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Edge-colorings avoiding rainbow and monochromatic subgraphs

Maria Axenovich, Perry Iverson

Department of Mathematics, Iowa State University, USA

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Abstract

For two graphs G and H, let the mixed anti-Ramsey numbers, $\max R(n; G, H)$, $(\min R(n; G, H))$ be the maximum (minimum) number of colors used in an edge-coloring of a complete graph with n vertices having no monochromatic subgraph isomorphic to G and no totally multicolored (rainbow) subgraph isomorphic to H. These two numbers generalize the classical anti-Ramsey and Ramsey numbers, respectively.

We show that $\max R(n; G, H)$, in most cases, can be expressed in terms of vertex arboricity of H and it does not depend on the graph G. In particular, we determine $\max R(n; G, H)$ asymptotically for all graphs G and H, where G is not a star and H has vertex arboricity at least 3.

In studying $\min R(n; G, H)$ we primarily concentrate on the case when $G = H = K_3$. We find $\min R(n; K_3, K_3)$ exactly, as well as all extremal colorings. Among others, by investigating $\min R(n; K_t, K_3)$, we show that if an edge-coloring of K_n in k colors has no monochromatic K_t and no rainbow triangle, then $n \leqslant 2^{kt^2}$.

Keywords: Mixed Ramsey; Edge-coloring; Monochromatic; Totally multicolored

1. Introduction

An edge-colored graph is called monochromatic if all its edges have the same color. An edge-colored graph is called rainbow or totally multicolored if all its edges have distinct colors. For graphs G and H, we say that an edge-coloring of K_n is (G, H)-*good* if it contains neither a monochromatic copy of G nor a rainbow copy of H. The following proposition, see [19] characterizes the pairs of graphs for which (G, H)-good colorings exist for arbitrary large n.

Proposition 1 (*Jamison, West [19]*). *For any large enough n, there is a (G, H)-good coloring of $E(K_n)$ if and only if the edges of G do not induce a star and H is not a forest.*

We call a (K_s, K_t)-good coloring simply (s, t)-good. Let $\max R(n; G, H)$ $(\min R(n; G, H))$ be the maximum (minimum) number of colors in a (G, H)-good coloring of K_n.

We call these two functions mixed Ramsey numbers. They are closely related to the classical anti-Ramsey function $AR(n, H)$ and the classical multicolor Ramsey function $R_k(G)$, respectively. Here $AR(n, H)$ is defined to be the largest number of colors in an edge-coloring of K_n not containing a rainbow copy of H. This function was introduced by

E-mail addresses: axenovic@math.iastate.edu (M. Axenovich), piverson@iastate.edu (P. Iverson).

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Erdős et al., see [13], see also [2,6,22]. The classical multicolor Ramsey function $R_k(G)$ is defined to be the smallest n such that any coloring of $E(K_n)$ in k colors contains a monochromatic copy of G, see for example [15,16]. Therefore, we see that studying max $R(n; G, H)$ is similar to studying $AR(n, H)$ and forbidding monochromatic G. Studying min $R(n; G, H)$ is similar to investigating $R_k(G)$ and forbidding rainbow H. Mixed Ramsey numbers are also related to various generalized Ramsey numbers. A (k, p, q)-*coloring* of $E(K_n)$ is a coloring such that each copy of K_k uses at least p and at most q colors. Thus, a $\left(k, 2, \binom{k}{2} - 1\right)$-coloring is simply a (K_k, K_k)-good coloring. The properties of (k, p, q) colorings with respect to maximum or minimum number of colors have been addressed in [3,5,8,11,24]. On the other hand, the problem of finding unavoidable rainbow H or monochromatic G in *any* coloring of K_n for n large enough, has been studied in [19], when H is a forest and G is a star, see also [18].

Observe that functions max $R(n; G, H)$ and min $R(n; G, H)$ are not defined for all graphs. To find all graphs for which these functions are defined, we shall need the following version of the Canonical Ramsey Theorem. Here, we say that c is a *lexical* edge-coloring of a graph F if its vertices can be ordered $v_1, \ldots, v_m$, and the colors can be renamed such that $c(v_i, v_j) = \min\{i, j\}$, for all $v_i v_j \in E(F)$.

Theorem 1 (*Deuber [10] and Erdős and Rado [12]*). *For any integers m, ℓ, r, there is an integer $n = n(m, \ell, r)$ such that any edge coloring of K_n contains either a monochromatic copy of K_m, a rainbow copy of K_r, or a lexically colored copy of K_ℓ.*

The smallest integer n, satisfying the conditions of Theorem 1 is called the Erdős–Rado number and is denoted $\mathrm{ER}(m, \ell, r)$. In general, the best bounds for symmetric Erdős–Rado numbers were provided by Lefmann and Rödl [21], in the following form:

$$2^{c_1 \ell^2} \leqslant \mathrm{ER}(\ell, \ell, \ell) \leqslant 2^{c_2 \ell^2 \log \ell}$$

for some constants c_1, c_2.

To state and prove our results we need the following definitions. The *vertex arboricity*, $a(H)$, of a graph H, is the smallest number of vertex sets partitioning $V(H)$, such that each of these sets induces a forest in H. The extremal function $\mathrm{ex}(n, H)$, for a graph H, is the largest number of edges in an n-vertex graph not containing H as a subgraph. The Turán graph $T(n, k)$ is an n-vertex complete k-partite graph with parts of almost equal sizes (different by at most one). The Turán theorem, [25], states that $\mathrm{ex}(n, K_{k+1}) = |E(T(n, k))|$. In general, the Erdős–Stone theorem, [14], states that $\mathrm{ex}(n, H) = |E(T(n, k))|(1 + \mathrm{o}(1))$, if the chromatic number of H, $\chi(H)$, is equal to $k + 1$, $k \geqslant 2$. For all other graph theoretic notions we refer the reader to [26].

Theorem 2. *Let G be a graph whose edges do not induce a star. Let H be a graph.*

(1) *If $a(H) \geqslant 3$ then*

$$\max R(n; G, H) = \frac{n^2}{2}\left(1 - \frac{1}{a(H) - 1}\right)(1 + \mathrm{o}(1)).$$

(2) *If $a(H) = 2$ then*

$$\max R(n; G, H) \leqslant c n^{2-(1/s)},$$

where $s = s(|V(G)|, |V(H)|)$.

(3) *If $a(H) = 1$ then* max $R(n; G, H)$ *is not defined.*

Thus, in particular, Theorem 2 determines max $R(n; G, H)$ for graphs H with vertex arboricity at least three. In the following theorem, we collect some partial results dealing with several classes of graphs H with vertex-arboricity 2. As follows from these results, max $R(n; G, H)$ can take a wide range of values, from linear to subquadratic; the value depends heavily on the structure of H.

Theorem 3. *Let G be a graph whose edges do not induce a star. Let H be a graph such that $a(H) = 2$.*

(1) *If each edge-cut of H contains a star with three edges or at least four vertices of degree* 2; *and* $\chi(G) \geqslant 3$ *then*

$$\tfrac{1}{4}\, n \log n \leqslant \max R(n;\, G,\, H).$$

(2) *If* $\chi(G) \geqslant 3$ *then*

$$\max R(n;\, G,\, C_k) = n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + \mathrm{O}(1).$$

(3) *If H contains a subgraph isomorphic to $K_{r,r}$, $r \geqslant 4$ then for a constant c,*

$$cn^{2-\frac{2}{r-2}} \leqslant ex(n,\, K_{r-2,r-2}) \leqslant \max R(n;\, G,\, H).$$

(4) *If $V(H)$ can be split into two sets inducing forests with one forest of order at most* 2 *then*

$$\max R(n;\, G,\, H) \leqslant n^{5/3}(1 + \mathrm{o}(1)).$$

Theorem 3 parts (1) and (4) implies, in particular, that $(\frac{1}{4})n \log n \leqslant \max R(n;\, K_4,\, K_4) \leqslant n^{5/3}(1+\mathrm{o}(1))$. Determining this mixed Ramsey number remains one of the most interesting open problems in this area.

