappa)} \, dm.
\end{align*}
\]

To continue with the proof, we need an elementary result:

**Claim.** Suppose that \( (z_\lambda)_{\lambda \in D} \) is a net in \( \mathbb{T}^\kappa \) converging to \( z_0 \). Then \( \lim_\lambda \| f_{z_\lambda} - f_{z_0} \| = 0 \).

Let us prove the claim. If it is not true, then there is an \( \epsilon > 0 \) such that, for every \( \lambda \in D \), there exists \( v \in D \), \( v \geq \lambda \), such that \( \| f_{z_\lambda} - f_{z_0} \| \geq \epsilon \). It is easy to see that the set \( E \) of all
$v \in D$ satisfying the above inequality is a directed set, and that $(z_v)_{v \in E}$ is a subnet of $(z_\lambda)_{\lambda \in D}$. Moreover there is a net $(x_v)_{v \in E}$ in $\mathcal{M}$ such that $|f(x_v, z_v) - f(x_v, z_0)| \geq \epsilon$ for every $v \in E$. Since $\mathcal{M} \times \mathbb{T}^k$ is compact, there exist a point $(x_0, z_0) \in \mathcal{M} \times \mathbb{T}^k$ and a subnet $(x_{\eta}, z_{\eta})_{\eta \in F}$ of $(x_v, z_v)_{v \in E}$ converging to $(x_0, z_0)$. Obviously $(z_\eta)_{\eta \in F}$ is a subnet of $(z_v)_{v \in E}$, so $z_0 = z'_0$. Consequently both $(x_\eta, z_\eta)_{\eta \in F}$ and $(x_v, z_v)_{v \in E}$ converge to $(x_0, z_0)$. Taking limits, this implies $|f(x_0, z_0) - f(x_0, z_0)| \geq \epsilon$, which is absurd.

Now, fix $(\alpha, w) \in \mathbb{T} \times \mathbb{T}^k$ and $\epsilon > 0$. We know that $(\alpha, w)$ belongs to the closure of both

$$N_j := \{(e^{in}, \rho^{-n}(v_\lambda)) : n \in 2\mathbb{N} - j\} \setminus \{(\alpha, w)\},$$

$j = 0, 1$. We first consider the case $j = 0$, and take a net $(y_\lambda)_{\lambda \in D} = (\rho^{-n_\lambda}(v_\lambda))_{\lambda \in \mathbb{N}}$ in $N_0$ converging to $(\alpha, w)$. Since $(e^{in_\lambda})_{\lambda \in D}$ converges to $\alpha$, there exists $\lambda_1 \in D$ such that

$$\left| \int_{A(\alpha, \alpha + 2\pi \phi)} f_w \, dm - \int_{A(n_\lambda, n_\lambda + 2\pi \phi)} f_w \, dm \right| < \frac{\epsilon}{2}$$

for every $\lambda \geq \lambda_1$.

On the other hand, by the claim, there exists $\lambda_2 \in D$ such that, if $\lambda \geq \lambda_2$, then

$$\|f_w - \rho^{-n_\lambda}(v_\lambda)\| < \epsilon/4\pi,$$

so

$$\left| \int_{A(n_\nu, n_\nu + 2\pi \phi)} f_w \, dm - \int_{A(n_\lambda, n_\lambda + 2\pi \phi)} f_{\rho^{-n_\lambda}(v_\lambda)} \, dm \right| < \frac{\epsilon}{4}$$

for every $\nu \in D$. We easily deduce that

$$\lim_{\lambda \to \lambda_2} \int_{A(n_\lambda, n_\lambda + 2\pi \phi)} f_{\rho^{-n_\lambda}(v_\lambda)} \, dm = \int_{A(\alpha, \alpha + 2\pi \phi)} f_w \, dm,$$

and consequently $2\pi \phi(f(\infty)) = \int_{A(\alpha, \alpha + 2\pi \phi)} f_w \, dm$. In a similar way, working with $N_1$, we see that $2\pi \phi(f(\infty)) = -\int_{A(\alpha, \alpha + 2\pi \phi)} f_w \, dm$. With the same arguments as in the proof of Theorem 3.1, we conclude that $f_w \equiv 0$, and finally $f \equiv 0$, as we wanted to prove. □

