Initial Boundary Value Problem for a Singularly Perturbed Parabolic Equation in Case of Exchange of Stability

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The theorem on passage to the limit is proved for a singularly perturbed parabolic problem in case when the degenerate equation has two intersecting roots. © 1999 Academic Press

1. INTRODUCTION: STATEMENT OF THE PROBLEM

Singularly perturbed problems for ordinary differential equations in the case when the degenerate equation has intersecting roots have been considered in [1–3]. Here we consider the initial boundary value problem

\[ L_\varepsilon u = \varepsilon^2 (u_t - u_{xx}) - f(u, x, t, \varepsilon) = 0, \]
\[(x, t) \in D = \{(0 < x < 1) \times (0 < t \leq T)\}, \]
\[ u(x, 0) = u^0(x), \]
\[ u_x(0, t) = u_x(1, t) = 0, \]

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where $\epsilon$ is a small positive parameter, $u$ is the unknown scalar function, and $f$ and $u_0$ are sufficiently smooth functions.

Suppose that the following assumptions hold.

(A$_1$) The equation $f(u, x, t, 0) = 0$ which is called the degenerate equation has exactly two real smooth roots with respect to $u$ in $\bar{D}$:

$$u = \varphi_1(x, t) \quad \text{and} \quad u = \varphi_2(x, t).$$

There exists a smooth function $t = \psi(x)$, $0 \leq x \leq 1$, such that

$$0 < \psi(x) < T \quad \text{for} \quad 0 \leq x \leq 1,$$

$$\varphi_1(x, t) = \varphi_2(x, t) \quad \text{for} \quad t = \psi(x),$$

$$\varphi_1(x, t) > \varphi_2(x, t) \quad \text{for} \quad 0 \leq t < \psi(x),$$

$$\varphi_1(x, t) < \varphi_2(x, t) \quad \text{for} \quad \psi(x) < t \leq T.$$  \hfill (4)

The condition (4) says that the surfaces $u = \varphi_1(x, t)$ and $u = \varphi_2(x, t)$ intersect in a curve whose projection into the domain $\bar{D}$ is described by $t = \psi(x)$.

$$f_u(\varphi_1(x, t), x, t, 0) \begin{cases} < 0 & \text{for} \ 0 \leq t < \psi(x), \\ > 0 & \text{for} \ \psi(x) < t \leq T, \end{cases}$$

$$f_u(\varphi_2(x, t), x, t, 0) \begin{cases} > 0 & \text{for} \ 0 \leq t < \psi(x), \\ < 0 & \text{for} \ \psi(x) < t \leq T. \end{cases}$$ \hfill (A$_2$)

We note that from (4) it follows that

$$f_u(\varphi_i(x, t), x, t, 0)|_{t=\psi(x)} = 0.$$ \hfill (6)

Under assumption (A$_2$) the rest point $v = \varphi_1(x, t)$ ($v = \varphi_2(x, t)$) of the associated equation

$$\frac{dv}{d\tau} = f(u, v, x, t, 0), \quad \tau \geq 0,$$ \hfill (7)

where $x$ and $t$ are considered as parameters is asymptotically stable (unstable) for $0 \leq t < \psi(x)$ and unstable (asymptotically stable) for $\psi(x) < t \leq T$. Thus, the exchange of stability of the two rest points takes place on the curve $t = \psi(x)$.

A simple example of a function $f(u, x, t, 0)$ satisfying the assumptions (A$_1$) and (A$_2$) is given by the quadratic function with respect to $u$

$$f(u, x, t, 0) = -(u - \varphi_1(x, t))(u - \varphi_2(x, t)).$$ \hfill (8)

if $\varphi_1$ and $\varphi_2$ satisfy the conditions (4) and (5).
(A_3) The initial function \( u^0(x) \) belongs to the basin of attraction of the rest point \( v = \varphi_1(x, 0) \) of the associated equation (7) for \( t = 0 \).

