Journal of Mathematical Analysis and Applications **240,** 249–279 (1999)
Article ID jmaa.1999.6619, available online at http://www.idealibrary.com on IDEAL®

Numerical Methods for Systems of Nonlinear Parabolic Equations with Time Delays

C. V. Pao

Submitted by William F. Ames

Received August 30, 1999

The purpose of this paper is to investigate some numerical aspects of a class of coupled nonlinear parabolic systems with time delays. The system of parabolic equations is discretized by the finite difference method which yields a coupled system of nonlinear algebraic equations. The mathematical analysis of the nonlinear system is by the method of upper and lower solutions and its associated monotone iterations. Three monotone iterative schemes are presented and it is shown that the sequence of iterations from each one of these iterative schemes converges monotonically to a unique solution of the finite difference system. A theoretical comparison result for the various monotone sequences and error estimates for the three monotone iterative schemes are obtained. It is also shown that the finite difference solution converges to the classical solution of the parabolic system as the mesh size decreases to zero.

1. INTRODUCTION

In many biochemical, physical, and ecological reaction processes the rate of change of various density functions, such as chemical concentrations, temperature, populations, etc., is affected by the present as well as the past state of the density functions. The mathematical problems of these kinds of reaction processes have traditionally been formulated in the framework of ordinary differential systems. In recent years, considerable attention has been given to systems where the effect of diffusion is taken into consideration. This leads to systems of reaction–diffusion equations with time delays which are often coupled through the reaction mechanism. However, most of the works in the current literature are devoted to the qualitative analysis of the system such as the existence and uniqueness of a

0022-247X/99 $30.00

solution and the dynamics of the system. Much less discussion is given to the numerical aspect of the system except in some special model problems (cf. [6–8, 16]). In this paper, we give a systematic treatment to a finite difference system of a general class of reaction–diffusion system with time delay, where the effect of convection is also taken into consideration. This treatment is an extension of an earlier article [12] for reaction–diffusion systems without time delays.

Suppose the reaction–diffusion–convection process occurs in a bounded medium Ω of $\mathbb{R}^p$ $(p = 1, 2, \dots)$ with boundary $\partial\Omega$. Then the system of equations may be written in the form

$$
\begin{aligned}
\partial u^{(l)}/\partial t - L^{(l)}u^{(l)} &= f^{(l)}(x, t, \mathbf{u}, \mathbf{u}_\tau) && (x \in \Omega, t > 0),\\
B^{(l)}u^{(l)} &= g^{(l)}(x, t) && (x \in \partial\Omega, t > 0),\\
u^{(l)}(x, t) &= \psi^{(l)}(x, t) && (x \in \Omega, -\tau_l \le t \le 0),\\
l &= 1, \dots, N,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{u} = (u^{(1)}(x, t), \dots, u^{(N)}(x, t))$, $\mathbf{u}_\tau = (u^{(1)}(x, t - \tau_1), \dots, u^{(N)}(x, t - \tau_N))$, and for each $l = 1, \dots, N$, $L^{(l)}u^{(l)}$ and $B^{(l)}u^{(l)}$ are given by

$$
\begin{aligned}
L^{(l)}u^{(l)} &= \nabla \cdot (D^{(l)} \nabla u^{(l)}) + \mathbf{v}^{(l)} \cdot \nabla u^{(l)}\\
B^{(l)}u^{(l)} &= \alpha^{(l)} \partial u^{(l)}/\partial \nu + \beta^{(l)} u^{(l)}.
\end{aligned}
$$

The constants $\tau_1, \dots, \tau_N$, representing the time delays in the vector function $\mathbf{u}_\tau$, are positive and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. It is assumed that the diffusion coefficient $D^{(l)} \equiv D^{(l)}(x, t)$ and the convection coefficient $\mathbf{v}^{(l)} = (v_1^{(l)}, \dots, v_p^{(l)})$ of $L^{(l)}$, where $v_\nu^{(l)} \equiv v_\nu^{(l)}(x, t)$ for $\nu = 1, \dots, p$, are continuous on $\overline{D}_T \equiv [0, T] \times \overline{\Omega}$, and $D^{(l)}$ is strictly positive on $\overline{D}_T$ for every finite $T > 0$. The coefficients $\alpha^{(l)}$ and $\beta^{(l)} \equiv \beta^{(l)}(x, t)$ of $B^{(l)}$ are continuous on $S_T \equiv \partial\Omega \times [0, T]$ with either $\alpha^{(l)} = 0$, $\beta^{(l)} > 0$ (Dirichlet condition) or $\alpha^{(l)} = 1$, $\beta^{(l)} \ge 0$ (Neumann or Robin condition), where $\overline{\Omega} = \Omega \cup \partial\Omega$. It is also assumed that $f^{(l)}$, $g^{(l)}$, and $\psi^{(l)}$ are continuous functions in their respective domains, and $f^{(l)}(\cdot, \mathbf{u}, \mathbf{u}_\tau)$ is, in general, nonlinear with respect to $\mathbf{u}$ and $\mathbf{u}_\tau$ (see [8] for more detailed discussions).

To discretize the system (1.1) for numerical solutions we formulate it as a finite difference system by the implicit method. This approach leads to a system of nonlinear algebraic equations which preserves many of the qualitative properties of the continuous system. As we have discussed in

[12] for the system without time delays, an explicit method or semi-implicit method may lead to incorrect or misleading information of the continuous solution while the implicit method preserves various qualitative behavior of the continuous solution (see Remark 3.1). However, because of the nonlinear nature of the problem it is necessary to use some kind of iteration process for the computation of the solution. In this paper we present three monotone iterative schemes, including an analytical comparison result among the sequences of iterations, by the method of upper and lower solutions. It is shown, by using upper and lower solutions as a pair of coupled initial iterations, that each of the three iterative schemes yields two monotone sequences which converge monotonically from above and below, respectively, to a unique solution of the finite difference system. This kind of monotone iteration has been widely used for both continuous and discrete parabolic and elliptic boundary value problems with or without time delays (e.g., see [2, 5−13]). A basic consequence of this method is that it gives improved upper and lower bounds of the solution in each iteration and that it leads to the existence and uniqueness of a solution of the nonlinear system. Moreover, since the iteration process involves linear uncoupled algebraic equations in the same fashion as in the case of linear systems, the numerical scheme is unconditionally stable with respect to the mesh size (in both time and space) and is suitable for parallel computations. Although the main concern of the present paper is to present some monotone computational schemes for numerical solutions of (1.1) rather than the best rate of convergence of the iterations it appears possible that the method developed in the recent paper [9] can be used to accelerate the rate of convergence of the iterations in these schemes without sacrificing the monotone property of the sequences of iterations. This aspect will be investigated in a future exploration.

The plan of the paper is as follows. In Section 2 we formulate a finite difference system of (1.1) and introduce the definition of coupled upper and lower solutions of the system for mixed quasimonotone functions. These upper and lower solutions are used as the initial iterations in an iteration process in Section 3, where two basic monotone sequences are constructed by a linear computational algorithm. Section 4 contains three improved monotone iterative schemes as well as an analytical comparison result among the sequences of iterations. An error estimate for each of the three monotone iterative schemes is obtained in Section 5. This estimate is given in relation to the effect of diffusion−convection and the strength of the reaction mechanism without explicit knowledge of the solution. Finally in Section 6 we show the convergence of the finite difference solution to the continuous solution using the basic monotone iterative scheme developed in Section 3.

2. UPPER AND LOWER SOLUTIONS

Let $\mathbf{x}_i = (x_{i_1}, \dots, x_{i_p})$ be an arbitrary mesh point in $\overline{\Omega}$, where $i = (i_1, \dots, i_p)$ is a multiple index with $i_\nu = 1, \dots, M_\nu$ and for each $\nu = 1, \dots, p$, M_ν is the total number of mesh points in the x_ν-direction. Denote by Ω_p, Λ_p, and $Q_p^{(l)}$ the set of mesh points in Ω, $\Omega \times (0, T]$, and $\Omega \times [-\tau_l, 0]$, respectively, and by $\partial\Omega_p$, S_p the set of mesh points in $\partial\Omega$ and $\partial\Omega \times (0, T]$. The set of all mesh points in $\overline{\Omega}$ and $\overline{\Omega} \times [0, T]$ are denoted by $\overline{\Omega}_p$ and $\overline{\Lambda}_p$, respectively. It is assumed that the domain Ω is connected. Let $k_n = t_n - t_{n-1}$ be the time increment and h_ν the spatial increment in the x_ν-direction. For each $l = 1, \dots, N$ we choose k_n such that $\tau_l = k_1 + \cdots + k_{s_l}$ for some integer $s_l > 0$. Define

$$u_{i,n}^{(l)} = u^{(l)}(\mathbf{x}_i, t_n), \qquad \mathbf{u}_{i,n} = \left(u_{i,n}^{(1)}, \dots, u_{i,n}^{(N)}\right),$$

$$u_{i,n-s_l}^{(l)} = u^{(l)}(\mathbf{x}_i, t_n - \tau_l), \qquad \mathbf{u}_{i,n-s} = \left(u_{i,n-s_1}^{(1)}, \dots, u_{i,n-s_N}^{(N)}\right),$$

$$f^{(l)}(\mathbf{u}_{i,n}, \mathbf{u}_{i,n-s}) = f^{(l)}(\mathbf{x}_i, t_n, \mathbf{u}_{i,n}, \mathbf{u}_{i,n-s}).$$

Define also the standard central difference operators,

$$\Delta^{(\nu)} u_{i,n} = h_\nu^{-2}\left[u(\mathbf{x}_i + h_\nu \mathbf{e}_\nu, t_n) - 2u(\mathbf{x}_i, t_n) + u(\mathbf{x}_i - h_\nu \mathbf{e}_\nu, t_n)\right],$$

$$\delta^{(\nu)} u_{i,n} = (2h_\nu)^{-1}\left[u(\mathbf{x}_i + h_\nu \mathbf{e}_\nu, t_n) - u(\mathbf{x}_i - h_\nu \mathbf{e}_\nu, t_n)\right],$$

and approximate the parabolic operators in (1.1) by

$$\mathscr{L}^{(l)}\left[u_{i,n}^{(l)}\right] = k_n^{-1}\left(u_{i,n}^{(l)} - u_{i,n-1}^{(l)}\right) - \sum_{\nu=1}^{p} \left(D_{i,n}^{(l)} \Delta^{(\nu)} u_{i,n} + \left(v_{i,n}^{(l)}\right)_\nu \delta^\nu u_{i,n}^{(l)}\right),$$

where $\mathbf{e}_\nu$ is the unit vector in $\mathbb{R}^p$ with νth component one and zero elsewhere, and $D_{i,n}^{(l)}$ and $(v_{i,n}^{(l)})_\nu$ are the standard approximations of $D^{(l)}(\mathbf{x}, t)$ and $(v^{(l)}(\mathbf{x}, t))_\nu$, respectively (cf. [1, 3]). Then we approximate the parabolic system (1.1) by the nonlinear finite difference system

$$\begin{aligned} \mathscr{L}^{(l)}\left[u_{i,n}^{(l)}\right] &= f^{(l)}(\mathbf{u}_{i,n}, \mathbf{u}_{i,n-s}) && \text{in } \Lambda_p, \\ \mathscr{B}^{(l)}\left[u_{i,n}^{(l)}\right] &= g_{i,n}^{(l)} && \text{on } S_p, \\ u_{i,n}^{(l)} &= \psi_{i,n}^{(l)} && \text{in } Q_p \qquad (l = 1, \dots, N), \end{aligned} \tag{2.1}$$

where $g_{i,n}^{(l)} = g^{(l)}(\mathbf{x}_i, t_n)$, $\psi_{i,n}^{(l)} = \psi^{(l)}(\mathbf{x}_i, t_n)$, and $\mathscr{B}^{(l)}[u_{i,n}^{(l)}]$ is a suitable approximation of the boundary operator $B^{(l)}$ (cf. [10, 11]). When the boundary operator is of Dirichlet type the boundary condition in (2.1) is reduced to $u_{i,n}^{(l)} = g_{i,n}^{(l)}$ on S_p. As in the case of scalar parabolic boundary value problems the above formulation by the implicit method is crucial for preserving the qualitative property of the solution of the continuous system (cf. [9, 11]).

To develop computational schemes as well as to show the existence and uniqueness of a solution to the discrete system (2.1) we find it more convenient to express the system in vector form. Let $M = M_1 M_2 \cdots M_p$ be the total number of mesh points in $\overline{\Omega}_p$ at which the value of the solution $\mathbf{u}(x_i, t_n)$ is to be computed. Define

$$U_n^{(l)} = \left(u_{1,n}^{(l)}, \ldots, u_{M,n}^{(l)}\right)', \qquad U_{n-s_l}^{(l)} = \left(u_{1,n-s_1}^{(l)}, \ldots, u_{M,n-s_l}^{(l)}\right)',$$

$$\mathbf{\Psi}_n^{(l)} = \left(\psi_{1,n}^{(l)}, \ldots, \psi_{M,n}^{(l)}\right)',$$

$$\mathbf{U}_n = \left(U_n^{(1)}, \ldots, U_n^{(N)}\right)', \qquad \mathbf{U}_{n-s} = \left(U_{n-s_1}^{(1)}, \ldots, U_{n-s_N}^{(N)}\right)',$$

$$F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) = \left(F^{(1)}(\mathbf{U}_n, \mathbf{U}_{n-s}), \ldots, F^{(N)}(\mathbf{U}_n, \mathbf{U}_{n-s})\right)',$$

where $(\cdot)'$ denotes the transpose of a row vector. Then we may express the system (2.1) in the form

$$\left(I + k_n A_n^{(l)}\right) U_n^{(l)} = U_{n-1}^{(l)} + k_n F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) + G_n^{(l)}, \qquad n = 1, 2, \ldots,$$
$$U_n^{(l)} = \Psi_n^{(l)},\ n = 0, -1, \ldots, -s_l\ (l = 1, \ldots, N), \tag{2.2}$$

where I is the identity matrix, $A_n^{(l)}$ is an M by M block matrix associated with the operators $\mathscr{L}^{(l)}$ and $\mathscr{B}^{(l)}$, and $G_n^{(l)}$ is a vector due to the boundary function $(g_{1,n}^{(l)}, \ldots, g_{M,n}^{(l)})$ (see [10, 11] for some detailed formulation).

To construct monotone sequences for the system (2.2) we require that the vector function $\mathbf{F}$: $\chi^q \times \chi^q \to \chi^q$ given by

$$\mathbf{F}(\mathbf{U}, \mathbf{V}) = \left(F^{(1)}(\mathbf{U}, \mathbf{V}), \ldots, F^{(N)}(\mathbf{U}, \mathbf{V})\right) \tag{2.3}$$

be mixed quasimonotone in a subset of $\chi^q \times \chi^q$, where $q = MN$ and

$$\chi^q = \left\{\mathbf{U} \equiv (U^{(1)}, \ldots, U^{(N)});\ U^{(l)} \in \mathbb{R}^M \text{ for } l = 1, \ldots, N\right\}.$$

Specifically, by writing the vectors $\mathbf{U}$, $\mathbf{V}$, and $F^{(l)}$ in the split form

$$\begin{aligned} \mathbf{U} &\equiv \left(U^{(l)}, [\mathbf{U}]_{a_l}, [\mathbf{U}]_{b_l}\right), \qquad \mathbf{V} \equiv \left([\mathbf{V}]_{c_l}, [\mathbf{V}]_{d_l}\right), \\ F^{(l)}(\mathbf{U}, \mathbf{V}) &\equiv F^{(l)}\left(U^{(l)}, [\mathbf{U}]_{a_l}, [\mathbf{U}]_{b_l}, [\mathbf{V}]_{c_l}, [\mathbf{V}]_{d_l}\right) \end{aligned} \tag{2.4}$$

for each $l = 1, \ldots, N$, where a_l, b_l, c_l, and d_l are nonnegative integers satisfying the relation

$$a_l + b_l = N - 1 \qquad \text{and} \qquad c_l + d_l = N \tag{2.5}$$

and $[\mathbf{U}]_\sigma$ denotes a vector with σ number of components of $\mathbf{U}$, we have the following definition.

