A Characterization of Weak-Monotone Matrices

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ABSTRACT

We provide some characterizations of weak-monotone matrices by using positive splittings. We prove that a matrix allowing a generalized $B$-splitting is weak-monotone iff the associated spectral radius is less than one. Next, we prove that weak-monotone matrices allow a generalized $B$-splitting. Weak-monotonicity is associated with strong positive solvability of a linear system $Ax = b$, $b \geq 0$.

1. INTRODUCTION

An $m \times n$ matrix $M$ is called (rectangular) monotone if, for all $x \in \mathbb{R}^n$ such that $Mx \geq 0$, one has $x \geq 0$. In case $m = n$, that is, when $M$ is a square matrix, monotonicity and inverse-positiveness turn out to be equivalent (see Collatz [8]). Some generalizations of the concept of monotonicity can be found in Berman and Plemmons [3, 6]. The most general (see [17] for graph and examples) is the class of weak-monotone matrices. We say that an $m \times n$ matrix is weak-monotone if, for all $x \in \mathbb{R}^n$ such that $Mx \geq 0$, there exists $y \in \mathbb{R}^n_+$ satisfying $Mx = My$. In Ben-Israel and Greville [2] weak-monotonicity is defined as

$$Mx \geq 0 \quad \Rightarrow \quad x \in \mathbb{R}^n_+ + N(M),$$

which is an equivalent condition. Obviously, any monotone matrix is weak-monotone, and in case $\text{rk}(M) = n$, the two conditions coincide.
The general interest of weak-monotone matrices comes from the equivalence between the weak-monotonicity of the matrix $M$ and the strong positive solvability of system

$$Mx = d, \quad d \in \text{Im}(M) \cap \mathbb{R}_+^n$$

[i.e., the existence of positive solutions $x \geq 0$ for any $d \geq 0, d \in \text{Im}(M)$].

The main purpose of this paper is to characterize weak-monotone matrices. A way of characterizing some types of monotonicity consists in the analysis of the existence of certain splittings for the matrix $M$. In Varga [18] and Ortega and Rheinboldt [13] regular splittings are introduced as

$$M = B - A$$

where $B$ is monotone and $T = B^{-1}A$ is nonnegative. In these works it is shown that if $M = B - A$ is a regular splitting, then $M$ is monotone if and only if $\lambda^*(T) < 1$, where $\lambda^*(C)$ stands for the Frobenius root of a matrix $C \geq 0$, i.e., the greatest eigenvalue of this matrix $C$. Conversely, any monotone matrix allows for a regular splitting. Extensions of regular splittings to rectangular matrices can be found in Berman and Neumann [4, 5].

In this paper we follow the approach developed in [15], where square monotone matrices are characterized in terms of a particular type of positive splittings (called $B$-splittings). In Section 2 we introduce the concept of generalized $B$-splittings for an $m \times n$ matrix $M$. Then we prove that if a matrix allows a generalized $B$-splitting whose spectral radius is less than one, it is weak-monotone. The relationship between the existence of a generalized $B$-splitting of $M$, satisfying the spectral radius condition, and weak-monotonicity of $M$ is also analyzed. Finally, in Section 3 we analyze the relationship between weak-monotonicity and the existence of a positive generalized inverse. Several final remarks close the paper.

All the matrices considered will be real. The rank of a matrix $M_{m \times n}$ is denoted by $\text{rk}(M)$, and the image and null space by $\text{Im}(M)$ and $N(M)$, respectively.

