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# Exact periodic cross-kink wave solutions and breather type of two-solitary wave solutions for the (3 + 1)-dimensional potential-YTSF equation☆

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ABSTRACT

In this paper, the (3 + 1)-dimensional potential-YTSF equation is investigated. Exact solutions with three-wave form including periodic cross-kink wave, periodic two-solitary wave and breather type of two-solitary wave solutions are obtained using Hirota's bilinear form and generalized three-wave approach with the aid of symbolic computation. Moreover, the properties for some new solutions are shown with some figures.


## 1. Introduction

As is well known that searching for exact solutions of nonlinear evolution equations arising in mathematical physics plays an important role in the study of nonlinear physical phenomena. Many effective and powerful methods to seek exact solutions were proposed, such as the inverse scattering method [1], the Hirota's bilinear form method [2], the two-soliton method [3], the homoclinic test technique [4–6], the Bäcklund transformation method [7], the Exp-function method [8–10], the auxiliary equation method [11,12], the Jacobian elliptic function method [13], the similarity transformation method [14] and so on.

Very recently, Dai et al. [15] proposed a new technique called "three-wave approach" to seek periodic solitary wave solutions for integrable equations, and this method has been used to investigate several equations [16,17]. The "three-wave approach" is an extension of the three-soliton method, the main difference is the selection of ansätz, by selecting and substituting a three-wave type of ansätz rather than three-soliton type of ansätz into the bilinear equation, one can effectively obtain exact solutions with three-wave form.

In this paper, a (3 + 1)-dimensional nonlinear evolution equation

$$-4u_{xt} + u_{xxxz} + 4u_xu_{xz} + 2u_{xx}u_z + 3u_{yy} = 0, \tag{1.1}$$

will be considered. Eq. (1.1) is called the potential-YTSF equation which was firstly introduced by Yu, Toda, Sasa and Fukuyama (YTSF) [18]. Recently, this system was studied and some single soliton and periodic solitary solutions were obtained [19,20]. In this work, we mainly apply the three-wave type of ansätz approach to determine three-wave solutions for Eq. (1.1). As a result, exact periodic cross-kink wave solutions, doubly periodic solitary wave solutions and breather type of two-solitary wave solutions are obtained.

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## 2. Exact solutions for the YTSF equation

In this section, the three-wave approach is applied to Eq. (1.1), exact periodic kink-wave, doubly periodic solitary wave solutions and breather type of two-solitary wave solutions are obtained.

We first make the variable transformation $\xi = x + \omega z$, whence Eq. (1.1) is changed into the following equation

$$-4u_{\xi t} + \omega u_{\xi\xi\xi\xi} + 6\omega u_{\xi} u_{\xi\xi} + 3u_{yy} = 0. \tag{2.1}$$

Obviously, an arbitrary constant $u_0$ is a solution of Eq. (2.1), we assume that the solution of Eq. (2.1) has the following form

$$u(\xi, y, t) = u_0 + 2(\ln \phi(\xi, y, t))_{\xi}, \tag{2.2}$$

where $\phi(\xi, y, t)$ is an unknown real function. Substituting (2.2) into (2.1) and using the bilinear form, we have

$$(-4D_{\xi}D_t + \omega D_{\xi}^4 + 3D_y^2)\phi \cdot \phi = 0, \tag{2.3}$$

where the bilinear operator "D" is defined as [2]

$$D_{\xi}^m D_y^n D_t^s f \cdot g = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^s f \cdot g|_{(\xi,y,t)=(\xi',y',t')}.$$

We now suppose that the real function $\phi(\xi, y, t)$ has the following ansätz:

$$\phi(\xi, y, t) = e^{-\eta_1} + L\cos(\eta_2) + H\cosh(\eta_3) + Ke^{\eta_1}, \tag{2.4}$$

where $\eta_i = a_i\xi + b_i y + d_i t + r_i,\ i = 1, 2, 3$ and $a_i,\ b_i,\ d_i$ are constants to be determined later, $r_i, i = 1, 2, 3$ are arbitrary constants.

