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On $\mathscr{F}$-Subnormal Subgroups and $\mathscr{F}$-Residuals of Finite Soluble Groups

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1. INTRODUCTION AND STATEMENT OF RESULTS

All groups that we consider are finite and soluble.

Recall that a formation is a class of groups which is closed under homomorphic images and subdirect products. Hence, if $\mathscr{F}$ is a formation and G is a group which is a direct product of the subgroups A and B, then G is in $\mathscr{F}$ if and only if A and B lie in $\mathscr{F}$. More generally, Doerk and Hawkes [4, IV, 1.18] proved that if G is a group such that $G = A \times B$, then $G^{\mathscr{F}} = A^{\mathscr{F}} \times B^{\mathscr{F}}$, where $G^{\mathscr{F}}$ is the $\mathscr{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathscr{F}$. The main purpose of this paper is the development of this result by means of the concept of $\mathscr{F}$-subnormal subgroup.

Suppose that $\mathscr{F}$ is a saturated formation. A maximal subgroup M of a group G is called $\mathscr{F}$-normal in G if $G/\mathrm{Core}_G(M) \in \mathscr{F}$. A subgroup H of G is called $\mathscr{F}$-subnormal in G if either $H = G$ or there exists a chain $H = H_0 \le H_1 \le \cdots \le H_n = G$ such that H_i is an $\mathscr{F}$-normal maximal subgroup of H_{i+1} for $0 \le i < n$. It is clear that if $\mathscr{F} = \mathscr{N}$, the saturated formation of all nilpotent groups, the $\mathscr{F}$-subnormal subgroups of G are exactly the subnormal subgroups of G.

Let $\mathscr{F}$ be a subgroup-closed saturated formation containing $\mathscr{N}$. It is rather easy to see that if $\mathscr{F}$ is closed under the product of normal subgroups, then $G^{\mathscr{F}} = A^{\mathscr{F}}B^{\mathscr{F}}$ for every pair of subnormal subgroups A and B such that $G = AB$. This result does not remain true if A and B are

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$\mathscr{F}$-subnormal in G. Consequently, a natural question arises:

Which are the subgroup-closed saturated formations $\mathscr{F}$ of full characteristic satisfying the following property: if $G = HK$ and H and K are $\mathscr{F}$-subnormal subgroups of G, then $G^{\mathscr{F}} = H^{\mathscr{F}}K^{\mathscr{F}}$? We have already some information about this question. The first author [3] proves the following result:

THEOREM A. *Let $\mathscr{F}$ be a subgroup-closed saturated formation of soluble groups containing $\mathscr{N}$. The following statements are pairwise equivalent*:

(1) *$\mathscr{F}$ satisfies the property*

(*) *If H and K are two $\mathscr{F}$-subnormal $\mathscr{F}$-subgroups of a group G and $G = HK$, then $G \in \mathscr{F}$.*

(2) *$\mathscr{F}$ satisfies the property*

(**) *If H is an $\mathscr{F}$-subnormal $\mathscr{F}$-subgroup of a group G and K is a normal $\mathscr{F}$-subgroup of G and $G = HK$, then $G \in \mathscr{F}$.*

(3) *If F is the integrated and full formation function defining $\mathscr{F}$, then for each prime number p, $F(p) = \mathscr{F} \cap \mathscr{S}_{\pi(p)}$ where $\pi(p) = \operatorname{char} F(p)$.*

(4) *For each prime number p, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathscr{F}$ is locally defined by the formation function f given by $f(p) = \mathscr{S}_{\pi(p)}$.*

Not every member of the family described in the aforementioned theorem satisfies the above property. In fact we prove:

THEOREM 1. *Let $\mathscr{F}$ be a subgroup-closed saturated formation containing $\mathscr{N}$. The following statements are pairwise equivalent*:

(1) *$\mathscr{F}$ satisfies the property*

(*) *If H and K are two $\mathscr{F}$-subnormal subgroups of a group G and $G = HK$, then $G^{\mathscr{F}} = H^{\mathscr{F}}K^{\mathscr{F}}$.*

(2) *$\mathscr{F}$ satisfies the property*

(**) *If H is an $\mathscr{F}$-subnormal subgroup of a group G and K is a normal subgroup of G and $G = HK$, then $G^{\mathscr{F}} = H^{\mathscr{F}}K^{\mathscr{F}}$.*

