Journal of Pure and Applied Algebra 215 (2011) 1958–1966

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Reynolds operator on functors

Amelia Álvarez [a,*], Carlos Sancho [b], Pedro Sancho [a]

[a] Departamento de Matemáticas, Universidad de Extremadura, Avenida de Elvas s/n, 06006 Badajoz, Spain

[b] Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain

ARTICLE INFO

Article history:
Received 9 February 2009
Received in revised form 22 October 2010
Available online 13 December 2010
Communicated by M. Broue

MSC:
Primary: 14L24
Secondary: 14L17

ABSTRACT

Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme. We prove that the category of dual functors (over the category of commutative R-algebras) of G-modules is equivalent to the category of dual functors of $\mathcal{A}^*$-modules. We prove that G is invariant exact if and only if $A^* = R \times B^*$ as R-algebras and the first projection $A^* \to R$ is the unit of A. If $\mathbb{M}$ is a dual functor of G-modules and $w_G := (1, 0) \in R \times B^* = A^*$, we prove that $\mathbb{M}^G = w_G \cdot \mathbb{M}$ and $\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}$; hence, the Reynolds operator can be defined on $\mathbb{M}$.

0. Introduction

Let R be a commutative ring with unit. An R-module M can be considered as a functor of $\mathcal{R}$-modules over the category of commutative R-algebras, which we will denote by $\mathcal{M}$ and we will call a quasi-coherent module, by defining $\mathcal{M}(S) := M \otimes_R S$. If $\mathbb{M}$ and $\mathbb{N}$ are functors of $\mathcal{R}$-modules, we will denote by $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ the functor of $\mathcal{R}$-modules

$$\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N})(S) := \mathrm{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{N}_{|S}),$$

where $\mathbb{M}_{|S}$ is the functor $\mathbb{M}$ restricted to the category of commutative S-algebras. The functor $\mathbb{M}^* := \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ is said to be a dual functor. For example, $\mathcal{M}$, $\mathcal{M}^*$ and $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$ are dual functors (see [1, 1.10] and Proposition 1.1).

An affine R-monoid scheme $G = \operatorname{Spec} A$ can be considered as a functor of monoids over the category of commutative R-algebras: $G^{\cdot}(S) := \mathrm{Hom}_{R\text{-}alg}(A, S)$, which is known as the functor of points of G.

It is well known that the theory of linear representations of algebraic groups can be developed, via their associated functors, as a theory of (abstract) groups and their linear representations. That is, the category of modules is equivalent to the category of quasi-coherent modules and the category of rational G-modules is equivalent to the category of quasi-coherent $G^{\cdot}$-modules. Moreover, it is sometimes necessary to consider some natural vector spaces via their associated functors. For example, if M and M' are two (rational) linear representations of $G = \operatorname{Spec} A$, then $\mathrm{Hom}_R(M, M')$ is not a (rational) linear representation of G, although $G^{\cdot}$ acts naturally on $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}')$; A^* is not a (rational) linear representation of $G = \operatorname{Spec} A$, although $G^{\cdot}$ acts naturally on $\mathcal{A}^*$.

$\mathcal{A}^*$ is a functor of algebras and there exists a natural and obvious morphism of functors of monoids $G^{\cdot} \hookrightarrow \mathcal{A}^*$. Then every functor of $\mathcal{A}^*$-modules is a functor of $G^{\cdot}$-modules (see Definition 1.3). In this paper, we prove the following theorem.

Theorem 1. *The category of dual functors of $G^{\cdot}$-modules is equal to the category of dual functors of $\mathcal{A}^*$-modules.*

In the classical situation observe that if we consider the rational points of G, $G^{\cdot}(R)$, and the inclusion $G^{\cdot}(R) \hookrightarrow A^*$, the category of (rational) $G^{\cdot}(R)$-modules it is not equivalent to the category of A^*-modules, even if G is a smooth algebraic group over an algebraically closed field (in this last case it is necessary to introduce topologies on A^* and on the modules).

* Corresponding author.
E-mail addresses: aalarma@unex.es (A. Álvarez), mplu@usal.es (C. Sancho), sancho@unex.es (P. Sancho).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.11.006

We say that $G = \operatorname{Spec} A$ is invariant exact if "taking invariants" is an exact functor (see Definition 2.2 and Theorem 2.13). If A is a projective R-module and G is invariant exact, then it is linearly reductive (Remark 3.4).

We prove that an affine R-group scheme $G = \operatorname{Spec} A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of $\mathcal{R}$-algebras (Corollary 2.6). If A is a projective R-module, $G = \operatorname{Spec} A$ is invariant exact if and only if $A^* = R \times C$ as R-algebras (Remark 2.7). If G is invariant exact, there exists an isomorphism $A^* = R \times B^*$ such that the first projection $A^* \to R$ is the unit of A. Let $w_G := (1, 0) \in R \times B^* = A^*$ be the "*invariant integral*" on G. We prove the following theorem.

Theorem 2. *Let $G = \operatorname{Spec} A$ be an invariant exact R-group and let $w_G \in A^*$ be the invariant integral on G. Let $\mathbb{M}$ be a dual functor of $G^{\cdot}$-modules. It holds that:*

(1) $\mathbb{M}^{G^{\cdot}} = w_G \cdot \mathbb{M}$.
(2) $\mathbb{M}$ *splits uniquely as a direct sum of $\mathbb{M}^{G^{\cdot}}$ and another subfunctor of $G^{\cdot}$-modules, explicitly*

$$\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

We call the projection $\mathbb{M} \to \mathbb{M}^{G^{\cdot}} = w_G \cdot \mathbb{M}$ the Reynolds operator. Taking sections we have $\mathbb{M}(R)^{G^{\cdot}} = w_G \cdot \mathbb{M}(R)$ and the morphism $\mathbb{M}(R) \to \mathbb{M}(R)^{G^{\cdot}}$, $m \mapsto w_G \cdot m$. The classical Reynolds operator is a particular case of Theorem 2, for $\mathbb{M} = \mathcal{M}$. The previous theorem still holds for any separated functor of $\mathcal{A}^*$-modules (see Definition 3.1). More generally, for every functor $\mathbb{N}$ of G-modules, we prove that there exists the maximal separated G-invariant quotient of $\mathbb{N}$ and that the dual of this quotient is $\mathbb{N}^{*G^{\cdot}}$ (Theorem 4.1).

In [4] it is proved that a Reynolds operator can be defined on $\operatorname{Hom}_B(M, M')$, where B is a G-algebra and M and M' are two BG-modules. Obviously $G^{\cdot}$ acts on $\mathbb{H}om_{\mathcal{B}}(\mathcal{M}, \mathcal{M}')$ and it is a separated functor by Proposition 3.8. Hence, the Reynolds operator can be defined on $\operatorname{Hom}_B(M, M')$ by Theorem 2. This is an example that shows that functorial treatment can clarify some problems.

Let $\chi : G \to G_m$ be a multiplicative character. Given a functor of G-modules $\mathbb{M}$, let $\mathbb{M}^{\chi}$ be the subfunctor of the χ-semi-invariant elements of $\mathbb{M}$ (see Definition 5.1). In Section 5, we extend the previous theorems about the invariant integral and the Reynolds operator to the semi-invariant integral and the Reynolds χ-operator. We apply these results to prove some results of [5] about generalized Cayley's Ω-processes in Example 5.6.