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1 Notice that in many applications the existence of nonnegative solutions for that kind of system turns out to be essential. That is the case, for instance, of some economic models, where $x$ represents quantities or prices [14, 16].
2. GENERALIZED B-SPLITTINGS

Square monotone matrices can be characterized in terms of a particular type of splitting, which we call B-splittings. A square matrix $M$ is said to allow a B-splitting if a pair of matrices $A > 0, B \geq 0$ exists such that

$$M = B - A,$$  \hspace{1cm} B regular

and

$$(1) \quad Bx \geq 0 \Rightarrow Ax \geq 0,$$

$$(2) \quad \begin{pmatrix} M \\ B \end{pmatrix} x \geq 0 \Rightarrow x \geq 0.$$  

In [15] it is proved that if $M$ allows a B-splitting, then $M$ is monotone if and only if $\lambda^*(AB^{-1}) < 1$. Conversely, any monotone matrix allows a B-splitting. Furthermore, it is proved in that work that if $M$ has a strictly positive column, then such a splitting does not exist for $M$.

The existence of a B-splitting for a matrix $M$ is an extension of the concept of Z-matrix, in the sense that any Z-matrix can be split in the form

$$M = sI - A,$$

and the matrices $B = sI$ and $A$ provide a B-splitting for $M$. Now, $\lambda^*(AB^{-1}) < 1$ means that $\lambda^*(A) < s$.

Let us start by extending the notion of B-splittings to general $m \times n$ matrices. Then we shall find conditions ensuring the weak-monotonicity of those matrices allowing a generalized B-splitting.

**Definition 1.** A positive splitting of an $m \times n$ matrix $M$,

$$M = B - A, \quad B \geq 0, \quad A \geq 0,$$

is said to be a generalized B-splitting of $M$ if

$$(1) \quad Bx \geq 0 \Rightarrow Ax \geq 0,$$

$$(2) \quad \begin{pmatrix} M \\ B \end{pmatrix} x \geq 0 \Rightarrow \text{there is } y \geq 0 \text{ such that } Mx = My.$$  

**Remark.** Note that, if $M$ is an $n \times n$ regular matrix, the notions of generalized B-splittings and B-splittings coincide. This definition extends the concept of B-splitting to nonsquare, or square nonregular, matrices. In this case, regularity of the matrix $B$ and monotonicity of the matrix $\begin{pmatrix} M \\ B \end{pmatrix}$ are not assumed.
The next example shows a matrix allowing a generalized $B$-splitting, but not a $B$-splitting.

**Example 1.** The matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

allows a generalized $B$-splitting: take

$$B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$ 

However, inasmuch as $M$ has a strictly positive column, it does not allow a $B$-splitting.

In the above definition, condition (1) is equivalent to the existence of a square matrix $T \geq 0$ such that $A = TB$ (see Mangasarian [10]). This fact allows us to express $M$ as the product of two matrices

$$M = B - A = (I - T)B$$

where $I - T$ is a $Z$-matrix and $B$ is a nonnegative matrix. References about $Z$-matrices, and conditions for monotonicity of a $Z$-matrix, can be found in [6].

Condition (2) in Definition 1 indicates that $M$ is weak-monotone on the set

$$S = \{ x \in \mathbb{R}^n \mid Bx \geq 0 \}.$$ 

If it is possible to take $B$ monotone, then this condition is obviously satisfied. Another sufficient condition that implies (2) is that the matrix $\begin{pmatrix} M \\ B \end{pmatrix}$ is weak-monotone (or another type of monotonicity). Note that weak-monotonicity of $\begin{pmatrix} M \\ B \end{pmatrix}$ does not imply that $M$ is weak-monotone, as the next example shows.

**Example 2.** The matrix

$$M = \begin{pmatrix} 1 & 2 & 2 \\ -4 & -6 & -6 \end{pmatrix}$$
is not weak-monotone, but in the splitting

\[
M = \begin{pmatrix}
2 & 4 & 4 \\
2 & 2 & 2 \\
6 & 8 & 8
\end{pmatrix}
- \begin{pmatrix}
1 & 2 & 2 \\
6 & 8 & 8
\end{pmatrix}
\]

the matrix \( \begin{pmatrix} M \\
B \end{pmatrix} \) is weak-monotone.

For those matrices allowing for generalized B-splittings, the following result holds.

**Theorem 1.** Let \( M \) be an \( m \times n \) matrix such that \( M = B - A \) is a generalized B-splitting, and suppose \( \text{Im}(M) \cap \text{int}(\mathbb{R}^n_+) \neq \emptyset \). Then the following conditions are equivalent:

(a) \( M \) is weak-monotone.