Assumption (A_3) means that the solution \( \Pi(x, \tau) \) of the initial problem \((s \) is considered as parameter) 

\[
\frac{d\Pi}{d\tau} = f(\varphi_1(x, 0) + \Pi, x, 0, 0), \quad \tau \geq 0; \quad \Pi(x, 0) = u^0 - \varphi_1(x, 0)
\]

exists for \( \tau \geq 0 \) and \( \Pi(x, \tau) \to 0 \) as \( \tau \to \infty \).

By assumption (A_3), for small \( \varepsilon \) the solution \( u(x, t, \varepsilon) \) of the problem (1),(2) has an exponentially fast change from the initial value \( u^0(x) \) to values close to \( \varphi_1(x, t) \) within a small time interval. After that the solution \( u(x, t, \varepsilon) \) will be close to \( \varphi_1(x, t) \) as long as the root \( \varphi_1(x, t) \) will be stable. But for \( t = \psi(x) \) the exchange of stability of the roots \( \varphi_1 \) and \( \varphi_2 \) takes place. The question arises about behavior of the solution \( u(x, t, \varepsilon) \) near the curve \( t = \psi(x) \) and for \( \psi(x) < t \leq T \).

Form the composed stable solution of the degenerate equation

\[
\hat{u}(x, t) = \begin{cases} 
\varphi_1(x, t), & 0 \leq t \leq \psi(x), \\
\varphi_2(x, t), & \psi(x) \leq t \leq T,
\end{cases} \quad 0 \leq x \leq 1.
\]

We note that \( \hat{u}(x, t) \) is a continuous function in \( \overline{D} \), but not smooth on the curve \( t = \psi(x) \).

Our aim is to prove that under some assumptions including (A_1)–(A_3) the limiting equality

\[
\lim_{\varepsilon \to 0} u(x, t, \varepsilon) = \hat{u}(x, t), \quad 0 \leq x \leq 1, \quad 0 < t \leq T
\]

holds.

2. THEOREM ON PASSAGE TO THE LIMIT

Introduce the notation

\[
\hat{f}_{uu}(x, t) = f_{uu}(\hat{u}(x, t), x, t, 0), \quad \hat{f}_e(x, t) = f_e(\hat{u}(x, t), x, t, 0)
\]

and assume

(A_4) \( \hat{f}_{uu}(x, \psi(x)) < 0 \) for \( 0 \leq x \leq 1 \).
Note that for the quadratic function (8) \( f_{uu} = -2 \), i.e., Assumption (A) holds. Assume further that

\[ (A_3) \quad f(x, \psi(x)) > 0. \]

In Section 3 we give an example which shows that if, instead of (A), the contrary inequality is fulfilled, then the limiting passage (10) may not be valid.

**THEOREM.** Under assumptions (A1)–(A5) and for sufficiently small \( \varepsilon \), the problem (1)–(3) has a solution \( u(x, t, \varepsilon) \) satisfying (10). Moreover, the representation holds

\[ u(x, t, \varepsilon) = \hat{u}(x, t) + \Pi(x, t/\varepsilon^2) + w(x, t, \varepsilon) \quad \text{in } \overline{D}, \quad (11) \]

where \( \Pi(x, \tau) \) is defined by (9), \( w(x, t, \varepsilon) = O(\varepsilon^{1/2}) \) in some small (but fixed as \( \varepsilon \to 0 \)) \( \delta \)-vicinity of the curve \( t = \psi(x) \) and \( w(x, t, \varepsilon) = O(\varepsilon) \) in the rest of \( \overline{D} \).

**Proof.** Let \( t_0 \) be any positive number independent of \( \varepsilon \) and such that

\[ t_0 < \min_{0 \leq x \leq 1} \psi(x). \]

It is well known ([4, Theorem 3.3]) that by virtue of assumption (A) for sufficiently small \( \varepsilon \) the solution \( u(x, t, \varepsilon) \) exists for \( 0 \leq t \leq t_0 \) and has the representation

\[ u(x, t, \varepsilon) = \varphi_1(x, t) + \Pi(x, t/\varepsilon^2) + O(\varepsilon), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq t_0. \quad (12) \]

Thus, (11) is fulfilled for \( 0 \leq t \leq t_0 \).

