Bounds for one-dimensional rings of finite Cohen–Macaulay type

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Abstract


Let $R$ be an integral domain finitely generated as an algebra over a field of characteristic not equal to 2 (or the localization of such a ring at some multiplicatively closed set); and assume that, for each maximal ideal $\mathcal{M}$, there is a bound on the ranks of the indecomposable finitely generated torsion-free $R_{\mathcal{M}}$-modules. We show that the only possible ranks for such indecomposable modules over $R$ are 1, 2, 3, 4, 5, 6, 8, 9 and 12. An example having indecomposables of each of these ranks is constructed over the field of rational numbers. Furthermore, over a broader class of reduced one-dimensional rings, the only possible ranks for indecomposable finitely generated torsion-free modules of constant rank are also 1, 2, 3, 4, 5, 6, 8, 9 and 12.

1. Introduction

In this paper all rings are commutative and Noetherian, and all modules are finitely generated. If $R$ is an integral domain with a bound on the ranks of the indecomposable torsion-free $R$-modules, then $R$ is at most one-dimensional [1, (1.2) and (1.4)]. We will restrict our attention to what Haefner and Levy [8] call ring-orders. A ring-order is a reduced ring of dimension one such that the normalization $\overline{R}$ (= integral closure in the classical quotient ring) is finitely generated as an $R$-module (equivalently, as an $R$-algebra). Over
a ring-order, a module $M$ is torsion-free (that is, $rm \neq 0$ whenever $r$ is a non-zero-divisor of $R$ and $m$ is a non-zero element of $M$) if and only if $M$ is a maximal (= one-dimensional) Cohen–Macaulay module.

Our main theorem (Theorem 5.1) says, essentially, that if $R$ is a ring-order admitting a bound on the ranks of the indecomposable maximal Cohen–Macaulay modules, then every maximal Cohen–Macaulay module of constant rank decomposes as a direct sum of modules whose ranks are at most 12. (The assumption that the rank be constant at the various minimal primes is important, as we point out later in this section. Also, there is a trifling separability condition we are forced to impose.)

The ring-order $R$ is said to have finite Cohen–Macaulay type provided there are only finitely many indecomposable maximal Cohen–Macaulay $R$-modules up to isomorphism. Among rings whose underlying additive groups are finitely generated, the ring-orders of finite Cohen–Macaulay type were characterized in [4] and [7] as exactly those rings satisfying the following conditions introduced by Drozd and Roiter [4]:

1. $\hat{R}$ is generated by 3 elements as an $R$-module.
2. $\text{rad}_R(\hat{R}/R)$ is cyclic as an $R$-module.

Here $\text{rad}_R(\hat{R}/R)$ denotes the intersection of the maximal $R$-submodules of $\hat{R}/R$. Since $\hat{R}/R$ has finite length, each of these conditions can be checked locally. For a local ring-order $R$ with maximal ideal $m$, these conditions can be restated as follows:

1. $R$ has multiplicity at most 3.
2. $(m\hat{R} + R)/R$ is cyclic as an $R$-module.

Ring-orders that are finitely generated as algebras over an infinite field never have finitely generated additive groups and rarely have finite Cohen–Macaulay type. In fact, the Picard group of isomorphism classes of rank-one projective modules is almost always infinite [14]. One way to analyze the module structure of these ring-orders is to localize. Suppose $R$ is the local ring of a point on a plane curve over an algebraically closed curve of characteristic 0. Greuel and Knörrer [6] showed that $R$ has finite Cohen–Macaulay type if and only if $R$ satisfies (dr). Moreover, they obtained explicit equations for the completions of these rings. These results were extended to all characteristics by Kiyek and Steinke [11] and to arbitrary perfect fields by R. Wiegand [15].

Another approach is to ask whether there is a bound on the ranks of the indecomposable maximal Cohen–Macaulay modules. In order to make this concept precise, we let $P_1, \ldots, P_s$ be the minimal prime ideals of $R$. Then each local ring $R_{P_i}$ is a field, and if $M$ is an $R$-module, we let $r_i(M)$ be the dimension of $M_{P_i}$ as a vector space over $R_{P_i}$. The rank of $M$ is
the s-tuple rank \((M) = (r_1(M), \ldots, r_s(M))\). We say \(R\) has bounded Cohen-Macaulay type provided there is an integer \(n\) such that the rank of every indecomposable Cohen-Macaulay module is less than or equal to the constant sequence \((n, \ldots, n)\).

We will see that these two approaches are essentially equivalent. Assume that \(R\) is connected (i.e., has no idempotents other than 0,1) and that \(R \neq \hat{R}\). We define the singular semilocalization of \(R\) to be the ring \(R_{\text{sing}} = S^{-1}R\), where \(S\) is the complement of the union of the singular maximal ideals. (These are the maximal ideals \(m\) for which \(R_m\) is not a discrete valuation ring.) Then \(R_{\text{sing}}\) is a semilocal ring-order. Clearly \(R\) satisfies (dr) if and only if \(R_{\text{sing}}\) satisfies (dr). There is an annoying situation we will avoid by imposing the following technical condition:

(2-sep) No residue field \(\hat{R}/M\) of \(\hat{R}\) is purely inseparable of degree 2 over \(R/(M \cap R)\).

The following theorem summarizes the current state of our knowledge regarding the classification of ring-orders of finite and bounded Cohen-Macaulay type:

**Theorem 1.1.** Let \(R\) be a connected ring-order, and assume \(R \neq \hat{R}\).

1. \(R\) has bounded Cohen-Macaulay type if and only if \(R_{\text{sing}}\) has bounded Cohen-Macaulay type.

2. If \(R\) does not satisfy (dr), then for every \(n\) there is an indecomposable maximal Cohen-Macaulay \(R\)-module of constant rank \(n\).

3. If \(R\) (equivalently \(R_{\text{sing}}\)) satisfies (dr) and (2-sep), then \(R_{\text{sing}}\) has finite Cohen-Macaulay type.

This theorem was proved in [13, (1.3), (2.1), (3.1)], except that (3) required every residue field of \(\hat{R}\) to be separable over the corresponding residue field of \(R\). In [16] it was shown that residue field extensions of degree 3 cause no problem. Now, in the presence of (dr), there can be no residue field extensions of degree greater than 3. Therefore the result holds under the weaker hypothesis (2-sep).

The goal of the present paper is to obtain uniform bounds on the ranks of indecomposable modules over ring-orders of bounded Cohen-Macaulay type. Some restrictions are necessary, in view of the examples constructed in [17]:

For every \(n \geq 1\) there is a semilocal ring-order \(R\) (depending on \(n\)) of finite Cohen-Macaulay type, together with an indecomposable maximal Cohen-Macaulay \(R\)-module \(M\) such that the rank of \(M\) at each minimal prime is at least \(n\). The module \(M\) does not have constant rank, however; in fact its ranks at the various minimal primes range from \(n\) to \(2n - 1\). (For the class of rings considered in [17], this is the narrowest range possible for the ranks.) We will
show, however, that 12 is the largest possible rank for an indecomposable of constant rank over a ring-order of bounded Cohen–Macaulay type satisfying (2-sep). Moreover, the only possible ranks are 1, 2, 3, 4, 5, 6, 8, 9, and 12. This result improves the bound of 39 obtained in [3].

In the course of the analysis in Section 3, we obtain an indecomposable module of rank 4 over a local ring-order of finite Cohen–Macaulay type. (Examples were already known with ranks 1, 2, and 3.) Section 4 contains a useful gluing technique for constructing semilocal rings with desired localizations. This technique is applied in Section 5 to obtain a ring-order of finite Cohen–Macaulay type with indecomposable modules of ranks 1, 2, 3, 4, 5, 6, 8, 9, and 12.

2. Artinian pairs

The following notation and assumptions will be in effect for the duration of this section: \( R \) is a local ring-order with residue field \( k \). Let \( \bar{R} \) be the normalization and \( c \) the conductor of \( R \) in \( \bar{R} \). To avoid discussing trivial special cases we always assume \( R \neq \bar{R} \). (Notice, for example, that item (2) of Theorem 2.1 would be false without this assumption.) Then \( R \) is represented as a pullback:

\[
\begin{array}{ccc}
R & \rightarrow & \bar{R} \\
\downarrow & & \downarrow \\
R/e & \rightarrow & \bar{R}/e
\end{array}
\]

The bottom line of the pullback is an example of what we call an Artinian pair, that is, a module-finite extension \( A \rightarrow B \) of Artinian rings. (They are Artinian because \( c \) contains a non-zero-divisor.) A module over the Artinian pair \( A \rightarrow B \) is a pair \( V \rightarrow W \), where \( W \) is a finitely generated projective \( B \)-module, \( V \) is an \( A \)-submodule of \( W \), and \( BV = W \). (We use arrows rather than ordered pairs, since the latter will be needed to represent elements of \( B \) in cases where \( B \) decomposes as a direct product of rings.) If \( V' \rightarrow W' \) is another module over the same Artinian pair, a morphism from \( V \rightarrow W \) to \( V' \rightarrow W' \) is by definition a \( B \)-module homomorphism from \( W \) to \( W' \) carrying \( V \) into \( V' \). The \( (A \rightarrow B) \)-modules form an additive category \( (A \rightarrow B)\)-mod, with well-defined notions of direct sums and indecomposables. We say \( (A \rightarrow B) \) has finite representation type provided there are, up to isomorphism, only finitely many indecomposable \( (A \rightarrow B) \)-modules.

We denote the Artinian pair corresponding to the ring-order \( R \) by \( R_{\text{art}} \). Given a maximal Cohen–Macaulay \( R \)-module \( M \), we denote by \( M_{\text{art}} \) the \( R_{\text{art}} \)-module
$M/cM \rightarrow \tilde{R}M/cM$. If we know $\tilde{R}M$ and $M_{\text{art}}$, we can recover $M$ as the pullback:

$$
\begin{array}{ccc}
M & \rightarrow & R M \\
\downarrow & & \downarrow \\
M/cM & \rightarrow & \tilde{R}M/cM
\end{array}
$$

Typically, one studies maximal Cohen–Macaulay $R$-modules by working in the category $R_{\text{art}}\text{-mod}$. The next theorem summarizes some of the basic facts about the relationship between $R$ and $R_{\text{art}}$.

**Theorem 2.1** (R. Wiegand [13, (1.6)–(1.9)]). Let $R$ be a local ring-order with $R \neq \tilde{R}$. Let $M$ and $N$ be maximal Cohen–Macaulay $R$-modules, and let $V \rightarrow W$ be an $R_{\text{art}}$-module.

1. $(M \oplus N)_{\text{art}} \cong M_{\text{art}} \oplus N_{\text{art}}$.
2. If $M_{\text{art}} \cong N_{\text{art}}$, then $M \cong N$.
3. $V \rightarrow W$ is isomorphic to $X_{\text{art}}$ for some maximal Cohen–Macaulay $R$-module $X$ if and only if $W \cong F/cF$ for some projective $\tilde{R}$-module $F$.
4. The Krull–Schmidt Theorem holds for direct sum decompositions in $R_{\text{art}}\text{-mod}$.
5. $R$ has finite Cohen–Macaulay type if and only if $R_{\text{art}}$ has finite representation type. \(\square\)

The conditions (dr) given in Section 1 can be checked by looking at the Artinian pair $R_{\text{art}}$. More generally, suppose $A \rightarrow B$ is an Artinian pair, and assume $A$ is local with maximal ideal $m$ and residue field $k$. Consider the following conditions:

1. $\dim_k B/mB \leq 3$.
2. $\dim_k (mB + A)/(m^2B + A) \leq 1$.

The ring-order $R$ clearly satisfies (dr) if and only if the Artinian pair $R_{\text{art}}$ satisfies (dr). Thus (dr) holds for Artinian pairs of finite representation type.