Substituting (2.4) into Eq. (2.3), and equating all the coefficients of $e^{-\eta_1}$, $e^{\eta_1}$, $\cos(\eta_2)$, $\cosh(\eta_3)$, $\sin(\eta_2)$, $\sinh(\eta_3)$ to zero yields a set of algebraic equations:

$$\begin{aligned}
&4K(4\omega a_1^4 + 3b_1^2 - 4a_1d_1) + L^2(4\omega a_2^4 - 3b_2^2 + 4a_2d_2) + H^2(4\omega a_3^4 + 3b_3^2 - 4a_3d_3) = 0,\\
&HL(-4\omega a_2^3a_3 + 4\omega a_2a_3^3 + 6b_2b_3 - 4a_3d_2 - 4a_2d_3) = 0,\\
&HL(\omega a_2^4 - 6\omega a_2^2a_3^2 + \omega a_3^4 - 3b_2^2 + 3b_3^2 + 4a_2d_2 - 4a_3d_3) = 0,\\
&L(-4\omega a_1^3a_2 + 4\omega a_1a_2^3 - 6b_1b_2 + 4a_2d_1 + 4a_1d_2) = 0,\\
&KL(4\omega a_1^3a_2 - 4\omega a_1a_2^3 + 6b_1b_2 - 4a_2d_1 - 4a_1d_2) = 0,\\
&L(\omega a_1^4 - 6\omega a_1^2a_2^2 + \omega a_2^4 + 3b_1^2 - 3b_2^2 - 4a_1d_1 + 4a_2d_2) = 0,\\
&KL(\omega a_1^4 - 6\omega a_1^2a_2^2 + \omega a_2^4 + 3b_1^2 - 3b_2^2 - 4a_1d_1 + 4a_2d_2) = 0,\\
&H(4\omega a_1^3a_3 + 4\omega a_1a_3^3 + 6b_1b_3 - 4a_3d_1 - 4a_1d_3) = 0,\\
&HK(-4\omega a_1^3a_3 - 4\omega a_1a_3^3 - 6b_1b_3 + 4a_3d_1 + 4a_1d_3) = 0,\\
&H(\omega a_1^4 + 6\omega a_1^2a_3^2 + \omega a_3^4 + 3b_1^2 + 3b_3^2 - 4a_1d_1 - 4a_3d_3) = 0,\\
&HK(\omega a_1^4 + 6\omega a_1^2a_3^2 + \omega a_3^4 + 3b_1^2 + 3b_3^2 - 4a_1d_1 - 4a_3d_3) = 0.
\end{aligned}$$

Solving the set of algebraic equations with the aid of Mathematica, we obtain:

*Case* (I).

$$H = 0, \qquad K = \frac{L^2(b^2 - \omega a^4)}{4b^2},$$

$$a_1 = 0, \qquad b_1 = b_2 = b, \qquad a_2 = a, \qquad d_1 = \frac{3b^2}{2a}, \qquad d_2 = -\frac{\omega a^3}{4}, \qquad r_1 = r_1, \qquad r_2 = r_2.$$

Substituting these results into (2.4), we have

$$\phi_1 = \frac{L\sqrt{b^2 - \omega a^4}}{b}\cosh\left(by + \frac{3b^2}{2a}t + r_1^*\right) + L\cos\left(a\xi + by - \frac{1}{4}\omega a^3 t + r_2\right), \quad \omega < \frac{b^2}{a^4},$$

$$\phi_2 = -\frac{L\sqrt{\omega a^4 - b^2}}{b}\sinh\left(by + \frac{3b^2}{2a}t + r_1^{**}\right) + L\cos\left(a\xi + by - \frac{1}{4}\omega a^3 t + r_2\right), \quad \omega > \frac{b^2}{a^4},$$

therefore we obtain the following solutions

$$u_1 = u_0 - 2ab\frac{\sin\left[a(x + \omega z) + by - \frac{\omega a^3}{4}t + r_2\right]}{\sqrt{b^2 - \omega a^4}\cosh\left[by + \frac{3b^2}{2a}t + r_1^*\right] + b\cos\left[a(x + \omega z) + by - \frac{\omega a^3}{4}t + r_2\right]}, \tag{2.5}$$