Finally, these results about the invariant integral, the Reynolds operator, etc., can be extended to functors of monoids with a reflexive functor of functions. We will explain it in detail in a next paper.

1. Preliminary results

[1] is the basic reference for reading this paper.

Let R be a commutative ring (associative with unit). All functors considered in this paper are covariant functors over the category of commutative R-algebras (associative with unit). We will say that $\mathbb{X}$ is a functor of sets (resp. monoids, etc.) if $\mathbb{X}$ is a functor from the category of commutative R-algebras to the category of sets (resp. monoids and so forth).

Let $\mathcal{R}$ be the functor of rings defined by $\mathcal{R}(S) := S$ for every commutative R-algebra S. A functor of sets $\mathbb{M}$ is said to be a functor of $\mathcal{R}$-modules if we have morphisms of functors of sets $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ and $\mathcal{R} \times \mathbb{M} \to \mathbb{M}$, so that $\mathbb{M}(S)$ is an S-module for every commutative R-algebra S. We will say that a functor of $\mathcal{R}$-modules $\mathbb{A}$ is a functor of $\mathcal{R}$-algebras if $\mathbb{A}(S)$ is a S-algebra with unit and S commutes with all the elements of $\mathbb{A}(S)$. If $\mathbb{M}$ and $\mathbb{N}$ are functors of $\mathcal{R}$-modules, we will denote by $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ the functor of $\mathcal{R}$-modules

$$\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N})(S) := \operatorname{Hom}_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{N}_{|S}),$$

where $\mathbb{M}_{|S}$ is the functor $\mathbb{M}$ restricted to the category of commutative S-algebras.

The functor $\mathbb{M}^* := \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathcal{R})$ is said to be a dual functor.

Proposition 1.1. *If $\mathbb{M}^*$ is a dual functor of $\mathcal{R}$-modules and $\mathbb{N}$ is a functor of $\mathcal{R}$-modules, then $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*)$ is a dual functor of $\mathcal{R}$-modules.*

Proof. Actually, $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}^*) = \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{N} \otimes \mathbb{M}, \mathcal{R})$. □

Given an R-module M, the functor of $\mathcal{R}$-modules $\mathcal{M}$ defined by $\mathcal{M}(S) := M \otimes_R S$ is called a quasi-coherent $\mathcal{R}$-module. The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of $\mathcal{R}$-modules and the category of quasi-coherent $\mathcal{R}$-modules [1, 1.12]. In particular, $\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{M}') = \operatorname{Hom}_R(M, M')$. The notion of quasi-coherente $\mathcal{R}$-module is stable under base change $R \to S$, that is, $\mathcal{M}_{|S}$ is equal to the quasi-coherent $\mathcal{S}$-module associated with the S-module $M \otimes_R S$.

The functor $\mathcal{M}^* = \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathcal{M}, \mathcal{R})$ is called an $\mathcal{R}$-module scheme. Specifically, $\mathcal{M}^*(S) = \operatorname{Hom}_S(M \otimes_R S, S) = \operatorname{Hom}_R(M, S)$. It is easy to check that given two functors of $\mathcal{R}$-modules $\mathbb{M}$ and $\mathbb{M}'$, then

$$(\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{M}'))_{|S} = \mathbb{H}om_{\mathcal{S}}(\mathbb{M}_{|S}, \mathbb{M}'_{|S}).$$

In particular, $(\mathcal{M}^*)_{|S}$ is an $\mathcal{S}$-module scheme. An $\mathcal{R}$-module scheme $\mathcal{M}^*$ is a quasi-coherent $\mathcal{R}$-module if and only if M is a projective R-module of finite type [2]. A basic result says that quasi-coherent modules and module schemes are reflexive, that is,

$$\mathcal{M}^{**} = \mathcal{M}$$

[1, 1.10]; thus, the functors $\mathcal{M} \rightsquigarrow \mathcal{M}^*$ and $\mathcal{M}^* \rightsquigarrow \mathcal{M}^{**} = \mathcal{M}$ establish an equivalence between the category of quasi-coherent modules and the category of module schemes. $\mathcal{M}$ and $\mathcal{M}^*$ are examples of dual functors.

Let $X = \operatorname{Spec} A$ be an affine R-scheme and let $X^\cdot$ be the functor of points of X, that is, the functor of sets

$$X^\cdot(S) = \operatorname{Hom}_{R\text{-sch}}(\operatorname{Spec} S, X) = \operatorname{Hom}_{R\text{-}alg}(A, S),$$

"points of X with values in S". Given another affine scheme $Y = \operatorname{Spec} B$, by Yoneda's lemma

$$\operatorname{Hom}_{R-sch}(X, Y) = \operatorname{Hom}(X^\cdot, Y^\cdot),$$

and $X^\cdot \simeq Y^\cdot$ if and only if $X \simeq Y$. We will sometimes denote $X^\cdot = X$.

Let $R \to S$ be a morphism of rings and let $X_S = \operatorname{Spec}(A \otimes_R S)$, then $(X^\cdot)_{|S} = (X_S)^\cdot$. Observe that $\operatorname{Hom}(X^\cdot, \mathcal{R}) = \operatorname{Hom}(X^\cdot, (\operatorname{Spec} R[X])^\cdot) = \operatorname{Hom}_{R\text{-}alg}(R[X], A) = A$, then $\mathbb{H}om(X^\cdot, \mathcal{R}) = \mathcal{A}$. There is a natural morphism $X^\cdot \to \mathcal{A}^*$, because $X^\cdot(S) = \operatorname{Hom}_{R\text{-}alg}(A, S) \subset \operatorname{Hom}_R(A, S) = \mathcal{A}^*(S)$.

Proposition 1.2. *Let $\mathbb{M}^*$ be a dual functor of $\mathcal{R}$-modules, any morphism of functors $X^\cdot \to \mathbb{M}^*$ factorizes via a unique morphism of functors of $\mathcal{R}$-modules $\mathcal{A}^* \to \mathbb{M}^*$.*

Proof. It is a consequence of the equalities

$$\begin{aligned}\operatorname{Hom}(X^\cdot, \mathbb{M}^*) &= \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathbb{H}om(X^\cdot, \mathcal{R})) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M}, \mathcal{A}) = \operatorname{Hom}_{\mathcal{R}}(\mathbb{M} \otimes_{\mathcal{R}} \mathcal{A}^*, \mathcal{R}) \\ &= \operatorname{Hom}_{\mathcal{R}}(\mathcal{A}^*, \mathbb{M}^*). \quad \square\end{aligned}$$

Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme, that is, $G^\cdot$ is a functor of monoids. $\mathcal{A}^*$ is an $\mathcal{R}$-algebra scheme, that is, besides from being an $\mathcal{R}$-module scheme it is a functor of $\mathcal{R}$-algebras. The natural morphism $G^\cdot \to \mathcal{A}^*$ is a morphism of functors of monoids. By [1, 5.3], given any dual functor of algebras $\mathbb{B}^*$ (that is, a dual functor of $\mathcal{R}$-modules which is a functor of $\mathcal{R}$-algebras), then any morphism of functors of monoids $G^\cdot \to \mathbb{B}^*$ factorizes via a unique morphism of functors of $\mathcal{R}$-algebras $\mathcal{A}^* \to \mathbb{B}^*$.