DEFINITION 2.1. The vector function $\mathbf{F}(\mathbf{U}, \mathbf{V})$ given by (2.3) is said to be mixed quasimonotone in $J_n \times J_{n-s}$ if, for each $l = 1, \dots, N$, there exist nonnegative integers a_l, b_l, c_l, and d_l satisfying (2.5) such that $F^{(l)}(\mathbf{U}, \mathbf{V})$ is nondecreasing in $[\mathbf{U}]_{a_l}$ and $[\mathbf{V}]_{c_l}$, and is nonincreasing in $[\mathbf{U}]_{b_l}$ and $[\mathbf{V}]_{d_l}$ for all $\mathbf{U} \in J_n$ and $\mathbf{V} \in J_{n-s}$. Similarly, $\mathbf{F}(\mathbf{U}, \mathbf{V})$ is said to be mixed quasimonotone in J_n if for each $\mathbf{V} \in J_{n-s}$ and each $l = 1, \dots, N$, $F^{(l)}(\mathbf{U}, \mathbf{V})$ is nondecreasing in $[\mathbf{U}]_{a_l}$ and is nonincreasing in $[\mathbf{U}]_{b_l}$ for all $\mathbf{U} \in J_n$.

The subsets J_n and J_{n-s} in the above definition are given by

$$
\begin{aligned}
J_n &\equiv \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n \rangle \equiv \left\{ \mathbf{U} \in \chi^q;\ \hat{\mathbf{U}}_n \le \mathbf{U} \le \tilde{U}_n \right\}, \\
J_{n-s} &\equiv \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle \equiv \left\{ \mathbf{V} \in \chi^q;\ \hat{\mathbf{U}}_{n-s} \le \mathbf{V} \le \tilde{\mathbf{U}}_{n-s} \right\},
\end{aligned}
\tag{2.6}
$$

where $\tilde{\mathbf{U}}_n$ and $\hat{\mathbf{U}}_n$ are a pair of coupled upper and lower solutions which are defined as follows:

DEFINITION 2.2. Two vectors $\tilde{\mathbf{U}}_n \equiv (\tilde{U}_n^{(1)}, \dots, \tilde{U}_n^{(N)})'$, $\hat{\mathbf{U}}_n \equiv (\hat{U}_n^{(1)}, \dots, \hat{U}_n^{(N)})'$ in χ^q are called coupled upper and lower solutions of (2.2) if $\tilde{\mathbf{U}}_n \ge \hat{\mathbf{U}}_n$ and

$$
\begin{aligned}
\left(I + k_n A_n^{(l)}\right)\tilde{U}_n^{(l)} &\ge \tilde{U}_{n-1}^{(l)} + k_n F^{(l)}\left(\tilde{U}_n^{(l)}, \left[\tilde{\mathbf{U}}_n\right]_{a_l}, \left[\hat{\mathbf{U}}_n\right]_{b_l}, \left[\tilde{\mathbf{U}}_{n-s}\right]_{c_l}, \left[\hat{\mathbf{U}}_{n-s}\right]_{d_l}\right) \\
&\quad + G_n^{(l)} \\
\left(I + k_n A_n^{(l)}\right)\hat{U}_n^{(l)} &\le \hat{U}_{n-1}^{(l)} + k_n F^{(l)}\left(\hat{U}_n^{(l)}, \left[\hat{\mathbf{U}}_n\right]_{a_l}, \left[\tilde{\mathbf{U}}_n\right]_{b_l}, \left[\hat{\mathbf{U}}_{n-s}\right]_{c_l}, \left[\tilde{\mathbf{U}}_{n-s}\right]_{d_l}\right) \\
&\quad + G_n^{(l)} \qquad \text{for } n = 1, 2, \dots \\
\tilde{U}_n^{(l)} \ge \Psi_n^{(l)} &\ge \hat{U}_n^{(l)} \qquad \text{for } n = 0, -1, \dots, -s_l \ (l = 1, \dots, N).
\end{aligned}
\tag{2.7}
$$

In the above definition inequalities between two vectors are in the sense of componentwise.

In proving the convergence of the finite difference solution of (2.1) to the classical solution of (1.1) as well as in the treatment of specific model problems of reaction−diffusion systems, it is more convenient to define upper and lower solutions to (2.1) directly as follows:

DEFINITION 2.3. A pair of functions $\tilde{\mathbf{u}}_{i,n} \equiv (\tilde{u}_{i,n}^{(1)}, \dots, \tilde{u}_{i,n}^{(N)})$, $\hat{\mathbf{u}}_{i,n} \equiv (\hat{u}_{i,n}^{(1)}, \dots, \hat{u}_{i,n}^{(N)})$, are called coupled upper and lower solutions of (2.1) if

$\tilde{\mathbf{u}}_{i,n} \geq \hat{\mathbf{u}}_{i,n}$ in $\overline{\Lambda}_p$, and if for each $l = 1, \ldots, N$,

$$\begin{aligned}
&\mathscr{L}^{(l)}\big[\tilde{u}_{i,n}^{(l)}\big] \geq f^{(l)}\Big(\tilde{u}_{i,n}^{(l)}, \big[\tilde{\mathbf{u}}_{i,n}\big]_{a_l}, \big[\hat{\mathbf{u}}_{i,n}\big]_{b_l}, \big[\tilde{\mathbf{u}}_{i,n-s}\big]_{c_l}, \big[\hat{\mathbf{u}}_{i,n-s}\big]_{d_l}\Big)\\
&\mathscr{L}^{(l)}\big[\hat{u}_{i,n}^{(l)}\big] \leq f^{(l)}\Big(\hat{u}_{i,n}^{(l)}, \big[\hat{\mathbf{u}}_{i,n}\big]_{a_l}, \big[\tilde{\mathbf{u}}_{i,n}\big]_{b_l}, \big[\hat{\mathbf{u}}_{i,n-s}\big]_{c_l}, \big[\tilde{\mathbf{u}}_{i,n-s}\big]_{d_l}\Big)\\
&\mathscr{B}^{(l)}\big[\tilde{u}_{i,n}^{(l)}\big] \geq g_{i,n}^{(l)} \geq \mathscr{B}^{(l)}\big[\hat{u}_{i,n}^{(l)}\big] \qquad \text{on } S_p\\
&\qquad \tilde{u}_{i,n}^{(l)} \geq \psi_{i,n}^{(l)} \geq \hat{u}_{i,n}^{(l)} \qquad\qquad \text{in } Q_p.
\end{aligned} \tag{2.8}$$

It is easy to verify that if $\tilde{\mathbf{u}}_{i,n}$ and $\hat{\mathbf{u}}_{i,n}$ satisfy (2.8) then the pair $\tilde{\mathbf{U}}_n \equiv (\tilde{U}_n^{(1)}, \ldots, \tilde{U}_n^{(N)})'$ and $\hat{\mathbf{U}}_n \equiv (\hat{U}_n^{(1)}, \ldots, \hat{U}_n^{(N)})'$ with

$$\tilde{U}_n^{(l)} \equiv \Big(\tilde{u}_{1,n}^{(l)}, \ldots, \tilde{u}_{M,n}^{(l)}\Big)', \qquad \hat{U}_n^{(l)} \equiv \Big(\hat{u}_{1,n}^{(l)}, \ldots, \hat{u}_{M,n}^{(l)}\Big)' \qquad (l = 1, \ldots, N)$$

satisfy the inequalities in (2.7). In the following discussion we assume that a pair of coupled upper and lower solutions $\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n$ (or $\tilde{\mathbf{u}}_{i,n}, \hat{\mathbf{u}}_{i,n}$) exist and that the following basic hypotheses on $A_N^{(l)}$ and $F^{(l)}$ are satisfied.

(H_1) For each $l = 1, \ldots, N$ and $n = 1, 2, \ldots,$ the matrix $A_n^{(l)} \equiv (a_n^{(l)})_{ij}$ is irreducible, and $(a_n^{(l)})_{ii} > 0$, $(a_n^{(l)})_{ij} \leq 0$ for $j \neq i$, and

$$\sum_{j=1}^{M} \big(a_n^{(l)}\big)_{ij} \geq 0 \qquad \text{for all } i = 1, \ldots, M. \tag{2.9}$$

(H_2) The vector function

$$\mathbf{F}(\mathbf{U}_n, \mathbf{U}_{n-s}) \equiv \big(F^{(1)}(\mathbf{U}_n, \mathbf{U}_{n-s}), \ldots, F^{(N)}(\mathbf{U}_n, \mathbf{U}_{n-s})\big)'$$

is a C^1-function and possesses a mixed quasimonotone property in $J_n \times J_{n-s}$, where $J_n \equiv \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n \rangle$ and $J_{n-s} \equiv \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle$.

Hypothesis (H_1) implies that $A_n^{(l)}$ is an M-matrix and its smallest eigenvalue $\mu_n^{(l)}$ is nonnegative. If strict inequality in (2.9) holds for at least one i then $\mu_n^{(l)} > 0$ and $(A_n^{(l)})^{-1}$ is a positive matrix (cf. [14, 15]). In either case, the inverse $(I + k_n A_n^{(l)})^{-1}$ exists and is a positive matrix. This positive property is critical in our construction of monotone convergent sequences for the system (2.2). It is to be noted that in the formulation of (2.2) if the convection term $\mathbf{v}^{(l)} \cdot \nabla u^{(l)}$ dominates the diffusion term $\nabla \cdot (D^{(l)} \nabla u^{(l)})$ then the hypothesis (H_1) can be satisfied either by taking h_ν suitably small or by using an upwind differencing scheme without any restriction on h_ν (cf. [1, 3]). The use of the central difference approximations for diffusion and convection and the connectness assumption on $\overline{\Omega}$ ensures that the matrix $A_n^{(l)}$ is irreducible (cf. [15]).

3. A MONOTONE ITERATION PROCESS

We first establish a basic monotone iteration process for the finite difference system (2.2) analogous to the treatment of parabolic systems without time delays in [12]. This iteration process is useful for proving the existence and uniqueness of a finite difference solution as well as its convergence to the continuous solution of (1.1). Let $\gamma_{i,n}^{(l)}$ be any nonnegative function in $\overline{\Lambda}_p$ such that

$$\gamma_{i,n}^{(l)} \geq \max\left\{ -\frac{\partial f^{(l)}}{\partial u^{(l)}}(\cdot, \mathbf{u}, \mathbf{w});\ \mathbf{u} \in \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n \rangle,\ \mathbf{w} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle \right\}, \quad (3.1)$$

where $u^{(l)}$ is the lth component of $\mathbf{u}$. Define

$$\mathscr{A}_n^{(l)} \equiv I + k_n\left(A_n^{(l)} + \Gamma_n^{(l)}\right), \qquad \Gamma_n^{(l)} \equiv \operatorname{diag}\left(\gamma_{1,n}^{(l)}, \ldots, \gamma_{M,n}^{(l)}\right)$$

$$\mathscr{F}^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) \equiv \Gamma_n^{(l)} U_n^{(l)} + F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) \qquad (l = 1, \ldots, N). \quad (3.2)$$

Then the system (2.2) is equivalent to

$$\mathscr{A}_n^{(l)} U_n^{(l)} = U_{n-1}^{(l)} + k_n \mathscr{F}^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) + G_n^{(l)}, \qquad n = 1, 2, \ldots,$$

$$U_n^{(l)} = \boldsymbol{\Psi}_n^{(l)}, \qquad n = 0, -1, \ldots, -s_l \qquad (l = 1, \ldots, N). \quad (3.3)$$

In view of (H_1), (H_2), and (3.1), the function $\mathscr{F}^{(l)}$ possesses the property

$$\mathscr{F}^{(l)}\left(U_n^{(l)}, [\mathbf{U}_n]_{a_l}, [\mathbf{V}_n]_{b_l}, [\mathbf{U}_{n-s}]_{c_l}, [\mathbf{V}_{n-s}]_{d_l}\right)$$

$$\geq \mathscr{F}^{(l)}\left(V_n^{(l)}, [\mathbf{V}_n]_{a_l}, [\mathbf{U}_n]_{b_l}, [\mathbf{V}_{n-s}]_{c_l}, [\mathbf{U}_{n-s}]_{d_l}\right)$$

$$\text{whenever } \tilde{\mathbf{U}}_n \geq \mathbf{U}_n \geq \mathbf{V}_n \geq \hat{\mathbf{U}}_n. \quad (3.4)$$

To show the existence and uniqueness of a solution to (3.3) we use $\overline{\mathbf{U}}_n^{(0)} = \tilde{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n^{(0)} = \hat{\mathbf{U}}_n$ as coupled initial iterations to construct two sequences

$$\left\{\overline{\mathbf{U}}_n^{(m)}\right\} \equiv \left\{\left(\overline{U}_n^{(1)}\right)^{(m)}, \ldots, \left(\overline{U}_n^{(N)}\right)^{(m)}\right\}, \quad \left\{\underline{\mathbf{U}}_n^{(m)}\right\} \equiv \left\{\left(\underline{U}_n^{(1)}\right)^{(m)}, \ldots, \left(\underline{U}_n^{(N)}\right)^{(m)}\right\}$$

from the linear iteration process

$$\mathscr{A}_n^{(l)}\left(\overline{U}_n^{(l)}\right)^{(m)} = \left(\overline{U}_{n-1}^{(l)}\right)^{(m)} + k_n \mathscr{F}^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{a_l},$$

$$\left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \left[\overline{\mathbf{U}}_{n-s}^{(m-1)}\right]_{c_l}, \left[\underline{\mathbf{U}}_{n-s}^{(m-1)}\right]_{d_l}\Big) + G_n^{(l)}$$

$$\mathscr{A}_n^{(l)}\left(\underline{U}_n^{(l)}\right)^{(m)} = \left(\underline{U}_{n-1}^{(l)}\right)^{(m)} + k_n \mathscr{F}^{(l)}\Big(\left(\underline{U}_n^{(l)}\right)^{(m-1)}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \quad (3.5)$$

$$\left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \left[\underline{\mathbf{U}}_{n-s}^{(m-1)}\right]_{c_l}, \left[\overline{\mathbf{U}}_{n-s}^{(m-1)}\right]_{d_l}\Big) + G_n^{(l)} \quad \text{for } n = 1, 2, \ldots$$

$$\left(\overline{U}_n^{(l)}\right)^{(m)} = \left(\underline{U}_n^{(l)}\right)^{(m)} = \boldsymbol{\Psi}_n^{(l)} \qquad \text{for} \qquad n = 0, -1, \ldots, -s_l$$

$$(l = 1, \ldots, N).$$

In the iteration process, n is fixed and the iteration is with respect to m starting from $m = 1$. Since $(\mathscr{A}_n^{(l)})^{-1}$ is a positive matrix and for each m the right-hand side of (3.5) is known, the sequences $\{\overline{\mathbf{U}}_n^{(m)}\}$ and $\{\underline{\mathbf{U}}_n^{(m)}\}$, called maximal and minimal sequences, respectively, are well-defined and can be computed by solving a linear algebraic system. The use of the implicit method in the formulation of the finite difference system ensures that each iteration is unconditionally stable with respect to the mesh sizes k_n and h_ν (cf. [1, 3]). In the following lemma we show the monotone property of the sequences.

