On zeros of characteristic $p$ zeta function

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Abstract
The location and multiplicity of the zeros of zeta functions encode interesting arithmetic information. We study the characteristic $p$ zeta function of Goss. We focus on “trivial” zeros and prove a theorem on zeros at negative integers, showing more vanishing than that suggested by naive analogies. We also compute some concrete examples providing the extra vanishing, when the class number is more than one.

Finally, we give an application of these results to the non-vanishing of certain class group components for cyclotomic function fields. In particular, we give examples of function fields, where all the primes of degree more than two are “irregular”, in the sense of the Drinfeld–Hayes cyclotomic theory.

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1. Introduction
The Riemann zeta function is defined by

$$\zeta_Q(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

where $s \in \mathbb{C}$ with $\Re s > 1$. We can analytically continue $\zeta_Q(s)$ to a meromorphic function on $\mathbb{C}$ with a pole of order 1 and residue 1 at 1. There is a rich special values
theory associated to $\zeta_{\mathbb{Q}}(s)$, which is intimately connected to the study of the Bernoulli numbers, $B_n$. If $n \geq 0$ we have

$$
\zeta_{\mathbb{Q}}(-n) = -\frac{B_{n+1}}{n+1}.
$$

Consequently, if $n \geq 1$, $\zeta_{\mathbb{Q}}(-2n) = 0$. Such zeros are called trivial zeros and they are simple zeros. With respect to the non-trivial zeros, the following conjecture is due to Riemann:

**Conjecture 1 (Riemann Hypothesis).** The non-trivial zeros of $\zeta_{\mathbb{Q}}(s)$ lie on the line $\Re s = \frac{1}{2}$.

All the zeros found so far have turned out to be simple zeros, so nowadays simplicity of zeros is also conjectured.

For $m = 2k$, $k > 0$ an integer we have

$$
\zeta_{\mathbb{Q}}(m) = -\frac{B_m (2\pi i)^m}{2m!}.
$$

There is no simple formula for $\zeta_{\mathbb{Q}}(2k+1)$ analogous to the previous one. It is not known whether $\zeta_{\mathbb{Q}}(2k+1)$ is rational or irrational, except for $k = 1$ when it is irrational. Also, divisibilities of $B_m$ by primes $p$ are closely related to information on components of the ideal class group of cyclotomic extensions $\mathbb{Q}(\zeta_p)$, where $\zeta_p$ is a primitive $p$th root of unity. For example, see Herbrand–Ribet Theorem in [Was1].

More generally the Dedekind zeta function $\zeta_K$ of a number field $K$ (a finite extension of $\mathbb{Q}$) is defined, for $s \in \mathbb{C}$ with $\Re s > 1$, by

$$
\zeta_K(s) := \sum_{\mathcal{I}} N(\mathcal{I})^{-s} = \prod_{\mathcal{P}} (1 - N(\mathcal{P})^{-s})^{-1},
$$

where the sum is taken over all non-zero ideals $\mathcal{I}$ of $\mathcal{O}_K$ (ring of integers of $K/I$). Here, $N(\mathcal{I}) = |\mathcal{O}_K/\mathcal{I}|$ is the norm of the ideal $\mathcal{I}$, and $\mathcal{P}$ runs through the prime ideals $\mathcal{P}$ of $\mathcal{O}_K$. Notice that for $K = \mathbb{Q}$, $\zeta_K = \zeta_{\mathbb{Q}}$ since $N(n\mathbb{Z}) = |\mathbb{Z}/n\mathbb{Z}| = n$. This function has a simple functional equation connecting $\zeta_K(s)$ to $\zeta_K(1-s)$. Let $r_1$ be the number of embeddings of $K$ in $\mathbb{R}$ and $r_2$ half the number of non-real embeddings of $K$ in $\mathbb{C}$. For $s > 1$, it is clear that there are no zeros and hence analyzing the poles of gamma factors in the functional equation, we can see that, at negative integers, the zeta function vanishes to order $r_1$ ($r_2$, respectively) if $s$ is even (odd). In addition, for $s$ a positive even integer, $(\zeta_K(s) / (2\pi i)^{r_1s})^2 \in \mathbb{Q}$, if $K$ is totally real. In general, orders of vanishing and special (leading) values encode a lot of interesting arithmetic information.

For a function field $K$ over the finite field of constants $\mathbb{F}_q$, $q = p^n$, we describe the Artin–Weil zeta function, but before that, we introduce some notation that will be used.
throughout this work. The order of the group of divisor classes of degree zero will be denoted by $h_K$, and is called the class number of $K/F$. For a divisor $D$, we put \[
 N (D) = q^\deg D. \]

Then the Artin–Weil zeta function of $K/F$ is defined, for $s \in \mathbb{C}$ with $\Re s > 1$, by

\[
\zeta_K (s) := \sum_D N (D)^{-s} = \prod_P (1 - N (P)^{-s})^{-1},
\]

where the sum is taken over the positive divisors $D$, and the product taken over all the places $P$ of $K/F$. The Riemann hypothesis for $\zeta_K (s)$ is known by Weil’s theorem, but it is just a rational function of $q^{-s}$. So, there cannot be an analogue of Euler’s Theorem connecting $\zeta_K (2k)$ to $(2\pi i)^{2k}$, for example.

We now introduce our characteristic $p$ zeta function for the rational function field $F_q (T)$, which will be a richer transcendental function, whose special values involve analogues of $2\pi i$. After mentioning some strong similarities, we study its zeros and orders of vanishing in more detail.

**Notation:**

- $\mathbb{F}_q$ is the finite field of $q = p^n$ elements.
- $A := \mathbb{F}_q [T]$, the analogue of $\mathbb{Z}$.
- $K := \mathbb{F}_q (T)$, the analogue of $\mathbb{Q}$.
- $K_\infty := \mathbb{F}_q ((T^{-1}))$, the field of Laurent series, which is the completion of $K$ with respect to the $T^{-1}$-adic valuation, $v (T^{-1}) = 1$, and is the analogue of $\mathbb{R}$.
- $\mathbb{C}_\infty := \widehat{K_\infty}$, is a completion of the algebraic closure of $K_\infty$ and is the analogue of $\mathbb{C}$.

Consider the following Carlitz zeta function: for $s \in \mathbb{N}$,

\[
\zeta_A (s) := \sum_{f \text{ monic}} f^{-s} = \prod_{g \text{ monic prime}} (1 - g^{-s})^{-1}.
\]

Here the requirement monic is playing the role of “positive” in the classical Riemann zeta function $\zeta_{\mathbb{Q}} (s)$. In other words, instead of the norm which just depends on the degree of the polynomial, Carlitz [C1] used the whole polynomial, paying the price of considering a smaller domain for $s$, since we do not know how to raise a polynomial to a complex power. More justification lies in the following Theorem [C1,C2]; [T3, p. 158], which we state without a complete explanation.

**Theorem 2.** If $m$ is a positive $A$-even integer,

\[
\zeta_A (m) = - B_m \tilde{\pi}^m / (q - 1) \Pi (m).
\]
In this case, $A$-even means a multiple of $q - 1$ which represents the number of signs in $A$ (units in $A$) replacing 2, the number of signs in $\mathbb{Z}$. Here, $B_m \in K$ is a Bernoulli analogue, $\Pi (m) \in A$ is analogue of factorial. Also, 

$$\tilde{\pi} = (T - T^q)^{\frac{1}{q-1}} \prod_{n=1}^{\infty} \left( 1 - \frac{T^q - T}{T^{q^{n+1}} - T} \right) \in \mathbb{C}_\infty, \quad \tilde{\pi}^q - 1 \in K_\infty,$$

plays the role of $2\pi i$ and is known to be transcendental over $K$. There is no functional equation known. But in fact, much is known about the nature of the special values at positive integers $[Y_1]$, in contrast to the classical case:

**Theorem 3** (Anderson-Thakur [A-T], Yu [Y1]). For $m$ positive (A-even or A-odd), $\zeta_A (m)$ is transcendental over $K$ and $\zeta_A (m) / \tilde{\pi}^m$ is also transcendental if $m$ is A-odd.