The next few results are concerned with function $\min R(n;\, G,\, H)$. The following result was proved in [9].

Theorem 4 (*Chung and Grinstead [9]*). *Let* $\min R(n;\, K_3,\, K_3) = k$, *then*

$$n = \begin{cases} \sqrt{5}^{k}, & k \text{ is even},\\ 2\sqrt{5}^{k-1}, & k \text{ is odd}. \end{cases}$$

The main contribution of this paper in studying $\min R$ is to describe **all** extremal colorings corresponding to these mixed Ramsey numbers (see Section 3) and give **all** corresponding extremal colorings. As a corollary of this classification, we obtain the following results.

Theorem 5. $\mathrm{ER}(3, n, 3) = \sqrt{5}^{n-1} + 1$ *if n is odd*. $\mathrm{ER}(3, n, 3) = 2\sqrt{5}^{n-2} + 1$ *if n is even.*

We also note that if one considers classical multicolor Ramsey problem and imposes an additional constraint of not having a rainbow triangle, then the modified Ramsey number will be relatively small. In our terminology, we have the following:

Theorem 6. *If* $\min R(n;\, K_t,\, K_3) = k$ *then* $n \leqslant 2^{kt^2}$.

We prove Theorem 2 in Section 2. We prove Theorems 4–6 in Section 3. Finally, in Section 4, we study some miscellaneous problems related to the mixed Ramsey numbers. We study the colorings avoiding rainbow K_4 and monochromatic K_3 by analyzing the lexically colored subgraphs and provide some bounds on $\max R(n;\, K_3,\, K_4)$. We also try to relate the classical multicolor Ramsey numbers for triangles with $\min R(n;\, K_3,\, K_3)$ in the last section. In doing so, we show that there are colorings of $E(K_n)$ where each subset of $\log n$ vertices contains a rainbow triangle. For an edge-coloring c of a graph G, we shall use the following notation: if $A, B \subseteq V(G)$, A and B disjoint, then $c(A)$ is the set of all colors spanned by a set A; $c(A, B)$ is a set of colors present on the edges between A and B under coloring c.

2. Proof of Theorems 2 and 3

We first need the following lemmas and constructions.

Lemma 1. *Let F be a forest on n vertices. Let c be a lexical coloring of $K = K_n$. Then K contains a rainbow copy of F under c.*

Proof. We use induction on n, which holds trivially for $n=2$. Assume that the statement holds true for any smaller n. Let $F' = F - v$, where v is a leaf or an isolated vertex of F. Let $v_1, \ldots, v_n$ be an ordering of vertices of K such that, without loss of generality, $c(v_i, v_j) = i$, $1 \leqslant i < j \leqslant n$. Then, we have a rainbow copy of F' in $K - v_1$ under coloring c. Note that this copy of F' does not use color 1. Now, consider a copy F^* of F in K formed by F' and v_1 corresponding to v. There is a single edge incident to v_1 in F^* and it has color 1. Thus F^* is rainbow. □

Lemma 2. *Let G be a graph on g vertices whose edges do not induce a star; let H be a graph on h vertices. Let one of the following hold*:

(1) *H is a graph with vertex arboricity a, $a \geqslant 2$, and c is an edge-coloring of K_n using at least* $\mathrm{ex}(n; T(qa, a)) + 1$ *colors*, $q = \mathrm{ER}(g, a(ah^2+2), h)$, *or*
(2) $V(H) = V_1 \cup V_2$, *where* $|V_1| \leqslant 2$, V_2 *induces a forest in H, and c is an edge-coloring of K_n using at least* $\mathrm{ex}(n; K_{3,t}) + 1$ *colors, where* $t = \mathrm{ER}(g, 3h+3, h) + 3$.

Then c contains either a monochromatic copy of G or a rainbow copy of H.

Proof. (1) Assume that there is a (G, H)-good coloring, c, of K_n with at least $\mathrm{ex}(n; T(qa, a)) + 1$ colors. Then there is a rainbow subgraph, R, of K_n such that $|E(R)| \geqslant \mathrm{ex}(n; T(qa, a)) + 1$. Thus R contains $T = T(qa, a)$ as a subgraph. Note that T is rainbow in c with partite sets $U_1, U_2, \ldots, U_a$, $|U_i| = q$, $1 \leqslant i \leqslant a$. By the Canonical Ramsey theorem, we have that each U_i induces either a monochromatic copy of K_g, thus containing a monochromatic copy of G; or a rainbow copy of K_h, thus containing a rainbow copy of H; or a lexically colored $K_{a(ah^2+2)}$. Since the first two options are not possible, we have that each U_i, $i = 1, \ldots, a$ contains a lexically colored $K_{a(ah^2+2)}$. Then there are subsets $V_i \subseteq U_i$, $|V_i| = ah^2+2$, $i = 1, \ldots, a$ such that V_is induce pairwise disjoint sets of colors. Let $T' = T[V_1 \cup V_2 \cup \cdots \cup V_a]$. Let B be the set of edges of T' whose colors appear on some edges inside V_i for some i, i.e.,

$$B = \{\{x, y\} \in E(T') : c(xy) = c(zz'), \text{ for some } z, z' \in V_k, 1 \leqslant k \leqslant a\}.$$

Claim. *There exist $W_i \subseteq V_i$ with $|W_i| \geqslant h$, $i = 1, \ldots, a$ such that $T'' = T'[W_1 \cup W_2 \cup \cdots \cup W_a]$ has no edges from B.*

Otherwise for any choice of subsets $W_i' \subseteq V_i$, $|W_i'| \geqslant h$, $T'[W_1' \cup \cdots \cup W_a']$ contains an edge from B. Let $s = |V_i| = ah^2 + 2$, $i = 1, \ldots, a$. Then we have that

$$|B| \geqslant \frac{\binom{s}{h}^a}{\binom{s-1}{h-1}^2 \binom{s}{h}^{a-2}} = \frac{s^2}{h^2}.$$

On the other hand, $|B| \leqslant a\,(s-1)$, (here the expression on the right corresponds to $|c(V_1) \cup c(V_2) \cup \cdots \cup c(V_a)|$). Thus $a\,(s-1) \geqslant s^2/h^2$, implying that $ah^2 \geqslant s$, a contradiction which proves the Claim.

Thus, T'' defined above has all edges between the parts W_i totally multicolored and the edges inside the parts W_is lexically colored with new, pairwise disjoint sets of colors, not used on edges of T''. Since $a(H) = a$, $V(H)$ can be partitioned into a parts each inducing forests $F_1, \ldots, F_a$. By Lemma 1 for any $i = 1, \ldots, a$, W_i contains a rainbow forest F_i, thus $K_n[W_1 \cup W_2 \cup \cdots \cup W_a]$ contains a rainbow copy of H, a contradiction.