Definition 1.3. A functor $\mathbb{M}$ of (left) G-modules is a functor of $\mathcal{R}$-modules endowed with an action of G, i.e., a morphism of functors of monoids $G^\cdot \to \mathbb{E}\mathrm{nd}_{\mathcal{R}}(\mathbb{M})$. A functor $\mathbb{M}$ of (left) $\mathcal{A}^*$-modules is a functor of $\mathcal{R}$-modules endowed with a morphism of functors of $\mathcal{R}$-algebras $\mathcal{A}^* \to \mathbb{E}\mathrm{nd}_{\mathcal{R}}(\mathbb{M})$.

The functors $M \rightsquigarrow \mathcal{M}$, $\mathcal{M} \rightsquigarrow \mathcal{M}(R) = M$ establish an equivalence between the category of rational G-modules and the category of quasi-coherent G-modules. The category of quasi-coherent G-modules is equal to the category of quasi-coherent $\mathcal{A}^*$-modules by [1, 5.5].

Notation 1.4. *For abbreviation, we sometimes use $g \in G$ or $m \in \mathbb{M}$ to denote $g \in G^\cdot(S)$ or $m \in \mathbb{M}(S)$ respectively. Given $m \in \mathbb{M}(S)$ and a morphism of $\mathcal{R}$-algebras $S \to T$, we still denote by m its image by the morphism $\mathbb{M}(S) \to \mathbb{M}(T)$.*

Definition 1.5. Let $\mathbb{M}$ be a functor of G-modules. We define

$$\mathbb{M}(S)^G := \{m \in \mathbb{M}(S), \text{ such that } g \cdot m = m \text{ for every } g \in G\}$$[1]

and we denote by $\mathbb{M}^G$ the subfunctor of $\mathcal{R}$-modules of $\mathbb{M}$ defined by $\mathbb{M}^G(S) := \mathbb{M}(S)^G$. We will say that $m \in \mathbb{M}$ is left G-invariant if $m \in \mathbb{M}^G$.

If $\mathbb{M}$ is a functor of G-modules (resp. of right G-modules), then $\mathbb{M}^*$ is a functor of right G-modules: $w * g := w(g \cdot -)$, for every $w \in \mathbb{M}^*$ and $g \in G$ (resp. of left G-modules: $g * w := w(- \cdot g)$). If $\mathbb{M}_1$ and $\mathbb{M}_2$ are two functors of G-modules and G is an affine R-group scheme, then $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}_1, \mathbb{M}_2)$ is a functor of G-modules, with the natural action $g * f := g \cdot f(g^{-1} \cdot -)$, and it holds that

$$\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}_1, \mathbb{M}_2)^G = \mathbb{H}om_G(\mathbb{M}_1, \mathbb{M}_2).$$

Let $\mathbb{M}$ be a functor of G-modules, then $(\mathbb{M}^G)_{|S} = (\mathbb{M}_{|S})^{G_S}$.

2. Invariant exact monoids

From now on, through out this paper $G = \operatorname{Spec} A$ is an affine R-monoid scheme.

Theorem 2.1. *The category of dual functors of G-modules is equivalent to the category of dual functors of $\mathcal{A}^*$-modules.*

[1] More precisely, $g \cdot m = m$ for every $g \in G(T)$ and every morphism of $\mathcal{R}$-algebras $S \to T$.

Proof. Let $\mathbb{M}$ be a dual functor of $\mathcal{R}$-modules. By Proposition 1.1, $\mathbb{E}\mathrm{nd}_{\mathcal{R}}(\mathbb{M})$ is a dual functor of $\mathcal{R}$-algebras. Hence,

$$\mathrm{Hom}_{mon}(G^{\cdot}, \mathbb{E}\mathrm{nd}_{\mathcal{R}}(\mathbb{M})) = \mathrm{Hom}_{\mathcal{R}-alg}(\mathcal{A}^*, \mathbb{E}\mathrm{nd}_{\mathcal{R}}(\mathbb{M})),$$

and giving a structure of functor of G-modules on $\mathbb{M}$ is equivalent to giving a structure of functor of $\mathcal{A}^*$-modules on $\mathbb{M}$.

Given two dual functors of G-modules $\mathbb{M}$ and $\mathbb{M}'$, it holds $\mathrm{Hom}_G(\mathbb{M}, \mathbb{M}') = \mathrm{Hom}_{\mathcal{A}^*}(\mathbb{M}, \mathbb{M}')$: observe that given a morphism of functors of $\mathcal{R}$-modules $L : \mathbb{M} \to \mathbb{M}'$ and $m \in \mathbb{M}$, the morphism $L_1 : G^{\cdot} \to \mathbb{M}'$, $L_1(g) := L(gm) - gL(m)$ is null if and only if the morphism $L_2 : \mathcal{A}^* \to \mathbb{M}'$, $L_2(a) := L(am) - aL(m)$ is null. □

Definition 2.2. An affine R-monoid scheme $G = \mathrm{Spec}\, A$ is said to be left invariant exact if for any exact sequence (in the category of functors of $\mathcal{R}$-modules) of dual functors of left G-modules

$$0 \to \mathbb{M}_1 \to \mathbb{M}_2 \to \mathbb{M}_3 \to 0$$

the sequence

$$0 \to \mathbb{M}_1^G \to \mathbb{M}_2^G \to \mathbb{M}_3^G \to 0$$

is exact. G is said to be invariant exact if it is left and right invariant exact.

If G is an affine R-group scheme and it is left invariant exact, then it is right invariant exact since every functor of right G-modules $\mathbb{M}$ can be regarded as a functor of left G-modules: $g \cdot m := m \cdot g^{-1}$.

Let $\Theta : G \to \mathcal{R}$, $g \mapsto 1$ be the trivial character, which induces the trivial representation $\Theta : \mathcal{A}^* \to \mathcal{R}$. Observe that $\Theta = 1 \in A$.

Theorem 2.3. *An affine R-monoid scheme $G = \mathrm{Spec}\, A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as $\mathcal{R}$-algebra schemes, where the projection $\mathcal{A}^* \to \mathcal{R}$ is Θ.*

Proof. Let us assume that G is invariant exact. The projection $\Theta : \mathcal{A}^* \to \mathcal{R}$ is a morphism of left and right G-modules (or $\mathcal{A}^*$-modules). Taking left invariants one obtains an epimorphism $\Theta : \mathcal{A}^{*G} \to \mathcal{R}$. Let $w_l \in A^*$ be left G-invariant such that $\Theta(w_l) = 1$. Likewise, taking right invariants let $w_r \in A^*$ be right G-invariant such that $\Theta(w_r) = 1$. Then $w = w_l \cdot w_r \in \mathcal{A}^*$ is left and right G-invariant and $\Theta(w) = 1$. Then, $w' \cdot w = w'(1) \cdot w = w \cdot w'$, because $g \cdot w = w = w \cdot g$. Hence, w is idempotent. Therefore, one finds a decomposition as a product of $\mathcal{R}$-algebra schemes $\mathcal{A}^* = w \cdot \mathcal{A}^* \oplus (1-w) \cdot \mathcal{A}^*$; moreover, the morphism $\mathcal{R} \to \mathcal{A}^*$, $\lambda \mapsto \lambda \cdot w$, is a section of Θ, $w \cdot \mathcal{A}^* = \mathcal{R} \cdot w$ and Θ vanishes on $(1-w) \cdot \mathcal{A}^*$.