**Proof of Theorem 3.3.** Notice first that $L^\infty(\mathbb{T})$ is isometrically isomorphic to $L^\infty(\mathbb{T} \cup \mathbb{T}_2)$, where $\mathbb{T}_i, i = 1, 2$, are disjoint copies of $\mathbb{T}$ endowed with the Lebesgue measure. It is not hard to see that this implies that $C(\mathcal{M})$ and $C(\mathcal{M} + \mathcal{M})$ are isometrically isomorphic, so $\mathcal{M}$ and $\mathcal{M} + \mathcal{M}$ are homeomorphic. Assume that $T = T[a, \phi, \Delta]$ is the isometric shift given in the proof of Theorem 3.1. We first define a homeomorphism $\chi : \mathcal{M} \times [0, 1] \to \mathcal{M} \times [0, 1]$ as $\chi(x, i) = (\phi(x), i + 1 \mod 2)$ for every $(x, i)$. For $i = 0, 1$, and $f \in C(\mathcal{M} \times [0, 1])$, denote by $f \times \{i\}$ its restriction to $\mathcal{M} \times \{i\}$, and put $\Delta_i(f) := \Delta(f \times \{i\})$.

Let $\rho_\kappa : \mathbb{T}^k \to \mathbb{T}^k, v_\kappa$, and $\Gamma_{v_\kappa}$ be as in the proof of Theorem 3.2.

Finally consider $X_\kappa := \mathcal{M} \times [0, 1] + \mathbb{T}^k + \mathcal{N} \cup \{\infty\}$, and define $T_\kappa : C(X_\kappa) \to C(X_\kappa)$ to be $T_\kappa := T[a_\kappa, \phi_\kappa, \Delta_\kappa]$, where

* $a_\kappa \equiv -1$ on $\mathcal{M} \times [0, 1] \cup \mathbb{T}^k$, and $a_\kappa \equiv 1$ everywhere else.
* $\phi_\kappa = \chi$ on $\mathcal{M} \times [0, 1]$, and $\phi_\kappa = \rho_\kappa$ on $\mathbb{T}^k$.
* $\Delta_\kappa := (\Delta_0 + \Delta_1 + \Gamma_{v_\kappa})/3$. 


As above, if \( f \in \bigcap_{n=1}^{\infty} T_k^n(C(X_\kappa)), k \in \mathbb{N}, \) and
\[
\tau(k) := \frac{k(k-1) \mod 4}{2},
\]
then
\[
3f(k) = 3(T_k^{-k+1} f)(1)
\]
\[
= \Delta_0(T_k^{-k} f) + \Delta_1(T_k^{-k} f) + \Gamma_{\kappa} (T_k^{-k} f)
\]
\[
= \Delta((T_k^{-k} f) \times \{0\}) + \Delta((T_k^{-k} f) \times \{1\}) + (T_k^{-k} f)(\kappa)
\]
\[
= (-1)^{\tau(k)} \Delta((f \times \{k \mod 2\}) \circ \phi^{-k})
\]
\[
+ (-1)^{\tau(k+1)} \Delta((f \times \{k+1 \mod 2\}) \circ \phi^{-k})
\]
\[
+ (-1)^k (f \circ \rho^{-k}_{\kappa})(\kappa)
\]
\[
= \frac{(-1)^{\tau(k)}}{2\pi} \int_{A(0,2\pi \Phi)} (f \times \{k \mod 2\}) \circ \psi^{-k} dm
\]
\[
+ \frac{(-1)^{\tau(k+1)}}{2\pi} \int_{A(0,2\pi \Phi)} (f \times \{k+1 \mod 2\}) \circ \psi^{-k} dm
\]
\[
+ (-1)^k f(\rho^{-k}_{\kappa}(\kappa))
\]
\[
= \frac{(-1)^{\tau(k)}}{2\pi} \int_{A(k,k+2\pi \Phi)} f \times \{k \mod 2\} dm
\]
\[
+ \frac{(-1)^{\tau(k+1)}}{2\pi} \int_{A(k,k+2\pi \Phi)} f \times \{k+1 \mod 2\} dm
\]
\[
+ (-1)^k f(\rho^{-k}_{\kappa}(\kappa)).
\]