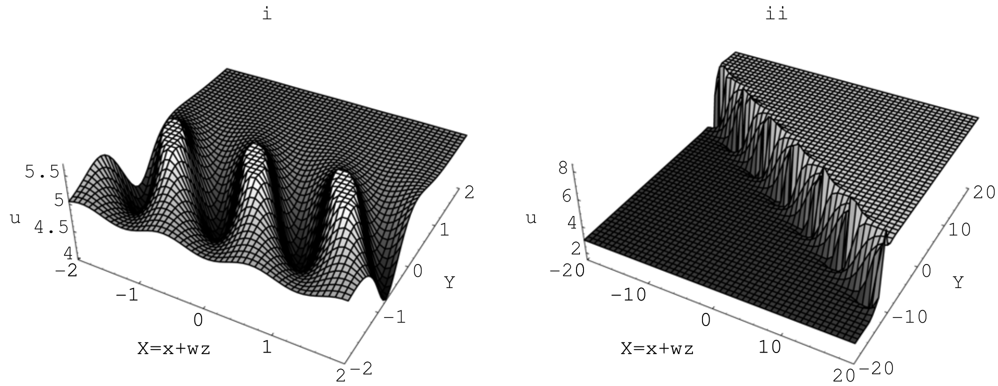


**Fig. 1.** (i) Mechanical feature of $u_1$: $u_0 = 5, a = 5, b = 2.5, \omega = -1, t = 1$. (ii) Mechanical feature of $u_3$: $u_0 = 5, a = 1, b = 1.5, \omega = -0.75, t = 1$.

and

$$u_2 = u_0 + 2ab \frac{\sin\left[a(x+\omega z) + by - \frac{\omega a^3}{4}t + r_2\right]}{\sqrt{\omega a^4 - b^2}\sinh\left[by + \frac{3b^2}{2a}t + r_1^{**}\right] - b\cos\left[a(x+\omega z) + by - \frac{\omega a^3}{4}t + r_2\right]}, \tag{2.6}$$

where $r_1^* = r_1 + \ln(\frac{L}{2b}\sqrt{b^2 - \omega a^4})$, $r_1^{**} = r_1 + \ln(\frac{L}{2b}\sqrt{\omega a^4 - b^2})$, $a,\ b$ and $\omega$ are free real constants.

Notice that (2.5) (Res. (2.6)) is singular periodic solitary wave solution for Eq. (1.1) which is periodic wave in $x,\ y,\ z$ direction, and a soliton in $y$–$t$ direction. In order to eliminate the singularity, we take $\omega < 0$ (Res. $\omega > 2b^2/a^4$) (see Fig. 1. (i)).

In addition, if in (2.5)–(2.6) we set

$$a_2 = \mathrm{i}a, \qquad b_1 = b_2 = \mathrm{i}b, \qquad r_1 = \mathrm{i}r_1, \qquad r_2 = \mathrm{i}r_2,$$

then we obtain

$$u_3 = u_0 + 2ab \frac{\sinh\left[a(x+\omega z) + by + \frac{\omega a^3}{4}t + r_2\right]}{\sqrt{b^2 + \omega a^4}\cos\left[by + \frac{3b^2}{2a}t + r_1\right] + b\cosh\left[a(x+\omega z) + by + \frac{\omega a^3}{4}t + r_2\right]}, \tag{2.7}$$

and

$$u_4 = u_0 - 2ab \frac{\sinh\left[a(x+\omega z) + by + \frac{\omega a^3}{4}t + r_2\right]}{\sqrt{\omega a^4 + b^2}\sin\left[by + \frac{3b^2}{2a}t + r_1\right] - b\cosh\left[a(x+\omega z) + by + \frac{\omega a^3}{4}t + r_2\right]}, \tag{2.8}$$

where $a,\ b$ and $\omega$ are free real constants and satisfy $\omega a^4 + b^2 > 0$.

Solution represented by (2.7) (Res. (2.8)) is also a periodic solitary wave solution which is periodic wave in $y$–$t$ direction, meanwhile it is solitary wave in $x, y, z$ direction (see Fig. 1. (ii)).

We notice that these four results are new solutions for Eq. (1.1). By verification, the solutions obtained above are indeed the solutions of the original $(3+1)$-dimensional YTSF equation.