The Krull–Schmidt Theorem does not always hold for maximal Cohen–Macaulay $R$-modules (see Example 3.3), but the only obstruction is, roughly speaking, incompatibility of ranks. To understand this point, we look more closely at item (3) of Theorem 2.1. The ring $\tilde{R}$, being semilocal and integrally closed, is a principal ideal ring, and hence is a direct product of finitely many principal ideal domains. Let $e_1, \ldots, e_s$ be the primitive idempotents of $\tilde{R}$. Letting $B = \tilde{R}/c$, we see that the projective $B$-module $W$ comes from a projective $\tilde{R}$-module if and only if $We_i$ is a free $Be_i$-module for each $i$. Now $B$ is a direct product of local principal ideal rings (one for each maximal ideal of $\tilde{R}$) so $W$ comes from a projective $\tilde{R}$-module if and only if each $We_i$ has
constant rank. In particular, this holds if each Be\textsubscript{i} is local, and Theorem 2.1 implies the following:

**Proposition 2.2.** With the notation above, suppose each Be\textsubscript{i} is local, equivalently, \( \bar{R} \) is a direct product of local rings.

1. The functor \( M \to M_{\text{art}} \) determines a bijection between the set of isomorphism classes of maximal Cohen–Macaulay \( R \)-modules and the set of isomorphism classes of \( R_{\text{art}} \)-modules.
2. \( M \) is indecomposable if and only if \( M_{\text{art}} \) is indecomposable.
3. The Krull–Schmidt Theorem holds for direct-sum decompositions of maximal Cohen–Macaulay \( R \)-modules.

We will draw heavily on the matrix decompositions of [7], which give explicit lists of the indecomposable \( R_{\text{art}} \)-modules under the following extra hypothesis: Each residue field of \( \bar{R} \) is equal to \( k \) (the residue field of \( R \)). There is a procedure for reducing to this case, which we now outline. (See [13, Section 3] and [3] for the details.)

### 2.1. Elimination of residue field growth

Let \( A \to B \) be any Artinian pair, with \( A \) local. Write \( B = B_{\text{e}_1} \times \cdots \times B_{\text{e}_t} \) as a direct product of local rings, and let \( K_j \) be the residue field of \( B_{\text{e}_j} \). We assume that each \( K_j \) is a simple extension of the residue field \( k \) of \( A \) and, after possibly renumbering the idempotents, that \( K_1 \) is a proper extension of \( k \). (The procedure we are about to describe will be needed only when \( A \to B \) satisfies conditions (dr). Then \( K_1 \) will have degree at most 3 over \( k \), and \( K_j = k \) if \( j > 1 \).) Write \( K = K_1 = k[u] \), let \( f \in k[X] \) be the minimal polynomial for \( u \) over \( k \), and lift \( f \) to a monic polynomial \( f' \in A[X] \). Put \( A' = A[X]/Af \) and \( B' = B[X]/Bf \). Pictorially:

\[
\begin{array}{ccc}
A' = A[X]/Af & \longrightarrow & B' = B[X]/Bf \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]

We record the result of this procedure:

**Theorem 2.3.** Let \( A \to B \) be a local Artinian pair satisfying (dr), and assume that \( B \) has a residue field \( K \) properly extending the residue field \( k \) of \( A \). Let \( A' \to B' \) be the Artinian pair constructed above.

1. \( A' \) is local with residue field \( K \).
2. \( A' \to B' \) satisfies (dr).
3. If \( K \) is a normal extension of \( k \), then every residue field of \( B' \) is isomorphic to \( K \).
(4) If $K$ is not a normal extension of $k$ (hence of degree 3), then $B'$ has two maximal ideals, with residue fields $L$ and $K$, where $[L : K] = 2$.

(5) As an $(A \to B)$-module, $A' \to B'$ is free of rank $[K : k]$.

(6) For every indecomposable $(A \to B)$-module $M$, there exist an indecomposable $(A' \to B')$-module $N$ and an integer $n \leq [K : k]$ such that $\bigoplus^n M \cong N$ as $(A \to B)$-modules.

**Proof.** For (1)-(5), we refer the reader to [13, (3.3) and (3.4)]. (The assumption in [13] that $K$ is separable over $k$ is not needed here.) To prove (6), let $M = (V \to W)$ be any indecomposable $(A \to B)$-module, and let $V' = A' \otimes_A V$, $W' = B' \otimes_B W$. Express $M' = (V' \to W')$ as a direct sum of indecomposable $(A' \to B')$-modules: $M' = M'_1 \oplus \cdots \oplus M'_m$. Now decompose each $M'_i$ as an $(A \to B)$-module, say $M'_i = M'_{i1} \oplus \cdots \oplus M'_{in_i}$. As an $(A \to B)$-module, $M'$ is a direct sum of $[K : k]$ copies of $M$. By applying the Krull–Schmidt Theorem [13, (1.5)] in the category $(A \to B)$-mod, we see that $M'_{ij} = M$ for each $i$, and $n_1 + \cdots + n_m = [K : k]$. In particular, $n_1 \leq [K : k]$ and we can take $N = M'_1$. □

After iterating this procedure a finite number of times (at most twice in the case of interest), we arrive at an Artinian pair with no residue field growth. There is, however, a slight problem: A crucial assumption in the matrix reductions in [7] is that $B$ be a principal ideal ring. (Of course this is automatic when $(A \to B) = R_{Art}.$) The problem is that $B'$ may not be a principal ideal ring even when $B$ is. However, we have the following:

**Lemma 2.4.** Assume $B$ is a principal ideal ring. If either $B$ is reduced or $K$ is separable over $k$, then $B'$ is a principal ideal ring.

**Proof.** We refer the reader to [13, (3.4)] in case $K/k$ is separable. If $B$ is reduced, then $B = K_1 \times \cdots K_s$, and it is clear that $B'$ is a principal ideal ring. □

In Section 3 we will need to know exactly what the map $(A \to B) \to (A' \to B')$ looks like modulo the radicals, in order to understand the behavior of $(A' \to B')$ as an $(A \to B)$-module. Suppose, first, that $K$ is separable over $k$ of degree 2. Then, from the diagram 1, since $B = B_{e_1} \times \cdots \times B_{e_t}$ and $B' = B'_{e_1} \times \cdots \times B'_{e_t}$, we have

\[
A' = A[X]/Af \to B'_{e_1} = B_{e_1}[X]/(f) \quad K \to K[X]/(f) \to K \times K
\]

\[
\begin{array}{c}
A \to B_{e_1} \\
A \to B_{e_1} \\
K \to K
\end{array}
\]

(modulo radicals)
Here \( \beta \) is not the diagonal embedding. Rather, it takes \( y \) to \((y, y')\), where \( \tau \) generates the Galois group of \( K \) over \( k \). (The reason is that \( B'\mathbb{e}_1/\text{rad}(B'\mathbb{e}_1) = K[X]/(\hat{f}) \) is identified with \( K \times K \) by sending \( u (= \text{the image of} \ X) \) to the two distinct roots of \( \hat{f} \).) Similarly, if \( K \) is Galois over \( k \) of degree 3, the map \( \gamma : K \to K \times K \times K \) takes \( y \) to \((y, y', y')\), where \( \{1, \tau, \sigma\} \) is the Galois group of \( K \) over \( k \). Finally, if \( K \) is separable over \( k \) but not Galois, let \( L \) be the normal closure of \( K \) over \( k \). Then the map \( \delta : K \to L \times K \) takes \( y \) to \((y', y)\), where \( \tau \) is an element of order 3 in the Galois group of \( L \) over \( k \).

2.2. Gorenstein rings do not matter

Our analysis in the case of residue field growth will make use of the following beautiful result proved by Bass [2, (6.2) and (7.2)]:

**Proposition 2.5.** Assume \( R \) is a Gorenstein ring (and local, as always), and let \( M \) be an indecomposable maximal Cohen–Macaulay \( R \)-module. Then either \( M \cong R \) or else \( M \) is an \( S \)-module for some ring \( S \) satisfying \( R \subset S \subset \bar{R} \). \( \square \)

Suppose, now, that \( M \) is an indecomposable maximal Cohen–Macaulay \( R \)-module, with \( R \) Gorenstein. Let \( \text{rank}(M) = (r_1, \ldots, r_s) \), and assume \( r_i > 1 \) for some \( i \). Then \( M \) is an \( S \)-module for some ring \( R \) with \( R \subset S \subset \bar{R} \). Obviously \( M \) is still an indecomposable maximal Cohen–Macaulay \( S \)-module, and its rank is unchanged. Moreover, in the cases we will treat, all such rings \( S \) will satisfy (dr). Therefore, if we have bounds on the ranks of indecomposable maximal Cohen–Macaulay \( S \)-modules, the same bounds will apply to \( R \). If \( S \) is Gorenstein, we repeat the process if necessary. Since \( \bar{R}/R \) has finite length, we must eventually reach a non-Gorenstein ring \( S \). (Note that this argument, which occurs in [2], shows that if all the rings between \( R \) and \( \bar{R} \) are Gorenstein, then no such \( M \) can exist, that is, indecomposables have rank \( \leq (1, \ldots, 1) \). Incidentally, we are temporarily ignoring a very minor point—that \( S \) might not be local. We will come back to this point later when we treat residue field growth of degree 2, just before the statement of Theorem 3.8. The idea, then, is to concentrate on the non-Gorenstein rings, which tend to have a much simpler structure, particularly when there is residue field growth.

2.3. The rings between \( R \) and \( \bar{R} \)

In Theorem 2.8 we will give an explicit description of the rings between \( R \) and \( \bar{R} \). The following criterion makes it very easy to identify which of our rings are Gorenstein:

**Proposition 2.6.** Let \( R \) satisfy (dr), and let \( A \to B \) be the Artinian pair associated to \( R \). Then \( l_A(B) = 2l_A(A) \) if \( R \) is Gorenstein, and \( l_A(B) = 2l_A(A) + 1 \)
if \( R \) is not Gorenstein. In particular, \( R \) is Gorenstein if and only if \( B \) has even length as an \( A \)-module.

**Proof.** By [9, Korollar 3.7] we know \( l_A(B) \geq 2l_A(A) \), with equality if and only if \( R \) is Gorenstein. (This much is true without (dr).) Therefore it will suffice to show that \( l_A(B) \leq 2l_A(A) + 1 \). Note that \( mB/m = mB/(A \cap mB) \cong (mB + A)/A \), which is cyclic by (dr). This cyclic module is annihilated by the non-zero ideal \( m^{e-1} \), where \( e \geq 1 \) is the index of nilpotency of \( m \); therefore it is shorter than \( A \). We have \( l_A(B) = l_A(A) + l_A(B/A) = l_A(A) + l_A(B/m) - 1 = l_A(A) + l_A(B/mB) + l_A(mB/m) - 1 \leq l_A(A) + 3 - 1 + l_A(A) - 1 = 2l_A(A) + 1. \)

**Notation.** Continuing and elaborating on the notation established above, we again let \( R \) denote a local ring-order with residue field \( k \). We assume that \( R \neq \tilde{R} \). Let \( R_{\text{art}} = (A \rightarrow B) \), let \( m \) be the maximal ideal of \( A \), and let \( J \) be the radical of \( B \). Let \( e_1, \ldots, e_s \) be the primitive idempotents of \( \tilde{R} \) and \( e_1, \ldots, t \), the primitive idempotents of \( B \). The \( e_j \) refine the \( e_i \), so \( t \geq s \); when \( s = t \) we are in the nice situation described by Proposition 2.2. (Notice that if \( R \) is complete, then \( s = t \).)

**Assumption.** We assume from now on that \( R \) satisfies (dr), equivalently, \( R_{\text{art}} = (A \rightarrow B) \) satisfies (dr).

Now \( t \leq 3 \), and at most one residue field of \( B \) can be a proper extension of \( k \). (Otherwise \( B/J \), and a fortiori \( B/mB \), would have \( k \)-dimension greater than 3.) We always number the idempotents so that \( (B/J)e_j = k \) for \( j > 1 \), and we put \( K = (B/J)e_1 \). Let \( d = [K:k] \), and let \( \mu = \dim_k(B/mB) \). (Then \( \mu \) is the number of generators required for \( \tilde{R} \) as an \( R \)-module, that is, the multiplicity of \( R \).) We have the following useful relation:

\[
t + d + \dim_k(J/mB) = \mu + 1.
\]

**Lemma 2.7.** Assume \( \mu = 3 \) and \( d > 1 \). Then \( mB = J \). If \( d = 2 \) then \( t = 2 \), and if \( d = 3 \) then \( t = 1 \).