Next fix \( \alpha \in \mathbb{T}, w \in \mathbb{T}_\kappa, \) and for \( j = 0, 1, 2, 3, \) take increasing sequences \( (n_j^k) \) in \( 4\mathbb{N} + j \) such that \( \lim_{k \to \infty} n_j^k = \alpha \mod 2\pi, \) and \( \lim_{k \to \infty} f(\rho^{-n_j^k}_{\kappa}(\kappa)) = f(w). \) Now put
\[
X^\alpha_i := \frac{1}{2\pi} \int_{A(\alpha,\alpha+2\pi \Phi)} f \times \{i\} dm
\]
for $i = 0, 1$. Taking into account that $\tau(n^j_i)$ is constant for each $j$, and that $\tau(2) = 1 = \tau(3)$, and $\tau(1) = 0 = \tau(4)$, we have that the following equalities hold:

$$3f(\infty) = X_0^\alpha - f(w) \quad (\text{case } j = 1)$$

$$= -X_0^\alpha - X_1^\alpha + f(w) \quad (\text{case } j = 2)$$

$$= -X_1^\alpha + X_0^\alpha - f(w) \quad (\text{case } j = 3)$$

$$= X_0^\alpha + X_1^\alpha + f(w) \quad (\text{case } j = 0).$$

We deduce that $X_0^\alpha = 0$ for every $\alpha \in \mathbb{T}$ and $i = 0, 1$, and that $f \equiv 0$ on $\mathbb{T}^\times$. As in the proof of Theorem 3.1, we easily conclude that $f \equiv 0$. \qed

5. Some differences between the real and complex cases

In this section we show that in the complex setting, it is possible to obtain nonseparable examples with arbitrary (finitely many) infinite connected components. For the different behavior in the real setting, see Example 5.3.

Our first result in this section is indeed given for separable examples. The idea of the proof is used in Theorem 5.2 to obtain nonseparable examples. In both cases $\mathbb{T}^0$ denotes the set $\{0\}$.

**Definition 5.1.** Let $X$ be compact and Hausdorff, and suppose that $T = T[a, \phi, \Delta]$: $C(X) \to C(X)$ is an isometric shift of type I. For $n \in \mathbb{N}$, we say that $T$ is $n$-generated if $n$ is the least number with the following property: There exist $n$ points $x_1, \ldots, x_n \in X \setminus \overline{X N}$ such that the set

$$\{\phi^k(x_j) : k \in \mathbb{Z}, j \in \{1, \ldots, n\} \}$$

is dense in $X \setminus \overline{X N}$.

Notice that isometries simultaneously of types I and II are always 1-generated (see [9, Theorem 2.5]), so the next theorem provides a way for constructing isometries that are not of type II.

**Theorem 5.1.** Let $\mathbb{K} = \mathbb{C}$. Suppose that $n \in \mathbb{N}$, and that $(\kappa_j)_{j=1}^n$ is a finite sequence of cardinals satisfying $0 \leq \kappa_j \leq \kappa$ for every $j$. Then there exists an $n$-generated isometric shift $T_n : C_{\mathbb{C}}(X_n) \to C_{\mathbb{C}}(X_n)$, where $X_n = \mathbb{T}^{\kappa_1} + \cdots + \mathbb{T}^{\kappa_n} + N \cup \{\infty\}$.

**Proof.** Let $\mathbb{P}_1, \ldots, \mathbb{P}_n$ be any pairwise disjoint subsets of $\mathbb{R}$ of cardinalities $\kappa_1, \ldots, \kappa_n$, respectively. Consider any family $\Lambda := \{\rho_\alpha : \alpha \in \mathbb{R}\}$ of real numbers linearly independent over $\mathbb{Q}$, and put $\sigma_j := \{\rho_{\alpha} : \alpha \in \mathbb{P}_j\}$ for each $j \leq n$ (in the case when $\kappa_j = 0$, that is, $\mathbb{P}_j = \emptyset$, $\sigma_j$ is the identity). Also let $v_j$ be a point in $\mathbb{T}^{\kappa_j}$.