(3) *For each prime number p, there exists a set of primes $\pi(p)$ with $p \in \pi(p)$ such that $\mathscr{F}$ is locally defined by the formation function f given by $f(p) = \mathscr{S}_{\pi(p)}$. These sets of primes satisfy the following property*: *If $q \in \pi(p)$, then $\pi(q) \subseteq \pi(p)$ for every pair of prime numbers p, q.*

Remark. The saturated formations $\mathscr{F}$ described in statement (3) of Theorem 1 are a generalization of the formations of all nilpotent and p-nilpotent groups (p a prime) in the following sense:

Let $\mathscr{F}$ be a saturated formation of soluble groups containing $\mathscr{N}$ and locally defined as in Theorem 1(3). Then a group $G \in \mathscr{F}$ if and only if G has a normal Hall $\pi(p)'$-subgroup for every prime number p.

Proof. Assume $G \in \mathscr{F} = \bigcap_{p \in P} \mathscr{S}_{p'}\mathscr{S}_p\mathscr{S}_{\pi(p)}$. Then as $p \in \pi(p)$, we have $G \in \bigcap_{p \in P} \mathscr{S}_{p'}\mathscr{S}_{\pi(p)}$. Now if $q \in \pi(p)$ then $\pi(q) \subseteq \pi(p)$ by the hypotheses. So for every prime number p, we have that $\mathscr{F} \subseteq \bigcap_{q \in \pi(p)} \mathscr{S}_{q'}\mathscr{S}_{\pi(p)}$. Consequently $O^{\pi(p)}(G) \leq \bigcap_{r \in \pi(p)} \mathscr{S}_{r'} = \mathscr{S}_{\pi(p)'}$ and we are done.

On the other hand consider $\mathscr{H} = \bigcap_{p \in P} \mathscr{S}_{\pi(p)'}\mathscr{S}_{\pi(p)}$ the class of groups such that $G \in \mathscr{H}$ if G possesses a normal $\pi(p)'$-Hall subgroup for all p. Take $G \in \mathscr{H} \setminus \mathscr{F}$ a group of minimal order. Then $G = NM$ is a primitive group with $\mathrm{Soc}(G) = N$ a minimal normal subgroup of G and M a maximal subgroup of G with $\mathrm{Core}_G(M) = 1$. Assume $p \mid |N|$. By minimality of G we have $M \in \mathscr{F}$. Now $G \in \mathscr{S}_{\pi(p)'}\mathscr{S}_{\pi(p)}$, so $G \in \mathscr{S}_{\pi(p)}$. In particular, $M \in \mathscr{S}_{\pi(p)} = f(p)$, and $G \in \mathscr{F}$, a contradiction.

Let $\mathscr{F}$ be a subgroup-closed saturated formation containing $\mathscr{N}$ and let F be the full and integrated local formation function defining $\mathscr{F}$. In [2, Theorem 3.3] it is proved that the set of all $\mathscr{F}$-subnormal subgroups is a lattice for every group if and only if F can be described in the following way:

There exists a partition $\{\pi_i\}_{i \in L}$ of the set of all prime numbers, such that $F(p) = \mathscr{S}_{\pi_i}$, for every prime number $p \in \pi_i$ and for every $i \in L$.

We refer to this family of saturated formations as lattice-formations. The members of this family also enjoy the following property [2, Theorem 4.1]: if H and K are two $\mathscr{F}$-subnormal $\mathscr{F}$-subgroups of G, then $\langle H, K \rangle \in \mathscr{F}$. This result motivates the following questions:

Suppose that $\mathscr{F}$ is a lattice-formation. Is it true that $G^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}} \rangle$ if H and K are two $\mathscr{F}$-subnormal subgroups of G such that $G = \langle H, K \rangle$? In the affirmative case, is it a characterization of the members of this family?

The following theorem contains the answer to the aforesaid questions.

THEOREM 2. *Let $\mathscr{F}$ be a subgroup-closed saturated formation containing $\mathscr{N}$, the class of all nilpotent groups. Then the following statements are pairwise equivalent*:

(1) *If H is an $\mathscr{F}$-subnormal subgroup of a group G then*:

(*) $\langle H, H^g \rangle^{\mathscr{F}} = \langle H^{\mathscr{F}}, (H^g)^{\mathscr{F}} \rangle$ *for all* $g \in G$.