**Proof.** If \( d = 3 \) both assertions are clear from (2); therefore we assume \( d = 2 \). It will suffice to show that \( t = 2 \). If this is not the case, then \( t = 1 \) and \( \dim_k(J/mB) = 1 \). It follows from Nakayama's lemma that \( J^2 \subseteq mB \); hence \( J/mB \) is a vector space over \( K \). But then \( \dim_k(J/mB) \) is even, the desired contradiction. \( \Box \)

**Types \( H_n \) and \( G_n \).** We are now ready to describe the rings between \( R \) and \( \tilde{R} \), when \( d = 2 \) or 3, in enough detail that we will be able to determine (in Section 3) the ranks of their indecomposables. The case \( d = 2 \) is considerably
harder, and to ease the pain we introduce yet another piece of notation. Suppose $d = 2$. Then $t = 2$, and we write $B_i = B_{e_i}$. Let $J_i$ be the radical of $B_i$, and let $v_i$ be the index of nilpotency of $J_i$. (Then $l_A(B) = 2v_1 + v_2$.) For each $n \geq 1$, we say $A \to B$ is of type $G_n$ ("G" for Gorenstein) provided $v_1 = n + 1$ and $v_2 = 2$. We say $A \to R$ is of type $H_n$ provided $v_1 = n$ and $v_2 = 1$. Also we say $R$ is of type $G_n$ or $H_n$ if $R_{\text{art}}$ is.

The next theorem gives us a good handle on the ring-orders satisfying (dr) and having residue field growth. The proof of the theorem is rather technical. The interested reader might compare it with the proof of [15, (4.3)], where similar conclusions were reached with the aid of a coefficient field.

**Theorem 2.8.** Assume $d > 1$ (that is, $\tilde{R}$ has a residue field $K$ properly extending $k$) and $R$ has multiplicity $\mu = 3$. Let $A \to B$ be the Artinian pair associated to $R$, and let $m$ and $J$ be the radicals of $A$ and $B$, respectively.

1. Suppose $d = 3$. If $J = 0$, then $(A \to B) = (k \to K)$. If $J \neq 0$ then $R$ is Gorenstein, $l_A(A) = 3$, and there is exactly one ring $S$ strictly between $R$ and $R$. Moreover, the Artinian pair associated to $S$ is $(k \to K)$.

2. Suppose $d = 2$. If $R$ is Gorenstein, then $R_{\text{art}}$ is of type $G_n$ for some $n \geq 1$. If $R$ is not Gorenstein, then $R_{\text{art}}$ is of type $H_n$ for some $n \geq 1$.

3. If $d = 2$, and $S$ is a local ring strictly between $R$ and $R$, then $S_{\text{art}}$ is of type $G_n$ or $H_n$, for some $n$.

4. If $(A \to B)$ is of type $H_n$, then $A$ is a principal ideal ring and $B = B_1 \times k$.

5. If $S$ is not local and $R \subset S \subset R$, then $S$ has exactly two maximal ideals, each with residue field $k$, and the localizations have multiplicities 2 and 1, respectively.

**Proof.** If $J = 0$, then $(A \to B)$ is either $(k \to K \times k)$ or $(k \to K)$. The first possibility arises only when $d$ is 2, and then $(A \to B)$ is of type $H_1$. ($R$ is not Gorenstein.)

Assume from now on that $J \neq 0$. If $d = 2$, suppose $J_1 = 0$. Since $mB = J$ (by Lemma 2.7), there is an element $x \in m - J^2$. When viewed as an element of $B = B_1 \times B_2$, $x = (0, u)$, and $B_2u = J_2$. Put $y = x^{v_2 - 1}$. Then $y \neq 0$, but $By \subseteq A$ (because $Ae_2 + J_2 = B_2$). But this is a contradiction, since the conductor of $A$ in $B$ is 0. Thus $J_1 \neq 0$. Now we know $\nu_1 \geq 2$.

If $J_2 = 0$, that is, $\nu_2 = 1$, then $A \to B$ is of type $H_n$.

For the rest of the proof we assume that if $d = 2$ then both components of the radical are non-zero. Again, we select $x \in m - J^2$; in addition, if $d = 2$ we choose $x$ outside $(J_1^2 \times J_2) \cup (J_1 \times J_2^2)$; now $Bx = J$ in either case.

Our next goal is to show that $m \not\subseteq (Ax + J^2)$ by carefully counting lengths. We know that $l_A(B/J^2) = 6$. Also, $(A + J)/J \cong A/m$ (length 1), and since $B/J$ has length 3, we have $l_A(B/(A + J)) = 2$. Now $(A + J)/(A + J^2)$ has length at most 1 by (dr2), and $(A + J^2)/(m + J^2)$ is simple. Therefore,
Finally, \((Ax + J^2)/J^2\) is simple, and it follows that the "gap" \((m + J^2)/(Ax + J^2)\) has length at least 1. A picture proof:

\[
\begin{array}{c|c}
B & \mathfrak{m}B + A \\
\mathcal{J} & \leftarrow \mathcal{A} + J^2 \\
& \leftarrow \mathfrak{m} + J^2 \\
& \mathfrak{A} + J^2 \mathcal{J} \\
& \leftarrow J = \mathfrak{m}B = Bx
\end{array}
\]

Choose \(f \in \mathfrak{m} - (Ax + J^2)\), write \(f = bx\), with \(b \in B\), and note that \(b \notin A + J\). Consider the element \(b \in B/J\). We have \(b \notin k\); therefore \(K = k + kb + kb^2\). It follows that \(B = A + Ab + Ab^2 + J\), and since \(J = \mathfrak{m}B\) we have \(B = A + Ab + Ab^2\) by Nakayama’s lemma. Now \(A\) contains \(x^2, bx^2 = fx\) and \(b^2x^2 = f^2\), and it follows that \(Bx^2 \subseteq A\). Therefore \(x^2 = 0\), and hence \(J^2 = 0\). Now we know that \(R\) is Gorenstein and \(A\) has length 3. The ring \(A + J\) lies strictly between \(A\) and \(B\); therefore its preimage \(S\) under the map \(\tilde{R} \to B\) is strictly between \(R\) and \(\tilde{R}\). The Artinian pair associated to \(S\) is obviously \(k \to k\). For the uniqueness assertion, it suffices to show that \(A + J\) is the only ring strictly between \(A\) and \(B\). Suppose \(A \subseteq C \subseteq B\), and let \(U\) be the radical of \(C\). If \(C/U = K\), then \(C = B\) by Nakayama’s lemma. Otherwise, \(C/U = k\), whence \(C = A + U\). In this case we have \(A \subseteq C \subseteq A + J\), and since \((A + J)/A\) has length 1, \(C\) has to be either \(A\) or \(A + J\).

Suppose from now on that \(d = 2\) (and \(B/J = K \times k\)). We first observe that \(\nu_1 \geq \nu_2\). If not, we put \(y = x^{\nu_2 - 1}\) and obtain a contradiction exactly as at the beginning of the proof. Next, we show \(\nu_2 \leq 2\). The element \(b \in k \times k\) is not in the image of \(A\), which is the diagonal of \(k \times k\). In particular, if we write \(b = (\beta, \gamma)\), with \(\beta \in K\), \(\gamma \in k\), then \(\beta \neq \gamma\). Select an element \(c \in A\) with image \(c = (\gamma, \gamma)\) and put \(g = (b - c)x \in \mathfrak{m}\). Then \(g \notin J^2\), yet \(g \in J_1^2\). Therefore, for each \(i, 1 \leq i < \nu_2\), we have \(g^i \in \mathfrak{m'} - \mathfrak{m'}^{i+1}\), yet \(g^i\) does not generate \(\mathfrak{m'}\). It follows that \(\mathfrak{m'}/\mathfrak{m'}^{i+1}\) has length at least 2, for \(i = 1, \ldots, \nu_2 - 1\). Therefore \(l_i(A) \geq 1 + 2(\nu_2 - 1) + \nu_1 - \nu_2 = \nu_1 + \nu_2 - 1\). Combining this with the inequality \(2\nu_2 + \nu_1 = l_A(A) \geq 2l_A(A)\) from Proposition 2.6, we get \(\nu_2 \leq 2\).

We have now proved (1) and (2). To prove (3), we note that \(S\) must have residue field \(k\), so its multiplicity is 3. Furthermore, since \(\mathfrak{m}B = J\), it is easy to see that \(S\) still satisfies (dr). Therefore (3) follows from (2). For (4), suppose \((A \to B)\) is of type \(H_n\). Then \(l_A(B) = 2n + 1\), and by Proposition 2.6, \(l_A(A) = n = \nu_1\). It follows that \(\mathfrak{m'}/\mathfrak{m'}^{i+1}\) is simple for each \(i\). In particular, \(\mathfrak{m}\) is principal, and it follows easily that the principal ideals \(\mathfrak{m}^i\) are the only ideals of \(A\).

For the proof of (5), let \(C\) be the image of \(S\) under the map \(\tilde{R} \to B\). Then \(A \subseteq C \subseteq B\), and \(C\) contains both idempotents of \(B\). Write \(C = C_1 \times C_2\), and examine the inclusions \(C_1 \subseteq B_1, C_2 \subseteq B_2\). Letting \(U = U_1 \times U_2\) be the radical of \(C\), we have \(UB = J\), since \(\mathfrak{m} \subseteq U\). Therefore \(U_1B = J_1\). Let \(\mu_i\),
be the multiplicity of $C_i$. Then $\mu_i = \dim(B_i/J_i)$ over $C_i/U_i$. Then $\mu_2 = 1$, so $C_1 = B_1$. Therefore $C_2 \neq B_2$, and we conclude that $\mu_1 = 2$ and $C_1/U_1 = k$.

**Remark 2.9.** We conjecture that if $d = 2$ or 3 and $\mu = 3$, then the lattice of over-rings of $R$ is always a chain: If $R$ is of type $H_n$, there is a unique overring of type $H_m$ for each $m \leq n$ (and no $G_m$s), together with a single non-local ring (as in (4)) at the top. If $R$ is $G_n$, there should be a unique minimal over-ring, and it will be of type $H_n$.

The simplest singularity of multiplicity 3 is the local ring of three coordinate axes, or what Bass calls a triad of discrete valuation rings. We will need the following fact [2, Section 7] about the corresponding Artinian pair:

**Proposition 2.10.** Let $k$ be any field, and let $(A \to B) = (k \to k \times k \times k)$. The only indecomposable $(A \to B)$-modules are the "trivial" modules

$$(A \to B), \quad (k \to k \times 0 \times 0), \quad (k \to k \times k \times 0)$$

and the four others obtained by symmetry. $\square$

3. Ranks of indecomposables in the local case

In this section we will give the possible ranks of the indecomposable maximal Cohen–Macaulay $R$-modules when $R$ is a local ring-order satisfying (dr), in terms of the parameters $d, s, t, \mu$ from Section 2. Recall that the rank of the maximal Cohen–Macaulay $R$-module $M$ is the $s$-tuple $\text{rank}(M) = (r_1, \ldots, r_s)$, where $r_i$ is the rank of the free $R e_i$-module $R e_i M$. Similarly, we define the rank of the $(A \to B)$-module $(V \to W)$ to be the $t$-tuple $(r_1, \ldots, r_t)$, where $r_j$ is the rank of the free $B e_j$-module $e_j W$.

Notice that every non-zero $s$-tuple consisting of 0s and 1s occurs as the rank of an indecomposable maximal Cohen–Macaulay $R$-module. For example, if $s = 3$, let $P_1, P_2, P_3$ be the minimal prime ideals of $R$. Then $R/P_1$ has rank $(1, 0, 0)$, $R/(P_1 \cap P_2)$ has rank $(1, 1, 0)$, etc. We call these the trivial sequences, and to avoid repetition we will usually list only the non-trivial sequences that can occur as ranks of indecomposable maximal Cohen–Macaulay $R$-modules. Similarly, all the trivial $t$-tuples occur as ranks of indecomposable $(A \to B)$-modules. If, for example, $t = 3$, then $A \to B e_1 \oplus B e_2$ is indecomposable and has rank $(1, 1, 0)$.