*Case* (II).

$$H = 0, \qquad K = \frac{L^2}{4}\left(\frac{b^2 - 4\omega a^4}{b^2 + 4\omega a^4}\right),$$

$$a_1 = a_2 = a, \qquad b_1 = 0, \qquad b_2 = b, \qquad d_1 = -\frac{4\omega a^4 + 3b^2}{8a}, \qquad d_2 = \frac{4\omega a^4 + 3b^2}{8a}, \qquad r_1 = r_1, \qquad r_2 = r_2.$$

By substituting these results into (2.4), we obtain

$$\phi_3 = L\left[\cos\left(a\xi + by + \frac{4\omega a^4 + 3b^2}{8a}t + r_2\right)\right] + \sqrt{A}\cosh\left(a\xi - \frac{4\omega a^4 + 3b^2}{8a}t + r_1^*\right), \quad A > 0,$$

$$\phi_4 = L\left[\cos\left(a\xi + by + \frac{4\omega a^4 + 3b^2}{8a}t + r_2\right) - \sqrt{-A}\sinh\left(a\xi - \frac{4\omega a^4 + 3b^2}{8a}t + r\right)\right], \quad A < 0.$$

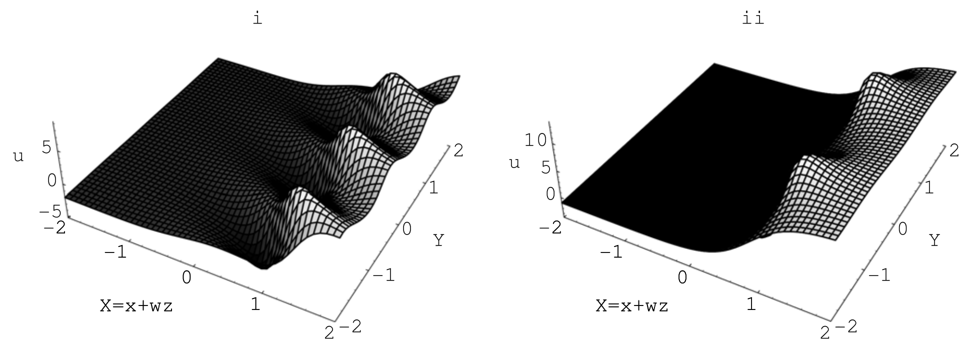


**Fig. 2.** (i) Mechanical feature of $u_5$: $u_0 = 2, a = 2, b = 4.5, \omega = -0.42, t = 1$. (ii) Mechanical feature of $u_{14}$: $u_0 = 5, a = 3, b = 3, \omega = -1/18, t = 1$.

Then, we obtain periodic kink-wave solutions

$$u_5 = u_0 + 2a\frac{\sqrt{A}\sinh\left[a(x+\omega z) - \frac{4\omega a^4+3b^2}{8a}t + r_1^*\right] - \sin\left[a(x+\omega z) + by + \frac{4\omega a^4+3b^2}{8a}t + r_2\right]}{\sqrt{A}\cosh\left[a(x+\omega z) - \frac{4\omega a^4+3b^2}{8a}t + r_1^*\right] + \cos\left[a(x+\omega z) + by + \frac{4\omega a^4+3b^2}{8a}t + r_2\right]}, \tag{2.9}$$

$$u_6 = u_0 + 2a\frac{\sqrt{-A}\cosh\left[a(x+\omega z) - \frac{4\omega a^4+3b^2}{8a}t + r\right] + \sin\left[a(x+\omega z) + by + \frac{4\omega a^4+3b^2}{8a}t + r_2\right]}{\sqrt{-A}\sinh\left[a(x+\omega z) - \frac{4\omega a^4+3b^2}{8a}t + r\right] - \cos\left[a(x+\omega z) + by + \frac{4\omega a^4+3b^2}{8a}t + r_2\right]}, \tag{2.10}$$

which are periodic waves in forward direction, meanwhile they are kink-waves in backward direction, where $r_1^* = r_1 + \ln(\frac{L}{2}\sqrt{A})$, $r = r_1 + \ln(\frac{L}{2}\sqrt{-A})$, $A = \frac{b^2-4\omega a^4}{b^2+4\omega a^4}$ and $a,\ b,\ \omega$ are free constants (see Fig. 2(i)).