LEMMA 3.1. *For each $n = 1, 2, \ldots,$ the maximal and minimal sequences $\{\overline{\mathbf{U}}_n^{(m)}\}, \{\underline{\mathbf{U}}_n^{(m)}\}$ possess the monotone property*

$$\hat{\mathbf{U}}_n \leq \underline{\mathbf{U}}_n^{(m)} \leq \underline{\mathbf{U}}_n^{(m+1)} \leq \overline{\mathbf{U}}_n^{(m+1)} \leq \overline{\mathbf{U}}_n^{(m)} \leq \tilde{\mathbf{U}}_n \qquad (m = 1, 2 \ldots). \quad (3.6)$$

Proof. Let $(\overline{W}_n^{(l)})^{(0)} = (\overline{U}_n^{(l)})^{(0)} - (\overline{U}_n^{(l)})^{(1)} = \tilde{U}_n^{(l)} - (\overline{U}_n^{(l)})^{(1)}$ for $l = 1, \ldots, N$. By (2.7), (3.2), and (3.5) with $m = 1$,

$$\begin{aligned}
\mathscr{A}_n^{(l)}\left(\overline{W}_n^{(l)}\right)^{(0)} &= \mathscr{A}_n^{(l)}\tilde{U}_n^{(l)} - \mathscr{A}_n^{(l)}\left(\overline{U}_n^{(l)}\right)^{(1)} \\
&\geq \Big[\tilde{U}_{n-1}^{(l)} + k_n\mathscr{F}^{(l)}\Big(\tilde{U}_n^{(l)}, \big[\tilde{\mathbf{U}}_n\big]_{a_l}, \big[\hat{\mathbf{U}}_n\big]_{b_l}, \\
&\qquad\qquad \big[\tilde{\mathbf{U}}_{n-s}\big]_{c_l}, \big[\hat{\mathbf{U}}_{n-s}\big]_{d_l}\Big) + G_n^{(l)}\Big] \\
&\quad - \Big[\left(\overline{U}_{n-1}^{(l)}\right)^{(1)} + k_n\mathscr{F}^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(0)}, \big[\overline{\mathbf{U}}_n^{(0)}\big]_{a_l}, \\
&\qquad\qquad \big[\underline{\mathbf{U}}_n^{(0)}\big]_{b_l}, \big[\overline{\mathbf{U}}_{n-s}^{(0)}\big]_{c_l}, \big[\underline{\mathbf{U}}_{n-s}^{(0)}\big]_{d_l}\Big) + G_n^{(l)}\Big] \\
&= \tilde{U}_{n-1}^{(l)} - \left(\overline{U}_{n-1}^{(l)}\right)^{(1)} = \left(\overline{W}_{n-1}^{(l)}\right)^{(0)} \qquad \text{for} \qquad n = 1, 2, \ldots,
\end{aligned}$$

and $(\overline{W}_n^{(l)})^{(0)} = \tilde{\mathbf{U}}_n^{(l)} - \boldsymbol{\Psi}_n^{(l)} \geq \mathbf{0}$ for $n = 0, -1, \ldots, -s_l$. The positive property of $(\mathscr{A}_n^{(l)})^{-1}$ implies that

$$\left(\overline{W}_n^{(l)}\right)^{(0)} \geq \left(\mathscr{A}_n^{(l)}\right)^{-1}\left(\overline{W}_{n-1}^{(l)}\right)^{(0)}, \qquad n = 1, 2, \ldots .$$

Since $(\overline{W}_0^{(l)})^{(0)} \geq \mathbf{0}$, an induction argument leads to $(\overline{W}_n^{(l)})^{(0)} \geq \mathbf{0}$ for $n = 1, 2, \ldots$. This proves $\overline{\mathbf{U}}_n^{(0)} \geq \overline{\mathbf{U}}_n^{(1)}$. A similar argument using the property of a lower solution and the second equation in (3.5) gives $\underline{\mathbf{U}}_n^{(1)} \geq \underline{\mathbf{U}}_n^{(0)}$. Moreover, by (3.5) and (3.4), the vector $(\overline{W}_n^{(l)})^{(1)} \equiv (\overline{\mathbf{U}}_n^{(l)})^{(1)} - (\underline{\mathbf{U}}_n^{(l)})^{(1)}$ satisfies the relation

$$\begin{aligned}\mathscr{A}_n^{(l)}\left(\overline{W}_n^{(l)}\right)^{(1)} = \left(\overline{W}_{n-1}^{(l)}\right)^{(1)} + k_n\Big[\mathscr{F}^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(0)}, \left[\overline{\mathbf{U}}_n^{(0)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(0)}\right]_{b_l}, \\ \left[\overline{\mathbf{U}}_{n-s}^{(0)}\right]_{c_l}, \left[\underline{\mathbf{U}}_{n-s}^{(0)}\right]_{d_l}\Big) + G_n^{(l)}\Big] \\ - k_n\Big[\mathscr{F}^{(l)}\Big(\left(\underline{U}_n^{(l)}\right)^{(0)}, \left[\underline{\mathbf{U}}_n^{(0)}\right]_{a_l}, \left[\overline{\mathbf{U}}_n^{(0)}\right]_{b_l}, \\ \left[\underline{\mathbf{U}}_{n-s}^{(0)}\right]_{c_l}, \left[\overline{\mathbf{U}}_{n-s}^{(0)}\right]_{d_l}\Big) + G_n^{(l)}\Big] \\ \geq \left(\overline{W}_{n-1}^{(l)}\right)^{(1)} \qquad \text{for } n = 1, 2, \ldots,\end{aligned}$$

and $(\overline{\mathbf{W}}_n^{(l)})^{(1)} = \mathbf{0}$ for $n = 0, -1, \ldots, -s_l$. This leads again to $(\overline{W}_n^{(l)})^{(1)} \geq 0$ for $n = 1, 2, \ldots$. The above conclusions imply that $\overline{\mathbf{U}}_n^{(0)} \geq \overline{\mathbf{U}}_n^{(1)} \geq \underline{\mathbf{U}}_n^{(1)} \geq \underline{\mathbf{U}}_n^{(0)}$ for $n = 1, 2, \ldots$. The monotone property (3.6) follows by an induction argument similar to that in [12]. ∎

The monotone property (3.6) implies that the limits

$$\lim_{m\to\infty} \overline{\mathbf{U}}_n^{(m)} \equiv \overline{\mathbf{U}}_n \qquad \text{and} \qquad \lim_{m\to\infty} \underline{\mathbf{U}}_n^{(m)} \equiv \underline{\mathbf{U}}_n \tag{3.7}$$

exist and satisfy $\hat{\mathbf{U}}_n \leq \underline{\mathbf{U}}_n \leq \overline{\mathbf{U}}_n \leq \tilde{\mathbf{U}}_n$ for every n. Letting $m \to \infty$ in (3.5) and using the relation (3.2) show that $\overline{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n$ satisfy the equations

$$\begin{aligned}\left(I + k_n A_n^{(l)}\right)\overline{U}_n^{(l)} &= \overline{U}_{n-1}^{(l)} + k_n F^{(l)}\Big(\overline{U}_n^{(l)}, \left[\overline{\mathbf{U}}_n\right]_{a_l}, [\underline{\mathbf{U}}_n]_{b_l}, \left[\overline{\mathbf{U}}_{n-s}\right]_{c_l}, [\underline{\mathbf{U}}_{n-s}]_{d_l}\Big) \\ &\quad + G_n^{(l)} \\ \left(I + k_n A_n^{(l)}\right)\underline{U}_n^{(l)} &= \underline{U}_{n-1}^{(l)} + k_n F^{(l)}\Big(\underline{U}_n^{(l)}, [\underline{\mathbf{U}}_n]_{a_l}, \left[\overline{\mathbf{U}}_n\right]_{b_l}, [\underline{\mathbf{U}}_{n-s}]_{c_l}, \left[\overline{\mathbf{U}}_{n-s}\right]_{d_l}\Big) \\ &\quad + G_n^{(l)} \\ &\qquad\qquad (n = 1, 2, \ldots) \qquad (3.8) \\ \overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n &= \mathbf{\Psi}_n \qquad (n = 0, -1, \ldots, -s_l).\end{aligned}$$

Hence $\overline{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n$ are solutions of (2.2) if $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$. To show this we define

$$\begin{aligned}\sigma_{l,n}^{(l)} &\equiv \max\left\{\frac{\partial f^{(l)}}{\partial u^{(l)}}(\cdot, \mathbf{u}, \mathbf{w});\ \mathbf{u} \in \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n\rangle, \mathbf{w} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s}\rangle\right\}, \\ \sigma_{j,n}^{(l)} &= \max\left\{\left|\frac{\partial f^{(l)}}{\partial u^{(j)}}(\cdot, \mathbf{u}, \mathbf{w})\right|;\ \mathbf{u} \in \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n\rangle, \mathbf{w} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s}\rangle\right\}, \\ &\qquad \text{when } j \neq l \qquad (l = 1, \ldots, N). \qquad (3.9)\end{aligned}$$

Then for any $\mathbf{W}_{n-s}$ in $\langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle$, the mean-value theorem implies that

$$F^{(l)}\big(U_n^{(l)}, [\mathbf{U}_n]_{a_l}, [\mathbf{V}_n]_{b_l}, \mathbf{W}_{n-s}\big) - F^{(l)}\big(V_n^{(l)}, [\mathbf{V}_n]_{a_l}, [\mathbf{U}_n]_{b_l}, \mathbf{W}_{n-s}\big)$$
$$\le \sigma_{l,n}^{(l)}\big(U_n^{(l)} - V_n^{(l)}\big) + \sum_{j \ne l}^{N} \sigma_{j,n}^{(l)}\big(U_n^{(j)} - V_n^{(j)}\big)$$
$$\text{whenever } \tilde{\mathbf{U}}_n \ge \mathbf{U}_n \ge \mathbf{V}_n \ge \hat{\mathbf{U}}_n. \quad (3.10)$$

Let $\mu_n^{(l)} \ge 0$ be the smallest eigenvalue of $A_n^{(l)}$ and define

$$\omega_n^{(l)} \equiv \max\left\{ \sigma_{l,n}^{(l)} - \mu_n^{(l)}, \sum_{j \ne l}^{N} \left[\sigma_{l,n}^{(j)} / \big(1 + k_n\big(\mu_n^{(j)} - \sigma_{l,n}^{(j)}\big)\big)\right] \right\}$$
$$\overline{\omega}_n \equiv \max\big\{\omega_n^{(l)};\ l = 1, \dots, N\big\}. \quad (3.11)$$

In the following theorem we show that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$ and $\overline{\mathbf{U}}_n$ is the unique solution of (2.2).

THEOREM 3.1. *Let $\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n$ be a pair of coupled upper and lower solutions of* (2.2) *and let hypotheses* (H_1) *and* (H_2) *hold. If*

$$k_n \overline{\omega}_n < 1 \qquad (n = 1, 2, \dots) \quad (3.12)$$

then the sequences $\{\overline{\mathbf{U}}_n^{(m)}\}, \{\underline{\mathbf{U}}_n^{(m)}\}$ *given by* (3.5) *with* $\overline{\mathbf{U}}_n^{(0)} = \tilde{\mathbf{U}}_n$ *and* $\underline{\mathbf{U}}_n^{(0)} = \hat{\mathbf{U}}_n$ *converge monotonically to a unique solution* $\mathbf{U}_n^*$ *of* (2.2). *Moreover*

$$\hat{\mathbf{U}}_n \le \underline{\mathbf{U}}_n^{(m)} \le \underline{\mathbf{U}}_n^{(m+1)} \le \mathbf{U}_n^* \le \overline{\mathbf{U}}_n^{(m+1)} \le \overline{\mathbf{U}}_n^{(m)} \le \tilde{\mathbf{U}}_n \qquad (m, n = 1, 2, \dots). \quad (3.13)$$

Proof. For the existence of a solution and the relation (3.13) it suffices to show that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$. Let $\mathbf{W}_n = \overline{\mathbf{U}}_n - \underline{\mathbf{U}}_n$, $W_n^{(l)} = \overline{U}_n^{(l)} - \underline{U}_n^{(l)}$, and $s^* = \min\{s_l;\ l = 1, \dots, N\}$, where $\mathbf{W}_n \equiv (W_n^{(1)}, \dots, W_n^{(N)})'$. Clearly, $W_n^{(l)} \ge \mathbf{0}$ for every l, and by (3.8),

$$\big(I + k_n A_n^{(l)}\big) W_n^{(l)} = W_{n-1}^{(l)}$$
$$+ k_n\Big[F^{(l)}\Big(\overline{U}_n^{(l)}, \big[\overline{\mathbf{U}}_n\big]_{a_l}, [\underline{\mathbf{U}}_n]_{b_l}, \big[\overline{\mathbf{U}}_{n-s}\big]_{c_l}, [\underline{\mathbf{U}}_{n-s}]_{d_l}\Big)$$
$$- F^{(l)}\Big(\underline{U}_n^{(l)}, [\underline{\mathbf{U}}_n]_{a_l}, \big[\overline{\mathbf{U}}_n\big]_{b_l}, [\underline{\mathbf{U}}_{n-s}]_{c_l}, \big[\overline{\mathbf{U}}_{n-s}\big]_{d_l}\Big)\Big]. \quad (3.14)$$

Consider the case $n = 1, \dots, s^*$. Since $\overline{\mathbf{U}}_{n-s} = \underline{\mathbf{U}}_{n-s} = \mathbf{\Psi}_{n-s}$ for $n = 1, \dots, s^*$, the relations (3.10) and (3.14) imply that

$$\big(I + k_n A_n^{(l)}\big) W_n^{(l)} \le W_{n-1}^{(l)} + k_n\left[\sigma_{l,n}^{(l)} W_n^{(l)} + \sum_{j \ne l}^{N} \sigma_{j,n}^{(l)} W_n^{(j)}\right],$$

which is equivalent to

$$\left[I + k_n\left(A_n^{(l)} - \sigma_{l,n}^{(l)} I\right)\right] W_n^{(l)} \leq W_{n-1}^{(l)} + k_n \sum_{j \neq l}^{N} \sigma_{j,n}^{(l)} W_n^{(j)}. \quad (3.15)$$

In view of (3.12), $1 + k_n(\mu_n^{(l)} - \sigma_{l,n}^{(l)}) > 0$. This relation and hypothesis (H_1) ensure that the inverse matrix $B_n^{(l)} \equiv [I + k_n(A_n^{(l)} - \sigma_{l,n}^{(l)} I)]^{-1}$ exists and is positive. Hence

$$W_n^{(l)} \leq B_n^{(l)} \left[W_{n-1}^{(l)} + k_n \sum_{j \neq l}^{N} \sigma_{j,n}^{(l)} W_n^{(j)} \right], \qquad n = 1, \ldots, s^*.$$

It follows from the same argument as that in [12] without time delays that $W_n^{(l)} = \mathbf{0}$ for every l. This shows that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$ for $n = 1, \ldots, s^*$. Using $\overline{\mathbf{U}}_{n-s} = \underline{\mathbf{U}}_{n-s}$ in (3.14) for $n = s^* + 1, \ldots, 2s^*$ the above reasoning leads to $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$ for $n = s^* + 1, \ldots, 2s^*$. A continuation of the same process yields $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$ for all n. This shows that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n$ $(\equiv \mathbf{U}_n^*)$ and $\mathbf{U}_n^*$ is a solution of (2.2) and satisfies (3.13).

To show the uniqueness of the solution we observe that if $\mathbf{U}_n \equiv (U_n^{(1)}, \ldots, U_n^{(N)})'$ is a solution of (2.2) in $\langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n \rangle$, then by the mean-value theorem, $\mathbf{V}_n = \mathbf{U}_n^* - \mathbf{U}_n$ possesses the property $V_{n-s} = \mathbf{0}$ for $n = 1, \ldots, s^*$ and satisfies the relation

$$\begin{aligned} \left(I + k_n A_n^{(l)}\right) V_n^{(l)} &= V_{n-1}^{(l)} + k_n \left[F^{(l)}(\mathbf{U}_n^*, \mathbf{U}_{n-s}^*) - F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}) \right] \\ &= V_{n-1}^{(l)} + k_n F_u^{(l)}(\boldsymbol{\xi}_n, \mathbf{U}_{n-s}) \mathbf{V}_n \qquad (n = 1, \ldots, s^*) \end{aligned} \quad (3.16)$$

where $F_u^{(l)}(\boldsymbol{\xi}_n, \mathbf{U}_{n-s}) \equiv (\partial f^{(l)} / \partial u^{(j)})(\boldsymbol{\xi}_n, \mathbf{U}_{n-s})$ is the Jacobi matrix of $F^{(l)}$ (with $\mathbf{U}_{n-s}$ fixed) and $\boldsymbol{\xi}_n$ is an intermediate value between $\mathbf{U}_n^*$ and $\mathbf{U}_n$. It follows again from the argument in [12] that $\mathbf{U}_n^* = \mathbf{U}_n$ for $n = 1, \ldots, s^*$. A continuation of the same reasoning for $n \geq s^* + 1$ leads to the conclusion $\mathbf{U}_n^* = \mathbf{U}_n$ for every n. This proves the theorem. ∎

Remark 3.1. From a computational point of view it is most convenient to formulate the finite difference system of (1.1) either by the explicit method or by a "semi-implicit" method in the sense that the function $F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s})$ is replaced by $F^{(l)}(\mathbf{U}_{n-1}, \mathbf{U}_{n-s})$. This formulation yields a finite difference approximation whose solution can be computed by a marching process with respect to n without iterations. Unfortunately, this approach not only imposes some severe restrictions on the mesh sizes k_n and h_ν for numerical stability (by explicit method) a more serious drawback is that it may lead to misleading or incorrect information of the solution. Some examples for this situation can be found in [9, 11].