Divisibilities of $B_m$ by primes of $A$ are closely related to information about $p$-primary components ($q = p^n$) of the class groups of rings of integers of “cyclotomic” extensions of $K$ in analogous fashion to $\mathbb{Q}$ case. For more details, see [T1].

Next we explain an analogue of exponentiation due to Goss. If $n \in \mathbb{N}$ and $s = x + yi \in \mathbb{C}$, 

$$n^s = e^{x \log n} = e^{x \log n} e^{iy \log n}, \quad \text{where} \quad |n^s| = n^x \quad \text{and} \quad |e^{iy \log n}| = 1.$$

In the function field case, Goss [G1] extended the definition of the characteristic $p$ zeta function to a bigger domain by defining exponentiation of monic polynomials $f \in A$ doing something similar to the previous exponentiation by complex numbers. The exponentiation of monic polynomials is defined as follows:

- Let $\mathbb{Z}_p$ be the ring of $p$-adic integers.
- Let $s = (x, y) \in S_\infty := \mathbb{C}_\infty \times \mathbb{Z}_p$.
- Let $f \in \mathbb{F}_q [T]$ be a monic polynomial.
- One sets

$$\langle f \rangle := f T^{-\deg f}.$$

Since $f$ is monic $\langle f \rangle \equiv 1 \pmod{T^{-1}}$ and so, $\langle f \rangle$ may be raised to the $y$th power for $y \in \mathbb{Z}_p$ by the binomial theorem.

- Finally set

$$f^s := x^{\deg f} \langle f \rangle^y.$$

We note that for $m \in \mathbb{Z}$, $f^m = f^s$ when $s = (T^m, m)$. So, we will write $\zeta_A (m)$ for $\zeta_A (s)$ when $s = (T^m, m)$. 

The Goss zeta function is defined in the same way that the Carlitz zeta function was defined, but now for \( s \in S_\infty \),
\[
\zeta_A(s) := \sum_{f \text{ monic}} f^{-s} = \prod_{g \text{ monic prime}} (1 - g^{-s})^{-1}.
\]

The function \( \zeta_A(s) \) has a natural “half-plane” of convergence, given by all \( s = (x, y) \in S_\infty \) with \( \deg(x) > 0 \), where \( \deg \) is the canonical extension to \( C_\infty \) of the \( \deg \) map defined over \( K_\infty \), and by grouping together terms of the same degree, can be analytically continued to an entire function on \( S_\infty \),
\[
\zeta_A(s) = \sum_{d=0}^{\infty} (1/x)^d \left( \sum_{\deg f = d \atop f \text{ monic}} (f)^{-y} \right).
\]

Then \( \zeta_A(s) \) is a continuous function on \( S_\infty \). For each fixed \( y \in \mathbb{Z}_p \), define \( \zeta_y(x) := \zeta_A(x, -y) \). Then \( \zeta_y(x) \) is an entire function of \( x^{-1} \), i.e., it is represented by a power series with infinite radii of convergence. For a negative integer \( s \), we have the following analogue of the classical result [G1].

**Theorem 4 (Goss [G1]).**
\[
\zeta_A(-(q-1)k) = 0, \text{ } k > 0 \text{ an integer}.
\]

These are the so called trivial zeros, since multiples of \( q-1 \) are analogous to even numbers.

In the next section, we will consider more general function fields and focus on “trivial” zeros and their order of vanishing. Let \( K \) be a function field of one variable with field of constants \( \mathbb{F}_q \) of characteristic \( p \), \( \infty \) be a place of \( K \) of degree \( d_\infty \), \( K_\infty \) be the completion of \( K \) at \( \infty \) with \( \mathbb{F}_\infty \subset K_\infty \), the field of constants, and \( A \) be the ring of elements of \( K \) having no pole outside \( \infty \). We take \( C_\infty \) to be the completion of an algebraic closure of \( K_\infty \). Recall that \( h_K \) is the divisor class number of \( K \), so \( h \), the class number of \( A \) as a Dedekind domain is \( h_K d_\infty \). Using sophisticated ideal exponentiation Goss [G3;G4, Chapter 8] defined the zeta function in general. The exponentiation of ideals is defined as follows: first, we introduce a function field version of the notion of “sign of a number.” A sign function on \( K_\infty^* \) is a homomorphism \( \text{sgn} : K_\infty^* \rightarrow \mathbb{F}_\infty^* \) which is the identity on \( \mathbb{F}_\infty^* \). We also set \( \text{sgn}(0) = 0 \). Sign functions may be easily constructed and any sign function must be trivial on \( U_1 \subset K_\infty^* \), the units of the valuation ring which are congruent to one modulus the maximal ideal. The element \( x \in K_\infty^* \) is positive (or monic) if and only if \( \text{sgn}(x) = 1 \). They form a subgroup of \( K_\infty^* \). Let \( \pi \in K_\infty^* \) be a fixed positive (i.e., \( \text{sgn}(\pi) = 1 \)) uniformizer. Then \( x \in K_\infty^* \) can be written uniquely as
\[
x = \text{sgn}(x) \pi^j \langle x \rangle,
\]
where \( j \in \mathbb{Z} \) and \( \langle x \rangle \in K_\infty^\times \) is a 1-unit, i.e., \( \langle x \rangle \in U_1 \). Let \( \mathfrak{I} \) be the group of fractional ideals of \( A \subset K \), the ring of functions regular outside \( \infty \). We let \( \mathfrak{P} \) be the subgroup of principal ideals and \( \mathfrak{P}^+ \subset \mathfrak{P} \) the subgroup generated by positive elements. Let \( \widehat{U}_1 \supset U_1 \) be the group of all 1-units in \( C_\infty \). The natural action of \( \mathbb{Z}_p \) on \( \widehat{U}_1 \) may be extended uniquely to an action of \( \mathbb{Q}_p \).

For a non-zero ideal \( \mathcal{I} \subset A \) and \( s = (x, y) \in S_\infty = C_\infty^\times \times \mathbb{Z}_p \), \( \mathcal{I}^s \) can be computed as follows: Let \( e \) be the order of \( \mathcal{I} \) in \( \mathcal{I} / \mathcal{P}^+ \); so \( \mathcal{I}^e = (\lambda) \) with positive \( \lambda \in K_\infty^\times \). Then,

\[
\mathcal{I}^s = x^{\deg(\mathcal{I})} \langle \lambda \rangle^{y/e},
\]

where \( y/e \in \mathbb{Q}_p \), the field of \( p \)-adic numbers. This definition of exponentiation of ideals, coincides with the one given previously in the case \( A = \mathbb{F}_q[T] \), because all the ideals have order \( e = 1 \). In this general context, the Goss zeta function \( \zeta_A \) of \( A \) is defined by

\[
\zeta_A(s) := \sum_{\mathcal{I}} \mathcal{I}^{-s}
\]

for \( s \in S_\infty \), where the sum is over all the ideals \( \mathcal{I} \) of \( A \). Using the infinite prime and “double congruences”, Goss was able to show that

**Theorem 5 (Goss [G4]).** If \( m \) is a positive \( A \)-even integer,

\[
\zeta_A(-m) = 0.
\]

In this case, “\( A \)-even” means a multiple of \( q^{d_\infty} - 1 \) (see [G4, Example 8.13.9]). These are also called “trivial zeros.”