Next write $X_n := \mathbb{T}^{\kappa_1} + \cdots + \mathbb{T}^{\kappa_n} + N \cup \{\infty\}$, and define $\phi_n : X_n \to X_n$ as $\sigma_j$ on each $\mathbb{T}^{\kappa_j}$. For $j \leq n$, let $z_j \in \mathbb{C} \setminus \{0\}$, with $|z_j| \leq 1/2^j$, and $\xi_j := e^{i\pi/2^{j-1}}$. Define a codimension 1 linear isometry $T_n$ on $C_{\mathbb{C}}(X_n)$ as $T_n := [a_n, \phi_n, \Delta_n]$, where $a_n \equiv \xi_j$ on $\mathbb{T}^{\kappa_j}$ for each $j \leq n$, and $a_n \equiv 1$ on $N \cup \{\infty\}$, and where $\Delta_n(f) := \sum_{j=1}^n z_j f(v_j)$ for every $f$.

Of course, the construction of $T_n$ depends on our choice of the sets $\mathbb{P}_j$ and $\Lambda$, the points $v_j$, and the numbers $z_j$. We will prove that for any choices, the operator $T_n$ satisfies the theorem.
We will do it inductively on \( n \). We start at \( n = 1 \). It is easy to see that \( T_1 : C\mathbb{C}(X_1) \rightarrow C\mathbb{C}(X_1) \) is an isometric shift (both of type I and type II). Now let us show that if \( T_n \) is an \( n \)-generated isometric shift for \( n = l \in \mathbb{N} \), then \( T_{l+1} \) is an \((l + 1)\)-generated isometric shift.

Suppose that \( f \in \bigcap_{m=1}^{\infty} T_{l+1}^m (C\mathbb{C}(X_{l+1})) \). It is easy to check that

\[
f(k) = (T_{l+1}^{k+1} f)(1) = \sum_{j=1}^{l+1} z_j (T_{l+1}^{-k} f)(v_j) = z_{l+1} \zeta_{l+1}^{-k} (f \circ \sigma_{l+1}^{-k})(v_{l+1}) + \sum_{j=1}^{l} z_j \zeta_j^{-k} (f \circ \sigma_j^{-k})(v_j),
\]

whenever \( k \in \mathbb{N} \).

Fix \( x_1 \in \mathbb{T}^{\mathbb{N}}_1, \ldots, x_{l+1} \in \mathbb{T}^{\mathbb{N}}_{l+1} \). For \( j = 0, 1 \), we can take increasing sequences \((n_j^k)\) in \( 2^{l+1} \mathbb{N} \) and \( 2^{l+1} \mathbb{N} + 2^j \), respectively, such that the sequences

\[
((f \circ \sigma_1^{-n_j^k})(v_1), \ldots, (f \circ \sigma_{l+1}^{-n_j^k})(v_{l+1}))_{k \in \mathbb{N}}
\]

converge to \((f(x_1), \ldots, f(x_{l+1})) \in \mathbb{C}^{l+1} \) for \( j = 0, 1 \).

This means, on the one hand, that

\[
f(\infty) = \lim_{k \to \infty} f(n_j^0) = z_{l+1} f(x_{l+1}) + \sum_{j=1}^{l} z_j f(x_j).
\]

And, on the other hand,

\[
f(\infty) = \lim_{k \to \infty} f(n_j^1) = -z_{l+1} f(x_{l+1}) + \sum_{j=1}^{l} z_j f(x_j).
\]

We deduce that \( f(x_{l+1}) = 0 \), that is, \( f \equiv 0 \) on \( \mathbb{T}^{\mathbb{N}}_{l+1} \), and consequently \( f \in \bigcap_{m=1}^{\infty} T_l^m (C\mathbb{C}(X_{l+1})) \). Since \( T_l \) is a shift, we conclude that \( f \equiv 0 \) on \( X_{l+1} \). It is also easy to see that \( T_{l+1} \) is \((l + 1)\)-generated.