In some cases, the lack of symmetry may be disturbing. For example, when $s = 3$ our data lists $(1, 1, 2)$ as a possible rank, but not $(1, 2, 1)$ or $(2, 1, 1)$. What this means is that for every local ring-order $R$ of finite Cohen–Macaulay type and with $s = 3$, one can order the primitive idempotents of $\tilde{R}$ in such a
way that there is no indecomposable of rank (1, 2, 1) or (2, 1, 1); further, there
exist such a ring-order that does have an indecomposable of rank (1, 1, 2).
The reordering business may raise further doubts, since we have already made
the convention that if there is residue field growth then it occurs in the first
component. But if \(d > 1\), then \(s \leq t \leq 2\), and we promise to stick with the
convention that the residue fields of \(\bar{R}e_1\) and \(Be_1\) are proper extensions of \(k\).

The multiplicity \(\mu\) is either 2 or 3, since we are not interested in the case
\(R = \bar{R}\). As far as ranks are concerned, the case \(\mu = 2\) is equally boring:

**Proposition 3.1 (Bass [2]).** If \(S\) is a ring-order (not necessarily local) such that
\(\bar{S}\) is generated by two elements as an \(S\)-module, then every maximal Cohen-
Macaulay \(S\)-module is isomorphic to a direct sum of ideals. In particular, if
\(\mu(R) = 2\), only the trivial sequences occur as ranks of indecomposable maximal
Cohen–Macaulay modules. \(\Box\)

The reader may find it helpful to consult [15] while going through these
technical details.

From now on we assume that \(\mu = 3\). Then (2) becomes
\[
t + d + \dim_k(J/mB) = 4.
\]

3.1. The case \(d = 1\) (no residue field growth)

**Theorem 3.2.** If \(d = s = 1\), then every indecomposable maximal Cohen-
Macaulay \(R\)-module has rank 1, 2 or 3. Each of these ranks occurs over
the local ring-order \(F[T^3, T^5]\), where \(F\) is any field. Moreover, this ring
has finite Cohen–Macaulay type.

**Proof.** Suppose first that \(t = 1\). Then we are in the situation described by
\[7, (5.4)\]. The complete list of possibly indecomposable \((A \rightarrow B)\)-modules
on pages 76 and 77 of [7] tells us that all the indecomposables have rank
at most 3. Since \(s = t\), Theorem 1.1 implies that the same bounds apply for
indecomposable maximal Cohen–Macaulay \(R\)-modules.

If \(R = F[T^3, T^5]\), the associated Artinian pair is \(F[\tau^3, \tau^5] \rightarrow F[\tau]\),
with \(\tau^8 = 0\), which has finite representation type by [7]. Therefore \(R\) has finite
Cohen–Macaulay type by Theorem 2.1. Since there are no elements of degree 2
or 4, there are indeed indecomposables of ranks 2 and 3 by the comments
after Corollary 5.9 of [7]. (The more degenerate ring \(F[T^3, T^5, T^7]\),
also works. On the other hand, \(F[T^3, T^4]\) has no indecomposables of
rank 3, by [7, (5.8)].)

Suppose next that \(t = 2\). According to the proof of [7, (5.11)], the possible
non-trivial ranks of the indecomposable \((A \rightarrow B)\)-modules are (1, 2), (2, 1)
and (2, 2). Suppose \(M\) is a maximal Cohen–Macaulay \(R\)-module of rank at
least 4. By writing \(M_{\text{art}}\) as a direct sum of indecomposables (allowing some
of ranks \((1,0), (0,1), (1,1)\) of course, one sees quickly that \(M_{\text{art}}\), a module of constant rank, has to have a direct summand \((V \to W)\) of rank \((2,2)\) or \((3,3)\). By Theorem 2.1, \(M\) has a direct summand of rank 2 or 3.

Finally we suppose \(t = 3\). Then the worst-case scenario (after possibly renumbering the primitive idempotents of \(B\)) gives \((1,1,2)\) as the only non-trivial possibility for the rank of an indecomposable \((A \to B)\)-module. If, now, \(M\) is a maximal Cohen–Macaulay \(R\)-module, write \(M_{\text{art}} = (V_1 \to W_1) \oplus \cdots \oplus (V_r \to W_r)\), with the \((V_i \to W_i)\) indecomposable. Now \(\text{rank}(M_{\text{art}}) = (r, r, r)\), for some \(r \geq 1\), and the values for \(\text{rank}(V_i \to W_i)\) are trivial or \((1,1,2)\). The direct sum of some of the \((V_i \to W_i)\) must have rank \((1,1,1)\) or \((2,2,2)\), and by (2.1), this direct summand of \(M_{\text{art}}\) comes from a direct summand of \(M\). It follows that the indecomposable maximal Cohen–Macaulay \(R\)-modules have rank \(\leq 2\) in this case. \(\Box\)

Returning to the case \(t = 2\) above, we show how to get Krull–Schmidt to fail over a local ring-order. Let \(F\) be any field, and let \(B = F[t] \times F[u]\) with the relations \(t^3 = u^5 = 0\). Let \(A\) be the \(F\)-subalgebra of \(B\) generated by \((t, u^2)\) and \((0, u^3)\). Then \(A \to B\) is the Artinian pair for the local ring at the origin of the union of the cusp \(y^2 = x^3\) and the \(x\)-axis. Of course this local ring satisfies Krull–Schmidt since \(s = t (\geq 2)\). We want to replace the local ring by an integral domain with the same associated Artinian pair. (Assuming \(F\) is perfect, the new ring will be analytically isomorphic to the old one. See the Appendix of [18].) This can be done painlessly as follows: Map \(F[X]\) onto \(B\) by sending \(X\) to \((t + 1, u)\), and let \(D\) be the localization of \(F[X]\) at the union of the maximal ideals \((X - 1)\) and \(X\). Take \(R\) to be the pullback of \(D\) and \(A\) over \(B\). Then [18, (3.1)], \(R\) is a local ring-order, \(\bar{R} = D\), and \(R_{\text{art}} = (A \to B)\).

**Example 3.3.** Let \(R\) be the local ring-order constructed above. The Krull–Schmidt Theorem fails for direct-sum decompositions of maximal Cohen–Macaulay \(R\)-modules.

**Proof.** Let \(Z_1, Z_2, Z_3, Z_4\) be \((A \to B)\)-modules of ranks \((2,1), (1,2), (0,1)\) and \((1,0)\), respectively. (See the list on p.81 of [7]. We note that the easier case discussed in [7, (5.10)] corresponds to the union of a (possibly higher order) cusp \(y^2 = x^{2n+1}\) and the \(y\)-axis. The Artinian pair for this singularity admits only \((2,1)\) as a non-trivial rank.) Using Theorem 2.1, define maximal Cohen–Macaulay \(R\)-modules \(M_1, \ldots, M_4\) by

1. \((M_1)_{\text{art}} = Z_1 \oplus Z_2\), of rank \((3,3)\).
2. \((M_2)_{\text{art}} = Z_1 \oplus Z_3\), of rank \((2,2)\).
3. \((M_3)_{\text{art}} = Z_2 \oplus Z_4\), of rank \((2,2)\).
4. \((M_4)_{\text{art}} = Z_3 \oplus Z_4\), of rank \((1,1)\).
It follows from Theorem 2.1 that the $M_i$ are indecomposable and that $M_1 \oplus M_4 \cong M_2 \oplus M_3$. Since the $M_i$ have ranks 3, 2, 2 and 1, respectively, these decompositions are not equivalent.

**Theorem 3.4.** Suppose $d = 1$ and $s = 2$. The only non-trivial sequences occurring as ranks of maximal Cohen-Macaulay $R$-modules are $(2,1)$, $(1,2)$ and $(2,2)$. Each of these ranks occurs for the ring $(F[X,Y]/Y(Y^2 - X^3))_{(x,y)}$, where $F$ is an arbitrary field. Moreover, this ring has finite Cohen-Macaulay type.

**Proof.** Suppose first that $t = 2$. Then every $R_{\text{art}}$-module comes from an $R$-module, and we appeal again to the list on p. 81 of [7]. Conversely, the ring $(F[X,Y]/Y(Y^2 - X^3))_{(x,y)}$ is the non-degenerate case of Case (II) in Section 5 of [7], that is, $\Gamma y = r_3^2$, and the same list provides indecomposables of each of these ranks.

If $t = 3$ we number the primitive idempotents $e_j$ so that there are no indecomposable $(A \to B)$-modules of rank $(1,2,1)$ or $(2,1,1)$, but possibly one of rank $(1,1,2)$. (See Case (III) of Section 5 of [7], in particular, the list on p. 83. We have switched the ordering of the coordinates, so that the “2” appears in last place.) Now $\hat{R} = \hat{R}e_1 \times \hat{R}e_2$, and exactly one of the $\hat{R}e_i$ is local.

Suppose first that $\hat{R}e_1$ is local. We write the rank of an $(A \to B)$-module as $(r_1, r_2)$, using the vertical bar to separate the two components of $\hat{R}e_1$. If there is an indecomposable $(A \to B)$-module of rank $(1,1,2)$, then its direct sum with a module of rank $(1,1,0)$ might correspond to an indecomposable maximal Cohen-Macaulay $R$-module of rank $(2,2)$. If, on the other hand, $\hat{R}e_2$ is local, we write ranks as $(r_1, r_2, r_3)$. This time we might get an indecomposable maximal Cohen-Macaulay $R$-module of rank $(1,2)$ as well, if there were an indecomposable $(A \to B)$-module of rank $(1,1,2)$. It is easy to see that no other non-trivial sequences can occur as ranks of indecomposables.

If $s = 3$, then $t = 3$ as well, and we know that $M$ is indecomposable if and only if $M_{\text{art}}$ is indecomposable. Referring again to the list on p. 83 of [7], we have the first half of the following theorem:

**Theorem 3.5.** Assume $d = 1$ and $s = 3$. One can number the three primitive idempotents of $\hat{R}$ in such a way that $(1,1,2)$ is the only non-trivial sequence occurring as the rank of an indecomposable maximal Cohen-Macaulay module. For any field $F$, the ring-order $(F[X,Y]/XY(Y - X^2))_{(x,y)}$ has finite Cohen-Macaulay type and has an indecomposable of rank $(1,1,2)$, provided the components of the normalization are listed in this order: $F[X,Y]/(Y) \times F[X,Y]/(Y - X^2) \times F[X,Y]/(X)$ (suitably localized).
Proof. The ring-order in question is the local ring at the origin of the union of the two coordinate axes and a parabola tangent to the x-axis. By the discussion following the proof of Theorem 5.1 of [15], we see that this is the curve denoted there by \( P_2 \). This ring definitely has an indecomposable with non-trivial rank sequence, since it does not satisfy the criteria of [8]. (The local ring of a curve at a point with three analytic branches has the property that every maximal Cohen–Macaulay module is isomorphic to a direct sum of ideals if and only if all three branches are smooth and the three tangent directions are distinct.) The only question is whether or not we have numbered the components correctly. One way to see that we have is to appeal to symmetry, and notice that the y-axis (\( X = 0 \)) is distinguished from the other two branches by virtue of not being tangent to any of the other branches. If this argument makes the reader queasy, we can work with the following parametrization: Take the subalgebra of \( F[T]\times F[U]\times F[V] \) generated by \( x = (T, U, 0) \) and \( y = (0, U^2, V) \). (Note the encouraging fact that \( xy(y - x^2) = 0 \).) This parametrization gives the Artinian pair

\[
F[(t, u, 0), (0, u^2, v)] \to F[t] \times F[u] \times F[v],
\]
with \( t^3 = u^3 = v^2 = 0 \).

(See [15] for more detail.) Now we know the order is correct, because the component where the rank is 2 has the parameter with the lowest degree of nilpotency. (See the sentence “Further, if...” near the bottom of p. 71 of [7], but note that the order of the components used by Green and Reiner is different from ours.)