4. THREE COMPUTATIONAL ITERATIVE SCHEMES

In the iterative scheme (3.5) the vectors $[\overline{\mathbf{U}}_{n-s}^{(m)}]$, $[\underline{\mathbf{U}}_{n-s}^{(m)}]$ and $(\overline{U}_{n-1}^{(l)})^{(m)}, (\underline{U}_{n-1}^{(l)})^{(m)}$ at earlier time steps are included in the process of iterations. The inclusion of these iterations is useful in proving the convergence of the finite difference solution to the continuous solution in a later section. However, from a computational point of view it is more desirable to replace these vectors by the values of the solution. We do this in the present section and show the same kind of monotone convergence of the maximal and minimal sequences. Moreover, we introduce two additional monotone iterative schemes, called Gauss−Seidel and Jacobi iterations, which are useful for computing numerical solutions of (1.1) in multidimensional spatial domains. An important improvement of these iterative schemes is that the function $\mathbf{F}(\mathbf{U}, \mathbf{V})$, given by (2.3), is not required to be quasimonotone with respect to the components of $\mathbf{V}$. Specifically, we replaced hypothesis (H_2) by the following condition.

(H_2') For each $\mathbf{V} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle$, $\mathbf{F}(\mathbf{U}, \mathbf{V})$ is mixed quasimonotone in $\mathbf{U}$ for $\mathbf{U} \in J_n \equiv \langle \hat{\mathbf{U}}_n, \tilde{\mathbf{U}}_n \rangle$.

In the hypothesis (H_2') the pair $\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n$ are the generalized upper and lower solutions which are defined by the following.

Definition 4.1. A pair of vectors $\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n$ are called generalized upper and lower solutions of (2.2) if they satisfy the conditions in Definition 2.2 except that the first two inequalities in (2.7) are replaced by

$$\begin{gathered}
\left(I + k_n A_n^{(l)}\right)\tilde{U}_n^{(l)} \geq \tilde{U}_{n-1}^{(l)} + k_n F^{(l)}\left(\tilde{U}_n^{(l)}, \left[\tilde{\mathbf{U}}_n\right]_{a_l}, \left[\hat{\mathbf{U}}_n\right]_{b_l}, \mathbf{V}_{n-s}\right) + G_n^{(l)} \\
\left(I + k_n A_n^{(l)}\right)\hat{U}_n^{(l)} \leq \hat{U}_{n-1}^{(l)} + k_n F^{(l)}\left(\hat{U}_n^{(l)}, \left[\hat{\mathbf{U}}_n\right]_{a_l}, \left[\tilde{\mathbf{U}}_n\right]_{b_l}, \mathbf{V}_{n-s}\right) + G_n^{(l)} \\
\text{for every } \mathbf{V}_{n-s} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s} \rangle \ (l = 1, \dots, N). \qquad (4.1)
\end{gathered}$$

It is clear from the above definition that if $\mathbf{F}(\mathbf{U}, \mathbf{V})$ is mixed quasimonotone in $J_n \times J_{n-s}$ then the conditions in Definition 4.1 coincide with that in Definition 2.2. To describe the iterative schemes we write the matrix $A_n^{(l)}$ in the split form $A_n^{(l)} \equiv D_n^{(l)} - \mathscr{U}_n^{(l)} - \mathscr{L}_n^{(l)}$, where $\mathscr{D}_n^{(l)}$, $\mathscr{U}_n^{(l)}$, and $\mathscr{L}_n^{(l)}$ are the diagonal, upper-off-diagonal, and lower-off-diagonal matrices of

$A_n^{(l)}$, respectively. Define

$$\left(\mathscr{A}_n{}^{(l)}\right)_G \equiv I + k_n\left(\mathscr{D}_n^{(l)} - \mathscr{L}_n^{(l)} + \Gamma_n^{(l)}\right)$$
$$\left(\mathscr{A}_n{}^{(l)}\right)_J \equiv I + k_n\left(\mathscr{D}_n^{(l)} + \Gamma_n^{(l)}\right) \qquad (l = 1, \dots, N). \tag{4.2}$$

Then we have the following three iterative schemes.

(a) Pichard Iteration:

$$\mathscr{A}_n{}^{(l)}\left(\overline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\Gamma_n^{(l)}\left(\overline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$\mathscr{A}_n{}^{(l)}\left(\underline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\Gamma_n^{(l)}\left(\underline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F^{(l)}\Big(\left(\underline{U}_n^{(l)}\right)^{(m-1)}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$m = 1, 2, \dots \ (n = 1, 2, \dots,\ l = 1, \dots, N) \tag{4.3}$$

(b) Gauss–Seidel Iteration:

$$\left(\mathscr{A}_n^{(l)}\right)_G\left(\overline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\left(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\right)\left(\overline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F_n^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$\left(\mathscr{A}_n^{(l)}\right)_G\left(\underline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\left(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\right)\left(\underline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F_n^{(l)}\Big(\left(\underline{U}_n^{(l)}\right)^{(m-1)}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$m = 1, 2, \dots,\ (n = 1, 2, \dots,\ l = 1, \dots, N). \tag{4.4}$$

(c) Jacobi Iteration:

$$\left(\mathscr{A}_n^{(l)}\right)_J\left(\overline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\left(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)} + \mathscr{L}_n^{(l)}\right)\left(\overline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F_n^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$\left(\mathscr{A}_n^{(l)}\right)_J\left(\underline{U}_n^{(l)}\right)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\left(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)} + \mathscr{L}_n^{(l)}\right)\left(\underline{U}_n^{(l)}\right)^{(m-1)}$$
$$+F_n^{(l)}\Big(\left(\underline{U}_n^{(l)}\right)^{(m-1)}, \left[\underline{\mathbf{U}}_n^{(m-1)}\right]_{a_l}, \left[\overline{\mathbf{U}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)},$$
$$m = 1, 2, \dots \ (n = 1, 2, \dots,\ l = 1, \dots, N). \tag{4.5}$$

In each of the above iterations the initial iteration $(\overline{\mathbf{U}}_n^{(0)}, \underline{\mathbf{U}}_n^{(0)})$ is $(\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n)$ and the initial condition is given by

$$\left(\overline{U}_n^{(l)}\right)^{(m)} = \left(\underline{U}_n^{(l)}\right)^{(m)} = \Psi_n^{(l)},$$
$$m = 1, 2, \dots \ \ (n = 0, -1, \dots, -s,\ l = 1, \dots, N). \quad (4.6)$$

Since by (H_1) the diagonal elements of $D_n^{(l)}$ are positive and the elements of $\mathscr{U}_n^{(l)}$ and $\mathscr{L}_n^{(l)}$ are nonnegative, the inverse matrices $(\mathscr{A}_n^{(l)})_G^{-1}$ and $(\mathscr{A}_n^{(l)})_J^{-1}$ exist and are positive (cf. [14, 15]). Hence the sequences given by the above iteration processes are well-defined and can be computed by the standard computational algorithm in the same fashion as that for linear algebraic systems. Moreover, for fixed m each of the linear equations in the iteration processes is unconditionally stable with respect to the mesh sizes k_n and h_ν. As in the case of linear problems, the Gauss–Seidel and Jacobi iterative schemes have the advantage than the Picard iterations when dealing with the parabolic system (1.1) with two or higher dimensional spatial domains. In the following theorem we show that the maximal and minimal sequences obtained from any one of the iterative schemes converge monotonically to the unique solution $\mathbf{U}_n^*$ of (2.2).

THEOREM 4.1. *Let* $\tilde{\mathbf{U}}_n, \hat{\mathbf{U}}_n$ *be a pair of generalized upper and lower solutions of* (2.2), *and let Hypotheses* (H_1) *and* $(H_2)'$ *and condition* (3.12) *hold. Then the sequences* $\{\overline{\mathbf{U}}_n^{(m)}\}, \{\underline{\mathbf{U}}_n^{(m)}\}$ *given by any one of the iterative schemes in* (4.3), (4.4), *and* (4.5) *converge monotonically from above and below, respectively, to the unique solution* $\mathbf{U}_n^*$ *of* (2.2). *Moreover, relation* (3.13) *holds*.

Proof. (a) Picard iteration. It is easy to see from (4.1), (4.3), and $\hat{\mathbf{U}}_n \le \mathbf{U}_n^* \le \tilde{\mathbf{U}}_n$ that

$$\begin{aligned}
&\mathscr{A}_n^{(l)}\left[\tilde{U}_n^{(l)} - \left(\overline{U}_n^{(l)}\right)^{(1)}\right] \\
&\quad \ge \left[\tilde{U}_{n-1}^{(l)} + k_n\left(\Gamma_n^{(l)}\tilde{U}_n^{(l)} + F^{(l)}\left(\tilde{U}_n^{(l)}, \left[\tilde{\mathbf{U}}_n\right]_{a_l}, \left[\hat{\mathbf{U}}_n\right]_{b_l}, \mathbf{U}_{n-s}^*\right)\right) + G_n^{(l)}\right] \\
&\qquad - \left[(U_{n-1}^*)^{(l)} + k_n\left(\Gamma_n^{(l)}\left(\overline{U}_n^{(l)}\right)^{(0)}\right.\right. \\
&\qquad\qquad \left.\left. + F^{(l)}\left(\left(\overline{U}_n^{(l)}\right)^{(0)}, \left[\overline{\mathbf{U}}_n^{(0)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(0)}\right]_{b_l}, \mathbf{U}_{n-s}^*\right)\right) + G_n^{(l)}\right] \\
&\quad = \tilde{U}_{n-1}^{(l)} - (U_{n-1}^*)^{(l)} \ge 0 \qquad (l = 1, \dots, N).
\end{aligned}$$

By the positive property of $(\mathscr{A}_n^{(l)})^{-1}$ we have $(\overline{U}_n^{(l)})^{(0)} \ge (\overline{U}_n^{(l)})^{(1)}$. A similar argument gives $(\underline{U}_n^{(l)})^{(1)} \ge (\underline{U}_n^{(l)})^{(0)}$. Moreover, by (4.3) (with $m = 1$), (3.4), and $\overline{\mathbf{U}}_n^{(0)} \geq \underline{\mathbf{U}}_n^{(0)}$,

$$\mathscr{A}_n^{(l)}\Big[\big(\overline{U}_n^{(l)}\big)^{(1)} - \big(\underline{U}_n^{(l)}\big)^{(1)}\Big] = k_n\Big[\Gamma_n^{(l)}\Big(\big(\overline{U}_n^{(l)}\big)^{(0)} - \big(\underline{U}_n^{(l)}\big)^{(0)}\Big) + F^{(l)}\Big(\big(\overline{U}_n^{(l)}\big)^{(0)}, \big[\overline{\mathbf{U}}_n^{(0)}\big]_{a_l}, \big[\underline{\mathbf{U}}_n^{(0)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big) - F^{(l)}\Big(\big(\underline{U}_n^{(l)}\big)^{(0)}, \big[\underline{\mathbf{U}}_n^{(0)}\big]_{a_l}, \big[\overline{\mathbf{U}}_n^{(0)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] \geq \mathbf{0}.$$

This yields $(\overline{U}_n^{(l)})^{(1)} \geq (\underline{U}_n^{(l)})^{(1)}$. The above conclusions show that $\underline{\mathbf{U}}_n^{(0)} \leq \underline{\mathbf{U}}_n^{(1)} \leq \overline{\mathbf{U}}_n^{(1)} \leq \overline{\mathbf{U}}_n^{(0)}$. It follows by an induction argument that $\{\overline{\mathbf{U}}_n^{(m)}\}$ and $\{\underline{\mathbf{U}}_n^{(m)}\}$ possess the monotone property (3.6). Letting $m \to \infty$ in (4.3) and (4.6) shows that the limits $\overline{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n$ in (3.7) exist and satisfy the relation

$$\big(I + k_n A_n^{(l)}\big)\overline{U}_n^{(l)} = (U_{n-1}^*)^{(l)} + k_n F^{(l)}\Big(\overline{U}_n^{(l)}, \big[\overline{\mathbf{U}}_n\big]_{a_l}, [\underline{\mathbf{U}}_n]_{b_l}, \mathbf{U}_{n-s}^*\Big) + G_n^{(l)}$$
$$\big(I + k_n A_n^{(l)}\big)\underline{U}_n^{(l)} = (U_{n-1}^*)^{(l)} + k_n F^{(l)}\Big(\underline{U}_n^{(l)}, [\underline{\mathbf{U}}_n]_{a_l}, \big[\overline{\mathbf{U}}_n\big]_{b_l}, \mathbf{U}_{n-s}^*\Big) + G_n^{(l)}$$
$$(n = 1, 2, \dots)$$
$$\overline{U}_n^{(l)} = \underline{U}_n^{(l)} = \Psi_n^{(l)} \qquad (n = 0, -1, \dots, -s_l). \tag{4.7}$$

This implies that $W_n^{(l)} \equiv \overline{U}_n^{(l)} - \underline{U}_n^{(l)}$ satisfies the equation

$$\big(I + k_n A_n^{(l)}\big)W_n^{(l)} = k_n\Big[F^{(l)}\Big(\overline{U}_n^{(l)}, \big[\overline{\mathbf{U}}_n\big]_{a_l}, [\underline{\mathbf{U}}_n]_{b_l}, \mathbf{U}_{n-s}^*\Big) - F^{(l)}\Big(\underline{U}_n^{(l)}, [\underline{\mathbf{U}}_n]_{a_l}, \big[\overline{\mathbf{U}}_n\big]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big].$$

In view of (3.9), we have

$$\Big[I + k_n\Big(A_n^{(l)} - \sigma_{l,n}^{(l)} I\Big)\Big]W_n^{(l)} \leq k_n \sum_{j \neq l}^{n} \sigma_{j,n}^{(l)} W_n^{(j)}.$$

It follows again from [12] that $W_n^{(l)} = \mathbf{0}$, $l = 1, \dots, N$. This shows that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n \equiv \mathbf{U}_n$, and the components $U_n^{(l)}$ of $\mathbf{U}_n$ satisfy

$$\big(I + k_n A_n^{(l)}\big)U_n^{(l)} = (U_{n-1}^*)^{(l)} + k_n F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}^*).$$

Let $V_n^{(l)} \equiv U_n^{(l)} - (U_n^*)^{(l)}$, $l = 1, \dots, N$. Since U_n^* is the solution of (2.2) we see that

$$\big(I + k_n A_n^{(l)}\big)V_n^{(l)} = k_n\big[F^{(l)}(\mathbf{U}_n, \mathbf{U}_{n-s}^*) - F^{(l)}(\mathbf{U}_n^*, \mathbf{U}_{n-s}^*)\big] = k_n F_u^{(l)}(\boldsymbol{\xi}_n, \mathbf{U}_{n-s}^*)\mathbf{V}_n.$$

This is in a form similar to that in (3.16) which ensures that $V_n^{(l)} = 0$ for $n = 1, 2, \ldots$. This proves $\mathbf{U}_n = \mathbf{U}_n^*$ and thus the theorem for the Picard iteration.