(b) There exists \( x > 0 \) such that \( Mx > 0 \).

(c) \( \lambda^*(T) < 1 \).

**Proof.** (a) \( \Rightarrow \) (b): Since a positive vector \( d \in \text{Im}(M) \) exists,

\[
d = My > 0 \quad \text{for some} \quad y \in \mathbb{R}^n;
\]

then weak-monotonicity implies that there exists \( x \geq 0 \) such that

\[
Mx = d > 0.
\]

(b) \( \Rightarrow \) (c): By hypothesis there is some \( x \geq 0 \) such that

\[
Mx = (I - T)Bx > 0.
\]

As \( I - T \) is a Z-matrix and \( Bx \geq 0 \), this condition implies \( \lambda^*(T) < 1 \) (see, for instance, [6, Chapter 6]).

(c) \( \Rightarrow \) (a): Let \( x \in \mathbb{R}^n \) with \( Mx \geq 0 \). Then \( (I - T)(Bx) \geq 0 \), and, as \( I - T \) is a Z-matrix with \( \lambda^*(T) < 1 \), \( I - T \) is monotone, which implies \( Bx \geq 0 \). Finally, by condition (2) for generalized B-splittings, \( y > 0 \) exists such that \( My = Mx \), and consequently \( M \) is weak-monotone.

Note that condition (b) in Theorem 1 only makes sense when \( \text{Im}(M) \cap \text{int}(\mathbb{R}^n_+) \neq \emptyset \). The result obviously holds when \( \text{Im}(M) = \mathbb{R}^m \), i.e., when \( \text{rk}(M) = m \).
It is well established that the monotonicity of a square matrix $M$ implies that its transpose $M^t$ is monotone. The next theorem proves that the same result holds for weak-monotone matrices.

**Theorem 2.** Let $M$ be an $m \times n$ matrix. Then $M$ is weak-monotone if and only if $M^t$ is weak-monotone.

**Proof.** (We shall only prove one implication, since the other one follows analogously.)

Suppose that $M$ is weak-monotone. If $M^t$ is not weak-monotone, there exists $x \in \mathbb{R}^m$ such that $c = M^tx \geq 0$, and the system

$$M^ty = c, \quad y \geq 0$$

has no solution. By Farkas's lemma\(^2\) there exists some vector $z \in \mathbb{R}^n$ such that

$$Mz \leq 0, \quad c'z > 0.$$

Now, as $M(-z) \geq 0$ and $M$ is weak-monotone, there exists $u \geq 0$ such that $Mu = M(-z)$. Then

$$M(z + u) = 0 \quad \text{and} \quad (z + u) \in N(M).$$

Since $N(M)$ and $\text{Im}(M^t)$ are orthogonal supplementary subspaces,

$$c'(z + u) = 0,$$

and hence

$$c'z = -c'u < 0$$

because $c \geq 0, u \geq 0$, which contradicts $c'z > 0$. Therefore, $M^t$ is weak-monotone. $\blacksquare$

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\(^2\) Farkas's lemma (see, for instance, [1, Chapter 2]): Let $M$ be an $m \times n$ matrix, and $c$ be an $n$-vector. Then exactly one of the following two systems has a solution: $Mx \leq 0$ and $c'x > 0$ for some $x \in \mathbb{R}^n$; $M^ty = c$ and $y \geq 0$ for some $y \in \mathbb{R}^m$. 
In the following results we analyze the relationships between weak-monotonicity and the existence of a generalized B-splitting for a matrix $M$. It is necessary to introduce an additional condition on the null space $N(M)$.