(2) Let c be a (G, H)-good coloring of K_n with more than $\mathrm{ex}(n, K_{3,t})$ colors. Then, there is a rainbow $K_{3,t}$ with parts A, B of sizes 3 and t, respectively. There are at most three edges of colors from $c(A)$ between A and B, and thus, by deleting at most three vertices from B, one can find a set $B' \subseteq B$, such that $c(A) \cap c(A, B') = \emptyset$, $|B'| \geqslant |B| - 3$. Since $|B'| \geqslant \mathrm{ER}(g, 3h+3, h)$ and c has no monochromatic G and no rainbow H, we have that B' contains a set B'' spanning a lexically colored K_{3h+3}. Again, by deleting at most three vertices from B'', one can find $B''' \subseteq B''$, $|B'''| \geqslant |B''| - 3$, such that $c(B''') \cap c(A) = \emptyset$. Since all edges between A and B''' have distinct colors, there are two vertices, say $x, y \in A$ incident to at most $2|c(B''')|/3 = 2(|B'''| - 1)/3$ edges of color from $c(B''')$ in $K_n[B''' \cup A]$. Thus there is a set of vertices, $B^* \subseteq B'''$, $|B^*| \geqslant |B'''|/3$,such that $c(\{x, y\}, B^*) \cap \left(c(A) \cup c(B''')\right) = \emptyset$. Since $|B^*| \geqslant h$, and B^* spans any rainbow forest on h vertices, $B^* \cup \{x, y\}$ spans a rainbow H, a contradiction.

This argument can be made more precise for $H = K_4$ allowing t to be smaller by a somewhat tedious case analysis, see for example [5]. This concludes the proof of Lemma 2. □

Consider a graph G whose edges do not span a star. Let H be a graph, let $a = a(H)$, be the vertex arboricity of H, let $h = |V(H)|$, $g = |V(G)|$.

Construction 1. For $a \geqslant 3$, let Q be a Turán graph $T(n, a-1)$ with parts $V_1, \ldots, V_{a-1}$. Construct an edge-coloring, c, of K_n by totally multicoloring the edges of Q and lexically coloring the complete graph induced by each V_i, $i = 1, \ldots, a-1$ with pairwise disjoint sets of new colors.

For any copy of H in K_n, there is $i \in \{1, \ldots, a-1\}$ such that $H[V_i]$ contains a cycle. This cycle has at least two edges of the same color under c. Thus there is no rainbow copy of H in coloring c. On the other hand, there is no monochromatic copy of a graph G since all color classes in c are stars. Using the following: $|E(T(n,k))| \geqslant n^2/2(1-(1/k)) - n/8$, we have that the total number of colors in this coloring is $|E(T(n, a-1))| + n - a + 1 \geqslant n^2/2(1 - \frac{1}{a-1}) - (n/8) + n - a + 1 = n^2/2(1 - \frac{1}{a-1})(1 + \mathrm{o}(1))$.

Construction 2. Let k be the smallest integer such that $n \leqslant 2^k$. We shall first describe a coloring, c, of K_N, where $N = 2^k$. Label the vertices of K_N by binary vectors of length k, color the edges corresponding to the edges of a hypercube (i.e., the ones with Hamming distance one on corresponding vectors), with distinct colors from the set $\{0, -1, -2, \ldots\}$. For any other edge, let its color be the smallest index of the position where the vectors corresponding to the endpoints differ.

This coloring uses exactly $\frac{1}{2} N \log N + \log N$ colors. Each color class in this coloring corresponds to a bipartite graph thus there is no monochromatic G with $\chi(G) \geqslant 3$. On the other hand, one can show by induction on k that this coloring does not have any rainbow H such that each of its edge-cuts has either a vertex of degree at least three or four vertices of degree at least two. Indeed, the basis is trivial; for a step, assume that there is a rainbow copy of H with $V(H) = S_0 \cup S_1$, such that the vertices in S_i have corresponding binary vectors with i in the first position, $i = 0, 1$. If one of S_is is empty, we apply induction, otherwise, we see that there are at least two edges of color 1 between S_0 and S_1. Thus H is not rainbow, a contradiction. Now, consider a complete subgraph of K_N on n vertices using the largest number, s, of colors under c. Since $N/2 \leqslant n \leqslant N$, we have that $s \geqslant N \log N/4 \geqslant (n \log n)/4$.

The following construction was given in [13], we include it here for completeness.

Construction 3. Let $t = \lfloor n/(k-1) \rfloor$. Let the vertex set, V, of a complete graph G be a disjoint union of $V_0, V_1, \ldots, V_t$, such that $|V_1| = \cdots = |V_t| = k-1$, $0 \leqslant |V_0| < k-1$. Let the color of any edge with one endpoint in V_i and another endpoint in V_j, for $i < j$, be i. Color the edges spanned by each V_i, $i = 0, \ldots, t$, with new distinct colors using pairwise disjoint sets of colors for each V_i.

The total number of colors in this coloring is at least

$$\binom{k-1}{2} t + \binom{n - t(k-1)}{2} + t - 1 = \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + \mathrm{O}(1).$$

This coloring does not have any rainbow cycle of length at least k and does not have any monochromatic graph G, $\chi(G) \geqslant 3$.

Construction 4. This construction is similar to Construction 2. Let k be the smallest integer such that $n \leqslant 2^k$. We shall first describe a coloring, c, of K_N, where $N = 2^k$. Let r be an integer, $r \geqslant 4$. Label the vertices of K_N by binary vectors of length k, let the color of an edge be the smallest index of the position where the vectors corresponding to its endpoints differ. Now, let S_0 and S_1 be the sets of vertices whose first positions are encoded 0 and 1, respectively. Let Q be a bipartite graph with parts S_0 and S_1 which contains no subgraph isomorphic to $K_{r-2,r-2}$ and having largest possible number of edges. Lets recolor the edges of Q with new distinct colors.

The total number of colors used is $\mathrm{ex}(N, K_{r-2,r-2})(1 + \mathrm{o}(1))$. Since each color class corresponds to a bipartite graph, the coloring does not contain any monochromatic G with $\chi(G) \geqslant 3$. Let H be a graph containing $K_{r,r}$ as a subgraph. Assume that the coloring has a rainbow copy of H, then it has a rainbow copy of $K_{r,r}$ with parts A and B. Let $A = A_0 \cup A_1$, $B = B_0 \cup B_1$, where $A_0, B_0 \subseteq S_0$, $A_1, B_1 \subseteq S_1$. Lets assume without loss of generality that $|A_0| \geqslant r/2 \geqslant 2$. Since the complete subgraphs induced by S_i, $i = 0, 1$ do not have rainbow cycles, we see that $|B_0| \leqslant 1$.

Thus $|B_1| \geqslant r-1$ and $|A_1| \leqslant 1$, so $|A_0| \geqslant r-1$. Since Q does not contain a subgraph isomorphic to $K_{r-2,r-2}$, there are at least two edges colored 1 between A_0 and B_1. Thus H is not rainbow, a contradiction. As in Construction 2 we can choose a complete subgraph on n vertices which, in this coloring, has at least $c_1 \operatorname{ex}(n, K_{r-2,r-2})$ colors, for a constant c_1.

Proof of Theorem 2. *Case* 1: $a(H) \geqslant 3$. Construction 1 gives the lower bound. The upper bound follows from Lemma 2 and the Erdős–Stone theorem stating that, for fixed s and a,

$$\operatorname{ex}(n, T(s,a)) = \binom{n}{2}\left(1 - \frac{1}{a-1}\right)(1 + \mathrm{o}(1)).$$

Case 2: $a(H) = 2$.