Returning to the zeta function for \( A = \mathbb{F}_p[T] \), Wan [Wan1] found that for a fixed \( y \in \mathbb{Z}_p \) if \( \zeta(x, y) = 0 \), then \( x \) lies on the real “line” \( K_\infty \) and the zeros are simple zeros (because the Newton polygon associated to each \( y \in \mathbb{Z}_p \) had slopes whose horizontal projections had length 1). This clearly looks like a “Riemann hypothesis.” We gave a simpler proof [Di1] of Wan’s result. Jeffrey Sheats in [S1] proved the same result for \( A = \mathbb{F}_q[T] \) for all \( q \). For general \( A \), all zeros need not be in \( K_\infty \) (see [T3, Example 5.8.3]).

On the other hand, let \( L(s) \), \( s = (x, y) \in S_\infty \) be an \( L \)-series of arithmetic type, defined by Goss. The trivial zeros then arise discretely in \( S_\infty \) (in fact, the zeros are at \( -m = (n^m, -m) \in S_\infty \)). But, you cannot ignore them when dealing with a given \( y \in \mathbb{Z}_p \). There are situations where the trivial zeros have higher multiplicity, that you can construct an element \( y \in \mathbb{Z}_p - \mathbb{N} \) (with \( \mathbb{N} \) being the non-negative integers) where the associated Newton polygon at has infinitely many slopes of length greater than 1. The interest is in the effect of trivial zeros at \( -m \) for those \( A \)-even \( m \) approaching \( y \). Those zeros which are influenced by trivial zeros, were called by Goss [G6] “near trivial”. It is explained there how trivial and near-trivial zeros should arise in general, and how they might ultimately be handled via Hensel’s Lemma (whereas classically
one uses Gamma-functions). Goss breaks up all the zeros into two classes, the near-trivial zeros and the “critical zeros” (= all other zeros). But then there is a remarkable surprise: all zeros computed by Wan et al. are near-trivial (this was established by Goss when $q = p$ and is expected to hold for all $q$).

For a fixed $y \in \mathbb{Z}_p$, $L(x, y)$ is a power series in $x^{-1}$ with coefficients in a finite extension $K_\infty(y)$ of $K_\infty$. We let $K_\infty^{\text{tot}}$ be the extension of $K_\infty(y)$ obtained by adjoining the critical zeros of $L(x, y)$; we let $K_\infty^{\text{tot}, s}$ be its maximal separable subfield (over $K_\infty(y)$). We have the following conjectures for the critical zeros, due to Goss. Given the limited amount of experience we have with these functions, these “conjectures” are really more like “educated guesses” meant to begin to give a framework in which to discuss these issues.

**Conjecture 6** (Goss [G5]). The field $K_\infty^{\text{tot}, s}$ is a finite extension of $K$.

There is an obvious $v$-adic analogue of the above conjecture.

**Conjecture 7** (Goss [G5]). There exists a positive real number $b = b(y)$ such that if $\delta \geq b$, then there exists at most one critical zero in $x^{-1}$ of $L(x, y)$ of absolute value $\delta$.

Conjecture 7 is based on the examples of Wan, Sheats, etc., and plays a role similar to the classical generalized Riemann hypothesis. Indeed, in [G5], Goss showed how it leads to a variant of the classical generalized Riemann hypothesis for number fields. It implies Conjecture 6 because one can then easily show that almost all the zeros of $L(s)$ are totally inseparable over $K_\infty(y)$.

We will use the much simpler, but specialized approach of [T2], where it is suggested using all the ideals $I$ of $A$, but to restrict $s$ to be a multiple of the exponent $e$ of the finite abelian ideal class group of $A$ and letting $a^{-s}$ be the generator of $I^{-s}$. In the next section we describe, in a more explicit way, this zeta function and investigate the orders of vanishing of its zeros for negative integers $s$. In this work, we consider only the case when $d_\infty = 1$.

2. Orders of vanishing

From now on, since we only look at values and orders of vanishing at integers, we introduce and follow simpler notation of [T2,T3], which differ from our earlier notation.

First, consider

**Definition 8.** For $s \in \mathbb{Z}$, define the absolute zeta function:

\[
\zeta(s, X) := \zeta_A(s, X) := \sum_{d=0}^{\infty} X^d \sum_{\substack{\deg a = d \\text{monic}}} a^{-s} \in K[[X]],
\]

\[
\zeta(s) := \zeta_A(s) := \zeta(s, 1) \in K_\infty.
\]
When we say the order of vanishing of $\zeta(s)$ at negative $s$, we mean the order of vanishing of $\zeta(s, X)$ at $X = 1$. The drawback of Definition 8 is that we only consider principal ideals of $A$. So, this gives a full zeta function only for class number 1.

Let $H$ be the Hilbert class field of $A$, i.e., the maximal abelian unramified extension of $K$ in which $\infty$ splits completely. Let $L$ be a finite separable extension of $K$ and let $\mathcal{O}_L = \mathcal{O}$ denote the integral closure of $A$ in $L$. Assume now that $L$ contains $H$, then it is known that the norm of an ideal $\mathcal{I}$ of $\mathcal{O}$ is principal. Let $N\mathcal{I}$ denote the monic generator of this principal ideal. Now, we define the relative zeta function in this situation. Put, for $s \in \mathbb{Z}$,

$$\zeta_{\mathcal{O}}(s, X) := \sum_{d=0}^{\infty} X^d \sum_{\deg N\mathcal{I} = d} N\mathcal{I}^{-s} \in K[[X]].$$

$$\zeta_{\mathcal{O}}(s) := \zeta_{\mathcal{O}/A}(s) := \zeta_{\mathcal{O}}(s, X) := \zeta_{\mathcal{O}}(s, 1) \in K_\infty.$$

The last assertion, $\zeta_{\mathcal{O}}(s) \in K_\infty$ [G1], follows, for example, from the next very useful result. For a non-negative integer $k = \sum k_i q^i$, with $0 \leq k_i < q$, $l(k) := \sum k_i$, hence $l(k)$ is the sum of base $q$ digits of $k$.

**Proposition 9 (Thakur [T2]).** If $d > \frac{l(k)}{q-1}$, then

$$\sum_{a_1, \ldots, a_d \in \mathbb{F}_q} (f + a_1 w_1 + \cdots + a_d w_d)^k = 0,$$

for any $f, w_1, \ldots, w_d$.

**Remark 10.** Note that in particular, when $q = 2$, the sum vanishes if $d$, the number of parameters is greater than $l(k)$.

This proposition is also useful for proving the following result for the relative zeta functions, which is similar to the one mentioned in the introduction for the absolute zeta function.

**Theorem 11 (Goss [G3]).** For a negative integer $s$, $\zeta_{\mathcal{O}}(s) \in A$ and

$$\zeta_{\mathcal{O}}((q - 1)s) = 0.$$

Now we turn to the question of the order of vanishing. Let $\mathcal{P}$ be a prime ideal of $A$ and let $W$ be the Witt ring of $A/\mathcal{P}$. The identification $W/pW \cong A/\mathcal{P}$ provides us with the Teichmüller character $w : (A/\mathcal{P})^\times \rightarrow W^\times$ satisfying $w^k(a \mod \mathcal{P}) = (a^k \mod \mathcal{P}) \mod p$. Now let $\Lambda_{\mathcal{P}}$ denote the $\mathcal{P}$-torsion of the rank one, sgn-normalized Drinfeld module $\rho$ of generic characteristic. Let $G$ be the Galois group of $L(\Lambda_{\mathcal{P}})$ over $L$. 
Then $G$ can be thought of as a subgroup of $(A/P)\times$ and hence $w$ can be thought of as a $W$-valued character of $G$. Let $L(w^{-s}, u) \in \mathbb{W}(u)$ be the classical $L$-series of Artin and Weil in $u := q^{-sm}$, where $m$ is the extension degree of the field of constants of $L$ over $\mathbb{F}_q$. Let $S_\infty := \{\infty_j\}$ denote the set of infinite places of $L$ and let $G_j$ denote the Galois group of $L_{\infty_j}(\mathbb{A}/P)$ over $L_{\infty_j}$. Then $G_j \subset \mathbb{F}_q^\times$. Given $s$, let $S_s \subset S_\infty$ be the subset of infinite places at which $w^{-s}$ is an unramified character of $G$. Then $S_s$ does not depend on $P$. Put

$$\zeta_O(s, X) := \zeta_O(s, X) \prod_{\infty_j \in S_s} \left(1 - w^{-s}(\infty_j) X^{\deg(\infty_j)}\right)^{-1}.$$ 

**Theorem 12** (Goss [G3]). Let $L$ contain $H$ and let $s$ be a negative integer. Then $\zeta_O(s, X) \in A[X]$.

This immediately gives the following lower bound for the order of vanishing.