**Theorem 3.** Let $M$ be an $m \times n$ matrix such that $N(M) \cap \mathbb{R}_+^n = \{0\}$. Then the following conditions are equivalent:

(a) $M$ is weak-monotone;
(b) $M$ allows a generalized B-splitting

$$M = B - A, \quad A = TB \text{ with } \lambda^*(T) < 1.$$  

**Proof.** (a) $\Rightarrow$ (b): Since $N(M) \cap \mathbb{R}_+^n = \{0\}$ and $\text{Im}(M^t)$ is an orthogonal supplementary subspace of $N(M)$, there exists some $u \in \mathbb{R}^m$ such that $uM > 0$. As $M$ is weak-monotone, then there exists some $y \geq 0$ such that

$$z = uM = yM > 0.$$  

Consider

$$v = \left( \frac{1}{\sum y_k + \delta} \right) y, \quad \text{with } \delta \in \mathbb{R}, \ \delta > 0.$$  

Then the matrix

$$T = \begin{pmatrix}
    v_1 & v_2 & \cdots & v_m \\
    v_1 & v_2 & \cdots & v_m \\
    \vdots & \vdots & \ddots & \vdots \\
    v_1 & v_2 & \cdots & v_m
\end{pmatrix}_{m \times m}$$  

is nonnegative, and

$$\lambda^*(T) = \sum v_k < 1.$$  

Therefore, $I - T$ is a Z-matrix, and $(I - T)^{-1}$ exists and is nonnegative,

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots.$$
Postmultiplying by $M$, we obtain

$$
(I - T)^{-1}M = M + TM + T^2M + \cdots = M + \frac{1}{\delta}H,
$$

where

$$
H = \begin{pmatrix}
    z_1 & z_2 & \cdots & z_n \\
    \bar{z}_1 & z_2 & \cdots & \bar{z}_n \\
    \vdots & \vdots & \ddots & \vdots \\
    \bar{z}_1 & \bar{z}_2 & \cdots & \bar{z}_n
\end{pmatrix}
$$

Taking $\delta$ small enough, we can obtain that

$$
B = (I - T)^{-1}M \geq 0,
$$

and we can split $M$ as

$$
M = (I - T)B.
$$

Defining $A = TB$, the matrix $M$ allows a generalized $B$-splitting

$$
M = B - A
$$

with $\lambda^*(T) < 1$.

(b) $\Rightarrow$ (a): Let $x \in \mathbb{R}^n$ be such that $Mx \geq 0$. Then

$$
(I - T)Bx \geq 0,
$$

and, since $I - T$ is a $Z$-matrix with $\lambda^*(T) < 1$, we have

$$
Bx \geq 0.
$$

By condition (2) for generalized $B$-splittings, there exists some vector $y \geq 0$ such that $Mx = My$, and $M$ is weak-monotone. \hfill \blacksquare
From the previous theorem we can obtain the following result.

**Corollary 1.** Let $M$ be an $m \times n$ matrix. Then the following conditions are equivalent:

(a) $M$ is weak-monotone and $N(M) \cap \mathbb{R}^+_n = \{0\}$.
(b) $M$ allows a generalized $B$-splitting

$$M = B - A, \quad A = TB \quad \text{with} \quad \lambda^*(T) < 1,$$

and $B$ has no zero columns.

**Proof.** (a) $\Rightarrow$ (b): It is sufficient to observe that in the proof of Theorem 3, the matrix $B$ can be taken strictly positive.

(b) $\Rightarrow$ (a): It is sufficient to prove that there exists $u \in \mathbb{R}^m$ with $uM > 0$. For that, as $(I - T)^{-1} \succ 0$, there exists some $u \in \mathbb{R}^m$ such that $u(I - T) > 0$. Postmultiplying by $B$, we have

$$uM = u(I - T)B > 0,$$

since $B \succ 0$ and no column of $B$ is zero. $\blacksquare$

The existence of a generalized $B$-splitting for a matrix $M$ does not guarantee that the same occurs for its transpose.\footnote{The following matrix allows a generalized $B$-splitting, though $M^t$ does not:}

$$M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$ Note that as $M$ is regular, the concepts of $B$-splitting and generalized $B$-splitting coincide. As $M^t$ has a strictly positive column, it does not allow a $B$-splitting.
equivalent:

(a) $M$ is weak-monotone;
(b) $M'$ is weak-monotone;
(c) $M$ allows a generalized $B$-splitting

$$M = (I - T)B \quad \text{with} \quad \lambda^*(T) < 1;$$

(d) $M'$ allows a generalized $B$-splitting

$$M' = (I - S)C \quad \text{with} \quad \lambda^*(S) < 1.$$

The additional condition introduced can be taken as a "positive regularity," in the sense that

$$Mx = 0, \quad x \geq 0 \quad \Rightarrow \quad x = 0$$

Then the above results characterize weak-monotonicity of positive regular matrices.

3. GENERALIZED INVERSE POSITIVENESS.
FULL-RANK MATRICES

When $M$ is a full-rank matrix, weak-monotonicity can be characterized in terms of the positiveness of a generalized inverse of $M$. We have the following results:

1. Let $M$ be a square matrix $(n \times n)$ with $\text{rk}(M) = n$. Then the following conditions are equivalent:
   (a) $M$ is weak-monotone.
   (b) $M^{-1} \succeq 0$.

2. Let $M$ be an $m \times n$ matrix with $\text{rk}(M) = n$. Then the following conditions are equivalent:
   (a) $M$ is weak-monotone.
   (b) There is a nonnegative left inverse of $M$,

   $$G \succeq 0, \quad GM = I.$$
(3) Let \( M \) be an \( m \times n \) matrix with \( \text{rk}(M) = m \). Then the following conditions are equivalent:

(a) \( M \) is weak-monotone.

(b) There is a nonnegative right-inverse of \( M \),

\[
G \geq 0, \quad MG = I.
\]

Results (1) and (2) are in Collatz [8] and Mangasarian [9], respectively. Result (3) follows directly from Theorem 2 and result (2) above.

In general, no specific generalized inverse of \( M \) exists such that \( M \) is weak-monotone iff this generalized inverse is nonnegative. Suppose however that there exists a matrix \( M^\# \) such that

\[
MM^\#M = M
\]

(we call such a matrix a \((1)\)-inverse), and \( M^\# \) is nonnegative on \( \text{Im}(M) \), i.e.,

\[
y \geq 0, \quad y \in \text{Im}(M) \quad \Rightarrow \quad M^\#y \geq 0.
\]

Then \( M \) is weak-monotone.

To see this, notice that if \( Mx \geq 0 \), then, as \( Mx \in \text{Im}(M) \), we have \( M^\#(Mx) \geq 0 \) and \( Mx = M[M^\#(Mx)] \). Hence \( M \) is weak-monotone.

Now, we can obtain a first result on the existence of a \((1)\)-inverse nonnegative on \( \text{Im}(M) \). For that, let us define a group inverse of \( M \) as a matrix \( M^* \) satisfying

\[
MM^*M = M,
\]

\[
M^*MM^* = M^*,
\]

\[
MM^* = M^*M.
\]

Clearly, \( M^* \) is only meaningful when \( M \) is a square matrix; moreover, \( M^* \) exists iff \( N(M) = N(M^2) \) (see Campbell and Meyer [7, Chapter 7]).
THEOREM 4. For a square matrix $M$ the following conditions are equivalent and, obviously, imply weak-monotonicity of $M$:

(a) $Mx \geq 0$, $x \in \text{Im}(M) \Rightarrow x \geq 0$.
(b) The group inverse $M^\#$ exists, and it is nonnegative on $\text{Im}(M)$.

Proof. (a) $\Rightarrow$ (b): Let $y \in \mathbb{R}^n$ be such that $M^2y = 0$. Then, by letting

$$z = My,$$

we have

$$z \in \text{Im}(M), \quad Mz = M^2y \geq 0,$$

and, by hypothesis, $z \geq 0$. Analogously, $-z \geq 0$ and $My = 0$. Then $N(M) = N(M^2)$ and $M^\#$ exists.