(b) Gauss–Seidel iteration. By (4.2), (4.4), (4.1), and $\hat{\mathbf{U}}_{n-s} \le \mathbf{U}_{n-s}^* \le \tilde{\mathbf{U}}_{n-s}$, we have

$$
\begin{aligned}
&\left(\mathscr{A}_n^{(l)}\right)_G \left[\left(\overline{U}_n^{(l)}\right)^{(0)} - \left(\overline{U}_n^{(l)}\right)^{(1)}\right] \\
&\quad = \left[I + k_n\left(\mathscr{D}_n^{(l)} - \mathscr{L}_n^{(l)} + \Gamma_n^{(l)}\right)\right]\left(\overline{U}_n^{(l)}\right)^{(0)} \\
&\qquad - \Big\{\left(U_{n-1}^*\right)^{(l)} + k_n\Big[\left(\mathscr{U}_n^{(l)} + \Gamma_n^{(l)}\right)\left(\overline{U}_n^{(l)}\right)^{(0)} \\
&\qquad\qquad + F^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(0)}, \left[\overline{\mathbf{U}}_n^{(0)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(0)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)}\Big\} \\
&\quad = \left(I + k_n A_n\right)\left(\overline{U}_n^{(l)}\right)^{(0)} \\
&\qquad - \Big\{\left(U_{n-1}^*\right)^{(l)} + k_n F^{(l)}\Big(\left(\overline{U}_n^{(l)}\right)^{(0)}, \left[\overline{\mathbf{U}}_n^{(0)}\right]_{a_l}, \left[\underline{\mathbf{U}}_n^{(0)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big) + G_n^{(l)}\Big\} \\
&\quad \ge \tilde{U}_{n-1}^{(l)} - \left(U_{n-1}^*\right)^{(l)} \ge \mathbf{0}.
\end{aligned}
$$

The positivity of $(\mathscr{A}_n^{(l)})_G^{-1}$ implies that $(\overline{U}_n^{(l)})^{(0)} \ge (\overline{U}_n^{(l)})^{(1)}$. A similar argument using the relations in (4.4), (3.4), and $\mathscr{U}_n \ge 0$ leads to $(\overline{U}_n^{(l)})^{(1)} \ge (\underline{U}^{(l)})^{(1)} \ge (\underline{U}^{(l)})^{(0)}$. It follows by an induction argument that the sequences $\{\overline{\mathbf{U}}_n^{(m)}\}, \{\underline{\mathbf{U}}_n^{(m)}\}$ governed by (4.4) possess the monotone property (3.6), and therefore their limits $\overline{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n$ given by (3.7) exist. Letting $m \to \infty$ in (4.4) and using the definition of $(\mathscr{A}_n^{(l)})_G$ in (4.2) shows that $\overline{\mathbf{U}}_n$ and $\underline{\mathbf{U}}_n$ satisfy the equations in (4.7). By the proof of the Picard iteration we conclude that $\overline{\mathbf{U}}_n = \underline{\mathbf{U}}_n = \mathbf{U}_n^*$. This proves the theorem for the Gauss–Seidel iteration.

(c) Jacobi iteration. The proof is similar to that for Gauss–Seidel iteration and is omitted. ∎

An interesting consequence of the above monotone convergence of the maximal and minimal sequences is the following comparison result.

THEOREM 4.2. *Let the conditions in Theorem* 4.1 *be satisfied, and let* $(\{\overline{\mathbf{U}}_n^{(m)}\}_p, \{\underline{\mathbf{U}}_n^{(m)}\}_p)$, $(\{\overline{\mathbf{U}}_n^{(m)}\}_G, \{\underline{\mathbf{U}}_n^{(m)}\}_G)$, *and* $(\{\overline{\mathbf{U}}_n^{(m)}\}_J, \{\underline{\mathbf{U}}_n^{(m)}\}_J)$ *be the respective maximal–minimal sequences given by* (4.3), (4.4), *and* (4.5) *under the same initial condition* (4.6). *Then*

$$
\begin{aligned}
&\left(\overline{\mathbf{U}}_n^{(m)}\right)_p \le \left(\overline{\mathbf{U}}_n^{(m)}\right)_G \le \left(\overline{\mathbf{U}}_n^{(m)}\right)_J \\
&\left(\underline{\mathbf{U}}_n^{(m)}\right)_p \ge \left(\underline{\mathbf{U}}_n^{(m)}\right)_G \ge \left(\underline{\mathbf{U}}_n^{(m)}\right)_J \qquad (m = 1, 2, \ldots,\ n = 1, 2, \ldots,).
\end{aligned}
\tag{4.8}
$$

Proof. Let

$$\overline{\mathbf{V}}_n^{(m)} = \left(\overline{\mathbf{U}}_n^{(m)}\right)_G - \left(\overline{\mathbf{U}}_n^{(m)}\right)_p, \qquad \underline{\mathbf{V}}_n^{(m)} = \left(\underline{\mathbf{U}}_n^{(m)}\right)_p - \left(\underline{\mathbf{U}}_n^{(m)}\right)_G,$$

$$\left(\overline{V}_n^{(l)}\right)^{(m)} = \left(\overline{U}_n^{(l)}\right)_G^{(m)} - \left(\overline{U}_n^{(l)}\right)_p^{(m)}, \qquad \left(\underline{V}_n^{(l)}\right)^{(m)} = \left(\underline{U}_n^{(l)}\right)_p^{(m)} - \left(\underline{U}_n^{(l)}\right)_G^{(m)} \qquad (l = 1, \ldots, N),$$

where $(\overline{\mathbf{U}}_n^{(0)})_G = (\overline{\mathbf{U}}_n^{(0)})_p = \tilde{\mathbf{U}}_n$ and $(\underline{\mathbf{U}}_n^{(0)})_G = (\underline{\mathbf{U}}_n^{(0)})_p = \hat{\mathbf{U}}_n$. Since $(\mathscr{A}_n^{(l)})_G = \mathscr{A}_n^{(l)} + k_n \mathscr{U}_n^{(l)}$, a subtraction of the respective first equation in (4.3) and (4.4) and using the property $\mathscr{U}_n^{(l)}((\overline{U}_n^{(l)})_G^{(m-1)} - (\overline{U}_n^{(l)})_G^{(m)}) \geq \mathbf{0}$ yields

$$\begin{aligned} \mathscr{A}_n^{(l)}\left(\overline{V}_n^{(l)}\right)^{(m)} \geq k_n \Big[& \Gamma_n^{(l)}\left(\left(\overline{U}_n^{(l)}\right)_G^{(m-1)} - \left(\overline{U}_n^{(l)}\right)_p^{(m-1)}\right) \\ & + F^{(l)}\left(\left(\overline{U}_n^{(l)}\right)_G^{(m-1)}, \left[\left(\overline{\mathbf{U}}_n^{(m-1)}\right)_G\right]_{a_l}, \left[\left(\underline{\mathbf{U}}_n^{(m-1)}\right)_G\right]_{b_l}, \mathbf{U}_{n-s}^*\right) \\ & - F^{(l)}\left(\left(\overline{U}_n^{(l)}\right)_p^{(m-1)}, \left[\left(\overline{\mathbf{U}}_n^{(m-1)}\right)_p\right]_{a_l}, \left[\left(\underline{\mathbf{U}}_n^{(m-1)}\right)_p\right]_{b_l}, \mathbf{U}_{n-s}^*\right) \end{aligned} \tag{4.9}$$

Similarly, a subtraction of the respective second equation in (4.3) and (4.4) and using the relation $\mathscr{U}_n^{(l)}((\underline{U}_n^{(l)})_G^{(m)} - (\underline{U}_n^{(l)})_G^{(m-1)}) \geq \mathbf{0}$ leads to

$$\begin{aligned} \mathscr{A}_n^{(l)}\left(\underline{V}_n^{(l)}\right)^{(m)} \geq k_n \Big[& \Gamma_n^{(l)}\left(\left(\underline{U}_n^{(l)}\right)_p^{(m-1)} - \left(\underline{U}_n^{(l)}\right)_G^{(m-1)}\right) \\ & + F^{(l)}\left(\left(\underline{U}_n^{(l)}\right)_p^{(m-1)}, \left[\left(\underline{\mathbf{U}}_n^{(m-1)}\right)_p\right]_{a_l}, \left[\left(\overline{\mathbf{U}}_n^{(m-1)}\right)_p\right]_{b_l}, \mathbf{U}_{n-s}^*\right) \\ & - F^{(l)}\left(\left(\underline{U}_n^{(l)}\right)_G^{(m-1)}, \left[\left(\underline{\mathbf{U}}_n^{(m-1)}\right)_G\right]_{a_l}, \left[\left(\overline{\mathbf{U}}_n^{(m-1)}\right)_G\right]_{b_l}, \mathbf{U}_{n-s}^*\right). \end{aligned} \tag{4.10}$$

Since $(\overline{\mathbf{U}}_n^{(0)})_G = (\overline{\mathbf{U}}_n^{(0)})_p = \tilde{\mathbf{U}}_n$ and $(\underline{\mathbf{U}}_n^{(0)})_G = (\underline{\mathbf{U}}_n^{(0)})_p = \hat{\mathbf{U}}_n$, the relations in (4.9), (4.10), and (3.4) (with $m = 1$) imply that

$$\mathscr{A}_n^{(l)}\left(\overline{V}_n^{(l)}\right)^{(1)} \geq \mathbf{0} \quad \text{and} \quad \mathscr{A}_n^{(l)}\left(\underline{V}_n^{(l)}\right)^{(1)} \geq \mathbf{0}, \qquad l = 1, \ldots, N.$$

This leads to $(\overline{U}_n^{(l)})_G^{(1)} \geq (\overline{U}_n^{(l)})_p^{(1)}$ and $(\underline{U}_n^{(l)})_G^{(1)} \leq (\underline{U}_n^{(l)})_p^{(1)}$. An induction argument as that in [9, 12] shows that the first inequalities in (4.8) between $(\mathbf{U}_n^{(m)})_p$ and $(\mathbf{U}_n^{(m)})_G$ hold. The proof for the second inequalities in (4.8) between $(\mathbf{U}_n^{(m)})_G$ and $(\mathbf{U}_n^{(m)})_J$ is similar. ∎

It is seen from Theorem 4.2 that if the initial iteration in the iterative schemes (4.3), (4.4), and (4.5) is taken as a pair of generalized upper and lower solutions, and the initial functions for the iterations are the same, then the sequences of Picard iteration converges faster than the sequence of Gauss–Seidel iteration which in turn converges faster than the sequence of Jacobi iteration. This property holds for both the maximal sequences $\{\overline{\mathbf{U}}_n^{(m)}\}$ and the minimal sequence $\{\underline{\mathbf{U}}_n^{(m)}\}$.

5. ERROR ESTIMATES

The conclusion in Theorem 4.1 ensures that given any $\epsilon > 0$ there exist an integer $m^* = m^*(\epsilon)$ and a norm in $\mathbb{R}^q$ such that

$$\|\overline{\mathbf{U}}_n^{(m)} - \underline{\mathbf{U}}_n^{(m)}\| < \epsilon \qquad \text{for} \qquad m \geq m^*,\ n = 1, 2, \ldots,$$

where $q = MN$ and $(\overline{\mathbf{U}}_n^{(m)}, \underline{\mathbf{U}}_n^{(m)})$ is the mth iteration from any one of the iterative schemes in (4.3), (4.4), and (4.5). This shows that the error between the true solution $\overline{\mathbf{U}}_n^*$ and the mth approximation $\overline{\mathbf{U}}_n^{(m)}$ (or $\underline{\mathbf{U}}_n^{(m)}$) is bounded by ϵ when $m \geq m^*$. However, in actual computation the true values of $\mathbf{U}_{n-1}^*$ and $\mathbf{U}_{n-s}^*$ in the above iterative schemes are approximated by $\overline{\mathbf{U}}_{n-1}^{(m')}$ and $\overline{U}_{n-s}^{(m')}$ (or by $\underline{\mathbf{U}}_{n-1}^{(m')}$ and $\underline{\mathbf{U}}_{n-s}^{(m')}$), respectively, for some m'. This leads to an error between the theoretical and the computed mth approximation at n. Furthermore, there may also be round-off error in the computation of $\overline{\mathbf{U}}_n^{(m)}$ and $\underline{\mathbf{U}}_n^{(m)}$. In this section we give an estimate for the error between the true solution $\mathbf{U}_n^*$ and the computed mth approximation $\mathbf{V}_n^{(m)}$ for each of the iterative schemes (4.3), (4.4), and (4.5). Our estimate is given in terms of the smallest eigenvalue $\mu_n^{(l)}$ of $A_n^{(l)}$ and the strength of the function $f^{(l)}(\cdot, \mathbf{u}, \mathbf{u}_\tau)$, and it is independent of the true solution. Throughout this section we denote the computed mth approximation from either one of the iterative schemes by

$$\overline{\mathbf{V}}_n^{(m)} \equiv \left(\left(\overline{V}_n^{(1)}\right)^{(m)}, \ldots, \left(\overline{V}_n^{(N)}\right)^{(m)}\right)' \qquad \text{and}$$

$$\underline{\mathbf{V}}_n^{(m)} \equiv \left(\left(\underline{V}_n^{(1)}\right)^{(m)}, \ldots, \left(\underline{V}_n^{(N)}\right)^{(m)}\right)'.$$

5.1. *Picard Iteration*

For the Picard iterative scheme (4.3) $\overline{\mathbf{V}}_n^{(m)}$ and $\underline{\mathbf{V}}_n^{(m)}$ satisfy the relation

$$\mathscr{A}_n^{(l)}\left(\overline{V}_n^{(l)}\right)^{(m)} = \left(U_{n-1}^*\right)^{(l)} + k_n\Big[\Gamma_n^{(l)}\left(\overline{V}_n^{(l)}\right)^{(m-1)}$$
$$+F^{(l)}\Big(\left(\overline{V}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{V}}_n^{(m-1)}\right]_{a_l}, \left[\underline{\mathbf{V}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)} + \left(\bar{e}_n^{(l)}\right)^{(m)}$$

$$\mathscr{A}_n^{(l)}\big(\underline{V}_n^{(l)}\big)^{(m)} = (U_{n-1}^*)^{(l)} + k_n\Big[\Gamma_n^{(l)}\big(\underline{V}_n^{(l)}\big)^{(m-1)}$$
$$+F^{(l)}\Big(\big(\underline{V}_n^{(l)}\big)^{(m-1)}, \big[\underline{\mathbf{V}}_n^{(m-1)}\big]_{a_l}, \big[\overline{\mathbf{V}}_n^{(m-1)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)} + \big(\underline{e}_n^{(l)}\big)^{(m)}$$
$$(l = 1, \dots, N), \quad (5.1)$$

where $(\bar{e}_n^{(l)})^{(m)}$ and $(\underline{e}_n^{(l)})^{(m)}$ denote the combined errors due to round-off and the replacement of $(\mathbf{U}_{n-1}^*, \mathbf{U}_{n-s}^*)$ by $(\overline{\mathbf{V}}_{n-1}^{(m)}, \overline{\mathbf{V}}_{n-s}^{(m)})$ (or by $(\underline{\mathbf{V}}_{n-1}^{(m)}, \underline{\mathbf{V}}_{n-s}^{(m)})$). Let ϵ_r, ϵ_a be the maximum round-off error and the maximum allowable error between the theoretical and the computed mth approximation, respectively, and let

$$\bar{e}_n^{(m)} \equiv \Big(\big(\bar{e}_n^{(1)}\big)^{(m)}, \dots, \big(\bar{e}_n^{(N)}\big)^{(m)}\Big)', \qquad \underline{e}_n^{(m)} \equiv \Big(\big(\underline{e}_n^{(1)}\big)^{(m)}, \dots, \big(\underline{e}_n^{(N)}\big)^{(m)}\Big)'.$$

Then by the monotone convergence result of Theorem 4.1 there exists an integer m^* such that

$$\|\bar{e}_n^{(m)}\| + \|\underline{e}_n^{(m)}\| < 2(\epsilon_r + \epsilon_a) \qquad \text{for all } m \geq m^*, \tag{5.2}$$

where $\|\cdot\|$ is a suitable norm in $\mathbb{R}^q$. Define

$$\begin{aligned} \beta_n^* &\equiv \max\Big\{\big[1 + k_n\big(\mu_n^{(l)} + \gamma_n^{(l)}\big)\big]^{-1}; \ l = 1, \dots, N\Big\}, \\ \bar{\sigma}_n^{(l)} &\equiv \max\Big\{\sigma_{j,n}^{(l)} + \delta_j^{(l)}\gamma_n^{(l)}; \ j = 1, \dots, N\Big\}, \\ \omega_n^* &\equiv \sum_{l=1}^{N}\Big[\bar{\sigma}_n^{(l)}/\big(1 + k_n\big(\mu_n^{(l)} + \gamma_n^{(l)}\big)\big)\Big], \end{aligned} \tag{5.3}$$

where $\delta_j^{(l)} = 1$ if $j = l$, and $\delta_j^{(l)} = 0$ if $j \neq l$, and $\sigma_{j,n}^{(l)}$ is given by (3.9). Then we have the following error estimate for the Picard iteration.