Make the dependent variable transformation in (2.9)–(2.10) as follows

$$a = \mathrm{i}a, \qquad b = \mathrm{i}b, \qquad r_1 = \mathrm{i}r_1, \qquad r_2 = \mathrm{i}r_2,$$

where $a$, $b$, $r_1$ and $r_2$ are real constants. We, then, obtain periodic kink-wave solutions

$$u_7 = u_0 - 2a\frac{\sqrt{B}\sin\left[a(x+\omega z) + \frac{4\omega a^4-3b^2}{8a}t + r_1\right] - \sinh\left[a(x+\omega z) + by - \frac{4\omega a^4-3b^2}{8a}t + r_2\right]}{\sqrt{B}\cos\left[a(x+\omega z) + \frac{4\omega a^4-3b^2}{8a}t + r_1\right] + \cosh\left[a(x+\omega z) + by - \frac{4\omega a^4-3b^2}{8a}t + r_2\right]}, \tag{2.11}$$

$$u_8 = u_0 + 2a\frac{\sqrt{B}\cos\left[a(x+\omega z) + \frac{4\omega a^4-3b^2}{8a}t + r_1\right] + \sinh\left[a(x+\omega z) + by - \frac{4\omega a^4-3b^2}{8a}t + r_2\right]}{\sqrt{B}\sin\left[a(x+\omega z) + \frac{4\omega a^4-3b^2}{8a}t + r_1\right] + \cosh\left[a(x+\omega z) + by - \frac{4\omega a^4-3b^2}{8a}t + r_2\right]}, \tag{2.12}$$

which are periodic waves in $x$–$z$ direction, meanwhile they are kink-waves in $x, y, z$ direction, where $B = \frac{b^2+\omega a^4}{b^2-\omega a^4}$ and $a, b, \omega$ are free constants.

Solution (2.11) is found reported in [19,20]. By verification, the solutions obtained above are indeed the solutions of original equation (1.1).

*Case* (III).

$$H = 0, \qquad K = \frac{L^2(b^2 - \omega a^4)}{4b^2},$$

$$a_1 = b_2 = d_1 = 0, \qquad a_2 = a, \qquad b_1 = b, \qquad d_2 = -\frac{\omega a^4 + 3b^2}{4a}, \qquad r_1 = r_1, \qquad r_2 = r_2.$$

Substituting these results into (2.4), we obtain

$$\phi_5 = \frac{L}{b}\left[\sqrt{b^2 - \omega a^4}\cosh(by + r_1^*) + b\cos\left(a\xi - \frac{\omega a^4 + 3b^2}{4a}t + r_2\right)\right], \qquad b^2 - \omega a^4 > 0,$$

$$\phi_6 = -\frac{L}{b}\left[\sqrt{\omega a^4 - b^2}\sinh(by + r_1^{**}) - b\cos\left(a\xi - \frac{\omega a^4 + 3b^2}{4a}t + r_2\right)\right], \qquad b^2 - \omega a^4 < 0.$$

Therefore, we obtain periodic solitary wave solutions

$$u_9 = u_0 - 2ab\frac{\sin\left[a(x+\omega z) - \frac{\omega a^4+3b^2}{4a}t + r_2\right]}{\sqrt{b^2 - \omega a^4}\cosh(by + r_1^*) + b\cos\left[a(x+\omega z) - \frac{\omega a^4+3b^2}{4a}t + r_2\right]}, \tag{2.13}$$

$$u_{10} = u_0 + 2ab\frac{\sin\left[a(x+\omega z) - \frac{\omega a^4+3b^2}{4a}t + r_2\right]}{\sqrt{\omega a^4 - b^2}\sinh(by + r_1^{**}) + b\cos\left[a(x+\omega z) - \frac{\omega a^4+3b^2}{4a}t + r_2\right]}, \tag{2.14}$$

which are periodic waves in $x$–$z$ direction, meanwhile they are solitons in $y$-direction, where $r_1^* = r_1 + \ln(\frac{L}{2b}\sqrt{b^2 - \omega a^4})$, $r_1^{**} = r_1 + \ln(\frac{L}{2b}\sqrt{-b^2 + \omega a^4})$ and $a, b, \omega$ are free constants.

In particular, for special choices of the constants $a, b, r_1^*, r_1^{**}, r_2$ in (2.13)–(2.14), we can also obtain some solutions for Eq. (1.1).