(2) *Let F be the full and integrated local formation function defining $\mathscr{F}$. Then there exists a partition $\{\pi_i\}_{i \in L}$ of the set of all primes, such that $F(p) = \mathscr{S}_{\pi_i}$, for every prime number $p \in \pi_i$ and for every $i \in L$.*

(3) *If H and K are two $\mathscr{F}$-subnormal subgroups of a group G, then*:

(**) $\langle H, K\rangle^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$.

We use conventional notions and notation. They have been taken from the book of Doerk and Hawkes [4].

In order to prove our theorems we need the following preliminary results:

LEMMA A [4, A, 14.3]. *If H is a subnormal subgroup of a finite group G, then* Soc(G) *normalizes H*.

LEMMA B [5, Lemma 1.1]. *Let $\mathscr{F}$ be a subgroup-closed saturated formation. If H is $\mathscr{F}$-subnormal in G and $H \leq U \leq G$, then H is $\mathscr{F}$-subnormal in U*.

LEMMA C [4, IV, 3.16]. *Let c be one of the closure operations s, s_n, or N_0. Let $\mathscr{F} = LF(F)$, and assume that $\mathscr{F} = c\mathscr{F}$. Then $F(p) = cF(p)$ for all prime numbers p*.

LEMMA D [1, Lemma 1.3.2]. *Let the finite group $G = AB$ be the product of two subgroups A and B. If A, B, and G are D_π-groups for a set π of primes, then there exist Hall π-subgroups A_0 of A and B_0 of B such that $A_0 B_0$ is a Hall π-subgroup of G*.

2. PROOFS

Proof of Theorem 1. It is clear that (1) implies (2) because $\mathscr{N}$ is contained in $\mathscr{F}$ and so every subnormal subgroup of a group G is $\mathscr{F}$-subnormal in G.

(2) implies (3). By Theorem A we have that for each prime p, there exists a set of primes $\pi(p)$, with $p \in \pi(p)$, such that $\mathscr{F}$ is locally defined by the formation function f given by $f(p) = \mathscr{S}_{\pi(p)}$. Assume now that p and q are two prime numbers such that $q \in \pi(p)$. Suppose that there exists a prime $r \in \pi(q) \setminus \pi(p)$ (notice that $p \neq r \neq q$). Let V_q be an irreducible and faithful C_r-module over $GF(q)$. Denote by $X = [V_q]C_r$ the corresponding semidirect product. Then $X \in \mathscr{F}$. By [4, B, 11.7] we can take an irreducible and faithful X-module V_p over $GF(p)$ such that the trivial C_r-module is a quotient module of $(V_p)_{C_r}$. If $G = [V_p]X$ denotes the corresponding semidirect product, we have $G^{\mathscr{F}} = V_p$. Consider the normal subgroup $K = [V_p]V_q$ of G and $H = [V_p]C_r$. It is clear that H is an $\mathscr{F}$-subnormal subgroup of G. Moreover since $q \in \pi(p)$, we have $K \in \mathscr{F}$. But $H^{\mathscr{F}} \leq [V_p, C_r] < V_p$, a contradiction with the fact $G^{\mathscr{F}} = H^{\mathscr{F}}K^{\mathscr{F}}$.

(3) implies (1). We see that $\mathscr{F}$ verifies condition (*).

Suppose not and take G of minimal order among the groups X having two $\mathcal{F}$-subnormal subgroups A and B with $X = AB$ such that $X^{\mathcal{F}} \neq A^{\mathcal{F}}B^{\mathcal{F}}$. Then there exist two $\mathcal{F}$-subnormal subgroups H and K of G such that $G = HK$ and $G^{\mathcal{F}} \neq H^{\mathcal{F}}K^{\mathcal{F}}$. We choose a pair (H, K) such that $|H| + |K|$ is maximal.