The bound follows from Lemma 2 and Kövari–Sós–Turán theorem, [20], see for example [7], that states that, for a fixed s,

$$\operatorname{ex}(n, T(s,2)) = \operatorname{ex}(n, K_{s,s}) \leqslant cn^{2-1/s}(1 + \mathrm{o}(1)).$$

Case 3: $a(H) = 1$. Proposition 1 shows that, for large n, there is no (G, H)-good coloring in this case, thus $\max R(n; G, H)$ is not defined.

This concludes the proof of Theorem 2. □

Proof of Theorem 3. (1) It follows from Construction 2.

(2) Construction 3 provides the lower bound. For the upper bound, we observe that $\max R(n; G, H) \leqslant AR(n; H) = AR(n; C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + \mathrm{O}(1)$, as was recently proved in [23].

(3) The bound follows from Construction 4.

(4) This bound follows from Lemma 2 and the fact that $\operatorname{ex}(n, K_{3,t}) \leqslant cn^{5/3}$ for a constant c, see, for example, [7]. □

3. Colorings with small number of colors avoiding rainbow and monochromatic triangles

Recall that a (3, 3)-good coloring is simply a (K_3, K_3)-good coloring. For convenience, instead of minimizing the number of colors in a (3, 3)-good coloring of K_n, we shall investigate a dual problem of determining $f'(k)$, where $f'(k) = \max\{n :$ there is a (3, 3)-good coloring of $E(K_n)$ using k colors$\}$. Observe that if $f'(k) = n$ then $\min R(n; 3, 3) = k$. At the same time we shall study the following function: $f(k) = \max\{n :$ there is a (3, 3)-good coloring of $E(K_n)$ with no lexically colored $K_{k+1}\}$. Observe that $f(k) = \mathrm{ER}(3, k+1, 3) - 1$. Below, we provide two sets of edge-colorings of complete graphs, which we shall prove to be all extremal colorings corresponding to the functions f and f'.

3.1. Construction of $\mathscr{G}(n)$, $\mathscr{G}'(n)$

We define **Pent**, to be the set of all 2-edge colorings of K_5 such that each color class induces a 5-cycle. We define **Bip** to be the set of all edge-colorings of K_2. Next we define two products of sets of colorings. Let $\mathscr{C}$ and $\mathscr{C}'$ be two sets of edge-colorings of complete graphs. We say that a coloring c of a complete graph G is in

$$\mathscr{C} \times \mathscr{C}'$$

if there are, for some m:

(a) a partition of vertices $V(G) = V_1 \cup V_2 \cup \cdots \cup V_m$,
(b) $c' \in \mathscr{C}'$, a coloring of a complete graph on vertices $v_1, \ldots, v_m$, such that all edges between V_i and V_j have color $c'(v_i, v_j)$, $1 \leqslant i < j \leqslant m$,
(c) $c_1, c_2, \ldots, c_m \in \mathscr{C}$ such that c restricted to $G[V_i]$ is equal to c_i, $i = 1, \ldots, m$,
(d) $c(V_i) \cap c'(\{v_1, \ldots, v_m\}) = \emptyset$.

We define $\mathscr{C} \otimes \mathscr{C}'$, a set of edge-colorings of G similarly to $\mathscr{C} \times \mathscr{C}'$ with an additional requirement that $c(V_i) = c(V_j)$, $1 \leqslant i < j \leqslant m$.

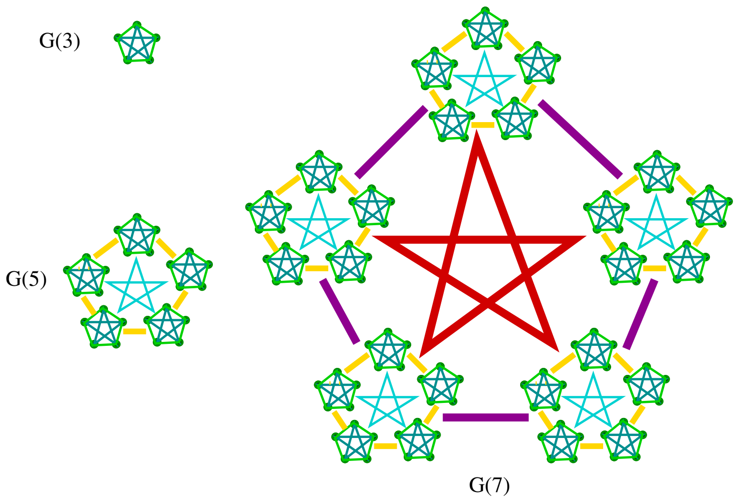

Fig. 1. $\mathscr{G}(n)$, n is odd.

Note that each coloring in $\mathscr{C} \times \mathscr{C}'$ is obtained by "blowing up" the vertices from some coloring in $\mathscr{C}'$ and using a coloring from $\mathscr{C}$ in each resulting part such that the colors inside the parts and between the parts do not overlap; each coloring in $\mathscr{C} \otimes \mathscr{C}'$ is obtained by "blowing up" the vertices from some coloring in $\mathscr{C}'$ and using some coloring from $\mathscr{C}$ in each resulting part such that each part uses the same set of colors and such that the colors inside the parts and between the parts do not overlap. Now we shall define the set of colorings $\mathscr{G}(n)$ recursively.

$$\mathscr{G}(n) = \begin{cases} \mathbf{Bip}, & n = 2; \\ \mathbf{Pent}, & n = 3; \\ (\mathscr{G}(n-2) \times \mathbf{Pent}) \cup (\mathscr{G}(n-1) \times \mathbf{Bip}), & n \text{ is even}, n \geqslant 4; \\ \mathscr{G}(n-2) \times \mathbf{Pent}, & n \text{ is odd}, n \geqslant 5. \end{cases}$$

Define $\mathscr{G}'(n)$ similarly. Let $\mathscr{G}'(i) = \mathscr{G}(i)$, $i = 2, 3$.

$$\mathscr{G}'(n) = \begin{cases} (\mathscr{G}'(n-2) \otimes \mathbf{Pent}) \cup (\mathscr{G}'(n-1) \otimes \mathbf{Bip}), & n \text{ is even}, n \geqslant 4; \\ \mathscr{G}'(n-2) \otimes \mathbf{Pent}, & n \text{ is odd}, n \geqslant 5. \end{cases}$$

See Figs. 1 and 2 for examples of colorings from $\mathscr{G}(n)$. Observe that all colorings in $\mathscr{G}(n)$ and $\mathscr{G}'(n)$ are defined on a complete graph with $N(n)$ vertices, where,

$$N(n) = \begin{cases} \sqrt{5}^{\,n-1} & \text{for } n \text{ odd}; \\ 2\sqrt{5}^{\,n-2} & \text{for } n \text{ even}. \end{cases} \tag{1}$$

Theorem 7. (1) *Let c be a* $(3, 3)$-*good coloring of K_N avoiding lexically colored K_{n+1} and N is as large as possible. Then $c \in \mathscr{G}(n)$.*

(2) *Let c be a* $(3, 3)$-*good coloring of K_N with n colors and N is as large as possible. Then $c \in \mathscr{G}'(n)$.*