THEOREM 5.1. *Let* $\mathbf{U}_n^*$ *be the true solution of* (2.2), *and let* $(\overline{\mathbf{V}}^{(m)}, \underline{\mathbf{V}}^{(m)})$ *be the computed mth approximation by the Picard iteration* (4.3). *Assume that hypotheses* (H_1) *and* (H_2') *hold and that*

$$k_n\omega_n^* < 1 \qquad \text{for} \qquad n = 1, 2, \dots. \tag{5.4}$$

Then there exist an integer m^* *and a norm in* $\mathbb{R}^q$ *such that*

$$\|\overline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*\| + \|\underline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*\| < \frac{2\beta_n^*}{1 - k_n\omega_n^*}(\epsilon_a + \epsilon_r)$$
$$\text{for all } m \geq m^* \ (n = 1, 2, \dots) \quad (5.5)$$

Proof. Let ϵ_0 be any constant satisfying $\|\bar{e}_n^{(m)}\| + \|\underline{e}_n^{(m)}\| < \epsilon_0 \leq 2(\epsilon_a + \epsilon_r)$, and let

$$\overline{\mathbf{E}}_n^{(m)} \equiv \overline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*, \qquad \underline{\mathbf{E}}_n^{(m)} \equiv \underline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*, \tag{5.6}$$

where $m \geq m^*$. By subtracting (3.3) with $(\mathbf{U}_n = \mathbf{U}_n^*)$ from (5.1), and using the positive property of $(\mathscr{A}_n^{(l)})^{-1}$, we see that the components $(\overline{E}_n^{(l)})^{(m)}$ and $(\underline{E}_n^{(l)})^{(m)}$ of $\overline{\mathbf{E}}_n^{(m)}$ and $\underline{\mathbf{E}}_n^{(m)}$ satisfy the relation

$$\begin{aligned}\left|\left(\overline{E}_n^{(l)}\right)^{(m)}\right| \leq \left(\mathscr{A}_n^{(l)}\right)^{-1}\Big\{k_n\Big[\Gamma_n^{(l)}\left|\left(\overline{E}_n^{(l)}\right)^{(m-1)}\right| \\ +\Big|F^{(l)}\Big(\left(\overline{V}_n^{(l)}\right)^{(m-1)}, \left[\overline{\mathbf{V}}_n^{(m-1)}\right]_{a_l}, \left[\underline{\mathbf{V}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big) \\ -F^{(l)}\Big((U_n^*)^{(l)}, [\mathbf{U}_n^*]_{a_l}, [\mathbf{U}_n^*]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big|\Big] + \left|\left(\bar{e}_n^{(l)}\right)^{(m)}\right|\Big\},\end{aligned}$$

$$\begin{aligned}\left|\left(\underline{E}_n^{(l)}\right)^{(m)}\right| \leq \left(\mathscr{A}_n^{(l)}\right)^{-1}\Big\{k_n\Big[\Gamma_n^{(l)}\left|\left(\underline{E}_n^{(l)}\right)^{(m-1)}\right| \\ +\Big|F^{(l)}\Big(\left(\underline{V}_n^{(l)}\right)^{(m-1)}, \left[\underline{\mathbf{V}}_n^{(m-1)}\right]_{a_l}, \left[\overline{\mathbf{V}}_n^{(m-1)}\right]_{b_l}, \mathbf{U}_{n-s}^*\Big) \\ -F^{(l)}\Big((U_n^*)^{(l)}, [\mathbf{U}_n^*]_{a_l}, [\mathbf{U}_n^*]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big|\Big] + \left|\left(\underline{e}_n^{(l)}\right)^{(m)}\right|\Big\},\end{aligned}$$

where $|W| = (|w_1|, \ldots, |w_M|)'$ for any $W = (w_1, \ldots, w_M)' \in \mathbb{R}^M$. Using the relation (5.3) in the above inequalities leads to

$$\begin{aligned}\left|\left(\overline{E}_n^{(l)}\right)^{(m)}\right| \leq \left(\mathscr{A}_n^{(l)}\right)^{-1}\Big\{k_n\Big[\hat{\sigma}_{l,n}^{(l)}\left|\left(\overline{E}_n^{(l)}\right)^{(m-1)}\right| + \sum_{j=1}^{a_l}\sigma_{j,n}^{(l)}\left|\left(\overline{E}_n^{(j)}\right)^{(m-1)}\right| \\ + \sum_{j=1}^{b_l}\sigma_{j,n}^{(l)}\left|\left(\underline{E}_n^{(j)}\right)^{(m-1)}\right|\Big] + \left|\left(\bar{e}_n^{(l)}\right)^{(m)}\right|\Big\} \qquad (l = 1, \ldots, N),\end{aligned}$$

$$\begin{aligned}\left|\left(\underline{E}_n^{(l)}\right)^{(m)}\right| \leq \left(\mathscr{A}_n^{(l)}\right)^{-1}\Big\{k_n\Big[\hat{\sigma}_{l,n}^{(l)}\left|\left(\underline{E}_n^{(l)}\right)^{(m-1)}\right| + \sum_{j=1}^{a_l}\sigma_{j,n}^{(l)}\left|\left(\underline{E}_n^{(j)}\right)^{(m-1)}\right| \\ + \sum_{j=1}^{b_l}\sigma_{j,n}^{(l)}\left|\left(\overline{E}_n^{(j)}\right)^{(m-1)}\right|\Big] + \left|\left(\underline{e}_n^{(l)}\right)^{(m)}\right|\Big\} \qquad (l = 1, \ldots, N),\end{aligned} \tag{5.7}$$

where $\hat{\sigma}_{l,n}^{(l)} \equiv \sigma_{l,n}^{(l)} + \gamma_n^{(l)} \geq 0$. Define

$$\begin{aligned}\left|\left(E_n^{(l)}\right)^{(m)}\right| = \left|\left(\overline{E}_n^{(l)}\right)^{(m)}\right| + \left|\left(\underline{E}_n^{(l)}\right)^{(m)}\right|, \qquad \left|\mathbf{E}_n^{(m)}\right| = \sum_{l=1}^{N}\left|\left(E_n^{(l)}\right)^{(m)}\right|, \\ \left|\left(e_n^{(l)}\right)^{(m)}\right| = \left|\left(\bar{e}_n^{(l)}\right)^{(m)}\right| + \left|\left(\underline{e}_n^{(l)}\right)^{(m)}\right|.\end{aligned} \tag{5.8}$$

Then adding the two inequalities in (5.7) and using the definition of $\overline{\sigma}_n^{(l)}$ in (5.3) yield

$$\left|\left(E_n^{(l)}\right)^{(m)}\right| \le \left(\mathscr{A}_n^{(l)}\right)^{-1}\left\{k_n \overline{\sigma}_n^{(l)}\left|\mathbf{E}_n^{(m)}\right| + \left|\left(e_n^{(l)}\right)^{(m)}\right|\right\} \qquad (l = 1, \ldots, N).$$

Since the spectral radius of $(\mathscr{A}_n^{(l)})^{-1}$ is bound by $\beta_n^{(l)} \equiv (1 + k_n(\mu_n^{(l)} + \gamma_n^{(l)}))^{-1}$ we see that given any $\epsilon_1 > 0$ there exists a norm in $\mathbb{R}^M$ such that

$$\left\|\left(E_n^{(l)}\right)^{(m)}\right\| \le \left(\beta_n^{(l)} + \epsilon_1\right)\left[k_n \overline{\sigma}_n^{(l)}\left\|\mathbf{E}_n^{(m-1)}\right\| + \left\|\left(e_n^{(l)}\right)^{(m)}\right\|\right]$$

(cf. [4]). Adding the above inequalities over l and choosing ϵ_1 sufficiently small, relation (5.2) and the choice of ϵ_0 ensure that

$$\|\mathbf{E}_n^{(m)}\| \le k_n \omega_n^* \|\mathbf{E}_n^{(m-1)}\| + \beta^* \epsilon_0. \tag{5.9}$$

It follows by an induction argument, using the fact that $\|\mathbf{E}^{(0)}\| = 0$ and $k_n \omega^* < 1$, that

$$\|\mathbf{E}_n^{(m)}\| < \beta_n^* \epsilon_0 (1 - k_n \omega_n^*)^{-1}, \qquad n = 1, 2, \ldots .$$

This leads to the estimate (5.5) which proves the theorem. ∎

5.2. *Gauss–Seidel and Jacobi Iterations*

To obtain an error estimate for the Gauss–Seidel and Jacobi iterations (4.4) and (4.5) we assume, for simplicity, that the diagonal elements of $(A_n^{(l)} + \Gamma_n^{(l)})$ are independent of i and are denoted by $(a_n^{(l)} + \gamma_n^{(l)})$. Define

$$\begin{aligned}
\tilde{\beta}_n &= \max\left\{1 + k_n\left(a_n^{(l)} + \gamma_n^{(l)}\right);\ l = 1, \ldots, N\right\} \\
\tilde{\sigma}_n &= \max\left\{\sigma_{j,n}^{(l)} + \left(a_n^{(l)} + \gamma_n^{(l)}\right);\ l = 1, \ldots, N\right\} \\
\tilde{\omega}_n &= \sum_{l=1}^{N} \left[\tilde{\sigma}_n^{(l)} / \left(1 + k_n\left(a_n^{(l)} + \gamma_n^{(l)}\right)\right)\right].
\end{aligned} \tag{5.10}$$

In terms of the above parameters we have the following.

THEOREM 5.2. *Let* $\mathbf{U}_n^*$ *be the true solution of* (2.2), *and let* $(\overline{\mathbf{V}}_n^{(m)}, \underline{\mathbf{V}}_n^{(m)})$ *be the computed mth approximation by either the Gauss–Seidel iteration* (4.4) *or the Jacobi iteration* (4.5). *Assume that* (H_1) *and* $(H_2)'$ *hold and that*

$$k_n \tilde{\omega}_n < 1 \qquad \text{for } n = 1, 2, \ldots . \tag{5.11}$$

Then there exist an integer m^* and a vector norm in $\mathbb{R}^q$ such that

$$\|\overline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*\| + \|\underline{\mathbf{V}}_n^{(m)} - \mathbf{U}_n^*\| \le \frac{2\tilde{\beta}_n}{1 - k_n\tilde{\omega}_n}(\epsilon_a + \epsilon_r) \qquad \textit{for all } m \ge m^*. \tag{5.12}$$

Proof. In the Gauss–Seidel iteration (4.4) the computed mth iterations $(\overline{V}_n^{(l)})^{(m)}$ and $(\underline{V}_n^{(l)})^{(m)}$ satisfy the relation

$$\begin{aligned}
\big(\mathscr{A}_n^{(l)}\big)_G\big(\overline{V}_n^{(l)}\big)^{(m)} &= (U_{n-1}^*)^{(l)} + k_n\Big[\big(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\big)\big(\overline{V}_n^{(l)}\big)^{(m-1)} \\
&\quad + F^{(l)}\Big(\big(\overline{V}_n^{(l)}\big)^{(m-1)}, \big[\overline{\mathbf{V}}_n^{(m-1)}\big]_{a_l}, \big[\underline{\mathbf{V}}_n^{(m-1)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)} + \big(\bar{e}_n^{(l)}\big)^{(m)}, \\
\big(\mathscr{A}_n^{(l)}\big)_G\big(\underline{V}_n^{(l)}\big)^{(m)} &= (U_{n-1}^*)^{(l)} + k_n\Big[\big(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\big)\big(\underline{V}_n^{(l)}\big)^{(m-1)} \\
&\quad + F^{(l)}\Big(\big(\underline{V}_n^{(l)}\big)^{(m-1)}, \big[\underline{\mathbf{V}}_n^{(m-1)}\big]_{a_l}, \big[\overline{\mathbf{V}}_n^{(m-1)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big)\Big] + G_n^{(l)} + \big(\underline{e}_n^{(l)}\big)^{(m)}
\end{aligned}$$

$$(l = 1, \dots, N). \tag{5.13}$$

Subtracting (3.3) from (5.13) and using the definition of $\overline{\mathbf{E}}_n^{(m)}$ and $\underline{\mathbf{E}}_n^{(m)}$ in (5.6) yield

$$\begin{aligned}
\big(\mathscr{A}_n^{(l)}\big)_G\big(\overline{E}_n^{(l)}\big)^{(m)} &= k_n\Big[\big(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\big)\big(\overline{E}_n^{(l)}\big)^{(m-1)} \\
&\quad + F^{(l)}\Big(\big(\overline{V}_n^{(l)}\big)^{(m-1)}, \big[\overline{\mathbf{V}}_n^{(m-1)}\big]_{a_l}, \big[\underline{\mathbf{V}}_n^{(m-1)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big) - F^{(l)}(\mathbf{U}_n^*, \mathbf{U}_{n-s}^*)\Big] \\
&\quad + \big(\bar{e}_n^{(l)}\big)^{(m)} \\
\big(\mathscr{A}_n^{(l)}\big)_G\big(\underline{E}_n^{(l)}\big)^{(m)} &= k_n\Big[\big(\Gamma_n^{(l)} + \mathscr{U}_n^{(l)}\big)\big(\underline{E}_n^{(l)}\big)^{(m-1)} \\
&\quad + F^{(l)}\Big(\big(\underline{V}_n^{(l)}\big)^{(m-1)}, \big[\underline{\mathbf{V}}_n^{(m-1)}\big]_{a_l}, \big[\overline{\mathbf{V}}_n^{(m-1)}\big]_{b_l}, \mathbf{U}_{n-s}^*\Big) - F^{(l)}(\mathbf{U}_n^*, \mathbf{U}_{n-s}^*)\Big] \\
&\quad + \big(\underline{e}_n^{(l)}\big)^{(m)}.
\end{aligned}$$

By the positive properties of $(\mathscr{A}_n^{(l)})_G^{-1}$ and $(\mathscr{U}_n^{(l)} + \Gamma_n^{(l)} + \sigma_{l,n}^{(l)}I)$ we obtain

$$\begin{aligned}
\Big|\big(\overline{E}_n^{(l)}\big)^{(m)}\Big| &\le \big(\mathscr{A}_n^{(l)}\big)_G^{-1}\Bigg\{k_n\Bigg[\big(\mathscr{U}_n^{(l)} + \Gamma_n^{(l)} + \sigma_{l,n}^{(l)}I\big)\Big|\big(\overline{E}_n^{(l)}\big)^{(m-1)}\Big| \\
&\quad + \sum_{j=1}^{a_l}\sigma_{j,n}^{(l)}\Big|\big(\overline{E}_n^{(j)}\big)^{(m-1)}\Big| + \sum_{j=1}^{b_l}\sigma_{j,n}^{(l)}\Big|\big(\underline{E}_n^{(j)}\big)^{(m-1)}\Big|\Bigg] + \Big|\big(\bar{e}_n^{(l)}\big)^{(m)}\Big|\Bigg\},
\end{aligned}$$

$$\left|\left(\underline{E}_n^{(l)}\right)^{(m)}\right| \le \left(\mathscr{A}_n^{(l)}\right)_G^{-1}\left\{k_n\left[\left(\mathscr{U}_n^{(l)} + \Gamma_n^{(l)} + \sigma_{l,n}^{(l)} I\right)\left|\left(\underline{E}_n^{(l)}\right)^{(m-1)}\right|\right.\right.$$
$$\left.\left. + \sum_{j=1}^{a_l}\sigma_{j,n}^{(l)}\left|\left(\underline{E}_n^{(j)}\right)^{(m-1)}\right| + \sum_{j=1}^{b_l}\sigma_{j,n}^{(l)}\left|\left(\overline{E}_n^{(j)}\right)^{(m-1)}\right|\right] + \left|\left(\underline{e}_n^{(l)}\right)^{(m)}\right|\right\}$$
$$(l = 1, \ldots, N). \quad (5.14)$$

Since the above relation is in exactly the same form as Eq. (5.12) of [12] we conclude from the argument in [12] that the error estimate (5.12) holds. This proves the theorem for the Gauss–Seidel iteration. The proof for the Jacobi iteration is similar and is omitted. ▌