Let M be an $\mathcal{F}$-normal maximal subgroup of G containing H. Then $M = H(K \cap M)$. Now H and $K \cap M$ are $\mathcal{F}$-subnormal subgroups of M. By minimality of G we have that $M^{\mathcal{F}} = H^{\mathcal{F}}(K \cap M)^{\mathcal{F}}$. Suppose that H is a proper subgroup of M. Since $G = MK$, the maximality of the pair (H, K) yields $G^{\mathcal{F}} = M^{\mathcal{F}}K^{\mathcal{F}} = H^{\mathcal{F}}K^{\mathcal{F}}$, a contradiction. So $H = M$ is an $\mathcal{F}$-normal maximal subgroup of G. Arguing in a similar way with K we obtain that both H and K can be assumed to be maximal $\mathcal{F}$-normal subgroups of G. Thus $G^{\mathcal{F}} \leq H \cap K$ and we have $H^{\mathcal{F}}K^{\mathcal{F}}$ is a normal subgroup of $G^{\mathcal{F}}$. On the other hand, by the minimality of G it follows that $\operatorname{Soc}(G) \leq G^{\mathcal{F}}$ and $G^{\mathcal{F}} = H^{\mathcal{F}}K^{\mathcal{F}}N$ for each minimal normal subgroup of G. So if N is a minimal normal subgroup of G and p is the prime dividing $|N|$, we can assume that $G^{\mathcal{F}}$ is a p-group. Otherwise $1 \neq O^p(G^{\mathcal{F}}) \leq H^{\mathcal{F}}K^{\mathcal{F}}$. Since $O^p(G^{\mathcal{F}})$ is normal in G, there would exist a minimal normal subgroup of G contained in $H^{\mathcal{F}}K^{\mathcal{F}}$ and we would be done. Moreover $\operatorname{Core}_G(H^{\mathcal{F}}K^{\mathcal{F}}) = 1$.

We reach the contradiction by proving $G \in \mathcal{F}$. To see this, we prove that G has a normal Hall $\pi(q)'$-subgroup for every prime q.

Let q be a prime such that $p \in \pi(q)$. Since $G/G^{\mathcal{F}} \in \mathcal{F}$ and Lemma D, we have that $T = G^{\mathcal{F}}H_{\pi(q)'}K_{\pi(q)'}$ is a normal subgroup of G, where $H_{\pi(q)'}$ is a Hall $\pi(q)'$-subgroup of H and $K_{\pi(q)'}$ is a Hall $\pi(q)'$-subgroup of K such that $H_{\pi(q)'}K_{\pi(q)'} = K_{\pi(q)'}H_{\pi(q)'}$ is a Hall $\pi(q)'$-subgroup of G. In particular $TK = H_{\pi(q)'}K$ and $TH = K_{\pi(q)'}H$ are subgroups of G. We distinguish two cases:

Case 1. $H_{\pi(q)'} \leq K$ or $K_{\pi(q)'} \leq H$.

If $H_{\pi(q)'} \leq K$, then T is a subgroup of K and $T \leq \operatorname{Core}_G(K)$. Hence $T^{\mathcal{F}} \leq (\operatorname{Core}_G(K))^{\mathcal{F}} \leq K^{\mathcal{F}}$. Since $(\operatorname{Core}_G(K))^{\mathcal{F}}$ is contained in $\operatorname{Core}_G(H^{\mathcal{F}}K^{\mathcal{F}}) = 1$, it follows that $T^{\mathcal{F}} = 1$ and so T belongs to $\mathcal{F}$. Therefore $H_{\pi(q)'}K_{\pi(q)'} = K_{\pi(q)'}$ is a characteristic subgroup of the normal subgroup T. In particular $G_{\pi(q)'} = K_{\pi(q)'}$ is normal in G and G has a normal Hall $\pi(q)'$-subgroup. If $K_{\pi(q)'} \leq H$, the result follows analogously.

Case 2. $H_{\pi(q)'}$ is not contained in K and $K_{\pi(q)'}$ is not contained in H.

In this case $G = TH = TK$ because H and K are maximal subgroups of G. Since $H/H^{\mathcal{F}}$ belongs to $\mathcal{F}$, we have that $H^{\mathcal{F}}H_{\pi(q)'}$ is a normal subgroup of H. Hence $[H^{\mathcal{F}}K^{\mathcal{F}}, H_{\pi(q)'}] \leq [G^{\mathcal{F}}, H_{\pi(q)'}] \leq G^{\mathcal{F}} \cap H^{\mathcal{F}}H_{\pi(q)'} = H^{\mathcal{F}}$ because $G^{\mathcal{F}}$ is a normal subgroup of G such that $H^{\mathcal{F}} \leq G^{\mathcal{F}} \leq H$ and $G^{\mathcal{F}}$ is a p-group. This means that $H_{\pi(q)'}$ normalizes $H^{\mathcal{F}}K^{\mathcal{F}}$. Analogously

$K_{\pi(q)'}$ normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$. Since $G^{\mathscr{F}}$ normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$, then so does T. Since $G = TK = TH$, it follows that $H^{\mathscr{F}}K^{\mathscr{F}}$ is a normal subgroup of G. This means that $H^{\mathscr{F}}K^{\mathscr{F}} = 1$ because $\mathrm{Core}_G(H^{\mathscr{F}}K^{\mathscr{F}}) = 1$. In particular H and K belong to $\mathscr{F}$. Applying Theorem A, we have $G \in \mathscr{F}$ and G has a normal Hall $\pi(q)'$-subgroup.