(i) Choosing

$$b = \mathrm{i}b, \qquad r_1^* = \mathrm{i}r,$$

then, from solution (2.13) we obtain the double periodic solution

$$u_{11} = u_0 - 2ab\frac{\sin\left[a(x+\omega z) - \frac{\omega a^4-3b^2}{4a}t + r_2\right]}{\sqrt{b^2 + \omega a^4}\cos(by + r) + b\cos\left[a(x+\omega z) - \frac{\omega a^4-3b^2}{4a}t + r_2\right]}, \tag{2.15}$$

where $a, b, \omega, r$ are real free constants and satisfy $b^2 + \omega a^4 > 0$.

(ii) Setting

$$a = \mathrm{i}a, \qquad r_2 = \mathrm{i}r,$$

then, from (2.13) and (2.14) we obtain cross kink-wave solutions

$$u_{12} = u_0 + 2ab\frac{\sinh\left[a(x+\omega z) + \frac{\omega a^4+3b^2}{4a}t + r\right]}{\sqrt{b^2 - \omega a^4}\cosh(by + r_1^*) + b\cosh\left[a(x+\omega z) + \frac{\omega a^4+3b^2}{4a}t + r\right]}, \quad b^2 - \omega a^4 > 0, \tag{2.16}$$

$$u_{13} = u_0 - 2ab\frac{\sinh\left[a(x+\omega z) + \frac{\omega a^4+3b^2}{4a}t + r\right]}{\sqrt{\omega a^4 - b^2}\sinh(by + r_1^{**}) + b\cosh\left[a(x+\omega z) + \frac{\omega a^4+3b^2}{4a}t + r\right]}, \quad b^2 - \omega a^4 < 0, \tag{2.17}$$

where $a, b, \omega, r$ are free real constants.

(iii) Choosing

$$a = \mathrm{i}a, \qquad b = \mathrm{i}b, \qquad r_1^* = \mathrm{i}r^*, \qquad r_1^{**} = \mathrm{i}r^{**}, \qquad r_2 = \mathrm{i}r,$$

we, then, obtain periodic kink-wave solutions

$$u_{14} = u_0 + 2ab\frac{\sinh\left[a(x+\omega z) + \frac{\omega a^4-3b^2}{4a}t + r\right]}{\sqrt{b^2 + \omega a^4}\cos(by + r^*) + b\cosh\left[a(x+\omega z) + \frac{\omega a^4-3b^2}{4a}t + r\right]}, \tag{2.18}$$

$$u_{15} = u_0 - 2ab\frac{\sinh\left[a(x+\omega z) + \frac{\omega a^4-3b^2}{4a}t + r\right]}{\sqrt{\omega a^4 + b^2}\sin(by + r^{**}) + b\cosh\left[a(x+\omega z) + \frac{\omega a^4-3b^2}{4a}t + r\right]}, \tag{2.19}$$

which are periodic waves in $y$-direction, meanwhile they are kink-waves in $x$–$z$ direction, where $a, b, \omega, r^*, r^{**}$ and $r$ are real constants (see Fig. 2. (ii)).

Solutions represented by (2.13), (2.15) and (2.16) are reported in [20] while others are new solutions for Eq. (1.1). By verification, the solutions obtained above are indeed the solutions of original $(3+1)$-dimensional YTSF equation.

*Case* (IV).

$$a_1 = 0, \qquad b_1 = b, \qquad d_1 = \frac{b\sqrt{3(\omega a^4 + 3b^2)}}{2a}, \qquad r_1 = r_1,$$

$$a_2 = a, \qquad b_2 = \frac{\sqrt{\omega a^4 + 3b^2}}{\sqrt{3}}, \qquad d_2 = 0, \qquad r_2 = r_2,$$

$$a_3 = -\mathrm{i}\frac{b}{a\sqrt{\omega}}, \qquad b_3 = -\mathrm{i}\frac{b\sqrt{\omega a^4 + 3b^2}}{a^2\sqrt{3\omega}}, \qquad d_3 = \mathrm{i}\frac{b(\omega a^4 - b^2)}{2a^3\sqrt{\omega}}, \qquad r_3 = \mathrm{i}r_3,$$

$$K = \frac{1}{4}(b^2 - \omega a^4)\left(\frac{L^2}{b^2} - \frac{H^2}{\omega a^4}\right).$$