3.2. The case \( d = 2 \) (residue field growth of degree 2)

Suppose now that \( d = 2 \). Then, by (3) and (2.8), we have \( s \leq t \leq 2 \) and \( A \to B \) is of type \( G_n \) or \( H_n \). We begin with an Artinian pair of type \( H_1 \), that is, \( (A \to B) = (k \to K \times k) \), a product of fields. This case will be used again when we treat non-Galois extensions of degree 3. We adjust the notation to fit the situation we will encounter there.

Proposition 3.6. Let \( K \subseteq L \) be fields with \( [L : K] = 2 \), and assume \( L \) is not purely inseparable over \( K \) of degree 2. Then the indecomposable \( (K \overrightarrow{\text{diag}} L \times K) \)-modules are:

1. \( K \to L \times K \) of rank \( (1, 1) \),
2. \( L \to L \times 0 \) of rank \( (1, 0) \),
3. \( K \to L \times 0 \) of rank \( (1, 0) \),
4. \( K \to 0 \times K \) of rank \( (0, 1) \),
5. \( L \to L \times L \) of rank \( (1, 2) \).
Proof. Write $L = K(u)$, and let $\tau$ be the non-trivial automorphism of $I/K$. Using the construction described in Section 2.1, we obtain:

$$
\begin{align*}
A' &= (L \times L) \times L = B' \\
A &= K \rightarrow (L \times K = B)
\end{align*}
$$

where the map $\beta : L \times K \rightarrow (L \times L) \times L$ takes $(x, y) \mapsto (x, x^\tau, y)$, for all $x \in L$, $y \in K$. By Theorem 2.3, the indecomposable $(A \rightarrow B)$-modules are $(A \rightarrow B)$-direct summands of indecomposable $(A' \rightarrow B')$-modules. The indecomposable $(L \rightarrow L \times L \times L)$-modules are just the seven trivial ones by Proposition 2.10. We will see how each of these modules decomposes as a $(K \rightarrow L \times K)$-module.

We claim that $L \rightarrow (L \times L \times L)$ decomposes as $(K \rightarrow W_1) \oplus (Ku \rightarrow W_2)$, where $W_1 = \{(x, x^\tau, y) \mid x \in L$, $y \in K\}$ and $W_2 = \{(xu, x^\tau u, yu) \mid x \in L$, $y \in K\}$. Clearly both of these summands are isomorphic to $K \rightarrow L \times K$, of rank $(1, 1)$. To see that the alleged decomposition works, let $(z, w, r) \in L \times L \times L$.

Of course $r = y_1 + y_2 u$ uniquely, with $y_1$ and $y_2 \in K$, and we need to represent $(z, w)$ in the form $(x_1, x_1^\tau) + (x_2 u, x_2 u^\tau)$. This amounts to solving $(x_1 + ux_2, x_1 + u^\tau x_2) = (z, w^\tau)$, which has a unique solution because the determinant $|1, u|_L$ is non-zero.

Next we have $L \rightarrow 0 \times 0 \times L$ of rank $(0, 2)$, which decomposes as a direct sum of two copies of $K \rightarrow 0 \times K$, each of rank $(0, 1)$.

The modules $L \rightarrow L \times 0 \times 0$ and $L \rightarrow 0 \times L \times 0$, when viewed as $(K \rightarrow L \times K)$-modules are indecomposable of rank $(1, 0)$ and isomorphic to $L \rightarrow L \times 0$.

Next, we consider $L \rightarrow L \times L \times 0$. The decomposition of $L \rightarrow L \times L \times L$ given above induces a decomposition of this module into a direct sum of two copies of $K \rightarrow L \times 0$.

Finally, we have the modules $L \rightarrow L \times 0 \times L$ and $L \rightarrow 0 \times L \times L$. Both are isomorphic to $L \rightarrow L \times L$ and are indecomposable as $(K \rightarrow L \times K)$-modules. For, suppose $\phi$ is an idempotent $(L \times K)$-isomorphism of $L \times L$, stabilizing the diagonal image of $L$. Then $\phi = (\mu, \theta)$, where $\mu$ is multiplication by some $e \in L$. Since $\phi$ is idempotent, $e = 0$ or 1. For each $z \in L$, we have $\phi(z, z) = (ez, \theta z)$, which forces $\theta z = ez$, as well. Thus $\phi = 0$ or $1$. \qed

Remark. There are two non-isomorphic modules of rank $(1, 0)$, given by Proposition 3.6(2) and (3).

For the next case, we are sloppy in the analysis of the ranks, only giving all possible ranks which might arise for indecomposable modules, since our bounds in Section 5 will not be affected. To say exactly which correspond to indecomposable modules would require a careful study of the modules in [7].
**Proposition 3.7.** Suppose that \( d = 2 \) and \( A \to B \) is an Artinian pair of type \( H_n \), \( n > 1 \). Assume that \( K \) is separable over \( k \). Then the ranks of all indecomposable \((A \to B)\)-modules are included in this list:

\[
(0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (2,4). \tag{4}
\]

**Proof.** By Theorem 2.8, we have \( B = B_1 \times k \), \( \text{radical}(B_1) = mB_1 \), \( (mB)^n = 0 \), \( B_1/mB_1 = K \), and \( A \) is a local principal ideal ring.

We use the construction described in Section 2.1:

\[
A' = A[X]/Af \longrightarrow B' = B[X]/Bf = B_1[X]/(f) \times K \xrightarrow{\cong} (B_1 \times B_1) \times K
\]

\[
A \quad \quad B \quad \quad B_1 \times k \quad \quad B_1 \times k
\]

By [7], the ranks of indecomposables over \((A' \to B')\) appear on this list:

\[
(0,0,1), (0,1,0), (1,0,0), (1,1,0), \tag{5}
\]

\[
(0,1,1), (1,0,1), (1,1,1), (1,1,2).
\]

The last item is listed as \((1,1,2)\) so that the “2” occurs in the coordinate of \( B' = B_1 \times B_1 \times K \), with the smallest index of nilpotency for its radical.

By inspecting the \((A' \to B')\)-modules with ranks on the list \((3.8.2)\), we get the following ranks for the same modules, considered as \((A \to B)\)-modules:

\[
(0,2), (1,0), (1,0), (2,0), (1,2), (1,2), (2,2), (2,4). \tag{6}
\]

Thus indecomposable \((A \to B)\)-modules have ranks less than or equal to \((2,4)\). Now by Theorem 2.3(6), if an indecomposable \((A \to B)\)-module \( M \) had rank \((2,1)\), then either \((2,1)\) or \((4,2)\) would have to appear on the list \((6)\), a contradiction. Therefore, \((2,1)\) cannot occur as the rank of an indecomposable \((A \to B)\)-module. By similar reasoning, neither can \((0,4), (1,4), (0,3), (1,3) \) or \((2,3)\).

We conclude that the only possible ranks for indecomposable \((A \to B)\)-modules in the case \( d = 2 \) are those on the list \((4)\). \( \Box \)

**Remark.** If \( R \) is of type \( H_n \) and \( R \) has two minimal primes, the list \((4)\) includes all possibilities for ranks of indecomposable \( R \)-modules. However, if \( R \) is a domain, \( R \)-modules come from \( R_{\text{art}} \)-modules of constant rank. If \( M \) were an \((A \to B)\)-module of constant rank, then \( M \) could be decomposed into a direct sum of modules of ranks in \((4)\). By inspection we see that there must be a direct summand \( N \) of \( M \) with constant rank 1, 2, 3 or 4. It follows that when the degree is 2 and \( R \) is a domain of type \( H_n \), the only possible ranks of indecomposable modules are 1, 2, 3, or 4.
The completion of the case \( d = 2 \). Now suppose that \( R \) is not of type \( H_n \) for any \( n \). Then \( R \) is of type \( G_n \)—that is, \( R \) is Gorenstein. For every indecomposable maximal Cohen–Macaulay \( R \)-module \( M \), either \( M \) is isomorphic to \( R \), or \( M \) is an \( S \)-module for some ring \( S \) with \( R \subseteq S \subseteq \hat{R} \), by Proposition 2.5. In the latter case, if \( S \) is local, we can apply Theorem 2.8 to \( S \) and repeat the process—either \( S \) is a \( G_n \) or an \( H_n \). If \( S \) is not local, then by Theorem 2.8(5) and Proposition 3.1, \( M \) is isomorphic to an ideal of \( S \). In summary:

**Theorem 3.8.** Let \( R \) be a local ring-order satisfying (dr). Assume \( d = 2 \) and \( K \) is separable over \( k \).

1. If \( s = 2 \), the only possible ranks of indecomposable maximal Cohen–Macaulay \( R \)-modules are those on the list (4).
2. If \( s = 1 \), the only possible ranks are 1, 2, 3 and 4. \( \square \)

3.3. The case \( d = 3 \) (residue field extension of degree 3)

When \( R \) satisfies (dr) and \( d = 3 \), then \( R \) is necessarily an integral domain by (3).

**Theorem 3.9.** Let \( R \) be a local ring-order satisfying (dr), and assume \( d = [K : k] = 3 \). Then every indecomposable maximal Cohen–Macaulay \( R \)-module has rank at most 4.

**Proof.** By Proposition 2.5 and Theorem 2.8(1), it suffices to investigate modules \((V \to W)\) over the Artinian pair \((k \to K)\). We consider the following cases: Case 1: \( K/k \) is Galois. Case 2: \( K/k \) is separable, but not Galois. Case 3: \( K/k \) is purely inseparable (and the characteristic is 3).

Case 1: \( K = k[u] \) and \( u \) is a root of an irreducible cubic polynomial \( f \) over \( k \) which has three distinct roots in \( K \). Let \( G = \{1, \tau, \sigma\} \) be the Galois group of \( K \) over \( k \). Applying the procedure of Section 2.1 yields the following diagram:

\[
\begin{align*}
A' & \xrightarrow{\text{diag}} K \times K \times K = B' \\
A & \xrightarrow{\text{incl}} k \xrightarrow{\gamma} K = B
\end{align*}
\]

where \( \gamma(x) = (x, x^\tau, x^\sigma) \), for each \( x \in K \). Thus the \( B \)-module structure in \( B' \) is the following: \( b(x, y, z) = (bx, b^\tau y, b^\sigma z) \), for all \( b \in B \), \((x, y, z) \in B' \). The indecomposable \((K \to K \times K \times K)\)-modules are the seven trivial ones by Proposition 2.10, and the indecomposable \((k \to K)\)-modules are exactly the indecomposable \((k \to K)\) summands of the \((K \to K \times K \times K)\)-modules.

We observe that \((K \to K \times K \times K)\) is the direct sum of these three \((k \to K)\)-modules:

1. \( k \to \{(x, x^\tau, x^\sigma) \mid x \in K \} \),

\[
\begin{align*}
A' & \xrightarrow{\text{diag}} K \times K \times K = B' \\
A & \xrightarrow{\text{incl}} k \xrightarrow{\gamma} K = B
\end{align*}
\]
(2) \( ku \rightarrow \{ (xu, xu^2, xu^3) \mid x \in K \} \),
(3) \( ku^2 \rightarrow \{ (xu^2, xu^2, xu^3) \mid x \in K \} \).

This is similar to the proof of Proposition 3.6. In order to see that every \((w, y, z) \in K \times K \times K\) is in the sum, note that the relevant equations and (Vandermonde) determinant are:

\[
\begin{align*}
w & = x_1 + x_2u + x_3u^2, & 1 & u & u^2 \\
y^\sigma & = x_1 + x_2u^\sigma + x_3u^2\sigma, & 1 & u^\sigma & (u^\sigma)^2 \neq 0. \\
z^\tau & = x_1 + x_2u^\tau + x_3u^2\tau, & 1 & u^\tau & (u^\tau)^2 \\
\end{align*}
\]

Thus \((K \rightarrow K \times K \times K)\) is the direct sum of three copies of \((k \rightarrow K)\), and the other six trivial indecomposables have ranks at most two over \((k \rightarrow K)\). We conclude that in Case 1, the indecomposable \((k \rightarrow K)\)-modules have rank 1 or 2. (In fact, there are indecomposables of rank 2, by [8].)