The function \( \Pi(x, \tau) \) has the estimate [4]

\[ |\Pi(x, \tau)| \leq C \exp(-\kappa \tau), \quad 0 \leq x \leq 1, \quad \tau \geq 0, \quad (13) \]

where \( C \) and \( \kappa \) are some positive constants. Hence, for \( t \geq t_0 \), we have \[ |\Pi(x, t/\varepsilon^2)| \leq C \exp(-\kappa t_0/\varepsilon^2) = o(\varepsilon^N), \]

where \( N \) is any positive number. Therefore, the representation (11) for \( t \geq t_0 \) may be written as

\[ u(x, t, \varepsilon) = \hat{u}(x, t) + \tilde{w}(x, t, \varepsilon), \quad (14) \]

where \( \tilde{w} \) has the same properties as \( w \) in (11).

In order to prove (14) for \( t_0 \leq t \leq T \) we will consider Eq. (1) for \( (x, t) \in D_0 = (0 < x < 1) \times (t_0 < t \leq T) \) with boundary conditions (3) and initial condition at \( t = t_0 \) which follows from (12):

\[ u(x, t_0, \varepsilon) = \varphi_1(x, t_0) + O(\varepsilon) = u_0(x, \varepsilon). \quad (15) \]
In the sequel, the concept of lower and upper solutions of the problem (1), (3), (15) plays a central role in our approach.

Following [5] we call the functions $\underline{U}(x, t, \varepsilon)$ and $\overline{U}(x, t, \varepsilon)$ the lower and upper solutions of the problem (1), (3), (15), respectively, provided that they satisfy the inequalities

$$\underline{U} \leq \overline{U} \quad \text{in } D_0, \quad \text{(16)}$$
$$L_\varepsilon U \leq 0, \quad L_\varepsilon \overline{U} \geq 0 \quad \text{in } D_0, \quad \text{(17)}$$
$$U_x \geq 0 \geq \overline{U}_x \quad \text{at } x = 0, \quad U_x \leq 0 \leq \overline{U}_x \quad \text{at } x = 1, \quad \text{(18)}$$
$$U \leq u_0(x, \varepsilon) \leq \overline{U} \quad \text{at } t = t_0, \quad \text{(19)}$$

where $u_0(x, \varepsilon)$ is defined by (15).

If we have constructed a pair of lower and upper solutions, then by a general theorem [5] we can conclude that there exists a solution $u(x, t, \varepsilon)$ of the problem (1), (3), (15) satisfying

$$\underline{U} \leq u \leq \overline{U} \quad \text{in } D_0.$$ 

For the construction of lower and upper solution we use the composed stable solution $\hat{u}(x, t)$. This function is not smooth on the curve $t = \psi(x)$. Therefore, we smooth $\hat{u}(x, t)$ by means of a known procedure (see, for example, [4]).

Using the function

$$\omega(\xi) = \pi^{-1/2} \int_{-\infty}^{\xi} \exp(-s^2) \, ds \quad \text{(20)}$$

where $\xi$ is defined by

$$\xi = (t - \psi(x))/\varepsilon^\alpha \quad \text{(21)}$$

and $\alpha$ is any number of the interval $(1/2, 1)$, we introduce the smooth function

$$\tilde{u}(x, t, \varepsilon) = \varphi_1(x, t) \omega(-\xi) + \varphi_2(x, t) \omega(\xi). \quad \text{(22)}$$