Now let a prime r be such that p does not belong to $\pi(r)$. Then p belongs to $\pi(r)'$ and since $G^{\mathscr{F}}$ is a p-group, it follows that G has a normal Hall $\pi(r)'$-subgroup.

Consequently G belongs to $\mathscr{F}$ and $1 = G^{\mathscr{F}} = H^{\mathscr{F}}K^{\mathscr{F}}$, final contradiction.

Proof of Theorem 2. It is clear that (3) implies (1). We prove (1) implies (2). We split the proof into the following steps.

(a) For each prime number p, every primitive group $G \in \mathscr{F} \cap b(F(p))$ is cyclic.

It is clear that G has a unique minimal normal subgroup N, and evidently N must be a q-group, where $q \neq p$, and q is a prime number. Therefore there exists an irreducible and faithful G-module V_p over $GF(p)$. We claim that G has a unique maximal subgroup M such that $\mathrm{Core}_G(M) = 1$, which provides the result.

Assume that M_1 and M_2 are maximal subgroups of G, $M_1 \neq M_2$ and $\mathrm{Core}_G(M_i) = 1$, $i = 1, 2$. Then $M_i \in F(p)$. Consider now the semidirect product $H = [V_p]G$ with respect to the action of G on V_p. Clearly H does not belong to $\mathscr{F}$, so $H^{\mathscr{F}} = V_p$. On the other hand, for $i = 1, 2$, V_pM_i is an $\mathscr{F}$-normal maximal subgroup of H and $V_pM_i \in \mathscr{S}_pF(p) = F(p) \subseteq \mathscr{F}$. Now $H = \langle V_pM_1, V_pM_2 \rangle = \langle V_pM_1, (V_pM_1)^g \rangle$ for some $g \in H$. So by (*) we have $H^{\mathscr{F}} = \langle (V_pM_1)^{\mathscr{F}}, ((V_pM_1)^g)^{\mathscr{F}} \rangle = 1$, a contradiction.

Arguing as in (2) and (3) of [2, Theorem 3.3] we obtain that if p and q are two prime numbers such that $p \in \mathrm{char}\, F(q)$, then $\mathrm{char}\, F(p) = \mathrm{char}\, F(q)$.

(b) If p, q are two prime numbers and $p \in \mathrm{char}\, F(q)$, then $\mathscr{S}_p \subseteq F(q)$. Assume there exists P a p-group such that P does not belong to $F(q)$ and suppose p^s is the exponent of the abelian p-group P/P'. Consider $Q = P \wr R$, where $R = \langle (1, 2, \ldots, p^s) \rangle$ is a cyclic group of order p^s regarded as a subgroup of the symmetric group of degree p^s. The above wreath product is taken with respect to the natural permutation representation of R of degree p^s. It is clear that Q is a p-group. Set $D = \{(a, a, \ldots, a)/a \in P\}$ the diagonal subgroup of $P^{\#}$, the base group of Q. Since $a^{p^s} \in P'$, we have that D is contained in $[P^{\#}, R]$ by [4, A, 18.4]. In particular $D \leq Q'$ and D is isomorphic to P. So what we have proved is that there exists a p-group Q such that $P \leq Q'$. Let us now consider the regular wreath product $W_1 = Q \wr C_p$ of Q with C_p and denote by V_q an arbitrary faithful W_1-module over $GF(q)$. If $W = [V_q]W_1$ denotes the corresponding semidirect product, it is clear that W does not belong to $\mathscr{F}$, otherwise we would have $P \in F(q)$, a contradiction. Since W_1 is p-group, we have that $W_1 \in \mathscr{F}$ and $W^{\mathscr{F}} \leq V_q$. Consider now the subgroup of W, $H = V_q C_p$. We have $O_{q'q}(H) = V_q$ and $p \in \operatorname{char} F(q)$, so $H \in \mathscr{F}$. If we take $g \in W$, we have that H and H^g are two $\mathscr{F}$-subnormal $\mathscr{F}$-subgroups of W. By condition (*), it follows that $\langle H, H^g \rangle^{\mathscr{F}} = \langle H^{\mathscr{F}}, (H^g)^{\mathscr{F}} \rangle = 1$, and $\langle H, H^g \rangle \in \mathscr{F}$.