**Case 2:** \(K/k\) is separable, but not Galois. Again write \(K = k[u]\), where \(u\) is a root of an irreducible cubic \(f\) over \(k\); say \(f(x) = (x - u)g(x)\), where \(g(x)\) is irreducible over \(K\). Let \(L\) be the splitting field of \(g\) over \(K\). Let \(\sigma\) generate the Galois group of \(L\) over \(K\) and let \(\tau\) be an element of order 3 in the Galois group of \(L\) over \(k\).

Applying the procedure of Section 2.1 twice, we obtain the following diagram:

\[
\begin{array}{c}
L \xrightarrow{\text{diag}} L \times L \times L \\
\uparrow\text{incl} & \uparrow\beta \\
K & \xrightarrow{\text{incl}} L \times K \\
\uparrow\text{incl} & \uparrow\delta \\
k & \xrightarrow{} K
\end{array}
\]

(7)

where \(\delta(y) = (y^\tau, y)\) and \(\beta(z, y) = (z, z^\sigma, y)\), for all \(y \in K, z \in L\).

In Proposition 3.6, we examined the indecomposables over the top line as modules over the second line of the diagram. To find the possible indecomposables over \(k \rightarrow K\), we need only examine the \((K \rightarrow L \times K)\)-modules listed in Proposition 3.6:

(1) \(K \rightarrow L \times K\) of rank 3,
(2) \(L \rightarrow L \times 0\) of rank 2,
(3) \(K \rightarrow L \times 0\) of rank 2,
(4) \(K \rightarrow 0 \times K\) of rank 1,
(5) \(L \rightarrow L \times L\) of rank 4.

Consequently every indecomposable \((k \rightarrow K)\)-module must have rank \(\leq 4\).

**Case 3:** \(K\) is purely inseparable over \(k\) of degree 3. In this case, there seems to be no way of avoiding actual matrix computations like those in [7]. To
Matrix notes. Suppose that \((A \to B)\) is any Artinian pair and \((V \to W)\) is a finitely generated \((A \to B)\)-module with \(W\) free of rank \(m\). We can assume \(W = B^m\), written as columns. Say \(V\) is generated as an \(A\)-module by \(v_1, \ldots, v_n\). Form the \(m \times n\) matrix \(G\) whose columns are \(v_1, \ldots, v_n\). Obviously, we can multiply \(G\) on the right by an invertible matrix with entries in \(A\) without changing the column space. Similarly, if we multiply \(G\) on the left by an invertible matrix over \(B\), the new matrix has column space isomorphic to \(V\). Thus the problem of decomposing the \((A \to B)\)-module \((V \to W)\) is equivalent to decomposing the \(m \times n\) matrix \(G\), allowing elementary column operations from \(A\) and elementary row operations from \(B\).

Now we return to our setting in this section and this case: \(A = k\), \(B = K\), \([K : k] = 3\), \(K = k[u]\), \(u^3 \in k\) and \(\text{char}(k) = 3\). Let \(f(X) = X^3 - u^3\) be the minimal polynomial for \(u\) over \(k\). We have the following diagram from the procedure in Section 2.1:

\[
\begin{align*}
A' = K & \twoheadrightarrow B' = K[X]/(X - u)^3 \twoheadrightarrow K[y], \ y^3 = 0 \\
A = k & \hookrightarrow B = K = k[u]
\end{align*}
\] where \(\gamma : u \mapsto y + u\).

The indecomposable modules over \(K \to K[y]\), the top line of the diagram, are determined in [7, p. 79]. (The element \(\pi\) there can safely be replaced by our \(y\), remembering that in fact \(y^3 = 0\).) Reproducing their list here, we consider:

\[
[1], \ [1y], \ [1y^2], \ [1y^2], \ G_1 = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & y^2 \end{bmatrix}, \ G_2 = \begin{bmatrix} 1 & 0 & y & y^2 \\ 0 & 1 & y^2 & 0 \end{bmatrix}.
\]

The first four matrices represent \((K \to K[y])\)-modules of rank 1, which means they have rank 3 as \((k \to K)\)-modules. We shall not worry about them. However, \(G_1\) and \(G_2\) each have rank 2 over \(K + K[y]\) and so rank 6 over \(k \to K\). We claim that the modules corresponding to these two matrices decompose over \((k \to K)\), and that the summands have rank at most 3.

Let \(\{\xi_1, \xi_2\}\) be the standard basis for \((B')^{(2)}\). Then \(G_1\) and \(G_2\) represent the modules

\[
M_1 = K\xi_1 + K\xi_2 + K(\nu \xi_1 + y^2 \xi_2) \to K[y]\xi_1 \oplus K[y]\xi_2 \\
M_2 = K\xi_1 + K\xi_2 + K(\nu \xi_1 + y^2 \xi_2) + Ky^2 \xi_1 \to K[y]\xi_1 \oplus K[y]\xi_2
\]
Changing these to \((k \to K)\)-modules involves replacing \(K\) by \(k + k(u + y) + k(u + y)^2\).

Let us work on \(M_1\) first. We are trying to save the reader the agony of too much detail (especially for \(M_2\)), but we assure you, we have checked this out.

\[
M_1 = (k + k(u + y) + k(u + y)^2)\xi_1 + (k + k(u + y) + k(u + y)^2)\xi_2 + (k + k(u + y) + k(u + y)^2)(y\xi_1 + y^2\xi_2)
\]

\[\to K\xi_1 \oplus Ky\xi_1 \oplus Ky^2\xi_1 \oplus K\xi_2 \oplus Ky\xi_2 \oplus Ky^2\xi_2.\]

Since \(y^3 = 0\), \(M_1\) corresponds to the following matrix:

\[
\begin{bmatrix}
1 & u & u^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2u & 0 & 0 & 0 & 1 & u & u^2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2u \\
0 & 0 & 0 & 1 & u & u^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2u & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & u^2
\end{bmatrix}
\]

After performing six elementary row operations and using characteristic 3, we obtain the following matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -u & 0 & u^3 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -u \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -u \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & u & u^2 \\
0 & 0 & 0 & 0 & 1 & 0 & u & u^2 & u^3 \\
0 & 0 & 0 & 0 & 1 & 0 & u & u^2 & u^3 \\
0 & 0 & 0 & 0 & 0 & 1 & 2u & 0 & 0 & 0
\end{bmatrix}
\]

Now we use column operations to eliminate all the 1s and \(u^3\) entries in the right-hand 6 x 3 block. (This is legal since \(u^3 \in k\).) Next, we use rows 1, 2 and 3 to clear out the \((5, 7), (6, 8)\) and \((4, 9)\) entries, respectively. Now three column operations will repair the damage done to the identity matrix, and we arrive at a matrix which decomposes as the direct sum of three copies of the following matrix:

\[
\begin{bmatrix}
1 & 0 & -u \\
0 & 1 & u^2
\end{bmatrix}
\]
For $M_2$, we perform similar operations on the following $6 \times 12$ matrix:

\[
\begin{pmatrix}
\xi_1 & 1 & u & u^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y\xi_1 & 0 & 1 & 2u & 0 & 0 & 0 & 1 & u & u^2 & 0 & 0 \\
y^2\xi_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2u & 1 & u & u^2 \\
\xi_2 & 0 & 0 & 0 & 1 & u & u^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
y\xi_2 & 0 & 0 & 0 & 1 & 2u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y^2\xi_2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & u & u^2 & 0 & 0 & 0 \\
\end{pmatrix}
\]

This matrix decomposes into a direct sum of three copies of the following matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & u^2 & 0 & u^4 \\
0 & 1 & 0 & u & u^2 & 0 \\
0 & 0 & 1 & 0 & u & u^2 \\
\end{pmatrix}
\]

We summarize the results we have obtained in Theorems 3.2, 3.4 and 3.5, Proposition 3.7, and Theorems 3.8 and 3.9:

**Theorem 3.10.** Suppose that $R$ is a local ring-order satisfying $(dr)$ and $(2\text{-}sep)$. Let $M$ be an indecomposable maximal Cohen–Macaulay $R$-module.

1. If $R$ is a domain, then $\text{rank}(M) \leq 4$.

2. If $R$ has two minimal primes and $d = 1$ (no residue field growth), then $\text{rank}(M)$ is one of the following:

   \[ (0, 1), (1, 1), (1, 0), (1, 2), (2, 1), (2, 2). \]

3. If $R$ has two minimal primes and $d = 2$, then $\text{rank}(M)$ is one of these:

   \[ (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (2, 4). \]

4. If $R$ has three minimal primes, then $\text{rank}(M)$ is either trivial or $(1, 1, 2)$ (with care taken that the “2” is in the coordinate with least index of nilpotency).

**Corollary 3.11.** Let $R$ be a local ring-order satisfying $(dr)$ and $(2\text{-}sep)$, and let $M$ be a maximal Cohen–Macaulay $R$-module of constant rank. Then $M$ has a direct summand of constant rank $r$ for some $r \leq 4$.

**Proof.** In each case described by Theorem 3.10, we write $M$ as a direct sum of indecomposables and consider the possibilities. If $R$ is a domain, we are done, and if $R$ has three minimal primes, it is easy to verify, using Theorem 3.10(4), that $M$ has a direct summand of constant rank 1 or 2. In the case covered by Theorem 3.10(2) we get a direct summand of rank at most 3, and in case (3)
it is not hard to see that we always get a summand of constant rank at most 4.
(What luck that there is no indecomposable of rank (2,1) in case (3)! For
otherwise we would have a module of constant rank 6 with no summand of
smaller constant rank.)

We conclude this section with an example showing that we do in fact get
indecomposables of rank 4.

Example 3.12. If $K = k[u]$ is separable, but not Galois and $[K : k] = 3$, then
there exist indecomposable $(k \to K)$-modules of ranks 2, 3 and 4.

Proof. Referring to the notation and diagram (7), we show $L \to L \times L$ (of
rank 4) is indecomposable. Recall that this arose from $L \to L \times 0 \times L$ and
$L \to 0 \times L \times L$ in Proposition 3.6. We will use the first of these, but drop
the middle (0) coordinate. Thus the action of $K$ on $L \times L$ is the following:
If $s \in K$, $(x,y) \in L \times L$, then $s(x,y) = (s^tx, sy)$. Suppose $\phi$ is a non-zero
idempotent $(k \to K)$-endomorphism of $L \to L \times L$. Then, for all $s \in K,$
$x,y \in L$, we have $s\phi(x,y) = \phi(s(x,y)) = \phi(s^tx, sy)$. Put $\phi = (\theta, \nu)$,
where $\theta : L \to L$, $\theta(s^tx) = s^t\theta(x)$, and $\nu \in \text{Hom}_K(L,L)$. Now $\phi$
stabilizes the diagonal, that is, $\theta(x) = \nu(x)$ for all $x \in L$. Then, for all $s \in K$ and
$x \in L$, we have $\nu(s^tx) = s^t\nu(x)$. Since $\nu \in \text{Hom}_K(L,L)$, it follows that
$\nu \in \text{Hom}_L(L,L) = L$. But $\nu$ is idempotent, so $\theta = \nu = \text{id}_L$.

Next we show that $K \to L \times K$ (of rank 3) is indecomposable as a $(k \to K)$-
module. Let $\phi = (\theta, \nu)$ be as before, except this time $\nu \in \text{Hom}_K(K,K) = K$.
Then $\nu = \text{id}_K$, and since $\phi$ stabilizes the diagonal of $K$, $\theta$ has to fix $K$.
Also, for every $x \in K$ we have $\theta(x^t) = x^t\theta(1) = x^t$, and again it follows that
$\theta = \text{id}_L$.