Let now $x \geq 0$, $x \in \text{Im}(M)$. Then $x = Mz$ with $z \in \mathbb{R}^n$. Consider $u = M^\#x$:

$$u = M^\#x = M^\#Mz = MM^\#z,$$

so $u \in \text{Im}(M)$. Furthermore

$$Mu = MM^\#x = MM^\#Mz = Mz = x \geq 0,$$

and, by hypothesis, $u \geq 0$. Then $M^\#$ is nonnegative on $\text{Im}(M)$.

(b) $\Rightarrow$ (a): Let $x \in \text{Im}(M)$ such that $Mx \geq 0$. Then we have

$$x = Mz, \quad z \in \mathbb{R}^n,$$

$$Mx \geq 0, \quad Mx \in \text{Im}(M).$$

By hypothesis

$$M^\#Mx \geq 0,$$

4 The conditions in Theorem 4 are not necessary, as can be seen by observing that the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is weak-monotone but does not satisfy the equivalent conditions of this theorem.
but
\[ M^*Mx = M^*MMz = Mz = x \geq 0. \]

Condition (a) in the above theorem says that, if \( M \) is monotone on \( \text{Im}(M) \), then \( M \) is weak-monotone. An interesting open problem consists of finding a \((I)\)-inverse \( M^w \) such that \( M \) is weak-monotone iff \( M^w \) is nonnegative on \( \text{Im}(M) \).

The class of MP matrices, introduced by Meyer and Plemmons [11], that is, the class of matrices having nonnegative nonsingular (full rank, in the case of rectangular matrices) \((I)\)-inverse, is in general contained in the class of weak-monotone matrices. In certain cases, the two conditions coincide.

**Theorem 5.** Let \( M \) be an \( m \times n \) matrix with \( \text{rk}(M) = m \). Then
\[
M \text{ is MP} \quad \iff \quad M \text{ is weak-monotone}.
\]

**Proof.** Only one implication need be proved, since if \( M \) has a nonnegative generalized inverse, then \( M \) is weak-monotone. To construct the generalized inverse, since \( \text{rk}(M) = m \), then \( \text{Im}(M) = \mathbb{R}^m \) and for each unit vector \( e_i = (0, \ldots, 1, \ldots, 0)^t \) we can take some vector \( x_i \in \mathbb{R}^n \) such that
\[
Mx_i = e_i.
\]

By weak-monotonicity, there exists \( u_i \geq 0 \) such that
\[
Mu_i = Mx_i = e_i.
\]

Then the matrix
\[
G = (u_1, u_2, \ldots, u_m)_{n \times m}
\]
is a nonnegative \((I)\)-inverse of \( M \), and \( \text{rk}(G) = m \). Now, \( M \) is an MP matrix.

As a consequence of the above result, and since weak-monotonicity and MP are conditions invariant under matrix transposition (Theorem 2 and Poole and Barker [17], respectively) we obtain the next result.
Corollary 3. Let $M$ be an $m \times n$ matrix of full rank. Then

$$M \text{ is weak-monotone } \iff M \text{ is } M^p.$$ 

4. FINAL REMARKS

In Section 2, a characterization of weak-monotone matrices has been established, in terms of a particular type of positive splittings (generalized $B$-splittings). It is not immediate how to identify whether a matrix $M$ allows a generalized $B$-splitting. We know that the existence of a strictly positive column does imply that a matrix cannot be decomposed by a $B$-splitting, but this does not imply that it cannot allow a generalized $B$-splitting (see Example 1).

However, some types of matrices do exist which allow a generalized $B$-splitting in a natural way. This is the case for $m \times n$ matrices $M$ such that

$$m_{ij} \leq 0 \quad \text{for all} \quad i \neq j, \quad i, j \leq \min \{m, n\}.$$ 

Of course, if $M$ is a square matrix ($m = n$), then it is a $Z$-matrix and it allows a $B$-splitting. If $m > n$, we can take

$$M = s \begin{pmatrix} I \\ J \end{pmatrix} - \begin{pmatrix} A \\ K \end{pmatrix}, \quad \text{with} \quad s > 0, \quad J, A, K > 0,$$

which provides a generalized $B$-splitting for $M$. For such a matrix (which we can call a generalized $Z$-matrix),

$$M \text{ is weak-monotone } \iff \lambda^*(A) < s.$$ 

Finally, if $m < n$, we can use its transpose $M'$ as the working matrix to characterize weak-monotonicity.