Repeated application of this argument shows that the normal closure of H in W, $\langle H^W \rangle$, belongs to $\mathscr{F}$. Now $\langle H^W \rangle = V_q \langle C_p^W \rangle$ implies that $\langle C_p^W \rangle \in F(q)$. Finally since $F(q)$ is a subgroup-closed formation and Q' is isomorphic to a subgroup of $\langle C_p^{W_1} \rangle$, we have $P \in F(q)$, a contradiction.

(c) If p, q are two prime numbers and $p \in \operatorname{char} F(q)$, then $\mathscr{S}_p F(q) = F(q)$. Assume that G is a group of minimal order in $\mathscr{S}_p F(q) \setminus F(q)$. Then G has a unique minimal normal subgroup N, $G/N \in F(q)$, and N is a p-group. Suppose now $G \in \mathscr{F}$. If M is a maximal subgroup of G and $g \in G$ such that $M \neq M^g$, we have M and M^g belong to $F(q)$. Now if V_q is an irreducible and faithful G-module over $GF(q)$ and $H = [V_q]G$ denotes the corresponding semidirect product, it is clear that H does not belong to $\mathscr{F}$. Therefore $H^{\mathscr{F}} = V_q$. Moreover $V_q M \in \mathscr{S}_q F(q) = F(q) \subseteq \mathscr{F}$. So $V_q M$ and $V_q M^g$ are two $\mathscr{F}$-normal $\mathscr{F}$-subgroups of H such that $H = \langle V_q M, (V_q M)^g \rangle$. By condition (*), $H \in \mathscr{F}$, a contradiction. Hence every maximal subgroup of G is normal in G. Hence G is a nilpotent group. Consequently G is a p-group and then $G \in F(q)$ because of (b).

Therefore G does not belong to $\mathscr{F}$. In particular, N is not contained in the Frattini subgroup of G. Thus there exists a maximal subgroup R of G such that $G = NR$, $R \in F(q)$, and $G^{\mathscr{F}} = N$. Now R must be again a nilpotent group by the above argument. So $R = P_1 \times P_2 \times \cdots \times P_r$, where $P_i \in \operatorname{Syl}_{p_i}(R)$ and $\pi(R) = \{p_1, \dots, p_r\}$. Now if $\pi(p) = \operatorname{char} F(p)$ for each prime p, we have $R \in \mathscr{S}_{\pi(q)} = \mathscr{S}_{\pi(p)}$. So $\pi(R)$ is contained in $\operatorname{char} F(p)$. Therefore, by (b), $R \in F(p)$ and $G \in \mathscr{S}_p F(p) = F(p) \subseteq \mathscr{F}$, a contradiction.

(d) If p is a prime number and $\pi = \operatorname{char} F(p)$, then $F(p) = \mathscr{S}_\pi$.

Since $F(p)$ is a subgroup-closed formation, it is clear that $F(p) \subseteq \mathscr{S}_\pi$. On the other hand, assume that $F(p) \neq \mathscr{S}_\pi$ and let $G \in \mathscr{S}_\pi \setminus F(p)$ be a group of minimal order. Let N be a minimal normal subgroup of G. Then N is a q-group, for some $q \in \pi$, and $G/N \in F(p)$. Thus $G \in \mathscr{S}_q F(p) = F(p)$ because of (c), a contradiction.

(2) implies (3). Suppose now there exists a partition $\{\pi_i\}_{i \in L}$ of the set of all prime numbers, such that $F(p) = \mathscr{S}_{\pi_i}$, for every prime $p \in \pi_i$ and for every $i \in L$. We see that $\mathscr{F}$ verifies condition (**).

Suppose not and take G a group of minimal order among the groups X having two $\mathscr{F}$-subnormal subgroups A and B such that $\langle A, B\rangle^{\mathscr{F}} \neq \langle A^{\mathscr{F}}, B^{\mathscr{F}}\rangle$.