Let us assume now that $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and $\pi_1 = \Theta$. Let $w = (1, 0) \in \mathcal{R} \times \mathcal{B}^* = \mathcal{A}^*$ and let us prove that G is invariant exact.

For any dual functor of G-modules $\mathbb{M}$, let us see that $w \cdot \mathbb{M} = \mathbb{M}^G$. One sees that $w \cdot \mathbb{M} \subseteq \mathbb{M}^G$, because $g \cdot (w \cdot m) = (g \cdot w) \cdot m = w \cdot m$, for every $g \in G$ and every $m \in \mathbb{M}$. Conversely, $\mathbb{M}^G \subseteq w \cdot \mathbb{M}$: Let $m \in \mathbb{M}$ be G-invariant. The morphism $G \to \mathbb{M}$, $g \mapsto g \cdot m = m$, extends to a unique morphism $\mathcal{A}^* \to \mathbb{M}$. The uniqueness implies that $w' \cdot m = w'(1) \cdot m$ and then $m = w \cdot m \in w \cdot \mathbb{M}$.

Taking invariants is a left exact functor. If $\mathbb{M}_2 \to \mathbb{M}_3$ is a surjective morphism, then the morphism $\mathbb{M}_2^G \to \mathbb{M}_3^G$ is surjective because so is the morphism $\mathbb{M}_2^G = w \cdot \mathbb{M}_2 \to w \cdot \mathbb{M}_3 = \mathbb{M}_3^G$. □

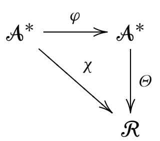
Remark 2.4. If a quasi-coherent $\mathcal{R}$-module $\mathcal{M}$ is isomorphic to a direct product $\mathbb{M} \times \mathbb{N}$ of functors of $\mathcal{R}$-modules, then $\mathbb{M}$ and $\mathbb{N}$ are quasi-coherent (specifically, they are the quasi-coherent modules associated with the modules $\mathbb{M}(R)$ and $\mathbb{N}(R)$). Dually, if $\mathcal{M}^*$ is isomorphic to a direct product $\mathbb{M} \times \mathbb{N}$ of functors of $\mathcal{R}$-modules, then $\mathbb{M}$ and $\mathbb{N}$ are $\mathcal{R}$-module schemes. If $\mathcal{A}^* = \mathbb{B} \times \mathbb{C}$ as functors of $\mathcal{R}$-algebras, then $\mathbb{B}$ and $\mathbb{C}$ are $\mathcal{R}$-algebra schemes.

Let $\chi : G \to G_m$ be a multiplicative character and let $\chi : \mathcal{A}^* \to \mathcal{R}$ be the induced morphism of functors of $\mathcal{R}$-algebras.

Corollary 2.5. *An affine R-monoid scheme $G = \mathrm{Spec}\, A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as $\mathcal{R}$-algebra schemes, where the projection $\mathcal{A}^* \to \mathcal{R}$ is χ.*

Proof. The character χ induces the morphism $G \to \mathcal{A}^*$, $g \mapsto \chi(g) \cdot g$, which induces a morphism of $\mathcal{R}$-algebra schemes $\varphi : \mathcal{A}^* \to \mathcal{A}^*$. This last morphism is an isomorphism because its inverse morphism is the morphism induced by χ^{-1}.

The diagram



is commutative. Hence, via φ, "$\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as $\mathcal{R}$-algebra schemes, where the projection $\mathcal{A}^* \to \mathcal{R}$ is Θ" if and only if "$\mathcal{A}^* = \mathcal{R} \times \mathcal{B}'^*$ as $\mathcal{R}$-algebra schemes, where the projection $\mathcal{A}^* \to \mathcal{R}$ is χ". Then, Theorem 2.3 proves this corollary. □

Corollary 2.6. *An affine R-group scheme $G = \operatorname{Spec} A$ is invariant exact if and only if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as $\mathcal{R}$-algebra schemes.*

Proof. Assume that $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and let $G \hookrightarrow \mathcal{A}^*, g \mapsto g$ be the natural morphism. The composite morphism

$$G \hookrightarrow \mathcal{A}^* = \mathcal{R} \times \mathcal{B}^* \stackrel{\pi_1}{\to} \mathcal{R}$$

is a multiplicative character and π_1 is the morphism induced by this character. Now it is easy to prove that this corollary is a consequence of Corollary 2.5. □

If M is an R-module and the natural morphism $M \to M^{**}$ is injective, for example if M is a projective module, then

$$\operatorname{Hom}_{\mathcal{R}}(\mathcal{M}^*, \mathcal{M}'^*) = \operatorname{Hom}_{\mathcal{R}}(\mathcal{M}', \mathcal{M}) = \operatorname{Hom}_R(M', M) \subseteq \operatorname{Hom}_R(M^*, M'^*).$$

Remark 2.7. Assume that A is a projective R-module. If $A^* = C_1 \times C_2$ as R-algebras, then the morphisms $\mathcal{A}^* \to \mathcal{A}^*$, $w \mapsto (1, 0) \cdot w - w \cdot (1, 0), (0, 1) \cdot w - w \cdot (0, 1)$ are null and $\mathcal{A}^* = \mathcal{A}^* \cdot (1, 0) \times \mathcal{A}^* \cdot (0, 1)$ as functors of $\mathcal{R}$-algebras. Then, an affine R-group scheme $G = \operatorname{Spec} A$ is invariant exact if and only if $A^* = R \times C$ as R-algebras.

Theorem 2.8. *An affine R-group scheme $G = \operatorname{Spec} A$ is invariant exact if and only if there exists a left G-invariant 1-form $w \in A^*$ such that $w(1) = 1$. Moreover, w is unique, it is right G-invariant and $*(w) = w$ (where $*$ is the morphism induced on $\mathcal{A}^*$ by the morphism $G \to G, g \mapsto g^{-1}$).*

Proof. If w_l is left invariant and $w_l(1) = 1$, then $*w =: w_r$ is right invariant, $w := w_l \cdot w_r$ is left and right invariant and $w(1) = 1$. Now we can proceed as in Theorem 2.3 in order to prove that G is invariant exact.

Let us only prove the last statement. We follow the notation used in the proof of the last theorem. We know that $\mathcal{A}^{*G} = (1, 0) \cdot \mathcal{A}^* = \mathcal{R} \times 0$, then $(1, 0) : A \to R$ is the only left G-invariant linear map $w : A \to R$ such that $(1, 0)(1) = 1$. As well, $(1, 0)$ is right invariant. Finally, $*(1, 0)$ is left invariant and $(*(1, 0))(1) = (1, 0)(1) = 1$, then $*(1, 0) = (1, 0)$. □

Remark 2.9. This result can be found in [3,7] for R being a field and G being a linearly reductive algebraic group (that is, every rational G-module M is direct sum of irreducible G-modules). If R is an algebraically closed field of characteristic zero, then G is a reductive group if and only if G is linearly reductive, by a theorem of H. Weyl. If R is a field of positive characteristic, then the monoid of matrices $M_n(R)$ is not a linearly reductive monoid (there exists rational representations of $M_n(R)$ not completely reducible); however, $M_n(R)$ is invariant exact (observe that given $0 \in M_n(R)$ and an $M_n(R)$-module $\mathbb{M}$, then $0 \cdot \mathbb{M} = \mathbb{M}^{M_n(R)}$)).

In the proof of Theorem 2.3, we have also proved Theorems 2.10 and 2.12.