Finally, we show that $K \to L \times 0$ is an indecomposable $(k \to K)$-module (of
rank 2). Let $\phi$ be a non-zero idempotent endomorphism of $L$, stabilizing $K$
and commuting with the action of $K$ (via the map $\beta$). Then $\phi \in \text{Hom}_E(L,L)$,
where $E$ is the image of $K$ under the automorphism $\tau$. Since $K = k(u)$,
$\nu = u^t \in E - K$. Then $L = K \oplus Kv$, and both $K$ and $Kv$ are stable under $\phi$.
Moreover, multiplication by $\nu$ is a $k$-isomorphism from $\phi(K)$ onto $\phi(Kv)$,
so $\phi(L)$ has even dimension as a vector space over $k$. Since $\phi(L)$ is a vector
space over $E$ and $[E : k] = 3$, it follows that $\phi(L) = L$. □

4. The gluing process

In this section we give a fairly general construction of a semilocal ring
with prescribed spectrum and localizations. The result is easiest to state using
the language of schemes. (But see Theorem 4.6 for an algebraic statement
of the main assertion of the theorem.) Given a finite partially ordered set
One-dimensional rings of finite Cohen-Macaulay type

Let $k$ be a commutative ring, and let $W_1, \ldots, W_m$ be reduced, one-dimensional local $k$-algebras. Let $X_i = \text{spec}(W_i)$, and let $Y$ be the disjoint union of the $X_i$. Let $\sim$ be an equivalence relation on the set of minimal elements of $Y$ satisfying the following condition: If $y \in X_i$, $z \in X_j$ and $y \sim z$, then $i \neq j$, and the fields $(W_i)_y$ and $(W_j)_z$ are $k$-isomorphic. Regard each $X_i$ as an open subset of $X = Y/\sim$. Then there is a scheme $(X, \mathcal{O}_X)$ with underlying topological space $X$, such that $(X_i, \mathcal{O}_{X_i})$ is $k$-isomorphic to $\text{Spec}(W_i)$ for $i = 1, \ldots, m$.

If, moreover, $k$ is an infinite field and all the fields $(W_i)_y$ are of the form $K(t)$, where $t$ is an indeterminate and $K/k$ is an algebraic extension (possibly depending on $y$), then the scheme $(X, \mathcal{O}_X)$ can be chosen to be affine.

The first assertion is clear: Just glue the schemes together by choosing isomorphisms on the overlaps. (Since we can glue on new open sets one at a time there is no requirement of compatibility of the isomorphisms.) Now assume the additional hypotheses are satisfied. We want to show that it is possible to glue the schemes $W_i$ in such a way that the resulting scheme is affine.

Before embarking on the proof, we will indicate what sort of things can go wrong when we glue improperly. Suppose for simplicity that $A$ and $B$ are local domains with the same quotient field $F$, and that we wish to build a domain $C$ with exactly two maximal ideals, and with (isomorphic copies of) $A$ and $B$ as localizations. The obvious thing to try is to let $C = A \cap B$. In fancy terms, we are gluing the schemes $\text{Spec}(A)$ and $\text{Spec}(B)$ over $\text{Spec}(F)$, using the identity map on $\text{Spec}(F)$. This obviously will not work if, for example, $A = B$. Thus we can try to replace $A$ by an isomorphic copy and then intersect with $B$. To be more specific, let $F = k(t)$, and let $A - B = k[t](t)$. If we replace $A$ by its isomorphic copy $A_1 = k[t](t-1)$, all is well: $A_1 \cap B$ does the job. What we have done is to glue the schemes $\text{Spec}(A)$ and $\text{Spec}(B)$ over $\text{Spec}(F)$, using the automorphism of $\text{Spec}(F)$ induced by the map $t \mapsto t - 1$. This is the essence of our construction: We will glue using carefully chosen automorphisms of the various quotient fields. This is always possible if $F = k(t)$ or, more generally, $K(t)$, where $K$ is an algebraic extension of an arbitrary infinite field $k$. The necessity of choosing the automorphisms carefully is shown also by an example due to Eakin [5], in which $A \cap B$ is not even Noetherian. (Eakin's example can easily be modified so that the quotient field is of the form $k(x, y)$.)

For certain quotient fields gluing is impossible. For example, it is well known [12] that there is no domain $C$ with two maximal ideals and with both localizations being complete discrete valuation rings. For a geometric example,
let $\Gamma$ be a smooth projective curve over $k$ with trivial automorphism group, and let $F$ be the function field of $\Gamma$. There cannot be a ring $C$ with quotient field $F$ and with two distinct maximal ideals $m, n$ such that $C_m$ and $C_n$ are $k$-isomorphic; for any isomorphism between the local rings would extend to a non-trivial automorphism of $F$. (This example was shown to us by Bill Heinzer.)

In order to clarify the gluing process, we will isolate the technical part of the argument. Let $A$ be any commutative ring. We denote the Jacobson radical and group of units of $A$ by $J(A)$ and $A^*$, respectively. We say $A$ is purely one-dimensional if every maximal ideal has height one. (Every connected ring of dimension one is purely one-dimensional.)

Let $A$ and $B$ be one-dimensional reduced semilocal $k$-algebras with the same total quotient ring $L$. Then $L = L_1 \times \cdots \times L_m$, where each $L_i$ is a field. Let $\tilde{A}$ and $\tilde{B}$ be the integral closures of $A$ and $B$, respectively, in $L$. Then $\tilde{A} = D_1 \times \cdots \times D_m$, where each $D_i$ is a semilocal principal ideal domain with quotient field $L_i$, and similarly $\tilde{B} = E_1 \times \cdots \times E_m$. We say that $A$ and $B$ are in general position provided, for $i = 1, \ldots, m$, no discrete valuation ring of $L_i$ contains both $D_i$ and $E_i$. The next theorem, adapted from a result due to Heinzer and Ohm [10, (2.9) and (2.10)], says that if $A$ and $B$ are in general position then the scheme obtained by gluing Spec$(A)$ and Spec$(B)$ over Spec$(L)$ is affine. The following lemma (implicit in the proof of [10, (2.9)]) will be used in the proof of the theorem.

**Lemma 4.2.** Let $U \subseteq V$ be an integral extension of one-dimensional, reduced Noetherian rings with the same total quotient ring, and let $f \in J(V)$. Then $f^n \in J(U)$ for $n \gg 0$.

**Proof.** First observe that for any integral extension $R \subseteq S$, $J(R) = J(S) \cap R$. Let $W = U[f]$, a module-finite extension of $U$. The conductor $c$ of $U$ in $W$ contains a non-zero-divisor and hence is not contained in the union of the minimal prime ideals of $W$. Since $f \in J(W)$, it follows that $f^n \in c$ for $n \gg 0$. Then $f^n \in J(W) \cap U = J(U)$. □

**Theorem 4.3.** Let $A$ and $B$ be purely one-dimensional reduced semilocal rings with the same total quotient ring $L = L_1 \times \cdots \times L_m$, and assume $A$ and $B$ are in general position. Then there exist elements $g \in A^* \cap J(B)$ and $h \in J(A) \cap B^*$.

**Proof.** Choose, for each $i = 1, \ldots, m$, a non-zero element $f_i \in L_i$ having positive value for each valuation overring of $D_i$ and negative value for each valuation overring of $E_i$. (Notation is as in the paragraph above Lemma 4.2.) Then $f = (f_1, \ldots, f_m) \in J(\tilde{A})$, and by Lemma 4.2 we have $f^n \in J(\tilde{A})$ for $n$ sufficiently large. For each $i$, $1 + f_i^n$ has negative value for each valuation overring of $E_i$. Therefore $(1 + f^n)^{-1} \in J(\tilde{B})$; and, enlarging $n$ if necessary,
we have \( g = (1 + f^n)^{-n} \in A^* \cap J(B) \). Similarly, there is an element \( h \in J(A) \cap B^* \).

In order to prove Theorem 4.1 we need to extend Theorem 4.3 so that it allows the semilocal rings \( A \) and \( B \) to be glued over arbitrary subsets of their minimal prime spectra.

**Theorem 4.4.** Let \( V \) and \( W \) be purely one-dimensional reduced semilocal \( k \)-algebras, and let \( a \in J(V) \), \( b \in J(W) \). Let \( K = V[a^{-1}] \), \( L = W[b^{-1}] \); and suppose there is a \( k \)-isomorphism \( \phi : K \to L \) such that the images \( A \) and \( B \) of \( V \) and \( W \), respectively, in \( L \) are in general position. Let \( U \) be defined by the following Cartesian square (pullback):

\[
\begin{array}{ccc}
U & \rightarrow & W \\
\downarrow & & \downarrow \\
V & \rightarrow & K & \overset{\phi}{\rightarrow} & L
\end{array}
\]

Then there exist \( p, q \in U \) such that \( U[p^{-1}] \cong V \), \( U[q^{-1}] \cong W \) (\( k \)-algebra isomorphisms), \( Up + Uq = U \), and \( pq \in J(U) \). In particular, \( U \) is a reduced, semilocal (Noetherian!) ring of dimension one.

**Proof.** Let \( C = A \cap B \), and let \( g \) and \( h \) be the elements of \( C \) provided by Theorem 4.3. Since \( V \) is semilocal we can lift \( g \) to a unit \( v \in V^* \). We can also lift \( g \) to an element \( w \in W \), but here we have to be careful. For one thing, we want to be sure that \( w \in J(W) \). Let \( I \) be the kernel of the map \( W \to L \), and let \( w_1 \) be any element of \( W \) mapping to \( g \) in \( L \). Since \( g \in J(B) \), \( w_1 \) belongs to every maximal ideal of \( W \) that contains \( I \). Let \( M_1, \ldots, M_t \) be the maximal ideals of \( W \) that do not contain \( I \). Choose \( c \in W \) such that \( c \equiv 1 \pmod{I} \) and \( c \equiv 0 \pmod{M_j} \) for \( j = 1, \ldots, t \). Then \( w_2 = cw_1 \in J(W) \) and \( w_2 \) maps to \( g \).

We also want \( W[w^{-1}] = W[b^{-1}] \), and for this we need to know that \( w \) belongs to every minimal prime containing \( g \), equivalently, not containing \( I \). Let \( I' \) be the intersection of these minimal primes. Noting that \( \sqrt{(I \cap J(W)) + (I' \cap J(W))} = J(W) \), we have, for some \( n \geq 1 \), \( w_2^n = x + w \), where \( x \in I \) and \( w \in I' \cap J(W) \). Replacing \( g \) by \( g^n \), we now have a preimage \( w \in J(W) \) such that \( W[w^{-1}] = W[b^{-1}] \).

The elements \( v \) and \( w \) yield an element \( p = (v, w) \in U \). Similarly, there is an element \( q = (y, z) \in U \), such that \( y \in J(V) \), \( V[y^{-1}] = V[a^{-1}] \) and \( z \in W^* \). We see, by tensoring (*) with \( U[p^{-1}] \) and \( U[q^{-1}] \), that \( U[p^{-1}] \cong V \) and \( U[q^{-1}] \cong W \).

Let \( P \) be any maximal ideal of \( U \). Since every element of \( P \) maps to a non-unit in either \( V \) or \( W \), and since \( A \) and \( B \) are semilocal, it follows that \( P \) is the contraction of a maximal ideal of either \( A \) or \( B \). The assertions that \( Cp + Cq = C \) and \( pq \in J(C) \) are now clear. \( \square \)
The final step in the proof of Theorem 4.1 is to show that under the hypotheses of Theorem 4.1 we can choose the isomorphism $\phi$ in Theorem 4.4 in such a way that the rings $A$ and $B$ are in general position. The following lemma is exactly what is needed.

**Lemma 4.5.** Let $k$ be an infinite field, and let $A$ and $B$ be purely one-dimensional reduced $k$-algebras with the same total quotient ring $L = L_1 \times \cdots \times L_m$. Assume $L_i = K_i(t)$, where $K_i$ is an algebraic extension of $k$, and $t$ is an indeterminate. Then there is a $k$-automorphism $\psi$ of $L$ such that $\psi(A)$ and $B$ are in general position.

**Proof.** Refer to the notation preceding Theorem 4.2. Since we need only work with a single $L_i$ at a time, we can reduce to the following situation: $L = K(t)$, where $K$ is an algebraic extension of $k$; and $D$ and $E$ are semilocal principal ideal domains with quotient field $L$. Our goal is to find a $k$-automorphism $\psi$ of $L$ such that $\psi(D)$ and $E$ share no valuation overrings. Let $V$ be the set of valuation overrings of $D$. Note first that for every $V \in V$, there is at most one constant $\alpha \in k$ such that $t + \alpha$ is in the maximal ideal of $V$. Choose $\alpha$ such that $t + \alpha$ has non-positive value for each $V \in V$, and make the change of variable $t \mapsto (t + \alpha)^{-1}$. Thus, we may assume that $K[t] \subseteq V$ for every $V \in V$. Since $K$ is algebraic over $k$, each $V$ actually contains $K[t]$.