It is easy to show [4] that

$$\tilde{u}(x, t, \varepsilon) = \hat{u}(x, t) + \sigma(x, t, \varepsilon), \quad \text{(23)}$$

where $\sigma(x, t, \varepsilon) \leq 0$, $\sigma = O(\varepsilon^\alpha)$ in any fixed $\delta$-vicinity of the curve $t = \psi(x)$ and $\sigma = o(\varepsilon^N)$ for any $N$ outside of such a $\delta$-vicinity.
Now we construct lower and upper solutions by using the smooth function \( \tilde{u}(x, t, \varepsilon) \) in the following form
\[
\begin{align*}
U &= \tilde{u}(x, t, \varepsilon) - \varepsilon A - \varepsilon z(x, \varepsilon), \\
\tilde{U} &= \tilde{u}(x, t, \varepsilon) + \varepsilon^{1/2}Ah(x, t, \varepsilon) + \varepsilon z(x, \varepsilon),
\end{align*}
\]
where \( h(x, t, \varepsilon) \) is a sufficiently smooth function such that \( h = 1 \) in some small \( \delta/2 \)-vicinity of the curve \( t = \psi(x) \), \( h = \varepsilon^{1/2} \) outside of the \( \delta \)-vicinity of the curve \( t = \psi(x) \) and, finally, \( h(x, t, \varepsilon) \) changes monotonically from 1 to \( \varepsilon^{1/2} \) in domains \( \delta/2 \leq |t - \psi(x)| \leq \delta \), \( \delta \) is such that \( t_0 < \psi(x) - \delta \) for \( 0 \leq x \leq 1 \); moreover \( z(x, \varepsilon) = \exp(-kx/\varepsilon) + \exp(-k(1 - x)/\varepsilon) \) and the positive constants \( A \) and \( k \) will be chosen in an appropriate way later.

It is obvious that (16) is fulfilled for any positive \( A \) and \( k \). Taking into account (15) and the relation (23) we get that (19) is fulfilled for sufficiently large \( A \).

By (20)--(22) we have
\[
\tilde{u}_x = \varphi_1 \cdot \omega(-\xi) + \varphi_2 \cdot \omega(\xi)
+ \varepsilon^{-\alpha} \pi^{-1/2} \left[ \varphi_2(x, t) - \varphi_1(x, t) \right] \exp(-\xi^2).
\]

From (4) it follows that
\[
\varphi_2(x, t) - \varphi_1(x, t) = O(|t - \psi(x)|)
\]
and consequently
\[
\varepsilon^{-\alpha} \left[ \varphi_2(x, t) - \varphi_1(x, t) \right] \exp(-\xi^2) = O(\xi) \exp(-\xi^2) = O(1).
\]
Therefore
\[
\tilde{u}_x = O(1).
\]

Analogously, we have
\[
\tilde{u}_t = O(1), \quad \tilde{u}_{xx} = O(\varepsilon^{-\alpha}).
\]

Differentiating (24) with respect to \( x \) at \( x = 0 \) we get
\[
\tilde{U}_x(0, t, \varepsilon) = \tilde{u}_x(0, t, \varepsilon) + k(1 - \exp(-k/\varepsilon)).
\]

Taking into account (27) we conclude that
\[
\tilde{U}_x(0, t, \varepsilon) > 0
\]
for sufficiently large \( k \) and sufficiently small \( \varepsilon \).

One can check that the other inequalities (18) are also fulfilled for sufficiently large \( k \) and sufficiently small \( \varepsilon \).
Now we check that $U$ and $\overline{U}$ satisfy the inequalities (17). By (24), (26), (27) and using the representation

$$f(U, x, t, \varepsilon) = f(\hat{u}(x, t) + \sigma(x, t, \varepsilon) - \varepsilon A - \varepsilon z(x, \varepsilon), x, t, \varepsilon)$$

$$= f(\hat{u}(x, t), x, t, 0) + \hat{f}_u(x, t)(\sigma - \varepsilon A - \varepsilon z) + \hat{f}_e(x, t)\varepsilon + O((\sigma + \varepsilon)^2)$$

$$= \hat{f}_u(x, t)(\sigma - \varepsilon A - \varepsilon z) + \hat{f}_e(x, t)\varepsilon + o(\varepsilon),$$

we get

$$L_\varepsilon U = \varepsilon^2(U_t - aU_{xx}) - f(U, x, t, \varepsilon)$$

$$= O(\varepsilon^{2-\alpha}) - \hat{f}_u(x, t)(\sigma - \varepsilon A - \varepsilon z) - \hat{f}_e(x, t)\varepsilon + o(\varepsilon). \quad (28)$$