Then there exist two $\mathscr{F}$-subnormal subgroups H and K of G such that $\langle H, K\rangle^{\mathscr{F}} \neq \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$. Choose H and K with $|H| + |K|$ maximal. Assume that $\langle H, K\rangle$ is a proper subgroup of G. Since H and K are also $\mathscr{F}$-subnormal subgroups of $\langle H, K\rangle$, it follows that $\langle H, K\rangle^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$, by minimality of G. So we may assume $G = \langle H, K\rangle$. Again by minimality of G, we can deduce easily that $\mathrm{Soc}(G) \leq G^{\mathscr{F}}$ and $G^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle N$ for each minimal normal subgroup N of G. So we may assume $\mathrm{Core}_G(\langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle) = 1$. On the other hand, since H and K are $\mathscr{F}$-subnormal subgroups of G, it follows by [2, Lemma 3.1] that $\langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$ is a subnormal subgroup of G. Therefore $N \leq N_G(\langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle)$ for each minimal normal subgroup of G. Consequently $\mathrm{Soc}(G)$ normalizes $\langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$. This implies that $\langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$ is a normal subgroup of $G^{\mathscr{F}}$.

Arguing as in Theorem 1 we also have that $G^{\mathscr{F}}$ is a p-group for some prime number p.

Assume now that H is a proper subgroup of $HG^{\mathscr{F}}$. Since $HG^{\mathscr{F}}$ and K are $\mathscr{F}$-subnormal subgroups of G and $|HG^{\mathscr{F}}| + |K| > |H| + |K|$ it follows that $G^{\mathscr{F}} = \langle (HG^{\mathscr{F}})^{\mathscr{F}}, K^{\mathscr{F}}\rangle$ by the choice of the pair (H, K). Now H and $G^{\mathscr{F}}$ are two $\mathscr{F}$-subnormal subgroups of $HG^{\mathscr{F}}$. Theorem 1 implies that $(HG^{\mathscr{F}})^{\mathscr{F}} = H^{\mathscr{F}}$ and $G^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle$, a contradiction.

We may assume that $G^{\mathscr{F}}$ is contained in $H \cap K$. In particular $H^{\mathscr{F}}K^{\mathscr{F}}$ is a subgroup of $G^{\mathscr{F}}$. Let X be a Hall $\pi(p)$-subgroup of G. Since $G/G^{\mathscr{F}} \in \mathscr{F}$, it follows that $X/G^{\mathscr{F}}$ is a normal subgroup of $G/G^{\mathscr{F}}$, by [2, Lemma 3.2], and so X is a normal subgroup of G.

Suppose that H is a proper subgroup of HX. Since HX is an $\mathscr{F}$-subnormal subgroup of G, it follows that $G^{\mathscr{F}} = \langle (HX)^{\mathscr{F}}, K^{\mathscr{F}}\rangle$ by the choice of the pair (H, K). Again $(HX)^{\mathscr{F}} = H^{\mathscr{F}}$ by Theorem 1 because X belongs to $\mathscr{F}$. This means that $G^{\mathscr{F}} = \langle H^{\mathscr{F}}, K^{\mathscr{F}}\rangle = H^{\mathscr{F}}K^{\mathscr{F}}$, a contradiction. Therefore we may assume that X is contained in $H \cap K$. In particular X normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$. On the other hand, by [2, Lemma 3.2], if $H_{\pi(p)'}$ is a Hall $\pi(p)'$-subgroup of H, it follows that $H_{\pi(p)'}H^{\mathscr{F}}$ is a normal subgroup of H. Consequently $H^{\mathscr{F}}K^{\mathscr{F}}$ is a Hall $\pi(p)$-subgroup of $T = (H^{\mathscr{F}}K^{\mathscr{F}})H_{\pi(p)'}$. Now, since $T/H^{\mathscr{F}}$ belongs to $\mathscr{F}$, we have that $H^{\mathscr{F}}K^{\mathscr{F}}$ is a normal subgroup of T by [2, Lemma 3.2]. Hence $H_{\pi(p)'}$ normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$. Analogously $K_{\pi(p)'}$, a Hall $\pi(p)'$-subgroup of K, normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$. Therefore $G = \langle H, K\rangle$ normalizes $H^{\mathscr{F}}K^{\mathscr{F}}$ and $H^{\mathscr{F}}K^{\mathscr{F}}$ is a normal subgroup of G. From the fact $\mathrm{Core}_G(H^{\mathscr{F}}K^{\mathscr{F}}) = 1$, we deduce that H and K belong to $\mathscr{F}$. By [2, Theorem 4.1], G belongs to $\mathscr{F}$, final contradiction.

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