**Theorem 5.2.** Let \( K = \mathbb{C} \). Suppose that \( n \in \mathbb{N} \), and that \((\kappa_j)_{j=1}^n\) is a finite sequence of cardinals satisfying \( 0 \leq \kappa_j \leq c \) for every \( j \). Then there exists an isometric shift \( T_n^{2\mathbb{N}} : C\mathbb{C}(X_n^{2\mathbb{N}}) \rightarrow C\mathbb{C}(X_n^{2\mathbb{N}}) \), where \( X_n^{2\mathbb{N}} = 2\mathbb{N} + \mathbb{T}^{\kappa_1} + \cdots + \mathbb{T}^{\kappa_n} + \mathbb{N} \cup \{\infty\} \).
Proof. The proof is similar to that of Theorem 5.1. We consider the homeomorphism $\phi$ on $\mathcal{M}$ coming from the rotation $\psi : \mathbb{T} \to \mathbb{T}$ given in the proof of Theorem 3.1. Fix $n \in \mathbb{N}$, and assume that $X_n$ and $T_n = [a_n, \phi_n, \Delta_n]$ are as in the proof of Theorem 5.1. Take $z_{n+1} \in \mathbb{C} \setminus \{0\}$ such that $|z_{n+1}| \leq 1/2^{n+1}$, and put $\zeta_{n+1} := e^{i\pi/2^n}$.

We are going to define an isometric shift on $X_n^{\mathcal{M}}$. First put

$$\Delta_n^{\mathcal{M}} := \frac{z_{n+1}}{2^n} \int_{\mathbb{T} \times \{0, 2\pi\phi\}} f \, dm + \Delta_n(f).$$

Obviously we are assuming that $1/2^n$ does not belong to the linear span (over $\mathbb{Q}$) of $\{\rho_\alpha : \alpha \in \mathbb{P}_1 \cup \cdots \cup \mathbb{P}_n\}$. Let $a_n^{\mathcal{M}} \in C_c(X_n^{\mathcal{M}})$ be equal to $\zeta_{n+1}$ on $\mathcal{M}$, and equal to $a_n$ on $X_n$, and let $\phi_n^{\mathcal{M}} : X_n^{\mathcal{M}} \to X_n^{\mathcal{M}}$ be defined as $\phi_n$ on $X_n$, and as $\phi$ on $\mathcal{M}$.

We consider $T_n^{\mathcal{M}} := [a_n^{\mathcal{M}}, \phi_n^{\mathcal{M}}, \Delta_n^{\mathcal{M}}]$. Following the same process as in the proof of Theorem 5.1, we easily obtain that $0 = z_{n+1} \int_{A(\alpha, 2\pi \phi + \alpha)} f \, dm$ for every $\alpha \in \mathbb{T}$. As in the proof of Theorem 3.1, we see that $f \equiv 0$ on $\mathbb{T}$, which is to say that $f \equiv 0$ on $\mathcal{M}$. We deduce that $f \in \bigcap_{m=1}^{\infty} T_n^{\mathcal{M}}(C_c(X_n))$, and consequently $f \equiv 0$. \hfill $\Box$

Remark 5.1. Notice that in both Theorems 5.1 and 5.2, we allow the possibility that $\kappa_j = \kappa_k$ for some (or all) $j \neq k$.

Our next example shows in fact that the procedure followed above is no longer valid when dealing with $\mathbb{K} = \mathbb{R}$.

Example 5.3. Let $\mathbb{K} = \mathbb{R}$. Suppose that $X = Y + X_1 + X_2 + X_3$ is compact, where each $X_j$ is connected and nonempty, and $\mathcal{N} \subset Y$. Let $T = [a, \phi, \Delta]$ be a codimension 1 linear isometry on $C(\mathbb{R})$, and assume that $\phi(X_j) = X_j$, $j = 1, 2, 3$. Let us see that $T$ is not a shift. First, there are $j, k, \ell \neq k$, with $a(X_j) = a(X_k) \in \{-1, 1\}$. There are also $\alpha_j, \alpha_k \in \mathbb{R}$ such that $|\alpha_j|, |\alpha_k| > 0$ and $\Delta(\alpha_j^3 \xi_{X_j} + \alpha_k \xi_{X_k}) = 0$, where $\xi_A$ denotes the characteristic function on $A$. It is easy to check that $\alpha_j^3 \xi_{X_j} + \alpha_k \xi_{X_k}$ belongs to $T^n(C(\mathbb{R}))$ for every $n \in \mathbb{N}$, and consequently $T$ is not a shift.

In particular, we see that neither $C(\mathbb{R})(\mathbb{T} + \mathbb{T}^2 + \mathbb{T}^3 + \mathcal{N} \cup \{\infty\})$ nor $C(\mathcal{M} + \mathbb{T} + \mathbb{T}^2 + \mathbb{T}^3 + \mathcal{N} \cup \{\infty\})$ admit an isometric shift.

References