Theorem 2.10. *An affine R-monoid scheme $G = \operatorname{Spec} A$ is invariant exact if and only if there exists a left and right G-invariant 1-form $w \in A^*$ such that $w(1) = 1$.*

Definition 2.11. Let $G = \operatorname{Spec} A$ be an invariant exact affine R-monoid scheme. The only 1-form $w_G \in \mathcal{A}^*$ that is left and right G-invariant and such that $w_G(1) = 1$ is called the invariant integral on G (influenced by the theory of compact Lie groups).

Theorem 2.12. *An affine R-group scheme $G = \operatorname{Spec} A$ is invariant exact if and only if for every exact sequence (in the category of functors of $\mathcal{R}$-modules) of G-module schemes*

$$0 \to \mathcal{M}_1^* \to \mathcal{M}_2^* \to \mathcal{M}_3^* \to 0$$

the sequence

$$0 \to \mathcal{M}_1^{*G} \to \mathcal{M}_2^{*G} \to \mathcal{M}_3^{*G} \to 0$$

is exact.

Theorem 2.13. *Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme. Assume that A is a projective R-module. G is invariant exact if and only if the functor "take invariants" is (left and right) exact on the category of quasi-coherent G-modules (or equivalently, the category of rational G-modules).*

Proof. Assume that the functor "take invariants" is (left and right) exact on the category of quasi-coherent G-modules. $\mathcal{A}^*$ is an inverse limit of quotients $\mathcal{B}_i$, which are coherent $\mathcal{R}$-algebras by [1, 4.12]. It can be assumed that the morphism $\Theta : \mathcal{A}^* \to \mathcal{R}$ factorizes via $\mathcal{B}_i$ for all i. Observe that $\mathcal{B}_i$ are (left and right) $\mathcal{A}^*$-modules, then they are G-modules. Now, as in Theorem 2.10, we can prove that $\mathcal{B}_i = \mathcal{R} \times \mathcal{B}'_i$ as coherent $\mathcal{R}$-algebras (where the projection onto the first factor is Θ). Then, taking inverse limit $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ and, by Theorem 2.3, G is invariant exact. □

3. Reynolds operator on separated functors

Let $\mathbb{M}$ be a functor of $\mathcal{R}$-modules and let $\mathbb{K}$ be the kernel of the natural morphism $\mathbb{M} \to \mathbb{M}^{**}$. One has that $\mathbb{K}(S) = \{m \in \mathbb{M}(S) : w(m) = 0$, for every $w \in \mathbb{M}^*(T)$ and every morphism of R-algebras $S \to T\}$. Moreover, $(\mathbb{M}/\mathbb{K})^* = \mathbb{M}^*$ (then $(\mathbb{M}/\mathbb{K})^{**} = \mathbb{M}^{**}$) and the morphism $\mathbb{M}/\mathbb{K} \to (\mathbb{M}/\mathbb{K})^{**}$ is injective.

Definition 3.1. We will say that $\mathbb{M}$ is a separated functor of $\mathcal{R}$-modules if the morphism $\mathbb{M} \to \mathbb{M}^{**}$ is injective, that is, $m \in \mathbb{M}$ is null if and only if $w(m) = 0$ for every $w \in \mathbb{M}^*$.

Dual functors of $\mathcal{R}$-modules are separated: given $0 \neq m \in \mathbb{M} = \mathbb{N}^*$ there exists $n \in \mathbb{N}$ such that $m(n) \neq 0$; if $\tilde{n}$ is the image of n by the morphism $\mathbb{N} \to \mathbb{N}^{**} = \mathbb{M}^*$, then $\tilde{n}(m) = m(n) \neq 0$. Every subfunctor of $\mathcal{R}$-modules of a separated functor of $\mathcal{R}$-modules is separated.

Proposition 3.2. *Let $G = \operatorname{Spec} A$ be an invariant exact R-monoid and let $w_G \in \mathcal{A}^*$ be the invariant integral on G. Let $\mathbb{M}$ be a separated functor of $\mathcal{A}^*$-modules. It holds that:*

(1) $\mathbb{M}^G = w_G \cdot \mathbb{M}$.
(2) $\mathbb{M}$ *splits uniquely as a direct sum of* $\mathbb{M}^G$ *and another subfunctor of G-modules, explicitly*

$$\mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

The morphism $\mathbb{M} \to \mathbb{M}^G$, $m \mapsto w_G \cdot m$ will be called the Reynolds operator of $\mathbb{M}$.

Proof. (1) One deduces that $w_G \cdot \mathbb{M} \subseteq \mathbb{M}^G$, because $g \cdot (w_G \cdot m) = (g \cdot w_G) \cdot m = w_G \cdot m$ for every $g \in G$ and every $m \in \mathbb{M}$. Conversely, let us see that $\mathbb{M}^G \subseteq w_G \cdot \mathbb{M}$. Let $m \in \mathbb{M}^G$. The morphism $G \to \mathbb{M} \hookrightarrow \mathbb{M}^{**}$, $g \mapsto g \cdot m = m$, extends to a unique morphism $\mathcal{A}^* \to \mathbb{M}^{**}$. The uniqueness implies that $w' \cdot m = w'(1) \cdot m$ and then $m = w_G \cdot m \in w_G \cdot \mathbb{M}$.
(2) Since $\mathcal{A}^* = w_G \cdot \mathcal{A}^* \oplus (1 - w_G) \cdot \mathcal{A}^*$, then

$$\mathbb{M} = \mathcal{A}^* \otimes_{\mathcal{A}^*} \mathbb{M} = w_G \cdot \mathbb{M} \oplus (1 - w_G) \cdot \mathbb{M}.$$

Let $\mathbb{M} = \mathbb{M}^G \oplus \mathbb{N}$ be an isomorphism of G-modules. The G-module structure of $\mathbb{N}$ extends to an $\mathcal{A}^*$-module structure, because $\mathbb{N} = \mathbb{M}/\mathbb{M}^G$. Moreover, $\mathbb{N}$ is separated because it is a subfunctor of $\mathcal{R}$-modules of $\mathbb{M}$. Now, every morphism of G-modules between separated $\mathcal{A}^*$-modules is a morphism of $\mathcal{A}^*$-modules, because the morphism between the double duals is of $\mathcal{A}^*$-modules by Theorem 2.1. Thus, multiplying by w_G one concludes that $w_G \cdot \mathbb{M} = \mathbb{M}^G \oplus w_G \cdot \mathbb{N}$; hence, $w_G \cdot \mathbb{N} = 0$ and $(1 - w_G) \cdot \mathbb{M} = (1 - w_G) \cdot \mathbb{N} = \mathbb{N}$. □

Proposition 3.3. *Let $G = \operatorname{Spec} A$ be an invariant exact affine R-group scheme and let $\mathbb{M}$ and $\mathbb{N}$ be dual functors of G-modules. If $\pi : \mathbb{M} \to \mathbb{N}$ is an epimorphism of functors of G-modules and $s : \mathbb{N} \to \mathbb{M}$ is a section of functors of $\mathcal{R}$-modules of π, then $w_G \cdot s$ is a section of functors of G-modules of π.*

Proof. Let us consider the epimorphism of functors of G-modules (then of $\mathcal{A}^*$-modules)

$$\pi_* : \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{N}, \mathbb{M}) \to \mathbb{H}om_{\mathcal{R}}(\mathbb{N}, \mathbb{N}),\ f \mapsto \pi \circ f.$$

Then, $\pi \circ (w_G \cdot s) = \pi_*(w_G \cdot s) = w_G \cdot \pi_*(s) = w_G \cdot \mathrm{Id} = \mathrm{Id}$. □

Likewise, it can be proved that if $\mathbb{M}$ and $\mathbb{N}$ are functors of G-modules, $\mathbb{M}$ is a dual functor, $i : \mathbb{M} \to \mathbb{N}$ is an injective morphism of G-modules and r is a retract of functors R-modules of i, then $w_G \cdot r$ is a retract of functors of G-modules of i.