**Theorem 13** (Goss [G3]). Let $L$ contain $H$ and let $s$ be a negative integer, then the order of vanishing of $\zeta_O(s)$ is at least

$$V_s := \text{ord}_{X=1} \prod_{\infty_j \in S_s} \left(1 - w^{-s}(\infty_j) X^{\deg(\infty_j)}\right).$$

**Example 14.** $L = K = H$. Then $V_s = 0$ or 1, according as $s$ is odd or even. Goss proved that for $A = O = \mathbb{F}_q[T]$, the order of vanishing is $V_s$.

These lower bounds are in fact obvious analogues of the exact order of vanishing for the Dedekind zeta function in the number field case. Goss mentioned as an open question whether these lower bounds are exact. In [T2] this was answered in the negative, and it was shown that the patterns of extra vanishing are quite surprising, involving the $q$-digits of $s$. The extra vanishing seems to happen at those $s$ with bounded sum of $p$-adic digits. See [G6] for more on this. More specifically, we have the following theorem.

**Theorem 15** (Thakur [T2]). If $d_\infty = 1$, $q = 2$ and $K$ is hyperelliptic, then the order of vanishing of $\zeta(s)$ at negative integers $s$ is 2 if $l(-s) \leq g$, where $g$ is the genus of $K$.

There is also a result [T2] for general $q$, involving more complicated conditions on Weierstrass gaps at $\infty$ for $A$.

A direct application of this theorem in the following example, shows that it is possible to have extra vanishing, i.e., order of vanishing greater than $V_s$, the lower bound for the order of vanishing.
Example 16. Let $A = \mathbb{F}_2 [x, y] / (y^2 + y + x^5 + x^3 + 1)$. Here, $H = K = L$ and the genus $g$ of $K$ is two. So, for $s = -(2^n + 2^m)$, $m \geq 0$, $n \geq 0$ (possibly with $m = n$) the order of vanishing is two rather than $V_s = 1$.

For more background and details on Drinfeld modules and zeta functions see [G4,T3].

Now, we turn to a zeta function that involves not only principal ideals, but all the ideals of $A$. Recall that $d_\infty = 1$. Let $s$ be a multiple of $e$, and define $I_s$ to be $a^{s/e}$, where $a$ is the monic generator of $I_e$. Then we define the zeta function as follows: for $s$, an integer multiple of $e$,

$$
\zeta(s, X) := \zeta_A(s, X) := \sum_{d=0}^{\infty} X^d \sum_{\deg I = d, I \text{ ideal of } A} I^{-s},
$$

$$
\zeta(s) := \zeta_A(s) := \zeta(s, 1).
$$

If $h = 1$, this definition coincides with the absolute zeta function defined earlier (see Definition 8). So below we concentrate only on $h > 1$. In [G4, Chapter 8, Section 13] it is proved that when we use the “generalized Teichmüller character” we get the same lower bound as in Theorem 13 defined without assuming that $L$ contains $H$, in particular for $L = K$. In order to understand the definition better we present an example, worked out by Thakur.

Example 17. Let $A = \mathbb{F}_2 [x, y] / (y^2 + x^2 y + x^5 + x^4 + x)$. Then $\deg x = 2$, $\deg y = 5$, and $g = \frac{5-1}{2}$. It is easy to see that the affine curve $y^2 + x^2 y = x^5 + x^4 + x^2$ is smooth, and the degree of the infinite place in $A$ is one. This is example $C_{12}$ of [Du1], with $h = 2$. Then, of course, $e = 2$. Consider $s$, a positive integer divisible by $e = 2$. Then, since $h = 2$, we only have two classes of ideals, the principal class and the non-principal class. Let $I_0 = (x, y)$ be the representative of the non-principal class, $I_0^2 = (x)$ and $\deg I_0 = 1$. Notice that $I_0$ is not a principal ideal, since in $A$ we do not have elements of degree 1. Then

$$
\zeta(-s, X) = \sum_{d=0}^{\infty} X^d \left( \sum_{\deg a = d, a \text{ monic}} a^s + \sum_{\deg I = d, I \sim I_0} I^s \right),
$$

where $I \sim I_0^{-1} \sim I_0$ means that $I$ is in the non-principal class, i.e., $\mathcal{I} I_0 = (b)$, for some $b \in A$ of $\deg = d + 1$ and $I^2 = b^2 / x^d = b^2 x^{d-2} \in A$. Conversely, if $b \in A$ with $b^2 x \in A$, then $(b^2) = (x) I_1 = I_0^2 I_1$ for some ideal $I_1$, so that $(b) = \mathcal{I} I_0$ for some ideal $I$ and $I$ is necessarily in the non-principal class.

Now, $b \in A$ implies that $b = f(x) + y g(x)$, where $f$ and $g \in \mathbb{F}_2 [x]$. But $b^2 = f(x^2) + y^2 g(x)^2 \equiv f(x) (\mod x)$ since $x$ divides $y^2$. So, $x | b^2$ if and only
if \( x \mid f(x) \). Therefore,

\[
\left\{ b \in A \mid \deg b = d + 1 \text{ and } \frac{b^2}{x} \in A \right\}
\]

\[
= \{ b \in A \mid \deg b = d + 1 \text{ and } b = xf(x) + yg(x) \}.
\]

Also, \( \frac{b^2}{x} = \frac{x^2 f(x)^2 + y^2 g(x)^2}{x} = xf(x)^2 + \frac{(x^2 y + x^4 + x)g(x)}{x} = xf(x)^2 + wg(x)^2 \), where \( w = xy + x^4 + x^3 + 1 \). So,

\[
\sum_{\deg I = d} I^s = \sum_{\deg b = d + 1 \atop b \text{ monic integral}} \left( \frac{b^2}{x} \right)^{s/2} = \sum_{\deg (xf(x) + yg(x)) = d + 1} \left( xf(x)^2 + wg(x)^2 \right)^{s/2}.
\]

Suppose now that \( l(s) = 1 \), i.e., \( s = 2^n, n \geq 1 \). For the zeta function, we distinguish the following contributions:

**Principal Part.** Because of the bound in the Proposition 9, \( \sum_{\deg a = d} a^s = 0 \) if \( d > 2 \). So, principal ideals contribute \( 1 + X^2 \left( x^s + (x + 1)^s \right) = 1 + X^2 \left( x^s + x^s + 1 \right) = 1 + X^2 \).

**Non-Principal Part.** For \( d \geq 4 \), \( \deg (xf(x) + yg(x)) \geq 5 \), we are summing over more than one parameter, so the contribution to the sum is zero. (See Remark 10). Hence, \( g(x) = 0 \). Therefore, the total non-principal contribution is \( X x^{s/2} + X^3 \left( (x^3)^{s/2} + (x^3 + x)^{s/2} \right) = X x^{s/2} + X^3 x^{s/2} = (1 + X^2) X x^{s/2} \).