6. CONVERGENCE OF THE FINITE DIFFERENCE SOLUTION

The method used in [12] for the convergence of finite difference solutions can be applied to the system (1.1) with time delays. To show this we assume that the coefficients of $\mathscr{L}^{(l)}$ and $B^{(l)}$ and the functions $f^{(l)}$, $g^{(l)}$, and $\psi^{(l)}$ are suitably smooth in their respective domains, and $\beta^{(l)}(x,t) \neq 0$ on S_T, where

$$\mathscr{L}_\gamma^{(l)} u^{(l)} \equiv u_t^{(l)} - L^{(l)} u^{(l)} + \gamma^{(l)} u^{(l)} \qquad (l = 1, \ldots, N)$$

and $\beta^{(l)}$ is the boundary coefficient of $B^{(l)}$ (cf. [7, 8]). Recall that for the mixed quasimonotone function $\mathbf{f}(\mathbf{u}, \mathbf{u}_\tau) \equiv (f^{(1)}(\mathbf{u}, \mathbf{u}_\tau), \ldots, f^{(N)}(\mathbf{u}, \mathbf{u}_\tau))$, coupled upper and lower solutions of (1.1), denoted by $\tilde{\mathbf{u}} = (\tilde{u}_1, \ldots, \tilde{u}_N)$ and $\hat{\mathbf{u}} \equiv (\hat{u}_1, \ldots, \hat{u}_N)$, are required to satisfy $\tilde{\mathbf{u}} \ge \hat{\mathbf{u}}$ and the relation

$$\tilde{u}_t^{(l)} - L^{(l)}\tilde{u}^{(l)} \ge f^{(l)}\left(\tilde{u}^{(l)}, [\tilde{\mathbf{u}}]_{a_l}, [\hat{\mathbf{u}}]_{b_l}, [\tilde{\mathbf{u}}_\tau]_{c_l}, [\hat{\mathbf{u}}_\tau]_{d_l}\right)$$
$$\hat{u}_t^{(l)} - L^{(l)}\hat{u}^{(l)} \le f^{(l)}\left(\hat{u}^{(l)}, [\hat{\mathbf{u}}]_{a_l}, [\tilde{\mathbf{u}}]_{b_l}, [\hat{\mathbf{u}}_\tau]_{c_l}, [\tilde{\mathbf{u}}_\tau]_{d_l}\right) \qquad \text{in } D_T$$
$$B^{(l)}[\tilde{u}^{(l)}] \ge g^{(l)}(x,t) \ge B^{(l)}[\hat{u}^{(l)}] \quad \text{on } S_T$$
$$\tilde{u}^{(l)}(x,t) \ge \psi^{(l)}(x,t) \ge \hat{u}^{(l)}(x,t) \quad \text{in } [-\tau_l, 0] \times \Omega$$
$$(l = 1, \ldots, N), \quad (6.1)$$

where $D_T = \Omega \times (0, T]$, $S_T = \partial\Omega \times (0, T]$. Define

$$\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \equiv \left\{\mathbf{u} \in \mathscr{C}\left(\overline{D}_T\right);\ \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\right\}$$

and choose any nonnegative function $\gamma^{(l)} \equiv \gamma^{(l)}(x,t)$ such that (3.1) is satisfied with respect to $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ in $\overline{D}_T \equiv \overline{\Omega} \times [0, T]$. It has been shown in

[8] that if $\overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ and $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ are used as a pair of coupled initial iterations in the iteration process

$$\begin{aligned}
\mathscr{L}_\gamma^{(l)}\big(\overline{u}^{(l)}\big)^{(m)} &= \gamma^{(l)}\big(\overline{u}^{(l)}\big)^{(m-1)} \\
&\quad + f^{(l)}\Big(\big(\overline{u}^{(l)}\big)^{(m-1)}, [\overline{\mathbf{u}}^{(m-1)}]_{a_l}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{b_l}, \\
&\qquad \big[\overline{\mathbf{u}}_\tau^{(m-1)}\big]_{c_l}, \big[\underline{\mathbf{u}}_\tau^{(m-1)}\big]_{d_l}\Big), \\
\mathscr{L}_\gamma^{(l)}\big(\underline{u}^{(l)}\big)^{(m)} &= \gamma^{(l)}\big(\underline{u}^{(l)}\big)^{(m-1)} \\
&\quad + f^{(l)}\Big(\big(\underline{u}^{(l)}\big)^{(m-1)}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{a_l}, [\overline{\mathbf{u}}^{(m-1)}]_{b_l}, \\
&\qquad \big[\underline{\mathbf{u}}_\tau^{(m-1)}\big]_{c_l}, \big[\overline{\mathbf{u}}_\tau^{(m-1)}\big]_{d_l}\Big), \\
B^{(l)}\big(\overline{u}^{(l)}\big)^{(m)} &= B^{(l)}\big(\underline{u}^{(l)}\big)^{(m)} = g^{(l)}(x,t), \\
\big(\overline{u}^{(l)}\big)^{(m)}(x,t) &= \big(\underline{u}^{(l)}\big)^{(m)}(x,t) = \psi^{(l)}(x,t) \qquad (l = 1,\dots,N), \quad (6.2)
\end{aligned}$$

where $m = 1, 2, \dots,$ then the corresponding sequences

$$\{\overline{\mathbf{u}}^{(m)}\} \equiv \Big\{\big(\overline{u}^{(1)}\big)^{(m)}, \dots, \big(\overline{u}^{(N)}\big)^{(m)}\Big\}, \{\underline{\mathbf{u}}^{(m)}\} \equiv \Big\{\big(\underline{u}^{(1)}\big)^{(m)}, \dots, \big(\underline{u}^{(N)}\big)^{(m)}\Big\}$$

exist and converge monotonically and uniformly to a unique solution $\mathbf{u} \equiv (u^{(1)}, \dots, u^{(N)})$ of (1.1) and satisfy the relation

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}^{(m)} \le \underline{\mathbf{u}}^{(m+1)} \le \mathbf{u} \le \overline{\mathbf{u}}^{(m+1)} \le \overline{\mathbf{u}}^{(m)} \le \tilde{\mathbf{u}} \qquad \text{on } \overline{D}_T. \qquad (6.3)$$

To prove the convergence of the solution $\mathbf{u}_{i,n} \equiv (u_{i,n}^{(1)}, \dots, u_{i,n}^{(n)})$ of (2.1) to the solution $\mathbf{u}(x_i, t_n)$ of (1.1) we consider a fixed discretized domain Λ_p^* and show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for each $l = 1, \dots, N$,

$$\big|u^{(l)}(x_i, t_n) - u_{i,n}^{(l)}\big| < \epsilon \quad \text{for all } (x_i, t_n) \in \Lambda_p^* \qquad \text{when } k_n + |h| < \delta, \qquad (6.4)$$

where $|h| = h_1 + \cdots + h_p$. Since for the convergence problem, k_n and h_ν can be made arbitrarily small, we may assume that $k_n = k$ and $h_\nu = h$ for all n and ν. Similarly the function $\gamma_{i,n}^{(l)}$ in (3.1) may be chosen independent of i so that $\Gamma_n^{(l)} = \gamma_n^{(l)} I$. In our convergence theorem to be given it is understood that $k/|h|^2$ remains fixed as $k \to 0$, $|h| \to 0$, and the set of mesh points in Λ_p^* is always contained in every refinement of Λ_p^*. Before proving the relation (6.4) we prepare the following lemma.

LEMMA 6.1. *Let* $\{r_n^{(m)}\}$ *be a sequence of positive numbers such that* $r_0^{(m)} \le \epsilon_0$ *for all m and*

$$r_n^{(m)} \le a r_{n-1}^{(m)} + b r_n^{(m-1)} + c \qquad \text{for } m = 0, 1, 2, \ldots, \quad n = 1, \ldots, N_T, \quad (6.5)$$

and let $\theta \equiv b(1-a)^{-1}$, $b^{(m)} \equiv 1 + \theta + \theta^2 + \cdots + \theta^{m-1}$, *and* $\bar{r}^{(m)} = \max_n(r_n^{(m)})$, *where a, b, and c are positive constants with* $a < 1$ *and* N_T *is any positive integer. Then*

$$\bar{r}^{(m)} \le \theta^m \bar{r}^{(0)} + b^{(m)}\left(\epsilon_0 + c(1-a)^{-1}\right) \qquad \text{for } m = 1, 2 \ldots. \quad (6.6)$$

Proof. For any fixed m in (6.5), an induction argument in n gives

$$\begin{aligned} r_1^{(m)} &\le a r_0^{(m)} + b r_1^{(m-1)} + c \le a\epsilon_0 + b\bar{r}^{(m-1)} + c \\ r_2^{(m)} &\le a\left(a\epsilon_0 + b\bar{r}^{(m-1)} + c\right) + b r_2^{(m-1)} + c \\ &\le a^2\epsilon_0 + (a+1)(b\bar{r}^{(m-1)} + c) \\ &\vdots \\ r_n^{(m)} &\le a^n\epsilon_0 + (a^{n-1} + a^{n-2} + \cdots + 1)(b\bar{r}^{(m-1)} + c). \end{aligned}$$

In view of $a < 1$ the above relation yields

$$\bar{r}^m \le \epsilon_0 + (1-a)^{-1}(b\bar{r}^{(m-1)} + c) = \theta\bar{r}^{(m-1)} + \left(\epsilon_0 + c(1-a)^{-1}\right).$$

The relation (6.6) follows by an induction argument in m. ∎

THEOREM 6.1. *Let* $(\tilde{\mathbf{u}}(x,t), \hat{\mathbf{u}}(x,t))$ *and* $(\tilde{\mathbf{u}}_{i,n}, \hat{\mathbf{u}}_{i,n})$ *be coupled upper and lower solutions of* (1.1) *and* (2.1), *respectively, and let Hypotheses* (H_1) *and* (H_2) *hold. Assume that* $\beta^{(l)}(x,t) \ne 0$ *and for any* $\epsilon^* > 0$ *there exists* $\delta > 0$ *such that*

$$\left|\tilde{\mathbf{u}}(x_i, t_n) - \tilde{\mathbf{u}}_{i,n}\right| \le \epsilon^*, \left|\hat{\mathbf{u}}(x_i, t_n) - \hat{\mathbf{u}}_{i,n}\right| \le \epsilon^* \qquad \text{whenever } k + |h| < \delta. \quad (6.7)$$

Then the solution $\mathbf{u}_{i,n}$ *of* (2.1) *converges to the solution* $\mathbf{u}(x_i, t_n)$ *of* (1.1) *on* Λ_p^* *as* $k + |h| \to 0$.

Proof. Let $((u^{(l)}(x,t))^{(m)}, (u_{i,n}^{(l)})^{(m)})$ represent either the pair $((\bar{u}^{(l)}(x,t))^{(m)}, (\bar{u}_{i,n}^{(l)})^{(m)})$ or the pair $((\underline{u}^{(l)}(x,t))^{(m)}, (\underline{u}_{i,n}^{(l)})^{(m)})$. Since

$$\left|u^{(l)}(x_i,t_n) - u_{i,n}^{(l)}\right| \le \left|u^{(l)}(x_i,t_n) - \left(u^{(l)}(x_i,t_n)\right)^{(m)}\right| + \left|\left(u^{(l)}(x_i,t_n)\right)^{(m)} - \left(u_{i,n}^{(l)}\right)^{(m)}\right| + \left|\left(u_{i,n}^{(l)}\right)^{(m)} - u_{i,n}^{(l)}\right|$$

for every m and $\{(u^{(l)}(x_i,t_n))^{(m)}\}$ and $\{(u_{i,n}^{(l)})^{(m)}\}$ converge uniformly to $u^{(l)}(x_i,t_n)$ and $u_{i,n}^{(l)}$, respectively, as $m \to \infty$, it suffices to show that given any $\epsilon > 0$ there exist $\delta > 0$ and an integer m^* such that for some $m \ge m^*$ and every $l = 1,\dots,N$,

$$\left|\left(u^{(l)}(x_i,t_n)\right)^{(m)} - \left(u_{i,n}^{(l)}\right)^{(m)}\right| < \epsilon \qquad \text{in } \Lambda_p^* \qquad \text{when } k + |h| < \delta. \quad (6.8)$$

Let $((\bar{u}^{(l)})^{(m)}, (\underline{u}^{(l)})^{(m)})$ be the solution of (6.2) and let

$$\left(\bar{V}_n^{(l)}\right)^{(m)} \equiv \left(\left(\bar{u}^{(l)}(x_1,t_n)\right)^{(m)}, \dots, \left(\bar{u}^{(l)}(x_M,t_n)\right)^{(m)}\right)'$$

$$\left(\underline{V}_n^{(l)}\right)^{(m)} \equiv \left(\left(\underline{u}^{(l)}(x_1,t_n)\right)^{(m)}, \dots, \left(\underline{u}^{(l)}(x_M,t_n)\right)^{(m)}\right)'$$

$$\bar{\mathbf{V}}_n^{(m)} \equiv \left(\left(\bar{V}_n^{(1)}\right)^{(m)}, \dots, \left(\bar{V}_n^{(N)}\right)^{(m)}\right)', \underline{\mathbf{V}}_n^{(m)} \equiv \left(\left(\underline{V}_n^{(1)}\right)^{(m)}, \dots, \left(\underline{V}_n^{(N)}\right)^{(m)}\right)'.$$

By the finite difference approximation of $L^{(l)}u^{(l)}$, $(\bar{u}^{(l)})^{(m)}$ and $(\underline{u}^{(l)})^{(m)}$ satisfy the relation

$$\begin{aligned}
\mathscr{L}_\gamma^{(l)}\left[(\bar{u}^{(l)})^{(m)}\right] &= \gamma^{(l)}(\bar{u}^{(l)})^{(m-1)} \\
&\quad + f^{(l)}\Big((\bar{u}^{(l)})^{(m-1)}, [\bar{\mathbf{u}}^{(m-1)}]_{a_l}, [\underline{\mathbf{u}}^{(m-1)}]_{b_l}, \\
&\qquad [\bar{\mathbf{u}}_\tau^{(m-1)}]_{c_l}, [\underline{\mathbf{u}}_\tau^{(m-1)}]_{d_l}\Big) + 0^{(m)}(k,|h|), \\
\mathscr{L}_\gamma^{(l)}\left[(\underline{u}^{(l)})^{(m)}\right] &= \gamma^{(l)}(\underline{u}^{(l)})^{(m-1)} \\
&\quad + f^{(l)}\Big((\underline{u}^{(l)})^{(m-1)}, [\underline{\mathbf{u}}^{(m-1)}]_{a_l}, [\bar{\mathbf{u}}^{(m-1)}]_{b_l}, \\
&\qquad [\underline{\mathbf{u}}_\tau^{(m-1)}]_{c_l}, [\bar{\mathbf{u}}_\tau^{(m-1)}]_{d_l}\Big) + 0^{(m)}(k,|h|) \quad \text{in } \Lambda_p^*, \\
B^{(l)}\left[(\bar{u}^{(l)})^{(m)}\right] &= B^{(l)}\left[(\underline{u}^{(l)})^{(m)}\right] = g^{(l)}(x_i,t_n) + 0^{(m)}(|h|) \qquad \text{on } S_p^*, \\
(\bar{u}^{(l)})^{(m)}(x_i,t_n) &= (\underline{u}^{(l)})^{(m)}(x_i,t_n) = \psi^{(l)}(x_i,t_n) \quad \text{in } Q_p \ (l = 1,\dots,N),
\end{aligned} \tag{6.9}$$

where $u^{(l)}$, $\mathbf{u}$, and $\mathbf{u}_\tau$ are evaluated at the mesh point (x_i, t_n) and $0^{(m)}(k, |h|)$ and $0^{(m)}(|h|)$ converge to 0 as $k_n + |h| \to 0$. To express (6.9) in vector form we write, for notational convenience,

$$F^{(l)}\big(\overline{\mathbf{V}}_n^{(m)}; \underline{\mathbf{V}}_n^{(m)}\big) \equiv F^{(l)}\Big(\big(\overline{V}_n^{(l)}\big)^{(m)}, \big[\overline{\mathbf{V}}_n^{(m)}\big]_{a_l}, \big[\underline{\mathbf{V}}_n^{(m)}\big]_{b_l}, \big[\overline{\mathbf{V}}_{n-s}^{(m)}\big]_{c_l}, \big[\underline{\mathbf{V}}_{n-s}^{(m)}\big]_{d_l}\Big),$$

$$F^{(l)}\big(\underline{\mathbf{V}}_n^{(m)}; \overline{\mathbf{V}}_n^{(m)}\big) \equiv F^{(l)}\Big(\big(\underline{V}_n^{(l)}\big)^{(m)}, \big[\underline{\mathbf{V}}_n^{(m)}\big]_{a_l}, \big[\overline{\mathbf{V}}_n^{(m)}\big]_{b_l}, \big[\underline{\mathbf{V}}_{n-s}^{(m)}\big]_{c_l}, \big[\overline{\mathbf{V}}_{n-s}^{(m)}\big]_{d_l}\Big)$$

$$(l = 1, \dots, N). \quad (6.10)$$

Then (6.9) may be written as

$$\mathscr{A}_n^{(l)}\big(\overline{V}_n^{(l)}\big)^{(m)} = \big(\overline{V}_{n-1}^{(l)}\big)^{(m)} + k\Big[\Gamma_n^{(l)}\big(\overline{V}_n^{(l)}\big)^{(m-1)} + F^{(l)}\big(\overline{\mathbf{V}}_n^{(m-1)}; \underline{\mathbf{V}}_n^{(m-1)}\big)\Big] + \mathbf{0}^{(m)}(k, |h|),$$

$$\mathscr{A}_n^{(l)}\big(\underline{V}_n^{(l)}\big)^{(m)} = \big(\underline{V}_{n-1}^{(l)}\big)^{(m)} + k\Big[\Gamma_n^{(l)}\big(\underline{V}_n^{(l)}\big)^{(m-1)} + F^{(l)}\big(\underline{\mathbf{V}}_n^{(m-1)}; \overline{\mathbf{V}}_n^{(m-1)}\big)\Big] + \mathbf{0}^{(m)}(k, |h|)$$

$$\text{for } n = 1, 2, \dots,$$

$$\big(\overline{V}_n^{(l)}\big)^{(m)} = \big(\underline{V}_n^{(l)}\big)^{(m)} = \Psi_n^{(l)} \qquad \text{for } n = 0, -1, \dots -s_l \ (l = 1, \dots, N), \quad (6.11)$$

where $\mathscr{A}_n^{(l)}$ is given by (3.2) with $\Gamma_n^{(l)} = \gamma_n^{(l)} I$.