Remark 3.4. We shall say a rational G-module M is simple if it does not contain any G-submodule $M' \subsetneq M$, such that M' is a direct summand of M as an R-module, this last condition is equivalent to the morphism of functors of $\mathcal{R}$-modules $\mathcal{M}^* \to \mathcal{M}'^*$ being surjective (see the paragraph previous to [1, 1.14]). If G is an invariant exact affine R-group scheme, M is a rational G-module and it is a noetherian R-module, then it is easy to prove, using the previous proposition, that M is a direct sum of simple G-modules.

Example 3.5. Let us give the proof of the famous *Finiteness Theorem of Hilbert*, [6], for its simplicity: "Let k be a field, let G be a linearly reductive affine k-group scheme and let us consider an operation of G over an algebraic variety $X = \operatorname{Spec} A$. Then $X/\sim := \operatorname{Spec} A^G$ is an algebraic variety".

Proof. Let $\xi_1, \dots, \xi_m$ be a system of generators of the k-algebra A. Let V be a finite dimensional G-submodule of A which contains $\xi_1, \dots, \xi_m$. The natural morphism $S^{\cdot} V \to A$ is surjective. We have to prove that A^G is an algebra of finite type. As G is invariant exact, it is sufficient to prove that $(S^{\cdot} V)^G = (k[x_1, \dots, x_n])^G$ is a k-algebra of finite type.

Let $I \subset k[x_1, \dots, x_n]$ be the ideal generated by $(x_1, \dots, x_n)^G$. Let $f_1, \dots, f_r \in (x_1, \dots, x_n)^G$ be a finite system of generators of I. We can assume f_i are homogeneous. Let us prove that $k[x_1, \dots, x_n]^G = k[f_1, \dots, f_r]$. Given a homogeneous $h \in k[x_1, \dots, x_n]^G$, we have to prove that $h \in k[f_1, \dots, f_r]$. We are going to proceed by induction on the degree of h. If $\operatorname{dg} h = 0$, then $h \in k \subseteq k[f_1, \dots, f_r]$. Let $\operatorname{dg} h = d > 0$. We can write $h = \sum_{i=1}^r a_i \cdot f_i$, where $a_i \in k[x_1, \dots, x_n]$ are homogeneous of degree $d - \operatorname{dg}(f_i)$ (which are less than d). Then

$$h = w_G \cdot h = \sum_{i=1}^{r} w_G \cdot (a_i \cdot f_i) \overset{*}{=} \sum_{i=1}^{r} (w_G \cdot a_i) \cdot f_i.$$

(Observe in $\overset{*}{=}$ that $g \cdot (a_i \cdot f_i) = (g \cdot a_i) \cdot (g \cdot f_i) = (g \cdot a_i) \cdot f_i$, then $w \cdot (a_i \cdot f_i) = (w \cdot a_i) \cdot f_i$ for all $w \in A^*$). By the induction hypothesis $w_G \cdot a_i \in k[f_1, \dots, f_r]$ and therefore $h \in k[f_1, \dots, f_r]$. □

Let us see more examples where this theory can be applied.

Let $G = \operatorname{Spec} A$ be an affine R-group scheme and let B be an R-algebra.

Definition 3.6. We say that B is a G-algebra if G acts on B by endomorphisms of R-algebras, that is, there exists a morphism of monoids $G^{\cdot} \to \mathbb{E}\mathrm{nd}_{\mathcal{R}-alg}(\mathcal{B})$.

We will say that a functor of $\mathcal{R}$-modules $\mathbb{M}$ is a functor of $\mathcal{B}$-modules if there exists a morphism of functors of $\mathcal{R}$-algebras $\mathcal{B} \to \mathbb{E}nd_{\mathcal{R}}(\mathbb{M})$.

Definition 3.7. Let B be a G-algebra and $\mathbb{M}$ a functor of $\mathcal{B}$-modules. We say that $\mathbb{M}$ is a $\mathcal{B}G$-module if it has a G-module structure which is compatible with the $\mathcal{B}$-module structure, that is,

$$g(b \cdot m) = g(b) \cdot g(m)$$

for every $g \in G$, $b \in \mathcal{B}$ and $m \in \mathbb{M}$.

If $\mathbb{M}$ and $\mathbb{N}$ are $\mathcal{B}G$-modules, then it is easy to check that $\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is a subfunctor of G-modules of $\mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$ and it coincides with the kernel of the morphism of G-modules

$$\begin{array}{ccc} \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathbb{M}, \mathbb{N}) & \overset{\varphi}{\to} & \mathbb{H}\mathrm{om}_{\mathcal{R}}(\mathcal{B} \otimes_{\mathcal{R}} \mathbb{M}, \mathbb{N}), \\ L & \mapsto & L_1 - L_2 \end{array}$$

where $L_1(b \otimes m) := L(b \cdot m)$ and $L_2(b \otimes m) := b \cdot L(m)$. Therefore, if $\mathbb{N}$ is a dual functor as well, then $\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is an $\mathcal{A}^*$-module. Moreover, $\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ is separated because it is an R-submodule of $\mathbb{H}om_{\mathcal{R}}(\mathbb{M}, \mathbb{N})$, and this latter is separated because it is a dual functor. Hence, if $G = \operatorname{Spec} A$ is an invariant exact R-group scheme, $\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G = w_G \cdot \mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$. We have proved the following proposition.

Proposition 3.8. *Let $\mathbb{N}$ be a dual functor of $\mathcal{B}G$-modules and let $\mathbb{M}$ be a functor of $\mathcal{B}G$-modules. Then,*

(1) $\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$ *is a separated functor of $\mathcal{A}^*$-modules.*
(2) *If $G = \operatorname{Spec} A$ is an invariant exact affine R-group scheme, then*

$$\mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G = w_G \cdot \mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})$$

and $w_G \cdot : \mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N}) \to \mathbb{H}\mathrm{om}_{\mathcal{B}}(\mathbb{M}, \mathbb{N})^G$ *is the Reynolds operator.*

4. Reynolds operator on functors

Let us generalize the Reynolds operator to all functors of G-modules.

Let us assume that $G = \operatorname{Spec} A$ is an invariant exact monoid.

Given a dual functor of G-modules $\mathbb{M}$, the dual morphism of $\mathbb{M}^G \hookrightarrow \mathbb{M}$ is the Reynolds operator of $\mathbb{M}^*$.