For each $\beta \in k$, let $\psi_{\beta}$ be the $K$-automorphism of $L$ taking $t$ to $t + \beta$. It will suffice to prove that $\psi_{\beta}(V) \neq \psi_{\gamma}(V)$ if $V \in V$ and $\beta \neq \gamma$. Since $\psi_{\beta-\gamma} = \psi_{\beta}^{-1}\psi_{\gamma}$ it is enough to show that $\psi_{\beta}(V) = V$ for at most finitely many $\beta \in k$. Since $V \supseteq K[t]$, we know $V = K[t]_{(p)}$, where $p$ is a monic irreducible polynomial in $K[t]$. Let $r$ be a root of $p$ (in the algebraic closure of $K$). Notice that $\psi_{\beta}(V) = K[t]_{(p(t + \beta))}$. If, now, $\psi_{\beta}(V) = V$, it follows that $p(t + \beta) = cp(t)$ for some non-zero constant $c \in K$. But then $r + \beta$ is also a root of $p$, and hence there are only finitely many such $\beta$. □

**Theorem 4.6.** Let $k$ be an infinite field, and $t$ an indeterminate over $k$. Let $(R_1, M_1), \ldots, (R_n, M_n)$ be reduced local $k$-algebras of dimension one, such that for each $i$ and each minimal prime $p$ of $(R_i)$, $(R_i)_p$ is $k$-isomorphic to $K(t)$, where $K/k$ is an algebraic extension, possibly depending on $i$ and $p$. Let $X$ be a finite one-dimensional partially ordered set with exactly $n$ maximal elements $x_1, \ldots, x_n$, all of them non-minimal. Assume

1. For each $i$, there is an order-embedding $\phi_i : \text{Spec } R_i \to X$ taking $M_i$ to $x_i$; that is, the number of elements of $X$ that are $\leq x_i$ is equal to the cardinality of $\text{spec}(R_i)$.
2. If $P \in \text{spec } R_i$ and $Q \in \text{Spec } R_j$ are such that $\phi_i(P) = \phi_j(Q)$, then $(R_i)_P \cong (R_j)_Q$ as $k$-algebras.

Then there is a semilocal $k$-algebra $R$ with maximal ideals $\{N_i \mid i = 1, \ldots, n\}$ such that $\text{spec } R$ is order-isomorphic to $X$ (with $N_i$ mapping to $x_i$) and $R_{N_i} \cong R_i$. 

for all $i$. Moreover, if each $R_i$ is a domain essentially of finite type over $k$, the same holds for $R$.

**Proof.** We may assume $n \geq 2$ and proceed by induction. Let $Y = \{ y \in X \mid y \leq x_i \text{ for some } i \leq n - 1 \}$. Choose a semilocal $k$-algebra $V$ with maximal ideals $N_1, \ldots, N_{n-1}$ such that $V N_i \cong R_i$ for $i = 1, \ldots, n-1$, and such that there is an order-isomorphism $\theta : \text{spec}(V) \to Y$ taking $N_i$ to $x_i$. Let $Z = \{ y \in Y \mid y < x_n \}$, and let $S = \theta^{-1}(Z) \subset \text{spec}(V)$. The primes in $S$ are minimal, so there is an element $a \in J(V)$ such that $a \notin \bigcup S$. Put $W = R_n$, let $\Gamma = \phi^{-1}_n(Z)$, and choose $b \in J(W)$ such that $b \notin \bigcup \Gamma$. Each of the rings $K = V[a^{-1}]$ and $L = W[b^{-1}]$ is a direct product of $|Z|$ fields, each of the form $k(t)$, and the compatibility condition (2) implies that there is a $k$-isomorphism $\xi : K \to L$.

Let $A$ and $B$ be the images of $V$ and $W$ under the maps $V \to K$ and $W \to L$, respectively. Choose, using Theorem 4.5, a $k$-automorphism $\psi$ of $L$ such that $\psi(\xi(A))$ and $B$ are in general position. Let $\phi = \psi \xi : K \to L$, and take $R$ to be the ring $U$ provided by Theorem 4.4.

To prove the last assertion, let $G_i$ be a finite subset of $R_{N_i}$ such that $R_{N_i}$ is a localization of $k[G_i]$. Write each $g \in G_i$ as a fraction $a/b$, with $a \in R$ and $b \in R - N_i$, and let $H$ be the set of all $as$ and $bs$ so obtained as $i$ ranges from 1 to $n$. Each $R_{N_i}$ is a localization of $k[H]$, necessarily at a prime ideal $P_i$ of $k[H]$. Then $R = k[H]_{P_1} \cap \cdots \cap k[H]_{P_n} = S^{-1}k[H]$, where $S = k[H] - (P_1 \cup \cdots \cup P_n)$. $\Box$

In the next section we will use this theorem in the following special situation: $n = 2$, and each $R_i$ is a local domain. In order to obtain indecomposables of various ranks, we will glue modules together, using the following theorem:

**Theorem 4.7** (R. Wiegand [13, (1.11)]). Let $R$ be a semilocal ring-order with maximal ideals $M_1, \ldots, M_n$, and let $A_i$ be an $R_{M_i}$-module for each $i$. Assume that for each $i, j$ and each minimal prime ideal $P \subset M_i \cap M_j$, $(A_i)_P \cong (A_j)_P$ (that is, they have the same dimension as vector spaces over $R_P$). Then there exists an $R$-module $A$, unique up to isomorphism, such that $A_{M_i} \cong A_i$ for every $i$. $\Box$

Finally, we will need a local-global theorem for splitting off direct summands. The following result is essentially the same as [13, (1.3)], but since the statement there is a little different, we will reproduce the proof here.

**Theorem 4.8.** Let $R$ be a ring-order, let $A$ be a maximal Cohen–Macaulay $R$-module, and let $n$ be a positive integer. Suppose, for every maximal ideal $M$ of $R$, $A_{M}$ has a direct summand of constant rank $n$. Then $A$ has a direct summand of constant rank $n$. 
Proof. We may assume \( R \) is connected, and that \( R \neq \tilde{R} \), since over a Dedekind domain every torsion-free module is isomorphic to a direct sum of ideals. Let \( M_1, \ldots, M_n \) be the singular maximal ideals, and let \( S = R - M_1 \cup \cdots \cup M_n \). We have, for each \( i \), \( A_{M_i} \cong F_i \oplus G_i \), with \( F_i \) of constant rank \( n \). Using Theorem 4.7, we easily obtain maximal Cohen–Macaulay \( R \)-modules \( F \) and \( G \) such that \( S^{-1}M \cong S^{-1}F \oplus S^{-1}G \). (The uniqueness assertion in Theorem 4.7 is needed here.) Since \( S^{-1}\text{Hom}_R(A, F) = \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}F) \) there is a homomorphism \( f : A \rightarrow F \) inducing a split surjection at each singular maximal ideal. Let \( H \) be the image of \( f \). The map \( A \rightarrow H \) is a split surjection everywhere, since \( H_M \) is free for every non-singular maximal ideal \( M \). The only question is whether or not \( H_M \) has the right rank. Since \( R \) is connected and \( R \neq \tilde{R} \), there exists a singular maximal ideal \( M_i \) and a minimal prime ideal \( P \) such that \( M > P \subset M_i \). Then \( \text{rank}(H_M) = \dim_{R_P}(H_P) = \text{rank}(F_i) = n \). \( \square \)

5. Global examples and bounds

Theorem 5.1 (Main theorem). (1) Suppose that \( R \) is a ring-order of bounded Cohen–Macaulay type satisfying (2-sep). Then every maximal Cohen–Macaulay module of constant rank has a direct summand of constant rank 1, 2, 3, 4, 5, 6, 8, 9, or 12.

(2) Let \( k \) be a field admitting a separable, non-Galois extension \( K \) of degree 3. There exists an integral domain \( R \), essentially of finite type over \( k \) such that

(a) \( R \) is a ring-order of finite Cohen–Macaulay type satisfying (2-sep),
(b) \( R \) has exactly two maximal ideals, each with residue field \( k \), and
(c) \( R \) has indecomposable maximal Cohen–Macaulay \( R \)-modules of ranks 1, 2, 3, 4, 5, 6, 8, 9, and 12.

Proof. For part (2), use Example 3.12 to see that \( k \rightarrow K \) has indecomposables of ranks 1, 2, 3, and 4. Let \( T \) be the pullback

\[
\begin{array}{ccc}
T & \rightarrow & K[X]_{(x)} \\
\downarrow & & \downarrow \\
k & \rightarrow & K
\end{array}
\]

Then \( T \) has finite Cohen–Macaulay type and has indecomposable torsion-free modules of ranks 1, 2, 3, and 4, by Proposition 2.2. By the glueing theorem (Theorem 4.6) there is a domain \( R \), essentially of finite type over \( k \), with exactly two maximal ideals \( M \) and \( N \), and such that \( R_M \cong R_N \cong T \). Moreover, \( R \) has finite Cohen–Macaulay type by Theorem 1.1. We use Theorem 4.7 to construct indecomposable torsion-free modules of ranks 1, 2, 3, 4, 5, 6, 8, 9, and 12. For example, to see that \( R \) has a module of rank 6, let \( A_3 \) be an indecomposable
torsion-free module over \( R_M \) of rank 3, and \( A_2 \) an indecomposable over \( R_N \) of rank 2. By Theorem 4.7, there is an \( R \)-module \( M \) such that \( M_M \cong (A_3)^2 \) and \( M_N \cong (A_2)^3 \), since \( ((A_3)^2)_0 \cong ((A_2)^3)_0 \). Now \( M \) is indecomposable, because \( M_{am} \) is indecomposable (by the Krull–Schmidt Theorem). Similarly, \( s = 4 + 1 = 3 + 2 \), \( 8 = 4 + 4 = 3 + 3 + 2 \), \( 9 = 4 + 4 + 1 = 3 + 3 + 1 \), and \( 12 = 4 + 4 + 4 = 3 + 3 + 3 + 3 \).

We remark that a different sort of example could be obtained by taking the second local ring to be \( K[Y^3, Y^5]_{(Y^3, Y^5)} \).

Part (1) is an immediate consequence of the following lemma:

**Lemma 5.2.** Suppose \( R \) is a ring-order of bounded Cohen–Macaulay type satisfying (2-sep), and let \( M \) be a maximal Cohen–Macaulay \( R \)-module of constant rank \( r \).

1. If \( r = 7, 11 \) or 15 then \( M \) has a direct summand of constant rank 3.
2. If \( r = 13 \) or 16 then \( M \) has a direct summand of constant rank 4.
3. If \( r = 10 \) or 14 then \( M \) has a direct summand of constant rank 6.
4. If \( r = 17 \) then \( M \) has a direct summand of constant rank 8.
5. If \( r \geq 18 \) the \( M \) has a direct summand of constant rank 12.

**Proof.** We may assume \( R \) is local, by Theorem 4.8. Now Corollary 3.11 implies that \( M \) is a direct sum of modules of constant rank 1, 2, 3 or 4. The proof is completed by looking at each case and analyzing how \( r \) can be expressed as a sum of positive integers less than or equal to 4. The details are rather uninspiring and are left to the reader. □

The results mentioned in the Abstract are easy consequences of the theorem. For, suppose \( k \) either is perfect or has characteristic different from 2. Let the domain \( R \) (not a field) be the localization of a finitely generated \( k \)-algebra at some multiplicative subset, and suppose for every maximal ideal \( M \) there is a bound on the ranks of the indecomposable torsionfree \( R_M \)-modules. By [1] \( R \) is one-dimensional, so \( R \) is a ring-order satisfying (sep) and (dr). The indecomposable torsionfree \( R \)-modules therefore have the advertised ranks, by (1) of the theorem. For the example, we use (2) of the theorem, with \( k = \mathbb{Q} \) and \( K = \mathbb{Q}(\sqrt{2}) \).

**References**