Theorem 7 provides a description of any (3, 3)-good coloring with restricted number of colors and any (3, 3)-good coloring with restricted size of the lexically colored complete subgraphs. It shows that the restriction of having a fixed number, s, of colors and a restriction on the order, t, of largest lexical subgraph in (3, 3)-good colorings gives a very similar extremal graph coloring and the same corresponding Ramsey-type numbers, when $t = s + 1$. In particular, we have the infinite family of exact Canonical Ramsey numbers as follows:

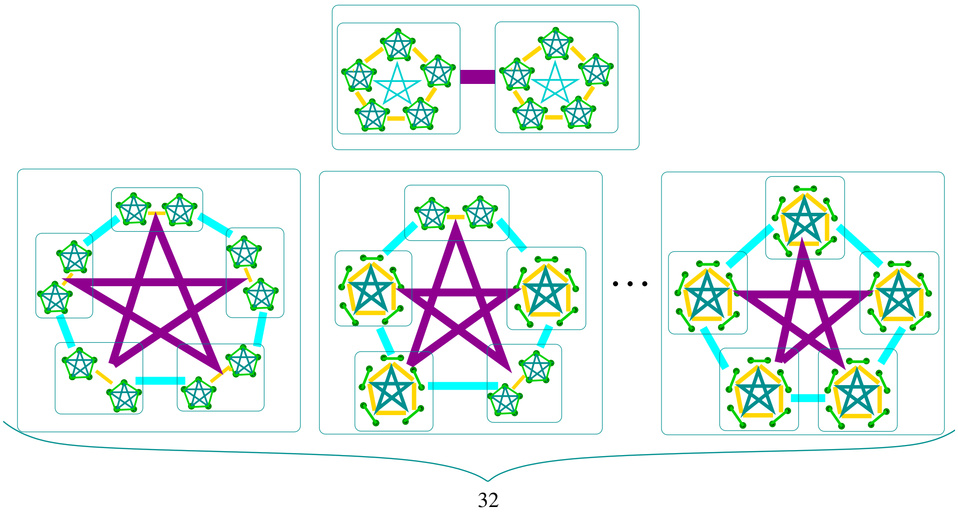

Fig. 2. $\mathcal{G}(n)$, n is even.

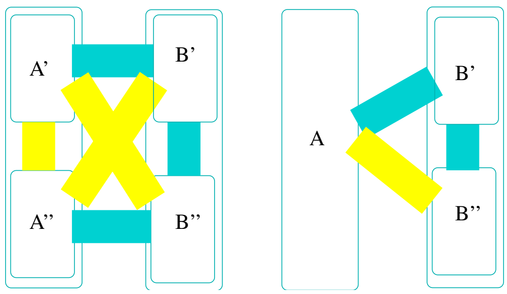

Fig. 3. A mixed pair (V_i, V_j).

Corollary 1. $\mathrm{ER}(3, n, 3) = \sqrt{5}^{\,n-1} + 1$ *if n is odd*, $\mathrm{ER}(3, n, 3) = 2\sqrt{5}^{\,n-2} + 1$, *if n is even. Moreover any coloring of K_N avoiding rainbow and monochromatic triangles and lexically colored K_n, with n as large as possible, is in $\mathcal{G}(n-1)$.*

While proving Theorem 7, we determine precisely the structure of the coloring between the monochromatic neighborhoods of a fixed vertex, giving a "local" perspective into the coloring. Note, that Theorem 7 can also be proved using a result of Gyárfás and Simonyi [17] stating that any coloring with no rainbow triangles can be obtained by "substituting" complete graphs with no rainbow triangles into vertices of 2-colored complete graphs thus describing Gallai colorings, see [16]. We prove Theorem 7 using a "local" argument in Section 3.2 and we prove it using the result of Gyárfás and Simonyi in Section 3.3. Both approaches are valuable: the first gives an understanding of a coloring structure in each vertex's neighborhood, which is promising for generalizations; on the other hand, the second proof is shorter.

3.2. *Proof of Theorem 7*

A pair (A, B) is monochromatic of color i if $c(A, B) = \{i\}$. A pair (A, B) is **mixed** of colors $\{i, j\}$ if $A = A' \cup A''$, $B = B' \cup B''$, $c(A', B') = c(B', B'') = c(B'', A'') = \{i\}$ and $c(A', B'') = c(A', A'') = c(A'', B') = \{j\}$. In a mixed pair, either A' or B' might be empty, but not both, see Fig. 3. If $c(A, B) = \{i\}$, we shall write $c(A, B) = i$. We can accurately describe the properties of (3, 3)-good colorings using monochromatic and mixed pairs as follows, see also Fig. 4.

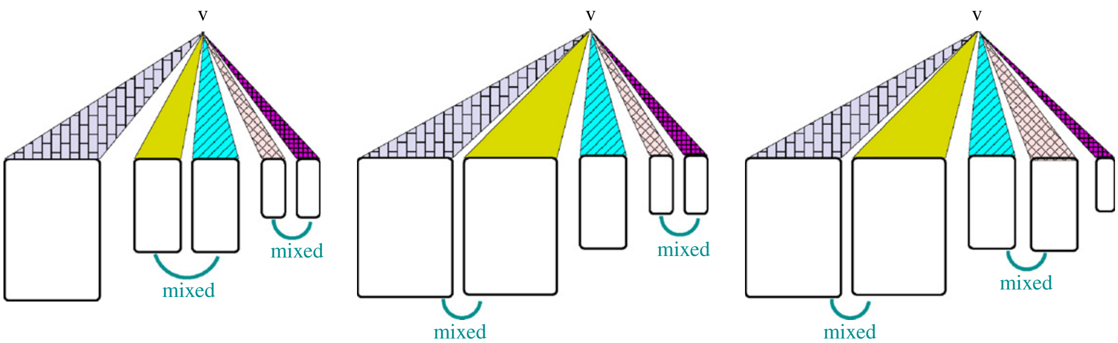

Fig. 4. All possible mixed pair configurations in extremal (3, 3)-good colorings with odd number of colors incident to v.

Lemma 3. *Let c be a $(3,3)$-good coloring of a complete graph G with colors $1, 2, \ldots$. Let $v \in V(G)$ and let $V_i = \{u \in V(G) : c(uv) = i\}$, $i = 1, 2, \ldots, k$. Assume that $V_i \neq \emptyset$, $i = 1, \ldots, k$. Then the following holds for an appropriate ordering of colors*:

(a) $c(V_i, V_j) \in \{i, j\}$ *and* (V_i, V_j) *is either monochromatic or mixed*,
(b) *if* (V_i, V_j) *is monochromatic then* $c(V_i, V_j) = i$, $i < j$,
(c) *if* (V_i, V_j) *is a mixed pair*, *then* $j = i + 1$,
(d) *if* (V_i, V_{i+1}) *is a mixed pair*, *then neither* (V_{i-1}, V_i) *nor* (V_{i+1}, V_{i+2}) *is a mixed pair.*

Proof. Part (a) of the lemma is easy and has been proved in [4], as well as the fact that if (V_i, V_j) is a mixed pair then (V_i, V_l) is not a mixed pair for any $l \neq j$, which immediately implies part (d). We prove part (b) by induction on k, which trivially holds for $k = 2$. Assume that sets $V_1, V_2, \ldots, V_{k-1}$ are ordered so that conclusion of part (b) holds. Observe that if $c(V_i, V_k) = i$ then for all j, $1 \leqslant j < i$, $c(V_j, V_k) = j$. Let i be the largest index such that $c(V_i, V_k) = i$. If no such i exists, define $i = 0$. If $i = k - 1$, we are done. Otherwise, for all $j > i$, we have $c(V_k, V_j) = k$ or (V_k, V_j) is a mixed pair. Let us relabel $V_{i+1}, V_{i+2}, \ldots, V_{k-1}$ to $V_{i+2}, V_{i+3}, \ldots, V_k$, and relabel V_k to V_{i+1}, respectively. The resulting ordering satisfies (b). To prove part (c), consider i, j, $1 \leqslant i < j - 1 \leqslant k - 1$ and assume that (V_i, V_j) is a mixed pair. Then there are vertices $v_i \in V_i$, $v_{i+1} \in V_{i+1}$, $v_j \in V_j$, such that $c(v_i, v_j) = j$, $c(v_i, v_{i+1}) = i$, $c(v_{i+1}, v_j) = i + 1$, giving a rainbow triangle, a contradiction. □

Lemma 3 implies that in a (3, 3)-good coloring, the coloring of the edges between the monochromatic neighborhoods of any vertex corresponds to a "blown up" lexical coloring with possible exceptional (mixed) pairs of consecutive sets.