Adding the principal and non-principal parts we obtain:

\[
\zeta(-s, X) = \left( 1 + X^2 \right) \left( 1 + X x^{s/2} \right),
\]

and we conclude that the order of vanishing is two.

Assume now that \( l(s) = 2 \), i.e., \( s = 2^m + 2^n, m > n > 0 \). Then

**Principal part.** As before, since \( l(s) = 2 \), for \( d > 4 \) the sum vanishes. Hence, the contribution is

\[
1 + X^2 \left( x^s + (x + 1)^s \right) + X^4 \left( \left( x^2 \right)^s + \left( x^2 + 1 \right)^s + \left( x^2 + x \right)^s + \left( x^2 + x + 1 \right)^s \right).
\]

**Non-principal part.** In this case, if \( d \geq 5 \) we are summing over more than two parameters. The contribution is

\[
X x^{s/2} + X^3 \left( \left( x^3 \right)^{s/2} + \left( x^3 + x \right)^{s/2} \right)
\]

\[
+ X^4 \left( \left( w + x (x) \right)^{s/2} + \left( w + x (x + 1) \right)^{s/2} + (w + x)^{s/2} + w^{s/2} \right).
\]
Then
\[
\zeta(-s, X) = 1 + X^{s/2} + X^2 \left( x^s + (x + 1)^s \right) + X^3 \left( x^{3s/2} + (x^3 + x)^{s/2} \right) \\
+ X^4 \left( x^2 + (x^2 + 1)^s + (x^2 + x)^s + (x^2 + x + 1)^s \right) \\
+ X^4 \left( w^{s/2} + (w + x)^{s/2} + (w + x^3)^{s/2} + (w + x^3 + x)^{s/2} \right).
\]

If the order of vanishing were 2, then \(\zeta(-s, X) = (1 + X^2) \left( 1 + x^{s/2}X + \theta X^2 \right)\), but \(X^3\) coefficients do not match. So, the order of vanishing is one.

We prove the following theorem generalizing the special phenomenon of this example, and Theorem 15.

**Theorem 18.** Let \(q = 2\). Let \(A\) given by \(y^2 + a(x)y = b(x)\) or \(y^2 + y = b(x)\), where \(\deg x = 2\) and \(\deg y = N\) is an odd number. Assume that \(I_k\), \(k = 1, \ldots, h - 1\) are integral ideals representing all non-trivial ideal classes with \(\deg I_k = d_k\) with order \(e_k\), as an element of the ideal class group. Further, assume that \(I_k^{e_k} = f_k\) is an irreducible polynomial and divides \(b(x)\). If \(N > 2\mu + e_k d_k\) for all \(k\), then \(\zeta(-es)\), where \(e\) is the exponent of the ideal class group and \(s\) is a positive integer, vanishes to order at least two, if \(l(es) \leq \mu\).

**Proof.** We have,
\[
\zeta(-es, X) = \sum_{d=0}^{\infty} X^d \sum_{\deg I = d} I^{es} = \sum_{d=0}^{\infty} X^d \sum_{\deg a = d} a^{es} + \sum_{k=1}^{h-1} \sum_{d=0}^{\infty} X^d \sum_{\deg I = d} I_k^{es}.
\]

Note that \(\mu < \frac{N-e_k d_k}{2} \leq \frac{N-1}{2} = g = \text{genus}\). Therefore, the principal part
\[
\sum_{d=0}^{\infty} X^d \sum_{\deg a = d} a^{es}
\]
contributes vanishing order two by Theorem 15. In fact,
\[
\sum_{d=0}^{\infty} X^d \sum_{\deg a = d} a^{es} = \zeta_{\mathbb{F}_2[x]}(-es, X^2).
\]
Also, \( T_k^{e_k} = f_k \) has deg = \( e_k d_k < N \), and so \( f_k = f_k(x) \), that is, \( f_k \in \mathbb{F}_2[x] \). Write \( \frac{e_k}{e_k} = s_k \). So, the \( k \)-term is

\[
\sum_{d=0}^\infty X^d \sum_{\text{deg } I = d} \mathcal{I}^{e_s} = \frac{1}{f_k^{s_k}} \sum_{d=0}^\infty X^d \sum_{\text{deg } I = d} (\mathcal{I} I)^{e_s}.
\]

(1)

We will show that this vanishes to order two for each \( k \) (hence, the total order of vanishing is at least two). Here \( f_k^{s_k} \) is independent of \( d \) and therefore, we can ignore it for vanishing considerations.

Now, \( \mathcal{I}_k \mathcal{I} \) are integers of degree \( d + d_k \) of the form \( f(x) + yg(x) \) if and only if \( f(x) + yg(x) \equiv 0 \pmod{f_k} \), that is,

\[
(f(x) + yg(x))^{e_k} = f^{e_k} + \sum_{j=1}^{e_k} \left( \binom{e_k}{j} f^{e_k-j} y^j g^j \equiv 0 \pmod{f_k} \right).\]

Then, if \( y^2 + a(x) y = b(x) \),

\[
y^2 \equiv a(x) y \pmod{f_k} \quad \text{therefore,} \quad y^j \equiv a(x)^j y \pmod{f_k} \quad (j \geq 1),
\]

but if \( y^2 + y = b(x) \) then

\[
y^2 \equiv y \pmod{f_k} \quad \text{therefore,} \quad y^j \equiv y \pmod{f_k} \quad (j \geq 1),
\]

and therefore

\[
\begin{cases}
(f(x) + yg(x))^{e_k} \equiv f^{e_k} + y \sum_{j=1}^{e_k} \binom{e_k}{j} f^{e_k-j} a^{j-1} g^j \pmod{f_k}, \\
n(x) + yg(x))^{e_k} \equiv f^{e_k} + y \sum_{j=1}^{e_k} \binom{e_k}{j} f^{e_k-j} g^j \pmod{f_k},
\end{cases}
\]

respectively. So, if this is congruent to zero \( \pmod{f_k} \) then \( f_k \mid f^{e_k} \) in both cases. But \( f_k \) being an irreducible polynomial in \( x \), this implies \( f_k \mid f \). Hence, \( f_k \mid yg^{e_k} \) and \( f_k \mid yg^{e_k} a^{e_k-1} \), respectively. When \( y^2 + y = b(x) \), this implies that \( f_k \mid g \) also. The same is true if \( y^2 + a(x) y = b(x) \), provided that gcd \( (f_k, a(x)) = 1 \) (note that \( e_k > 1 \)).

Under these circumstances, \( f(x)+yg(x) \) is an integer if and only if \( f_k \mid f(x) \). But, if \( f_k \mid a(x) \) then the quotient is an integer if and only if \( f_k \mid f(x) \) and \( g(x) \) can be anything. We consider \( f_k \mid g \) as the first case and when \( g \) arbitrary, the second case.

Therefore, in the two cases, we have, respectively,

\[
\sum_{\text{deg } I = d} (\mathcal{I}_k \mathcal{I})^{e_s} = \sum_{\text{deg } f_k(f+yg) = d+d_k} (f_k(f+yg))^{e_s},
\]
or,

\[
\sum_{\deg I = d} (I_kT)^{es} = \sum_{\deg (f_kT + y\bar{g}) = d+d_k} (f_kT + y\bar{g})^{es},
\]

where \( \bar{T} \) and \( \bar{g} \) are polynomials in \( x \).