Let $((\overline{U}_n^{(l)})^{(m)}, (\underline{U}_n^{(l)})^{(m)})$ be the solution of (3.5), and let

$$\big(\bar{z}_n^{(l)}\big)^{(m)} = \big(\overline{V}_n^{(l)}\big)^{(m)} - \big(\overline{U}_n^{(l)}\big)^{(m)}, \qquad \big(\underline{z}_n^{(l)}\big)^{(m)} = \big(\underline{V}_n^{(l)}\big)^{(m)} - \big(\underline{U}_n^{(l)}\big)^{(m)}.$$

Then (6.8) is proven if

$$\|\big(\bar{z}_n^{(l)}\big)^{(m)}\| + \|\big(\underline{z}_n^{(l)}\big)^{(m)}\| < \epsilon \quad \text{for } n = 1, 2, \dots, \text{ when } k + |h| < \delta. \quad (6.12)$$

By a subtraction of (3.5) from (6.11) we have

$$\mathscr{A}_n^{(l)}\big(\bar{z}_n^{(l)}\big)^{(m)} = \big(\bar{z}_{n-1}^{(l)}\big)^{(m)} + k\Big[\Gamma_n^{(l)}\big(\bar{z}_n^{(l)}\big)^{(m-1)} + F^{(l)}\big(\overline{\mathbf{V}}_n^{(m-1)}; \underline{\mathbf{V}}_n^{(m-1)}\big) - F^{(l)}\big(\overline{\mathbf{U}}_n^{(m-1)}; \underline{\mathbf{U}}_n^{(m-1)}\big)\Big] + \mathbf{0}^{(m)}(k, |h|)$$

$$\mathscr{A}_n^{(l)}\big(\underline{z}_n^{(l)}\big)^{(m)} = \big(\underline{z}_{n-1}^{(l)}\big)^{(m)} + k\Big[\Gamma_n^{(l)}\big(\underline{z}_n^{(l)}\big)^{(m-1)} + F^{(l)}\big(\underline{\mathbf{V}}_n^{(m-1)}; \overline{\mathbf{V}}_n^{(m-1)}\big) - F^{(l)}\big(\underline{\mathbf{U}}_n^{(m-1)}; \overline{\mathbf{U}}_n^{(m-1)}\big)\Big] + \mathbf{0}^{(m)}(k, |h|) \qquad \text{for } n = 1, 2, \dots,$$

$$\big(\bar{z}_n^{(l)}\big)^{(m)} = \big(\underline{z}_n^{(l)}\big)^{(m)} = \mathbf{0} \quad \text{for } n = 0, -1, \dots, -s \qquad (l = 1, \dots, N). \quad (6.13)$$

Since $k_n = k$ and $h_\nu = h$ for all n and ν we may write

$$k_n A_n^{(l)} = \left(k/h^2\right)\hat{A}_n^{(l)} \equiv r\hat{A}_n^{(l)}.$$

It is clear from the formulation of $A_n^{(l)}$ that the elements of $\hat{A}_n^{(l)}$ satisfy the properties in (H_1) and are bounded as $k + |h| \to 0$. Define $\mathscr{B}_n^{(l)} = (\mathscr{A}_n^{(l)})^{-1}$, where $\mathscr{A}_n^{(l)} = I + r\hat{A}_n^{(l)} + k\Gamma_n^{(l)}$. Then for any $\epsilon' > 0$ there exist a matrix norm and a vector norm in $\mathbb{R}^M$ such that

$$\begin{aligned} \|\mathscr{B}_n^{(l)}\| &\le \left(1 + r\hat{\mu}_n^{(l)} + k\gamma_n^{(l)} - \epsilon'\right)^{-1} \equiv \hat{\rho}_n^{(l)}, \\ \|\mathscr{B}_n^{(l)}\mathbf{V}\| &\le \|\mathscr{B}_n^{(l)}\|\,\|\mathbf{V}\| \quad \text{for } \mathbf{V} \in \mathbb{R}^M, \end{aligned} \tag{6.14}$$

where $\hat{\mu}_n^{(l)}$ is the smallest eigenvalue of $\hat{A}_n^{(l)}$ (cf. [4]). By the hypothesis $\beta_n^{(l)} \ne 0$, $\hat{\mu}_n^{(l)}$ is strictly positive for all k and h. Define

$$\kappa_n^{(l)} \equiv \max\left\{\left|\frac{\partial f^{(l)}}{\partial w^{(j)}}(\mathbf{u},\mathbf{w})\right|;\ \mathbf{u} \in \langle \hat{\mathbf{U}}_n; \tilde{\mathbf{U}}_n\rangle,\ \mathbf{w} \in \langle \hat{\mathbf{U}}_{n-s}, \tilde{\mathbf{U}}_{n-s}\rangle,\ j = 1,\dots,N\right\}, \tag{6.15}$$

where $w^{(j)}$, $j = 1,\dots,N$, are the components of $\mathbf{w}$. Then by (6.13), (6.14), and (5.3) we have

$$\begin{aligned} \left\|\left(\bar{z}_n^{(l)}\right)^{(m)}\right\| \le \hat{\rho}_n^{(l)}\Bigg\{&\left\|\left(\bar{z}_{n-1}^{(l)}\right)^{(m)}\right\| + k\Bigg[\bar{\sigma}_n^{(l)}\Bigg(\left\|\left(\bar{z}_n^{(l)}\right)^{(m-1)}\right\| + \sum_{j=1}^{a_l}\left\|\left(\bar{z}_n^{(j)}\right)^{(m-1)}\right\| \\ &+ \sum_{j=1}^{b_l}\left\|\left(\underline{z}_n^{(j)}\right)^{(m-1)}\right\|\Bigg) \\ &+ \kappa_n^{(l)}\Bigg(\sum_{j=1}^{c_l}\left\|\left(\bar{z}_{n-s}^{(j)}\right)^{(m-1)}\right\| + \sum_{j=1}^{d_l}\left\|\left(\underline{z}_{n-s}^{(j)}\right)^{(m-1)}\right\|\Bigg)\Bigg] + \|\mathbf{0}^{(m)}\|\Bigg\}, \\ \left\|\left(\underline{z}_n^{(l)}\right)^{(m)}\right\| \le \hat{\rho}_n^{(l)}\Bigg\{&\left\|\left(\underline{z}_{n-1}^{(l)}\right)^{(m)}\right\| + k\Bigg[\bar{\sigma}_n^{(l)}\Bigg(\left\|\left(\underline{z}_n^{(l)}\right)^{(m-1)}\right\| + \sum_{j=1}^{a_l}\left\|\left(\underline{z}_n^{(j)}\right)^{(m-1)}\right\| \\ &+ \sum_{j=1}^{b_l}\left\|\left(\bar{z}_n^{(j)}\right)^{(m-1)}\right\|\Bigg) \\ &+ \kappa_n^{(l)}\Bigg(\sum_{j=1}^{c_l}\left\|\left(\underline{z}_{n-s}^{(j)}\right)^{(m-1)}\right\| + \sum_{j=1}^{d_l}\left\|\left(\bar{z}_{n-s}^{(j)}\right)^{(m-1)}\right\|\Bigg)\Bigg] + \|\mathbf{0}^{(m)}\|\Bigg\}, \end{aligned} \tag{6.16}$$

where $\bar{\sigma}_n^{(l)}$ is given by (5.3).

Let $m \geq m^*$ be fixed and choose $\epsilon' < (r\hat{\mu}_n^{(l)} + k\gamma_n^{(l)})/2$ which is possible since $\hat{\mu}_n^{(l)} > 0$. Then by (6.14), $\hat{\rho}_n^{(l)} < 1$ for every l and n. Moreover, given any $\epsilon_1 > 0$ there exists $\delta > 0$ such that $\|\mathbf{0}^{(m)}\| < \epsilon_1$ when $k + |h| < \delta$. Define

$$\left\|\bar{\mathbf{z}}_n^{(m)}\right\| = \sum_{l=1}^{N} \left\|\left(\bar{z}_n^{(l)}\right)^{(m)}\right\|, \qquad \left\|\underline{\mathbf{z}}_n^{(m)}\right\| = \sum_{l=1}^{N} \left\|\left(\underline{z}_n^{(l)}\right)^{(m)}\right\|$$

$$\left\|\mathbf{z}_n^{(m)}\right\| = \left\|\bar{\mathbf{z}}_n^{(m)}\right\| + \left\|\underline{\mathbf{z}}_n^{(m)}\right\|$$

$$\hat{\sigma}_n = \sum_{l=1}^{N} \left(\sigma_n^{(l)}\right), \qquad \hat{\kappa}_n = \sum_{l=1}^{N} \left(\kappa_n^{(l)}\right), \tag{6.17}$$

and let $\hat{\rho}$, $\hat{\sigma}$, and $\hat{\kappa}$ be the respective maximum of $\hat{\rho}_n^{(l)}$, $\hat{\sigma}_n$, and κ_n over $n = 1, \ldots, N_T$ and $l = 1, \ldots, N$. Then addition of the two inequalities in (6.16) followed by summing up over $l = 1, \ldots, N$ yield the relation

$$\left\|\mathbf{z}_n^{(m)}\right\| \leq \hat{\rho}\left\{\left\|\mathbf{z}_{n-1}^{(m)}\right\| + k\left[\hat{\sigma}\left\|\mathbf{z}_n^{(m-1)}\right\| + \hat{\kappa}\left\|\mathbf{z}_{n-s}^{(m-1)}\right\|\right] + 2N\epsilon_1\right\}. \tag{6.18}$$

It is clear that $\hat{\rho} < 1$, and by (6.7) $\|\mathbf{z}_n^{(0)}\| < 2\epsilon^* \equiv \epsilon^{(0)}$ for all $n = 1, \ldots, N_T$.

Consider the case $n = 1, \ldots, s^*$. Since by (3.8) and (6.11) $\|\mathbf{z}_{n-s}^{(m)}\| = 0$, the relation (6.18) is reduced to

$$\left\|\mathbf{z}_n^{(m)}\right\| \leq \hat{\rho}\left[\left\|\mathbf{z}_{n-1}^{(m)}\right\| + k\hat{\sigma}\left\|\mathbf{z}_n^{(m-1)}\right\| + 2N\epsilon_1\right]. \tag{6.19}$$

In view of $\hat{\rho} < 1$ and $\|\mathbf{z}_0^{(m)}\| = 0$, an application of Lemma 6.1 with $r_n^{(m)} = \|\mathbf{z}_n^{(m)}\|$ gives

$$\left\|\mathbf{z}_n^{(m)}\right\| \leq \theta^m \left\|\mathbf{z}_n^{(0)}\right\| + b^{(m)}\left(2\hat{\rho}N\epsilon_1\right) < \theta^m \epsilon^{(0)} + c^{(m)}\epsilon_1,$$

where $\theta = k\hat{\rho}\hat{\sigma}(1 - \hat{\rho})^{-1}$, $b^{(m)} = 1 + \theta + \cdots + \theta^{m-1}$, and $c^{(m)} = 2\hat{\rho}Nb^{(m)}$. The arbitrariness of $\epsilon^{(0)}$ and ϵ_1 implies that for any $\epsilon_1' \leq \epsilon$ there exists $\delta > 0$ such that $\|\mathbf{z}_n^{(m)}\| < \epsilon_1'$ when $h + |h| < \delta$. This proves the relation (6.12) for $n = 1, \ldots, s^*$.

We next consider $n = s^* + 1, \ldots, 2s^*$. It is obvious from the above result for $n = 1, \ldots, s^*$ that $\|\mathbf{z}_{n-s}^{(m)}\| < \epsilon_1'$ for $n = s^* + 1, \ldots, 2s^*$. Using this relation in (6.18) leads to

$$\left\|\mathbf{z}_n^{(m)}\right\| \leq \hat{\rho}\left(\left\|\mathbf{z}_{n-1}^{(m)}\right\| + k\hat{\sigma}\left\|\mathbf{z}_n^{(m-1)}\right\| + \epsilon_2\right),$$

where $\epsilon_2 = k\hat{\kappa}\epsilon_1' + 2N\epsilon_1$. Since $\|\mathbf{z}_{s^*}^{(m)}\| < \epsilon_1'$, another application of Lemma 6.1 gives

$$\left\|\mathbf{z}_n^{(m)}\right\| \leq \theta^m \epsilon_1' + b^{(m)}\left(\epsilon_1' + (1 - \hat{\rho})^{-1}\epsilon_2\right).$$

Since ϵ_1' and ϵ_2 can be made arbitrarily small by taking $k + |h|$ small we conclude from the above inequality that for any $\epsilon_2' \leq \epsilon$ exists $\delta > 0$ such that $\|\mathbf{z}_n^{(m)}\| < \epsilon_2'$ when $k + |h| < \delta$. This proves the relation (6.12) for $n = s^* + 1, \ldots, 2s^*$. A continuation of the same process shows that (6.12) holds for $n = 1, \ldots, N_T$. This proves the theorem. ∎

REFERENCES

1. W. F. Ames, Numerical Methods for Partial Differential Equations (3rd edition), Academic Press, San Diego, 1992.
2. Y. Deng, G. Chen, W. Ni, and J. Zhou, ''Boundary element monotone iteration scheme for semilinear elliptic partial differential equations,'' *Math. Comp.* **65** (1996), 943–982.
3. C. A. Hall and T. A. Porsching, Numerical Analysis of Partial Differential Equations, Prentice Hall, Englewood Cliffs, NJ, 1990.
4. A. S. Householder, The Theory of Matrices in Numerical Analysis. Ginn (Blaisdell), Boston, 1964.
5. K. Ishihara, ''Monotone explicit iterations of the finite element approximation for the nonlinear boundary value problem,'' *Numer. Math.* **43** (1984), 419–437.
6. A. Leung, ''Monotone schemes for semilinear elliptic systems related to ecology,'' *Math. Methods Appl. Sci.* **4** (1982), 272–285.
7. C. V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.
8. C. V. Pao, ''Dynamics of nonlinear parabolic systems with time delays,'' *J. Math. Anal. Appl.*, **198** (1996), 751–779.
9. C. V. Pao, ''Accelerated monotone iterative methods for finite difference equations of reaction-diffusion,'' *Numer. Math.*, **79** (1998), 261–281.
10. C. V. Pao, ''Numerical methods for semilinear parabolic equations,'' *SIAM J. Numer. Anal.* **24** (1987), 24–35.
11. C. V. Pao, ''Positive solutions and dynamics of a finite difference reaction-diffusion systems,'' *Numer. Methods Partial Diff. Eqs.*, **9** (1993), 285–311.
12. C. V. Pao, ''Numerical analysis of coupled systems of nonlinear parabolic equations,'' *SIAM J. Numer. Anal.* **36** (1999), 394–416.
13. S. V. Parter, ''Mildly nonlinear elliptic partial differential equations and their numerical solution, I,'' *Numer. Math.*, **7** (1975), 113–128.
14. J. Schroder, Operator Inequalities, Academic Press, New York, 1980.
15. R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
16. J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.