Let $\mathbb{N}$ be a separated functor of G-modules. Let $\mathbb{N}_1 = \mathbb{N} \cap (1 - w_G) \cdot \mathbb{N}^{**}$. Then $\mathbb{N}_1 = \{n \in \mathbb{N} : w_G \cdot n = 0\}$, since $(1 - w_G) \cdot \mathbb{N}^{**} = \{n' \in \mathbb{N}^{**} : w_G \cdot n' = 0\}$. One deduces that $\mathbb{N}_1^G = \mathbb{N}_1 \cap \mathbb{N}^{**G} = 0$ and $(\mathbb{N}/\mathbb{N}_1)^G = \mathbb{N}/\mathbb{N}_1$, because $\mathbb{N}/\mathbb{N}_1$ injects into $\mathbb{N}^{**}/(1 - w_G) \cdot \mathbb{N}^{**} = \mathbb{N}^{**G}$. Moreover, $(\mathbb{N}/\mathbb{N}_1)^* = \mathbb{N}^{*G}$: $\mathbb{N}^{*G} = \mathbb{N}^* \cdot w_G$ vanishes on $\mathbb{N}_1$, then $\mathbb{N}^{*G} \subseteq (\mathbb{N}/\mathbb{N}_1)^*$, and $(\mathbb{N}/\mathbb{N}_1)^* \subseteq \mathbb{N}^{*G}$. Therefore, $\mathbb{N}^{**} \to (\mathbb{N}/\mathbb{N}_1)^{**}$ is the Reynolds operator of $\mathbb{N}^{**}$.

Theorem 4.1. *Let $G = \operatorname{Spec} A$ be an invariant exact R-monoid, let $\mathbb{N}$ be a functor of G-modules and let $\mathbb{N}_1 \subset \mathbb{N}$ be the subfunctor of G-modules defined by $\mathbb{N}_1 := \{n \in \mathbb{N} : w_G \cdot \tilde{n} = 0\}$, where $\tilde{n}$ denotes the image of n by the morphism $\mathbb{N} \to \mathbb{N}^{**}$. It holds that:*

(1) $\mathbb{N}/\mathbb{N}_1$ *is the maximal separated G-invariant quotient of $\mathbb{N}$.*
(2) *The double dual of the morphism $\mathbb{N} \to \mathbb{N}/\mathbb{N}_1$ is the Reynolds operator $\mathbb{N}^{**} \to \mathbb{N}^{**G}$ and one has the commutative diagram*

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{N}/\mathbb{N}_1 \\ \downarrow & & \downarrow \\ \mathbb{N}^{**} & \longrightarrow & \mathbb{N}^{**G} = (\mathbb{N}/\mathbb{N}_1)^{**}. \end{array} \qquad (1)$$

(3) *If $\mathbb{N}$ is a dual functor, then $\mathbb{N}/\mathbb{N}_1 = \mathbb{N}^G$ and the morphism $\mathbb{N} \to \mathbb{N}/\mathbb{N}_1$ is the Reynolds operator of $\mathbb{N}$.*

Proof. (2) $\mathbb{N}_1$ is the kernel of the composite morphism $\mathbb{N} \to \mathbb{N}^{**} \to \mathbb{N}^{**G}$, then $\mathbb{N}_1$ contains the kernel $\mathbb{K}$ of the morphism $\mathbb{N} \to \mathbb{N}^{**}$. Let $\mathbb{N}' = \mathbb{N}/\mathbb{K}$. Observe that $\mathbb{N}'^* = \mathbb{N}^*$, that $\mathbb{N}'$ is separated, $\mathbb{N}'_1 = \mathbb{N}_1/\mathbb{K}$ and $\mathbb{N}'/\mathbb{N}'_1 = \mathbb{N}/\mathbb{N}_1$. Therefore, the diagram (1) is commutative because is commutative for $\mathbb{N} = \mathbb{N}'$. In particular, $\mathbb{N}/\mathbb{N}_1$ is separated.

(1) We must prove that if $\mathbb{P} \subseteq \mathbb{N}$ is a subfunctor of G-modules such that $\mathbb{N}/\mathbb{P}$ is separated and G-invariant, then $\mathbb{N}_1 \subseteq \mathbb{P}$, i.e., $\mathbb{N}/\mathbb{P}$ is a quotient of $\mathbb{N}/\mathbb{N}_1$.

$\mathbb{N}/(\mathbb{P} \cap \mathbb{N}_1)$ is G-invariant and separated functor, because the morphism $\mathbb{N}/(\mathbb{P} \cap \mathbb{N}_1) \hookrightarrow \mathbb{N}/\mathbb{N}_1 \oplus \mathbb{N}/\mathbb{P}$, $\bar{n} \mapsto (\bar{n}, \bar{n})$ is injective. It is enough to prove that $\mathbb{N}_1 = \mathbb{P} \cap \mathbb{N}_1$. Let us denote $\mathbb{P}' = \mathbb{P} \cap \mathbb{N}_1$. From the composition of injections $\mathbb{N}^{*G} = (\mathbb{N}/\mathbb{N}_1)^* \hookrightarrow (\mathbb{N}/\mathbb{P}')^* \hookrightarrow \mathbb{N}^{*G}$ one concludes that $(\mathbb{N}/\mathbb{N}_1)^* = (\mathbb{N}/\mathbb{P}')^*$. Now, the commutative diagram

$$\begin{array}{ccc} \mathbb{N}/\mathbb{P}' & \longrightarrow & \mathbb{N}/\mathbb{N}_1 \\ \downarrow & & \downarrow \\ (\mathbb{N}/\mathbb{P}')^{**} & = & (\mathbb{N}/\mathbb{N}_1)^{**} \end{array}$$

implies that the morphism $\mathbb{N}/\mathbb{P}' \to \mathbb{N}/\mathbb{N}_1$ is injective; hence, $\mathbb{P}' = \mathbb{N}_1$.

(3) Recall that $\mathbb{N} = w_G \cdot \mathbb{N} \oplus (1 - w_G) \cdot \mathbb{N}$, $\mathbb{N}_1 = (1 - w_G) \cdot \mathbb{N}$ and $\mathbb{N}/\mathbb{N}_1 = w_G \cdot \mathbb{N} = \mathbb{N}^G$. □

Let us observe that $\mathbb{N}_1^0 := \{w \in \mathbb{N}^* : w(\mathbb{N}_1) = 0\} = (\mathbb{N}/\mathbb{N}_1)^* = \mathbb{N}^{*G}$. On the other hand, $(\mathbb{N}^{*G})^0 := \{n \in \mathbb{N} : \mathbb{N}^{*G}(n) = 0\} = \{n \in \mathbb{N} : \tilde{n}(\mathbb{N}^{*G} = \mathbb{N}^* \cdot w_G) = 0\} = \{n \in \mathbb{N} : (w_G \cdot \tilde{n})(\mathbb{N}^*) = 0\} = \{n \in \mathbb{N} : w_G \cdot \tilde{n} = 0\} = \mathbb{N}_1$.

5. Semi-invariants

Let $\chi : G = \operatorname{Spec} A \to \mathcal{R}$ be a multiplicative character and let $\chi : \mathcal{A}^* \to \mathcal{R}$ be the induced morphism.

Definition 5.1. Let $\mathbb{M}$ be a functor of G-modules. An element $m \in \mathbb{M}$ is said to be (left) χ-semi-invariant if $g \cdot m = \chi(g) \cdot m$ for every $g \in G$.

Definition 5.2. Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme and let $\chi : G \to \mathcal{R}$ be a multiplicative character. We will call the 1-form $w_\chi \in A^*$ which is left and right χ-semi-invariant and such that $w_\chi(\chi) = 1$, if it exists, a χ-semi-invariant integral on G.