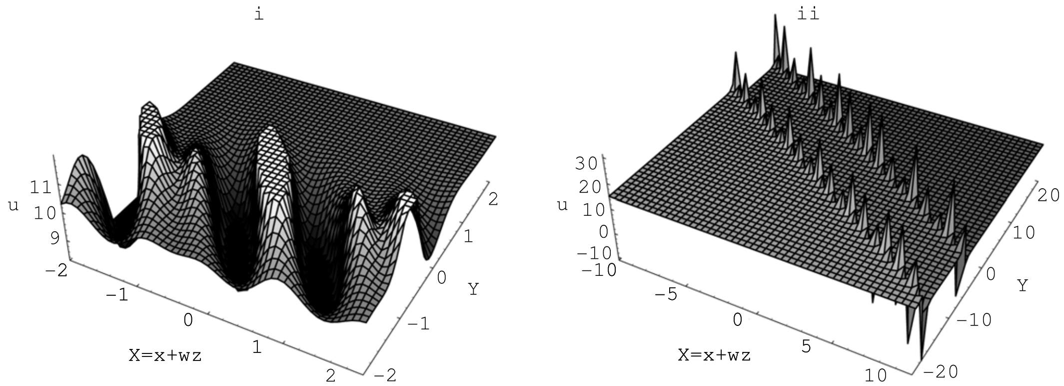


**Fig. 3.** (i) Mechanical feature of $u_{16}$: $u_0 = 10, L = 10, H = 20, a = 5, b = 1, \omega = 0.4, t = 1$. (ii) Mechanical feature of $u_{18}$: $u_0 = 10, L = 10, H = 20, a = 0.05, b = 0.04, \Omega = 0.01, t = 1$.

Substituting these into (2.4), we obtain

$$\phi_8 = 2\sqrt{K}\cosh(\eta_1) + L\cos(\eta_2) + H\cos(\eta_3), \quad K > 0, \tag{2.20}$$

$$\phi_9 = 2\sqrt{-K}\sinh(\eta_1) + L\cos(\eta_2) + H\cos(\eta_3), \quad K < 0, \tag{2.21}$$

where

$$\eta_1 = by + \frac{\sqrt{3}}{2}\frac{b\sqrt{\omega a^4 + 3b^2}}{a}t + (r_1 + \ln\sqrt{|K|}), \tag{2.22}$$

$$\eta_2 = a(x + \omega z) + \frac{\sqrt{\omega a^4 + 3b^2}}{\sqrt{3}}y + r_2, \tag{2.23}$$

$$\eta_3 = \frac{b}{a\sqrt{\omega}}(x + \omega z) + \frac{b\sqrt{\omega a^4 + 3b^2}}{a^2\sqrt{3\omega}}y - \frac{b(\omega a^4 - b^2)}{2a^3\sqrt{\omega}}t + r_3, \tag{2.24}$$

and therefore we obtain

$$u_{16} = u_0 - 2\cdot\frac{La\sin(\eta_2) + \frac{Hb}{a\sqrt{\omega}}\sin(\eta_3)}{2\sqrt{K}\cosh(\eta_1) + L\cos(\eta_2) + H\cos(\eta_3)}, \quad K > 0, \tag{2.25}$$

$$u_{17} = u_0 - 2\cdot\frac{La\sin(\eta_2) + \frac{Hb}{a\sqrt{\omega}}\sin(\eta_3)}{2\sqrt{-K}\sinh(\eta_1) + L\cos(\eta_2) + H\cos(\eta_3)}, \quad K < 0, \tag{2.26}$$

where $\eta_1, \eta_2, \eta_3$ are defined by (2.22)–(2.24), $K = \frac{1}{4}(b^2 - \omega a^4)(\frac{L^2}{b^2} - \frac{H^2}{\omega a^4})$ and $a, b, \omega, L, H$ are free constants.

Solution represented by (2.25) (Res. (2.26)) is a double periodic solitary wave solution which is soliton in $y, t$-direction, meanwhile it is periodic wave in $x$ (see Fig. 3. (i)).

Similarly, for special choices of $b, r_1, \omega$ in (2.25) and (2.26) we obtain the following solutions.