We examine now, sufficient conditions for the vanishing of the previous sums. If \( 2\mu + e_kd_k < d + d_k < N, d + d_k - e_kd_k > 2\mu \) implies that the sums have more than \( \mu \) parameters and so the sums vanish. So, without loss of generality \( d + d_k \leq 2\mu + e_kd_k < N, \) and these are the only terms which can give non-zero contributions. As \( d + d_k < N, \) \( \bar{g} = 0 \). Then \( \deg \bar{T} = d + d_k - e_kd_k \leq 2\mu. \)

Therefore, the \( k \)th term of the zeta sum is the following:

\[
\sum_{d=0}^{\infty} X^d \sum_{\deg I = d} (I_kT)^{es} = X^{e_kd_k-d_k} f_k^{es} \left( 1 + \cdots + X^{2\mu} \sum_{a_{\mu-1}, \ldots, a_0 \in \mathbb{F}_2} (x^{\mu} + a_{\mu-1}x^{\mu-1} + \cdots + a_0)^{es} \right) \\
= f_k^{es} X^{e_kd_k-d_k} \left( 1 + \cdots + X^{2\mu} \sum_{a_{\mu-1}, \ldots, a_0 \in \mathbb{F}_2} (x^{\mu} + a_{\mu-1}x^{\mu-1} + \cdots + a_0)^{es} \right) \\
= f_k^{es} X^{e_kd_k-d_k} \sum_{d=0}^{\mu} X^{2d} \sum_{\deg f} \bar{f}^{es} \\
= f_k^{es} X^{e_kd_k-d_k} \zeta_{\mathbb{F}_2[x]}^{es} (-es, X^2).
\]

The last step follows from Proposition 9, since \( \mu \geq l(es) \) implies that

\[
\sum_{d=0}^{\mu} X^{2d} \sum_{\deg f} \bar{f}^{es} = \sum_{d=0}^{\infty} X^{2d} \sum_{\deg f} \bar{f}^{es} = \zeta_{\mathbb{F}_2[x]}^{es} (-es, X^2).
\]

Now, since the order of vanishing of \( \zeta_{\mathbb{F}_2[x]}^{es} (-es, X) \) is one, by Example 14, we have that, for each \( k \), the order of vanishing of the \( k \)th term (1) is two. \( \Box \)

**Remark 19.** Notice that the same proof above works if we only assume that \( f_k \) is square free and \( \gcd(f_k, a(x)) = 1 \) or \( f_k \mid a(x) \).
Remark 20. From the proof of the Theorem 18, it follows that when $l (es) \leq \mu$ we have

$$
\zeta (-es, X) = \zeta_{F_2[x]} (-es, X^2) \left( 1 + \sum_{k=1}^{h-1} \frac{e_k (e_k-1)}{f_k} X^{e_k d_k - d_k} \right).
$$

Then the order of vanishing is exactly two when

$$
1 + \sum_{k=1}^{h-1} \frac{e_k (e_k-1)}{f_k} \neq 0.
$$

In particular, when $h = 2$ the order of vanishing is exactly two.

We tried to see whether there are examples satisfying the conditions of Theorem 18 (see below for these examples). We look for low genus and low class number or, at least, a low exponent. So, we tried with quadratic extension of $F_2(x)$, in which the infinite place of $F_2(x)$ ramifies and the ideal class group has exponent two, which corresponds to the lowest possible $e_k$ in Theorem 18. We have that, because of [R1, Theorem 10, p. 168], for quadratic extensions $F_2(x, y)/F_2(x)$ in which the infinite place of $F_2(x)$ ramifies, with $y^2 + y = b(x)$, $b(x) \in F_2[x]$, the class number $h$ is odd, and those with $y^2 + a(x) y = b(x)$, $a(x), b(x) \in F_2[x]$, $\frac{b(x)}{a(x)^2}$ has non-trivial denominator, have even class number. So, in the search for exponent two examples, we must restrict ourselves to curves of the form $y^2 + a(x) y = b(x)$. A full list of quadratic extensions of $F_2(x)$, in which the infinite place of $F_2(x)$ ramifies and the ideal class group has exponent two, is given in [B-D].

For $g = 2$, we have,

Proposition 21. Let $g = 2$. If the class group of $A$ has exponent two and satisfies the conditions of the Theorem 18, then $h = 2$.

Proof. Since $g = 2$ and $\mu < g$, we must have that $\mu = 1$. For $\mu = 1$ we must have that $2g + 1 > 2 + e_k d_k$. Since the exponent is two, $e_k = 2$. So, $2g - 1 > 2d_k$, i.e., $d_k = 1$. But then $h = N_1$, where $N_d$ denote the number of places of degree $d$ of $K$. For $g = 2$,

$$
h = \frac{2N_2 + N_1^2 + N_1 - 4}{2}.
$$

From this expression for $h$ we get, under the assumption that $h = N_1$, that $2h = 2N_2 + h^2 + h - 4$. Therefore, $h \geq h^2 - 4$ since $2N_2 \geq 0$. The inequality implies that $h \leq 2$. Therefore, $h = 2$. □

Example 22. Let $A = F_2[x, y]/(y^2 + xy + x(x^4 + x + 1))$. Then, consider

$$
\mathcal{I}_1 = (x, y), \quad \deg \mathcal{I}_1 = 1, \quad e_1 = 2, \quad \mathcal{I}_1^2 = (x).
$$
This is Example 23 from [B-D]; there, we found that \( g = 2, \ h = 2, \) so \( e = 2. \) \( \mathcal{I}_1 \) is not a principal ideal, since in \( A \) we do not have elements of degree 1. Then, the conditions of the theorem are satisfied with \( \mu = 1 \) and so the order of vanishing is two for \( s = 2^n, \ n \geq 1 \) (see Remark 20). In fact, we can decide just from the values of the \( N_d \)'s that the conditions of the theorem are satisfied. For example, here \( N_1 = 2, \) so there is an ideal of degree one. Now, \( h = 2 \) implies \( e_k d_k \leq 2, \) as required. (We arrange so that \( \mathcal{I}_{ek} \) divides \( b(x) \)).

A similar analysis applies to the next two examples.

**Example 23.** Let \( A = \mathbb{F}_2 [x, y] / (y^2 + xy + x(x^4 + x^3 + x^2 + x + 1)) \). Then, consider

\[
\mathcal{I}_1 = (x, y), \quad \deg \mathcal{I}_1 = 1, \quad e_1 = 2, \quad \mathcal{I}_1^2 = (x).
\]

The conditions of the theorem are satisfied with \( \mu = 1, \) and so the order of vanishing is two for \( s = 2^n, \ n \geq 1. \)

**Example 24.** Let \( A = \mathbb{F}_2 [x, y] / (y^2 + x^2 y + x(x^3 + x^2 + x + 1)) \). Then, consider

\[
\mathcal{I}_1 = (x, y), \quad \deg \mathcal{I}_1 = 1, \quad e_1 = 2, \quad \mathcal{I}_1^2 = (x).
\]

The conditions of the theorem are satisfied with \( \mu = 1, \) and so the order of vanishing is two for \( s = 2^n, \ n \geq 1. \)

**Example 25.** Let \( A = \mathbb{F}_2 [x, y] / (y^2 + (x^2 + x + 1) y + (x^2 + x + 1)(x^5 + x^2 + 1)) \). Then, consider

\[
\mathcal{I}_1 = (x^2 + x + 1, y), \quad \deg \mathcal{I}_1 = 2, \quad e_1 = 2, \quad \mathcal{I}_1^2 = (x^2 + x + 1).
\]

This is Example 25 from [B-D]; there, we found that \( g = 3, \ h = 2, \) so \( e = 2. \) Notice that \( \mathcal{I}_1 \) is not a principal ideal, since in \( A \) the only elements of degree two are \( x \) and \( x + 1 \) and none of them divide \( y. \) The conditions of the theorem are satisfied with \( \mu = 1, \) and so the order of vanishing is two for \( s = 2^n, \ n \geq 1 \) (see Remark 20).