Note that by assumption (A₂)

$$\hat{f}_u(x, t) \leq 0 \quad \text{in } D,$$

$$\hat{f}_u(x, t) = 0 \text{ on the curve } t = \psi(x) \text{ (see (6))},$$

$$\hat{f}_u(x, t) \leq -C_1 < 0 \text{ outside of the } \delta\text{-vicinity of the curve } t = \psi(x),$$

where $C_1 = C_1(\delta)$ is some constant independent of $\varepsilon$, and by assumption (A₂)

$$\hat{f}_e(x, t) \geq C_2 > 0$$

in any $\delta\text{-vicinity of the curve } t = \psi(x)$, if $\delta$ is sufficiently small. Taking into account that $\sigma \leq 0$ and $1/2 < \alpha < 1$, we get from (28) for sufficiently small $\varepsilon$

$$L_\varepsilon U \leq -\hat{f}_e(x, t)\varepsilon + o(\varepsilon) \leq -C_2\varepsilon + o(\varepsilon) < 0$$

in a sufficiently small $\delta\text{-vicinity of } t = \psi(x)$. Outside of the $\delta\text{-vicinity of the curve } t = \psi(x)$, the dominant term in (28) is $\varepsilon\hat{f}_u(x, t)A$, if $A$ is sufficiently large. This term provides the inequality

$$L_\varepsilon U < 0 \text{ outside of the } \delta\text{-vicinity of } t = \psi(x)$$

for sufficiently large $A$ and sufficiently small $\varepsilon$.

Analogously, using the representation

$$f(U, x, t, \varepsilon) = f(\hat{u}(x, t) + \sigma(x, t, \varepsilon) + \varepsilon^{1/2}Ah(x, t, \varepsilon)$$

$$+ \varepsilon z(x, \varepsilon), x, t, \varepsilon)$$

$$= \hat{f}_u(x, t)(\varepsilon^{1/2}Ah + o(\varepsilon^{1/2})) + \hat{f}_e(x, t)\varepsilon$$

$$+ \frac{1}{2}\hat{f}_{uu}(x, t)(\varepsilon a^2h^2 + o(\varepsilon)) + o(\varepsilon)$$

$$= \hat{f}_u(x, t)(\varepsilon^{1/2}Ah + o(\varepsilon^{1/2}))$$

$$+ \varepsilon \left[ \frac{1}{2}\hat{f}_{uu}(x, t)A^2h^2 + \hat{f}_e(x, t) \right] + o(\varepsilon),$$
we get

\[ L_\varepsilon \bar{U} = O(\varepsilon^{2-\alpha}) - \hat{f}_u(x,t)(\varepsilon^{1/2}Ah + o(\varepsilon^{1/2})) \]
\[ - \varepsilon \left[ \frac{1}{2} \hat{f}_{uu}(x,t) A^2 h^2 + \hat{f}_e(x,t) \right] + o(\varepsilon). \]  

(30)

Take into account that \( O(\varepsilon^{2-\alpha}) = o(\varepsilon) \), \(-\hat{f}_u(x,t)(\varepsilon^{1/2}Ah + o(\varepsilon^{1/2})) \geq 0 \) for sufficiently small \( \varepsilon \) in consequence of (29), and, finally, by assumption (A_4)

\[ \hat{f}_{uu}(x,t) \leq -C_3 < 0 \]

in any \( \delta \)-vicinity of the curve \( t = \psi(x) \), if \( \delta \) is sufficiently small. Then from (30) we have \( L_\varepsilon \bar{U} > 0 \) in the \( \delta \)-vicinity of the curve \( t = \psi(x) \) for sufficiently large \( \Lambda \) and sufficiently small \( \varepsilon \).