Let c be a (3, 3)-good coloring of a complete graph G such that it has no lexical subgraph of order larger than n and such that G has as many vertices as possible, i.e., $|V(G)| = f(n)$. Let c' be a (3, 3)-good coloring of a complete graph G' such that it uses n colors and G' has as many vertices as possible, i.e., $|V(G')| = f'(n)$. Let's choose vertices v, v' incident to the largest number, k, k', of colors in c, c', respectively. Let $V_1, V_2, \ldots, V_k$ be defined with respect to v and c, and $U_1, \ldots, U_{k'}$ be defined with respect to v' and c' as in Lemma 3. In the following lemma we shall analyze the structure of colorings induced by V_is and U_is in c and c', respectively.

Lemma 4.

(a) *If* $1 \leqslant i \leqslant k$ *and* V_i *is not a part of a mixed pair in* c *then* $|V_i| = |V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k \cup \{v\}| = f(n - i)$.
(a′) *If* $1 \leqslant i \leqslant k'$ *and* U_i *is not a part of a mixed pair in* c' *then* $|U_i| = |U_{i+1} \cup U_{i+2} \cup \cdots \cup U_{k'} \cup \{v'\}| = f'(n - i)$.
(b) *If* $2 \leqslant i \leqslant k$ *and* (V_{i-1}, V_i) *is a mixed pair in* c *then* $|V'_{i-1}| = |V''_{i-1}| = |V'_i| = |V''_i| = |V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k \cup \{v\}| = f(n - i)$.
(b′) *If* $2 \leqslant i \leqslant k'$ *and* (U_{i-1}, U_i) *is a mixed pair in* c' *then* $|U'_{i-1}| = |U''_{i-1}| = |U'_i| = |U''_i| = |U_{i+1} \cup U_{i+2} \cup \cdots \cup U_k \cup \{v'\}| = f'(n - i)$.

Proof. We shall prove part (a), the other parts can be proven in a very similar manner. Observe first that $c(V_i) \cap \{1, 2, \ldots, i\} = \emptyset$ and $c(V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k \cup \{v\}) \cap \{1, 2, \ldots, i\} = \emptyset$. Let $T_i \subseteq V_i$ be the largest set of vertices spanning a lexically colored complete subgraph. If $|T_i| > n - i$ consider $S_i = \{v_1, v_2, \dots, v_{i-1}, v\} \cup T_i$. We have that S_i is a set on more than n vertices spanning a lexically colored complete subgraph. On the other hand, if $|T_i| < n - i$ then we can enlarge V_i, thus contradicting the maximality of the number of vertices in the original graph. Thus, we have that $|V_i| = f(n - i)$. Similarly, we have that $|V_{i+1} \cup V_{i+2} \cup \cdots \cup V_k \cup \{v\}| = f(n - i)$. □

Observe that if V_k is not a part of a mixed pair then $|V_k| = 1$, otherwise a vertex in V_k will be incident to more than k colors, a contradiction. If (V_{k-1}, V_k) is a mixed pair, we have similarly, that $|V_k'| = |V_k''| = 1$. Using Lemma 4, we have that $|V_k| = 1 = n - k$ or that $|V_k'| = 1 = n - k$, i.e.,

$$n = k + 1.$$

Lemma 4(a) and (b) give us the following recursion: if V_1 is not a part of a mixed pair, we have that $f(n) = 2f(n - 1)$; if (V_1, V_2) is a part of a mixed pair, we have that $f(n) = 5f(n - 2)$.

$$f(n) = 5f(n-2) = 5\cdots 5f(n-2i) = 5\cdots 5\cdot 2f(n-2i-1) = \overbrace{5\cdots 5}^{m}\cdot\overbrace{2\cdots 2}^{s}.$$

This expression is clearly maximized when m is largest possible, namely equal to $\lfloor k/2 \rfloor$. Therefore, when k is even, all pairs $(V_1, V_2), (V_3, V_4), \dots, (V_{k-1}, V_k)$ being mixed. For k odd, exactly one set, V_i, for some i, is not a part of a mixed pair and $(V_1, V_2), (V_3, V_4), \dots, (V_{i-2}, V_{i-1}), (V_{i+1}, V_{i+2}), \dots, (V_{k-1}, V_k)$ are mixed pairs. This shows that $c \in \mathcal{G}(n)$.

A very similar argument shows that $c' \in \mathcal{G}'(n)$. This concludes the proof of Theorem 7.

3.3. Proof of Theorems 7 and 8 using structure of Gallai colorings

In [17], Gyárfás and Simonyi proved a theorem first suggested by the work of Gallai in [16] which we will restate here as follows.

Proposition 2 (*Gyárfás and Simonyi [17]*). *Let c be an edge coloring of K_n with no rainbow triangles. Then $c \in C \times C'$, where C is a set of all 2-colorings and C' is a set colorings of a complete graph with less than n vertices with no rainbow triangle*.

This Proposition gives us the following proof for Theorem 7.

Proof of Theorem 7. Let c be a $(3, 3)$-good coloring of a complete graph G with maximum number, N, of vertices such that it does not contain lexically colored K_{n+1}. We shall prove by induction on n, that $c \in \mathcal{G}(n)$. If $n = 2$ then $c \in \mathcal{G}(2)$; if $n = 3$, $G \in \mathcal{G}(3)$. Let $n \geqslant 4$. Proposition 2 implies that $c \in C_1 \times C_2$, where C_1 is a set of all 2-colorings with no monochromatic triangle and C_2 is the set of all $(3, 3)$-good colorings of complete graphs on less than N vertices. We have, for some $c_1 \in C_1$, a coloring of a complete graph on vertices $v_1, \dots, v_m$, $V(G) = V_1 \cup \cdots \cup V_m$, where $c(V_i, V_j) = c_1(v_i v_j)$, $1 \leqslant i < j \leqslant m$ and c defined on $G[V_i]$ is in C_2, $1 \leqslant i \leqslant m$. We have, in particular that $m \leqslant 5$ since c_1 is a 2-coloring with no monochromatic triangles.

Case 1: c_1 uses one color. Then $c_1 \in \mathcal{G}(2)$, and c defined on $G[V_i]$ has no lexically colored K_n and has as many vertices as possible, $i = 1, 2$. Thus c, defined on $G[V_i]$, is in $\mathcal{G}(n - 1)$, $i = 1, 2$.