Other matrices which allow a generalized $B$-splitting are the constant-column matrices, i.e., those matrices whose columns are the same:

$$M = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_2 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_m & t_m & \cdots & t_m \end{pmatrix}. $$
Then, taking 

\[ B = kE, \quad A = kE - M, \]

where \( k = \max\{|t_1|, |t_2|, \ldots, |t_m|\} \), and \( E \) is the matrix whose entries are all equal to 1, the matrix \( M \) allows a generalized \( B \)-splitting.

Finally, it must be remarked that positive and regular splittings are not compatible, in the sense that if a matrix \( M \) allows a regular splitting

\[ M = B - A, \quad B \text{ monotone}, \quad B^dA \geq 0, \]

where \( B^d \) stands for a left inverse of \( B \), and if the matrices \( B, A \) are nonnegative, then \( M \) is a generalized \( Z \)-matrix. This is an immediate consequence of the next theorem.

**Theorem 6.** Let \( B \) be a nonnegative monotone \( m \times n \) matrix. Then \( B \) contains an \( n \times n \) positive diagonal (or permutation of a diagonal) submatrix of rank \( n \). Conversely, if \( B \) contains a diagonal positive submatrix of rank \( n \), then \( B \) is monotone.

**Proof.** First we suppose that \( B \) has no zero rows. By monotonicity,

\[ \text{rk}(B) = n \leq m \]

and \( B' \) is an \( n \times m \) weak-monotone matrix of rank \( n \). By a previous result, a nonnegative right inverse exists:

\[ G^t \geq 0, \quad B'G^t = I, \quad \text{with} \ \text{rk}(G') = n. \]

We can take \( G \) decomposed, by permuting columns and permuting the correspondent rows in \( B \), in the form

\[ G = (G_1, G_2), \quad (G_1)_{n \times n}, \quad \text{rk}(G_1) = n, \]

\[ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad (B_1)_{n \times n}. \]

We prove that in each row and column of \( B_1 \) there is, at most, one nonzero
entry. Then $B_1$ must be a diagonal matrix, or a permutation of a diagonal one.

If there are two entries

$$b_{ik} \neq 0, \quad b_{jk} \neq 0, \quad i, j \leq n,$$

in the product

$$GB = I,$$

we have

$$g_r b^k = 0 \quad \forall r \neq k,$$

where

- $g_r$ = rth row of $G$,
- $b^k$ = kth column of $B$.

Then

$$g_{ri} - g_{rj} = 0 \quad \forall r \neq k,$$

and the $i$th and $j$th rows of $G$ are dependent, contradicting the condition $rk(G_1) = n$. Now, if there exist

$$b_{ki} \neq 0, \quad b_{kj} \neq 0,$$

then

$$g_r b^i = 0 \quad \forall r \neq i, \quad r \leq n$$

and

$$g_{rk} = 0 \quad \forall r \neq i.$$

Analogously,

$$g_{rk} = 0 \quad \forall r \neq j,$$

and $g^k = 0$, which contradicts $rk(G_1) = n$.

If $B$ has zero rows, we consider the submatrix $B^*$ where those rows are eliminated. By applying the above process to $B^*$ we obtain the desired result.
The converse is immediate. If we can write, by permuting rows and columns,

$$R = \begin{pmatrix} D \\ M_2 \end{pmatrix}$$

with $D$ a diagonal positive submatrix of rank $n$, then $B$ is monotone. 

This result is a generalization of a property that indicates that a monotone nonnegative regular matrix is diagonal, or a permutation of a diagonal one.

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REFERENCES


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