Outside of the \( \delta \)-vicinity of the curve \( t = \psi(x) \) we have \( \hat{f}_u(x,t) \leq -C_1 \) < 0, \( h = 1 \) and, hence, the dominant term in (30) is \(-\varepsilon^{1/2} \hat{f}_u(x,t)Ah\), since the other terms are \( o(\varepsilon^{1/2}) \). Due to this term, \( L_\varepsilon \bar{U} > 0 \) outside of the \( \delta \)-vicinity of \( t = \psi(x) \).

Thus, inequalities (17) are fulfilled in \( D_0 \), and consequently (24) and (25) are the lower and upper solutions, respectively, for the problem (1), (3), (15). Therefore, we can conclude that there exists a solution \( u(x,t,\varepsilon) \) satisfying

\[ \underline{U} \leq u(x,t,\varepsilon) \leq \bar{U} \quad \text{in } \overline{D}_0. \]

From these inequalities and relations (23)–(25) it follows that the representation (14) of \( u(x,t,\varepsilon) \) holds in \( \overline{D}_0 \). The relations (14) and (12) show that (11) is fulfilled. From (11) and (13), we get (10). The theorem is proved.

3. EXAMPLE

In this section we demonstrate the role of the assumption (A_3) by considering the following example:

\[ \varepsilon^2 (u_t - u_{xx}) = -u(u - t + \frac{1}{2}) - \varepsilon, \quad 0 < x < 1, \quad 0 < t \leq 1, \]  

(31)

\[ u(x,0) = 0, \quad u_x(0,t) = u_x(1,t) = 0. \]  

(32)

The degenerate equation has the two roots

\[ u = \varphi_1 = 0 \quad \text{and} \quad u = \varphi_2 = t - \frac{1}{2}, \]
which intersect at \( t = \psi(x) = 1/2 \). It is easy to check that inequalities (5) and assumption \((A_2)\) are fulfilled. The composed stable solution \( \hat{u}(x, t) \) reads

\[
\hat{u}(x, t) = \begin{cases} 
0, & 0 \leq t \leq 1/2, \\
1/2, & 1/2 \leq t \leq 1, 
\end{cases} 
0 \leq x \leq 1.
\]

The problem (9) for \( \Pi(x, \tau) \) has the form

\[
\frac{d\Pi}{d\tau} = -\Pi(\Pi + \frac{1}{2}), \quad \Pi(x, 0) = 0.
\]

From here we get \( \Pi \equiv 0 \). Thus, assumption \((A_3)\) is also fulfilled.

Finally, \( f_{uu} = -2 \), i.e., assumption \((A_4)\) holds, but \( f_\varepsilon = -1 < 0 \), i.e., instead of condition \((A_5)\), the contrary inequality takes place.

Let us prove that the solution \( u(x, t, \varepsilon) \) of the problem (31), (32) does not satisfy the condition

\[
\lim_{\varepsilon \to 0} u(x, t, \varepsilon) = \hat{u}(x, t) \quad \text{for } 0 \leq x \leq 1, 0 < t \leq 1. \tag{33}
\]

Indeed, it is obvious that the solution \( u(t, \varepsilon) \) of the problem

\[
\varepsilon^2 u_t = -u(\varepsilon - t + \frac{1}{2}) - \varepsilon, \quad u(0, \varepsilon) = 0 \tag{34}
\]

satisfies Eq. (31) and initial and boundary conditions (32). Thus, \( u(x, t, \varepsilon) = u(t, \varepsilon) \).

Problem (34) implies that

\[
u(t, \varepsilon) < 0 \tag{35}
\]

on the interval of its existence. But

\[
\hat{u}(x, t) = t - \frac{1}{2} > 0 \quad \text{for } \frac{1}{2} < t \leq 1. \tag{36}
\]

From (35) and (36) it follows that the limiting passage (33) cannot hold for \( \frac{1}{2} < t \leq 1 \).

It should be noted that actually we have a lack of global existence in problem (34). One can show that for sufficiently small \( \varepsilon \) the solution \( u(t, \varepsilon) \) does not exist on \([0, T]\) for \( T > \frac{1}{2} \) with \( T \) independent of \( \varepsilon \).

REFERENCES