Case 2: c_1 uses two colors. Then, since N is maximum, $c_1 \in \mathbf{Pent} = \mathcal{G}(3)$, and $m = 5$. We have also that c defined on $G[V_i]$ has no lexically colored K_{n-1}, $i = 1, \dots, 5$, thus, again by maximality of N, c, defined on $G[V_i]$, is in $\mathcal{G}(n - 2)$.

Using the number of vertices $N(n)$ in any coloring from $\mathcal{G}(n)$, see (1), we have that in Case 1,

$$|V(G)| = 2N(n-1) = \begin{cases} 2\cdot\sqrt{5}^{n-2}, & n \text{ is even;} \\ 4\cdot\sqrt{5}^{n-3}, & n \text{ is odd.} \end{cases}$$

In Case 2, we have

$$|V(G)| = 5N(n-2) = \begin{cases} \sqrt{5}^{n-1}, & n \text{ is odd;} \\ 2\cdot\sqrt{5}^{n-2}, & n \text{ is even.} \end{cases}$$

If n is odd, Case 2 gives more vertices, and we have $c \in \mathbf{Pent} \times \mathscr{G}(n-2)$. If n is even, both Cases give the same number of vertices and we have $c \in (\mathbf{Pent} \times \mathscr{G}(n-2)) \cup (\mathbf{Bip} \times \mathscr{G}(n-1))$. Therefore, $c \in \mathscr{G}(n)$ and $|V(G)| = N = N(n)$. This concludes the proof of the first part of the Theorem. $\square$

The proof of the second part is very similar and can be carried out using induction on the number of colors in the coloring.

Proof of Theorem 6. We shall prove the following stronger statement. Let $R(s,t)$ be a classical Ramsey number corresponding to the smallest number of vertices in a complete graph such that any coloring of its edges in two colors, Red and Blue, contains either a Red K_s or a Blue K_t. Let $\operatorname{mix} R(t_1, t_2, \ldots, t_k; 3)$ be the largest integer n such that there is a coloring of $E(K_n)$ with colors $\{1, 2, \ldots, k\}$ containing no rainbow triangle and no monochromatic K_{t_i} in color i, $t_i \geqslant 3$, $i = 1, \ldots, k$.

Claim. $\operatorname{mix} R(t_1, t_2, \ldots, t_k; 3) \leqslant \max_{1 \leqslant i < j \leqslant k} (R(t_i, t_j) - 1)^{(t_1+t_2+\cdots+t_k)/2}$.

To prove the Claim, consider such a coloring c. Since it does not have rainbow triangles, we have, applying Proposition 2, that the vertices of K_n are split into sets $V_1, \ldots, V_m$ such that all edges between any two sets have the same color and there are at most two colors, say, i and j, altogether used on them. Thus, we have that $m < R(t_i, t_j)$, where $R(t_i, t_j)$ is the classical two-color Ramsey number. If there is only one color, i, used between the sets $V_1, \ldots, V_m$, then $m = 2$ and neither V_1 nor V_2 induce $K_{t_i - 1}$ in color i, thus $n \leqslant 2 \operatorname{mix} R(t_1, \ldots, t_{i-1}, t_i - 1, t_{i+1}, \ldots, t_m; 3)$. If there are two colors, i and j, used between the sets $V_1, \ldots, V_m$, then because of maximality of n, we have that $m = R(t_i, t_j) - 1$ and each V_i does not induce $K_{t_i - 1}$ in color i and does not induce $K_{t_j - 1}$ in color j. Therefore $|V_\ell| \leqslant \operatorname{mix} R(t_1, t_2, \ldots, t_{i-1}, t_i - 1, t_{i+1}, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_k; 3)$, $1 \leqslant \ell \leqslant m$. Thus we have that $n \leqslant (R(t_i, t_j) - 1) \operatorname{mix} R(t_1, t_2, \ldots, t_{i-1}, t_i - 1, t_{i+1}, \ldots, t_{j-1}, t_j - 1, t_{j+1}, \ldots, t_k; 3)$. This recursion proves the claim.

Now, if we have a coloring of $E(K_n)$ with no monochromatic K_t and no rainbow K_3 using k colors, the Claim implies that

$$n \leqslant R(t,t)^{tk/2} \leqslant (4^t)^{tk/2} = 2^{t^2 k}. \qquad \square$$

Note: Since there is no general description of colorings with no rainbow K_s, $s \geqslant 4$ known to the best of our knowledge, the above technique does not extend to other graphs H, but K_3.

4. Miscellaneous

4.1. On (K_3, K_4)-colorings and the structure of lexically colored subgraphs

In this section, we investigate the structure of (3, 4)-good colorings with respect to lexically colored subgraphs. First, we establish that two complete lexically colored subgraphs in a (3, 4)-good colored complete graph can not have “too many” colors on the edges between them.

Lemma 5. *Let A,B be disjoint sets. Let c be a* (3, 4)*-good coloring of a complete graph G with vertex set $A \cup B$. Let A and B span lexically colored complete graphs using disjoint sets of colors in c. Then any rainbow cycle in c uses colors from $c(A) \cup c(B)$. Moreover, the number of colors in c on the edges between A and B which are different from the colors in $c(A) \cup c(B)$ is at most $|A| + |B| - 1$.*

Proof. Let C be a rainbow cycle in c not using colors from $c(A) \cup c(B)$. We shall show that this is impossible by induction on the length of C. Observe first that any such cycle has no edges in $G[A]$ or in $G[B]$. It is clear that C can not have length 4, otherwise, since $c(A) \cap c(B) = \emptyset$, the vertices of C will span a rainbow K_4. Suppose there is a rainbow cycle, C, of length $2n+2$, not using colors from $c(A) \cup c(B)$. Let $\{a_1, a_2, \ldots, a_{n+1}\} = A \cap V(C)$, and let $\{b_1, b_2, \ldots, b_{n+1}\} = B \cap V(C)$, in lexical order, i.e., such that $c(a_i a_j) = \alpha_i$, $c(b_i b_j) = \beta_i$, if $1 \leqslant i < j \leqslant n+1$, for distinct $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$. Now consider the smallest m such that $a_1 b_m \in E(C)$. Let $a_1 b_q$ and $a_p b_m$ be the other two edges of C incident to a_1 and b_m, respectively. Consider an edge $e = a_p b_q$. In order for $\{a_1, a_p, b_m, b_q\}$ not to induce a rainbow K_4, we must have $c(e) \in \{c(a_1 b_m), c(a_1 b_q), c(a_p b_m), \alpha_1, \beta_m\}$. Then $C' = C - \{a_1, b_m\} \cup e$ is rainbow cycle of length $2n$ with colors not used in $c(A\backslash a_1) \cup c(B\backslash b_m)$. Applying the induction hypothesis to a coloring c restricted to $G[A \cup B\backslash\{a_1, b_m\}]$ and a cycle C', we obtain a contradiction. Now, consider a maximal bipartite subgraph G' of G with partite sets A and B which does not have edges of colors from $c[A] \cup c[B]$. Since G' is acyclic, we have that $|E(G')| \leqslant |A| + |B| - 1$. $\square$

Lemma 6. *Let c be a* $(3, 4)$-*good coloring of a complete graph with vertex set $A \cup B$, such that A and B induce vertex-disjoint lexically colored graphs*, $|A| \leqslant |B|$. *Then* $|c(A, B)\backslash(c(A) \cup c(B))| \leqslant 6|A| + |B|$.