**Example 26.** Let \( A = \mathbb{F}_2 [x, y] / (y^2 + x(x + 1) y + x(x + 1)(x^5 + x^3 + x^2 + x + 1)) \). Then, consider

\[
\begin{aligned}
\mathcal{I}_1 &= (x, y), \quad \deg \mathcal{I}_1 = 1, \quad e_1 = 2, \quad \mathcal{I}_1^2 = (x), \\
\mathcal{I}_2 &= (x + 1, y), \quad \deg \mathcal{I}_2 = 1, \quad e_2 = 2, \quad \mathcal{I}_2^2 = (x + 1), \\
\mathcal{I}_3 &= \mathcal{I}_1 \mathcal{I}_2, \quad \deg \mathcal{I}_3 = 2, \quad e_3 = 2, \quad \mathcal{I}_3^2 = (x^2 + x).
\end{aligned}
\]

This is Example 32 from [B-D]; there, we found that \( g = 3, \ h = 4 \) and \( e = 2. \) Notice that \( \mathcal{I}_1 \) is not a principal ideal, since in \( A \) we do not have elements of degree 1. The
same comment applies to $I_2$. Also $I_3$ is not a principal ideal, since the only elements of $A$ of degree two are $x$ and $x + 1$. But $x + (x + 1)$ an element of $I_3$. The same is true for $x + 1$, $(x + 1)^3$ an element of $I_3$. Then, the conditions of the theorem are satisfied with $\mu = 1$, and so the order of vanishing is two for $s = 2^n$, $n \geq 1$(see Remark 20).

A similar analysis applies to the next example.

**Example 27.** Let $A = \mathbb{F}_2 [x, y] / (y^2 + x^2 + (x + 1)y + x(x + 1)(x^3 + x^2 + 1))$. Here $h = 4$, and $e = 2$ with

$$
\begin{align*}
I_1 &= (x, y) \quad \deg I_1 = 1, \quad e_1 = 2, \quad I_1^2 = (x), \\
I_2 &= (x + 1, y) \quad \deg I_2 = 1, \quad e_2 = 2, \quad I_2^2 = (x + 1), \\
I_3 &= I_1 I_2 \quad \deg I_3 = 2, \quad e_3 = 2, \quad I_3^2 = (x^2 + x).
\end{align*}
$$

Then, the conditions of the theorem are satisfied with $\mu = 1$, and so the order of vanishing is two for $s = 2^n$, $n \geq 1$.

**Remark 28.** By using the substitution $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{y}{x}$, the function fields corresponding to Examples 17, 22, and the function fields corresponding to Examples 23 and 24 are isomorphic and they involve a switch of the infinite places. But the respective rings $A$ of Examples 17 and 22 are not isomorphic, because their different exponents at the place at infinity are 2 and 4, respectively. The same comment applies to the rings $A$ of Examples 23 and 24. Their different exponents are 2 and 4, respectively. Now, using the substitution $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{y}{x}$ the function fields of Examples 26 and 27 are isomorphic and they involve a switch of the infinite places. But their respective rings $A$ are not isomorphic, because their different exponents at the infinite place are 4 and 2, respectively. We thank José Felipe Voloch for pointing this out.

**Remark 29.** If we consider the exponentiation of ideals defined by Goss and the corresponding definition of $\zeta_A (−s, X)$ then we have that its order of vanishing is the same that the order of vanishing of $\zeta_A (−ps, X)$. His definition coincides with ours when $s$ is a multiple of the exponent $e$. Hence, for the Examples 17, 22–27, we have that the order of vanishing is also 2 at $s = 1$.

Now, we include another example where extra vanishing also occurs. This example does not satisfy the hypothesis of the previous theorem.

**Example 30.** Let $A = \mathbb{F}_2 [x, y] / (y^2 + y + (x^2 + x + 1)(x^3 + x^2 + 1))$. Here $g = \frac{5 - 1}{2} = 2$. This is example $C_3$ of [Du1], with $h = 3$. Let

$$
\begin{align*}
I_1 &= (x^2 + x + 1, y) \quad \deg I_1 = 2, \quad I_1^2 = (x^3 + y + 1), \\
I_2 &= (x^2 + x + 1, y + 1) \quad \deg I_2 = 2, \quad I_2^2 = (x^3 + y).
\end{align*}
$$

Clearly, both ideals are not principal, since in $A$ the only elements of degree two are $x$ and $x + 1$ and none of them divide $y$. Notice that $I_1 \sim I_2^{-1}$. Assume that $I_1 \sim I_2$. 
Then $\mathcal{I}_1 \mathcal{I}_2$ and $\mathcal{I}_1 \mathcal{I}_2^{-1}$ are principals, and therefore also their product, a contradiction, because that would imply that the order of $\mathcal{I}_1 = 2$. So, $\mathcal{I}_1 \nmid \mathcal{I}_2$. Assume now that the ideal $\mathcal{I} \sim \mathcal{I}_1^{-1}$, then $\mathcal{I} = (f(x) + yg(x))$ and $\mathcal{I}^3 = \frac{(f(x)+yg(x))^3}{x^3+y+1} \in A$. But,

$$
\frac{(f(x) + yg(x))^3}{x^3+y+1} = \left( \frac{(x^2 + x + 1)^3 g(x)^3 + x (x^2 + x + 1)^2 f(x) g(x)^2}{x^2 + x + 1} \right) y \\
+ \left( \frac{(x+1) (x^2 + x + 1) f(x)^2 g(x) + f(x)^3}{x^2 + x + 1} \right) y \\
+ \left( \frac{x (x^2 + x + 1)^3 (x^3 + x^2 + 1) g(x)^3}{x^2 + x + 1} \right) \\
+ \left( \frac{(x+1) (x^2 + x + 1)^2 (x^3 + x^2 + 1) f(x) g(x)^2}{x^2 + x + 1} \right) \\
+ \left( \frac{(x^2 + x + 1) (x^3 + x^2 + 1) f(x)^2 g(x) + x^3 f(x)^3}{x^2 + x + 1} \right).
$$

Notice that the numerator of the previous expression is congruent to

$$
f(x)^3 \left( y + x^3 \right) \left( \mod (x^2 + x + 1) \right).
$$

Since $(x^2 + x + 1) \mid (y + x^3)$, then $\frac{(f(x)+yg(x))^3}{x^3+y+1}$ integral implies that $(x^2 + x + 1) \mid f(x)$. Conversely,

$$
\left( \frac{(x^2 + x + 1) f(x) + yg(x)}{x^3+y+1} \right)^3 = \left( \left( \frac{(x^2 + x + 1) g(x)^3 + xf(x) g(x)^2}{x^2 + x + 1} \right) y \\
+ \left( \frac{(x+1) f(x)^2 g(x) + f(x)^3}{x^2 + x + 1} \right) y \\
+ \left( \frac{x^4 + x^3 + x}{x^2 + x + 1} \right) g(x)^3 + x^3 f(x)^3 \\
+ \left( \frac{x^4 + x^2 + x + 1}{x^2 + x + 1} \right) f(x) g(x)^2 \\
+ \left( \frac{x^3 + x^2 + 1}{x^2 + x + 1} \right) f(x)^2 g(x) \right).
$$

Similarly, for $\mathcal{I}_2$, $\frac{(f(x)+(y+1)g(x))^3}{x^3+y}$ is integral if and only if $(x^2 + x + 1) \mid f(x)$. Now assume that $s = 3 \cdot 2^n$, $n = 0, 1, \ldots$. We are going to show that the order of vanishing of $\zeta_X(-s, X)$ for such an $s$ is 2.
**Principal part.** Notice that $\sum_{\deg f = d} f^s = 0$ when $\deg f > 4$ because of the Proposition 9. So the contribution to the zeta value is:

$$1 + X^2 (x^s + (x + 1)^s) + X^4 \left(x^{2s} + \left(x^2 + x\right)^s + \left(x^2 + x + 1\right)^s\right)$$

$$= 1 + X^2 \left(x^{s/3} (x + 1)^{s/3} + 1\right) + X^4 \left(x^{s/3} (x + 1)^{s/3}\right)$$

$$= \left(1 + X^2\right) \left(1 + X^2 \left(x^{s/3} (x + 1)^{s/3}\right)\right).$$