If a χ-semi-invariant integral exists, then it is unique: observe that $w \cdot w_\chi = w(\chi) \cdot w_\chi$, because $g \cdot w_\chi = \chi(g) \cdot w_\chi$ for every $g \in G$, since w_χ is left χ-semi-invariant. Likewise, $w_\chi \cdot w = w(\chi) \cdot w_\chi$. Given a left χ-semi-invariant $w \in A^*$ such that $w(\chi) = 1$, one concludes that $w = w_\chi(\chi) \cdot w = w_\chi \cdot w = w(\chi) \cdot w_\chi = w_\chi$.

Given a functor $\mathbb{M}$ of G-modules we will define $\mathbb{M}^\chi$ to be the functor $\mathbb{M}^\chi(S) := \{m \in \mathbb{M}(S) : g \cdot m = \chi(g) \cdot m$, for every $g \in G(T)$, and every S-algebra $T\}$.

Proposition 5.3. *Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme and let $\chi : G \to \mathcal{R}$ be a multiplicative character. $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of $\mathcal{R}$-algebras, where the projection onto the first factor is χ, if and only if there exists the χ-semi-invariant integral on G.*

Proof. If there exists the χ-semi-invariant integral on G, w_χ, then $\mathcal{A}^* = w_\chi \cdot \mathcal{A}^* \times (1 - w_\chi) \cdot \mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$ as functors of $\mathcal{R}$-algebras, where the projection onto the first factor is χ. Conversely, if $\mathcal{A}^* = \mathcal{R} \times \mathcal{B}^*$, where the first projection $\mathcal{A}^* \to \mathcal{R}$ is χ, then $w_\chi = (1, 0) \in \mathcal{R} \times \mathcal{B}^*$. □

By Corollary 2.5 we obtain the following theorem.

Theorem 5.4. *Let G be an affine monoid scheme and let $\chi : G \to G_m$ be a multiplicative character. G is invariant exact if and only if there exists the χ-semi-invariant integral on G.*

Likewise as in Proposition 3.2, we obtain the following result.

Proposition 5.5. *Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme and assume there exists the χ-semi-invariant integral on G, $w_\chi \in \mathcal{A}^*$. Let $\mathbb{M}$ be a separated functor of $\mathcal{A}^*$-modules. It holds that:*

(1) $\mathbb{M}^\chi = w_\chi \cdot \mathbb{M}$.

(2) $\mathbb{M}$ *splits uniquely as a direct sum of* $\mathbb{M}^\chi$ *and another subfunctor of G-modules, explicitly*

$$\mathbb{M} = w_\chi \cdot \mathbb{M} \oplus (1 - w_\chi) \cdot \mathbb{M}.$$

We call the morphism $\mathbb{M} \to \mathbb{M}^\chi$, $m \mapsto w_\chi \cdot m$, the Reynolds χ-operator.

Example 5.6. Let $G = \operatorname{Spec} A$ be an affine R-monoid scheme and let $\chi : G \to \mathcal{R}$ be a multiplicative character. An Ω-process associated with χ (see [5, 3.1]) is a nonzero linear operator $\Omega : A \to A$ such that

$$\Omega(a \cdot g) = \chi(g) \cdot (\Omega(a) \cdot g); \qquad \Omega(g \cdot a) = \chi(g) \cdot (g \cdot \Omega(a))$$

for all $a \in A$ and $g \in G$. The composite morphism

$$A \xrightarrow{\Omega} A \xrightarrow{\chi\cdot} A$$

is a morphism of left and right G-modules: $(\chi\cdot\circ\Omega)(g\cdot a) = \chi\cdot\chi(g)\cdot(g\cdot\Omega(a)) = (g\cdot\chi)\cdot(g\cdot\Omega(a)) = g\cdot(\chi\cdot\Omega(a)) = g\cdot((\chi\cdot\circ\Omega)(a))$; likewise, we prove that $\chi\cdot\circ\Omega$ is a morphism of right G-modules. Since

$$\mathrm{Hom}_{\text{left-right } G\text{-modules}}(A, A) = \mathrm{Hom}_{\text{left-right } \mathcal{A}^*\text{-modules}}(\mathcal{A}^*, \mathcal{A}^*) = Z(A^*)$$

($Z(A^*)$ is the center of A^*), then $\chi\cdot\circ\Omega = z\cdot$ for some $z \in Z(A^*)$. If G is a linearly reductive monoid and R is an algebraically closed field, then $\mathcal{A}^*$ is a semisimple algebra scheme and $A^* = \prod_{E_i\in I}\mathrm{End}_R(E_i)$ where I is the set of irreducible representations of G (up to isomorphism), by [1, 6.2, 6.8], hence $\chi\cdot\circ\Omega \in Z(A^*) = \prod_{i\in I} R$ (on the other hand, see [5, 4.4]).

Assume now that $0 \in G$ (that is an element such that $0\cdot g = g\cdot 0 = 0$ for all $g \in G$) and that $\Omega(\chi) = 1$ (generally $\chi\cdot\Omega(\chi) = z\cdot\chi = \chi(z)\cdot\chi = \chi\cdot\chi(z)$, $\chi(z) \in R$). The projection $w : A \to R$, $a \mapsto a(0)$ is left and right invariant and $w(1) = 1$, then G is an invariant exact R-monoid and $w = w_G$. The composite morphism $w' = w_G \circ \Omega$ is left and right χ-semi-invariant and $w'(\chi) = 1$, then $w' = w_\chi$. Given a rational G-module M, let us calculate the Reynolds χ-operator of M, that is, the morphism $M \to M$, $m \mapsto w_\chi\cdot m$ (on the other hand, see [5, 5.1]). The dual morphism of the multiplication morphism $\mathcal{M}^*\otimes\mathcal{A}^* \to \mathcal{M}^*$ is the comultiplication morphism $\mu\colon M \to M\otimes A$. If $\mu(m) = \sum_l m_l\otimes a_l$, then $g\cdot m = \sum_l a_l(g)\cdot m_l$, for all $g \in G$. Hence,

$$w_\chi\cdot m = \sum_l a_l(w_\chi)\cdot m_l = \sum_l w_\chi(a_l)\cdot m_l = \sum_l \Omega(a_l)(0)\cdot m_l.$$

Acknowledgements

The first and the third authors were partially supported by Junta de Extremadura and FEDER funds.

References

[1] A. Álvarez, C. Sancho, P. Sancho, Algebra schemes and their representations, J. Algebra 296 (1) (2006) 110–144.
[2] A. Álvarez, C. Sancho, P. Sancho, Characterization of quasi-coherent modules that are module schemes, Commun. Algebra 37 (5) (2009) 1619–1621.
[3] M. Brion, G.W. Schwarz, Théorie des invariants et géometrie des variétés quotients, in: Travaux en cours, vol. 61, Hermann, Paris, 2000.
[4] H. Chu, S.-J. Hu, M.-C. Kang, A variant of the Reynolds operator, Proc. Amer. Math. Soc. 133 (10) (2005) 2865–2871.
[5] W. Ferrer Santos, A. Rittatore, Generalizations of Cayley's Ω-process, Proc. Amer. Math. Soc. 135 (4) (2007) 961–968.
[6] D. Hilbert, Ueber die Theorie der algebraischen Formen, Math. Ann. 36 (4) (1890) 473–534.
[7] C. Sancho de Salas, Grupos algebraicos y teoría de invariantes, Sociedad Matemática Mexicana, México, 2001.