(i) Make the dependent variable transformation in (2.25) and (2.26) as follows:

$$b = \mathrm{i}b, \qquad r_1 = \mathrm{i}r,$$

where $b, r$ are real constants, and therefore we obtain

$$u_{18} = u_0 - 2\frac{La\sin(\eta_2) - \frac{Hb}{a\sqrt{\omega}}\sinh(\eta_3)}{2\sqrt{K_1}\cos(\eta_1) + L\cos(\eta_2) + H\cosh(\eta_3)}, \tag{2.27}$$

$$u_{19} = u_0 - 2\frac{La\sin(\eta_2) - \frac{Hb}{a\sqrt{\omega}}\sinh(\eta_3)}{-2\sqrt{K_1}\sin(\eta_1) + L\cos(\eta_2) + H\cosh(\eta_3)}, \tag{2.28}$$

where

$$\eta_1 = by + \frac{b\sqrt{3(\omega a^4 - 3b^2)}}{2a}t + r,$$

$$\eta_2 = a(x + \omega z) + \sqrt{(\omega a^4 - 3b^2)/3}y + r_2,$$

$$\eta_3 = \frac{b}{a\sqrt{\omega}}(x + \omega z) + \frac{b}{a^2}\sqrt{(\omega a^4 - 3b^2)/3\omega}y - \frac{b(\omega a^4 + b^2)}{2a^3\sqrt{\omega}}t + r,$$

$K_1 = \frac{1}{4}(b^2 + \omega a^4)(\frac{L^2}{b^2} + \frac{H^2}{\omega a^4})$ and $a, b, \omega, L, H$ are free constants.

(ii) Let $\omega = -\Omega$ satisfies $\Omega > 0$. Then, from (2.25), we obtain

$$u_{20} = u_0 - 2 \cdot \frac{La\sin(\eta_2) - \frac{Hb}{a\sqrt{\Omega}}\sinh(\eta_3)}{2\sqrt{K_2}\cosh(\eta_1) + L\cos(\eta_2) + H\cosh(\eta_3)}, \tag{2.29}$$

where $\eta_1, \eta_2, \eta_3$ are defined as

$$\eta_1 = by + \frac{b\sqrt{3(-\Omega a^4 + 3b^2)}}{2a}t + (r_1 + \ln\sqrt{K_2}),$$

$$\eta_2 = a(x - \Omega z) + \frac{\sqrt{-\Omega a^4 + 3b^2}}{\sqrt{3}}y + r_2,$$

$$\eta_3 = \frac{b}{a\sqrt{\Omega}}(x - \Omega z) + \frac{b\sqrt{-\Omega a^4 + 3b^2}}{a^2\sqrt{3\Omega}}y + \frac{b(\Omega a^4 + b^2)}{2a^3\sqrt{\Omega}}t + r_3,$$

and $K_2 = \frac{1}{4}(b^2 + \Omega a^4)(\frac{L^2}{b^2} + \frac{H^2}{\Omega a^4})$, $a, b, \Omega, L, H$ are free constants.

Solution represented by (2.27) (Res. (2.28) and (2.29)) is breather type of two-solitary wave solution which contains a periodic wave and two-solitary waves, whose amplitude periodically oscillates with the evolution of time (see Fig. 3. (ii)).

By verification, the solutions obtained above are indeed the solutions of original $(3 + 1)$-dimensional YTSF equation.

## 3. Conclusion

In this work, the three-wave type of ansätz approach is applied to the $(3 + 1)$-dimensional YTSF equation. New periodic solitary wave solutions are successfully obtained. These solutions include periodic solitary waves which are periodic waves in $x, y, z$ direction, meanwhile they are solitary waves in $y$–$t$ direction, periodic cross-kink waves which are periodic waves in forward direction, meanwhile they are kink-waves in backward direction. Moreover, a new periodic type of three-wave solutions including breather type of two-solitary solutions which are periodic waves in $y$–$t$ direction, meanwhile they are solitons in $x, y, z$, and doubly periodic solitary solution which is soliton in $y$–$t$ direction, meanwhile it is doubly periodic wave in $x, y, z$ direction are obtained as well. These results show that the three-wave type of ansätz approach is effective and simple method for seeking three-wave solutions and two-wave solutions of higher dimensional nonlinear evolution equations.

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