Proof. Let $c(A) \cap c(B) = I$. Let $A' \subseteq A$, $B' \subseteq B$ be the vertices "carrying" colors from I, i.e., $c(A') = c(B') = I$. Thus $c(A\backslash A') \cap c(B) = \emptyset$, $c(A') \cap c(B\backslash B') = \emptyset$. We also have that

$$c(A, B) = c(A\backslash A', B) \cup c(A', B\backslash B') \cup c(A', B'). \tag{2}$$

Using Lemma 5, we have that

$$|c(A\backslash A', B)\backslash(c(A) \cup c(B))| \leqslant |A\backslash A'| + |B| - 1 = |A| - 1,$$

$$|c(A', B\backslash B')\backslash(c(A) \cup c(B))| \leqslant |B\backslash B'| + |A'| - 1 = |B| - 1.$$

Now, we only need to estimate the number of colors between A' and B'. For a subset S, $S \subseteq A'$, let $S^* \subseteq B'$ such that $c(S) \cap c(S^*) = \emptyset$ and $|S| + |S^*| = |A'| + 1 = |B'| + 1$. Then again, from Lemma 5, we have that $|c(S, S^*)\backslash(c(A) \cup c(B))| \leqslant |S| + |S^*| - 1 = |A'|$. Thus, we can count over all such subsets S of A' of size $\lfloor |A'|/2 \rfloor$ to find an upper bound on the number of colors between A' and B'.

$$\begin{aligned}
|c(A', B')\backslash(c(A) \cup c(B))| &\leqslant \sum_{S \subseteq A', |S| = \lfloor |A'|/2 \rfloor} \frac{|c(S, S^*)\backslash(c(A) \cup c(B))|}{\binom{|A'|-2}{\lfloor |A'|/2 \rfloor - 1}} + |A'| \\
&= \binom{|A'|}{\lfloor |A'|/2 \rfloor} \frac{|A'| - 1}{\binom{|A'|-2}{\lfloor |A'|/2 \rfloor - 1}} + |A'| \leqslant 5|A'| \leqslant 5|A|.
\end{aligned}$$

Using all these bounds in (2), we have

$$|c(A, B)\backslash(c(A) \cup c(B))| \leqslant |A| - 1 + |B| - 1 + 5|A| \leqslant 6|A| + |B|. \quad \square$$

Theorem 8. *Let c be a* $(3, 4)$-*good coloring of K_n. Let $V(K_n) = L_1 \cup L_2 \cup \cdots \cup L_k$, where L_is are disjoint sets inducing lexically colored complete subgraphs. Then the total number of colors in c is at most $6kn$.*

Proof. By Lemma 6, $|c(L_i, L_j)|\backslash(c(L_i) \cup c(L_j))| \leqslant 6(|L_i| + |L_j|)$, $1 \leqslant i < j \leqslant k$. Moreover $|c(L_i)| = |L_i| - 1$, $i = 1, \ldots, k$. Therefore, the total number of colors is at most

$$\begin{aligned}
&\sum_{1 \leqslant i < j \leqslant k} 6(|L_i| + |L_j|) + \sum_{1 \leqslant i \leqslant k} (|L_i| - 1) \\
&\leqslant 6(k-1) \sum_{1 \leqslant i \leqslant k} |L_i| + \sum_{1 \leqslant i \leqslant k} |L_i| \leqslant 6k \sum_{1 \leqslant i \leqslant k} |L_i| = 6kn. \quad \square
\end{aligned}$$

4.2. Colorings containing a rainbow triangle induced by each subset of size at least $c \ln n$

In this section we show that there are colorings in which it is difficult to avoid rainbow triangles.

Lemma 7. *For any $k \geqslant 3$, n large enough, there is a coloring of $E(K_n)$ with k colors such that each subset of vertices of size at least $C \log n$ induces a rainbow triangle.*

Proof. Let G be a complete graph on n vertices. Consider a random k-coloring, c, of $E(G)$ with k colors $1, 2, \dots, k$ such that $\mathrm{Prob}(c(e) = i) = 1/k$ for any edge e and any color i, $1 \leqslant i \leqslant k$. We need to estimate the following:

$$\begin{aligned}
&\mathrm{Prob}(\forall S \subseteq V, |S| = s, G[S] \text{has a rainbow triangle in } c)\\
&\quad = 1 - \mathrm{Prob}(\exists S \subseteq V, |S| = s, G[S] \text{ has no rainbow triangle})\\
&\quad \geqslant 1 - \binom{n}{s} \mathrm{Prob}(\text{fixed } S, S \subseteq V, |S| = s, G[S] \text{has no rainbow triangle}). \qquad (3)
\end{aligned}$$

For a fixed subset, S, of s vertices, let $f(S) = \mathrm{Prob}(G[S]$ has no rainbow triangle in $c)$. Let $T_1, \dots, T_{\binom{s}{3}}$ be triples of vertices in S. Let B_i be the event that T_i induces a rainbow triangle in coloring c. Then, using generalized Janson's inequality, see for example [1],

$$f(S) = \mathrm{Prob}\left(\bigwedge_{i=1}^{\binom{s}{3}} \overline{B_i}\right) \leqslant \exp(-\mu^2/\Delta),$$

where

$$\mu = \sum_{i=1}^{\binom{s}{3}} \mathrm{Prob}(B_i), \qquad \Delta = \sum_{i=1}^{\binom{s}{3}} \sum_{i \sim j} \mathrm{Prob}(B_i \wedge B_j).$$

Here $i \sim j$ if B_i and B_j are not independent events, i.e., in the above situation, $B_i \sim B_j$ when T_i and T_j share two vertices.

$$\mathrm{Prob}(B_i) = \frac{(k-1)(k-2)}{k^2}, \quad \mathrm{Prob}(B_i \wedge B_j) = 2\frac{(k-1)^2(k-2)^2}{k^4},$$

if $i \sim j$. We have the following values of μ and Δ.

$$\mu = \sum_{i=1}^{\binom{s}{3}} \mathrm{Prob}(B_i) = \binom{s}{3} \frac{(k-1)(k-2)}{k^2},$$

$$\Delta = \sum_{i=1}^{\binom{s}{3}} \sum_{i \sim j} \mathrm{Prob}(B_i \wedge B_j) = \binom{s}{3} 3(s-3)2\frac{(k-1)^2(k-2)^2}{k^4}.$$

Therefore,

$$\frac{\mu^2}{2\Delta} \geqslant c_1 s^2$$

for a constant c_1, and

$$f(S) \leqslant \exp(-c_1 s^2).$$

Coming back to (3), we have

$$\begin{aligned}
&\mathrm{Prob}(\forall S \subseteq V, |S| = s, G[S] \text{ has a rainbow triangle in } c)\\
&\quad \geqslant 1 - \binom{n}{s} e^{-c_1 s^2}\\
&\quad > 0,
\end{aligned}$$

if $1 > \binom{n}{s} e^{-c_1 s^2}$, which holds if for $s > c' \ln n$, where $c' \geqslant 40$. $\square$

Note that this proof can be carried out using several other methods.

Remarks. We have determined max $R(n; G, H)$ in most cases. The only open problem left is to determine this function when the vertex arboricity of H is equal to two. In particular, one of the most intriguing problems is to find max $R(n; K_4, K_4)$. Recently Daniel Kral, Veselin Jungic and Tomas Kaiser announced that they have improved the upper bound to $n^{3/2}$. The problem of determining min $R(n; K_t, K_s)$ is wide open for all $t > 3, s > 3$.

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