**Non-principal part.** The contribution is:

$$\sum_{d=0}^{\infty} X^d \left(\sum_{\deg I = d} I^s + \sum_{\deg I = d} I^s\right),$$

where $I \sim I^{-1}$ means that $II_1 = \left((x^2 + x + 1) f (x) + yg (x)\right)$ and similarly for $I_2$. So, the contribution is

$$\frac{1}{(x^3 + y + 1)^{s/3}} \sum_{d=0}^{\infty} X^d \sum_{\deg I = d} \left((x^2 + x + 1) f (x) + yg (x)\right)^s$$

$$+ \frac{1}{(x^3 + y)^{s/3}} \sum_{d=0}^{\infty} X^d \sum_{\deg I = d} \left((x^2 + x + 1) f (x) + (y + 1) g (x)\right)^s.$$

Notice that in this last expression,

$$\deg \left((x^2 + x + 1) f (x) + yg (x)\right) = \deg \left((x^2 + x + 1) f (x) + (y + 1) g (x)\right)$$

$$= d + 2.$$

So, if $d > 4$, both sums are zero and using expression (2), we obtain that the non-principal contribution is:

$$X^2 + X^3 \sum_{x \in \mathbb{F}_2} \left(x^2 + x + 1\right)^{s/3}$$

$$+ X^4 \sum_{x, \beta \in \mathbb{F}_2} \left(x^2 + x + 1\right) \beta^3 + x (x + x) \beta^2 + (x + 1) (x + x)^2 \beta + (x + x)^3,$$
that is,

\[ X^2 + X^4 = X^2 \left( 1 + X^2 \right). \]

Therefore, adding the principal and non-principal part we get that the order of vanishing is two.

However, by a straight-forward calculation, which we omit, for \( s = 9 \cdot 2^n, 15 \cdot 2^n, 21 \cdot 2^n, n \geq 0 \) the order of vanishing is one.

**Remark 31.** Notice that for \( e = 2^n, \) \( l(es) = l(s) \) and the condition in Theorem 18 can be written as: \( \zeta_A(-es) \) has order two if \( l(s) \leq \mu \) rather than \( l(es) \leq \mu. \) But, for \( e = 3 \) it is no longer true that \( l(es) = l(s). \) Here, it seems that the order of vanishing is two if \( l(s) \leq \mu, \) and not just when \( l(es) \leq \mu \) as, in the previous Example, the cases \( e \cdot s = 3 \cdot 1 \) and \( e \cdot s = 3 \cdot 3 \) show for \( \mu = 1. \)

Here we finished our search for examples with exponent two and \( \mu = 1. \) For \( \mu = 2, \) since \( \mu < g \) we must look at \( g \geq 3. \) In fact, if you restrict to quadratic extension of \( \mathbb{F}_2(x), \) in which the infinite place of \( \mathbb{F}_2(x) \) ramifies and the ideal class group has exponent two, the search is unnecessary, because of:

**Proposition 32.** For \( \mu = 2, \) there are no \( A \)’s of quadratic extensions of \( \mathbb{F}_2(x), \) in which the infinite place of \( \mathbb{F}_2(x) \) ramifies and the ideal class group has exponent two that satisfy the conditions of the Theorem 18.

**Proof.** Let \( g = 3. \) For \( \mu = 2 \) we must have that \( 2g + 1 > 4 + e_k d_k. \) Since the exponent is two, \( e_k = 2. \) So, \( 2g - 3 > 2d_k, \) i.e., \( d_k = 1. \) But then \( h = N_1. \) For \( g = 3, \)

\[ h = \frac{6N_3 + 6N_1 N_2 + N_1^3 + 3N_1^2 - 10N_1}{6}. \]

So,

\[ 6h = 6N_3 + 6h N_2 + h^3 + 3h^2 - 10h, \]

implying that \( 6h \geq h^3 + 3h^2 - 10h \) since \( 6N_3 + 6h N_2 \geq 0. \) This implies that \( h \leq 2. \) But, for \( h = 2, N_1 \) cannot be two. Since there are no quadratic extension of \( \mathbb{F}_2(x), \) in which the infinite place of \( \mathbb{F}_2(x) \) ramifies and the ideal class group has exponent two of \( g > 3 \) (see [B-D]), the result follows. \( \square \)

Finally, we present interesting applications of extra vanishing to the class groups of cyclotomic function fields.

We refer to Hayes in [H1] for basics of function field cyclotomic theory, and to the already mentioned books of Goss [G4] and Thakur [T3]. For a prime \( \mathcal{P} \) of \( A, \) let
$K(\mathcal{P})$ denote the $\mathcal{P}$-cyclotomic extension of $K$. This is obtained by adjoining to $K$ the $\mathcal{P}$-torsion of a sgn-normalized rank one Drinfeld $A$-module.

Let $C$ be the $p$-primary component of the class group for $K(\mathcal{P})$. If $C$ is a non-trivial group, then $\mathcal{P}$ is called an irregular prime, following [G2] in the case $K = \mathbb{F}_q(T)$ (see also the remark at the end of this part). Let $w$ be the generalized Teichmüller character of [G4, Chapter 8, Section 11] and $C(w^{-i})$ be the $i$th isotypic component. We recall the results of Goss-Sinnott [G-S]; [G4, Theorem 8.14.4] (specialized to our situation when $q = 2$, which is sufficient for our purposes).

**Theorem 33.** If $0 < i < 2^{\deg \mathcal{P}} - 1$ then,

$$C(w^{-i}) \neq \{0\} \text{ if and only if } \mathcal{P} \text{ divides } \zeta(-i, X) \bigg|_{X=1} = \frac{1}{1 + X}.$$

Now, if $i$ is such that there is extra vanishing, then

$$\zeta(-i, X) \bigg|_{X=1} = 0,$$

so all the primes divide it.

**Corollary 34.** If $\zeta(-i, X)$ has extra vanishing, and $\mathcal{P}$ is such that $0 < i < 2^{\deg \mathcal{P}} - 1$, then $C(w^{-i}) \neq \{0\}$. In particular, $\mathcal{P}$ is an irregular prime.

In particular, for Example 16 all the $\mathcal{P}$ are irregular. In Examples 17, 22–27, all $\mathcal{P}$ of degree greater than 1 are irregular. In Example 30, all $\mathcal{P}$ of degree greater than 2 are irregular. (We have not checked the regularity of degree 1 and 2 primes in these examples.)

**Remark 35.** We have used the notion of irregularity as defined in [G2], where Goss showed that regular primes behave in a similar way for his analog of Fermat equation. But, it may be better to say that $\mathcal{P}$ is regular if and only if $p$ (the characteristic) does not divide the order of the class group of the ring of integers (above $A$) of $K(\mathcal{P})$. Then $\mathcal{P}$ is regular in the sense of [G2] implies that $\mathcal{P}$ is regular in the new sense, but the converse may not be true (see also [T1, p. 163] for more on the subject). In fact, Madan [M1] showed that if $L/K$ is a Galois extension, then $h_K$ divides $h_L$ (actually, being Galois is not necessary, see [T3, p. 17]). The cyclotomic extensions are Galois, so if $p|h_K$ (which is not the case in our examples when $h = 1$ or $h = 3$), then all $\mathcal{P}$ are automatically irregular, by definition. But even in that case, our result showing $\mathcal{P}$-divisibility of specific class group components is much stronger.
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