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# A dichotomy for the convex spaces of probability measures

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ABSTRACT

We show that every nonempty compact and convex space $M$ of probability Radon measures either contains a measure which has 'small' local character in $M$ or else $M$ contains a measure of 'large' Maharam type. Such a dichotomy is related to several results on Radon measures on compact spaces and to some properties of Banach spaces of continuous functions.



## 1. Introduction

Throughout this note $K$ denotes a compact Hausdorff space. By a *Radon measure* $\mu$ on $K$ we mean a finite measure defined on the Borel $\sigma$-algebra $Bor(K)$ of $K$ which is inner regular, that is

$$\mu(B) = \sup\{\mu(F)\colon\ F = \bar{F} \subseteq B\},$$

for any $B \in Bor(K)$. We denote by $P(K)$ the space of all probability Radon measures on $K$. The space $P(K)$ is always equipped with the *weak** topology inherited from $P(K) \subseteq C(K)^*$, where $C(K)$ is the Banach space of continuous functions on $K$. When we treat a given measure $\mu \in P(K)$ as a functional on $C(K)$ we write $\mu(g)$ rather than $\int_K g\,\mathrm{d}\mu$.

We say that a measure $\mu \in P(K)$ is of *Maharam type* $\kappa$ and write $\mathrm{mt}(\mu) = \kappa$ if $L_1(\mu)$ has density character $\kappa$. Except for some trivial cases, $\mathrm{mt}(\mu)$ can be defined as the minimal cardinality of a family $\mathcal{D} \subseteq Bor(K)$ such that for any $\varepsilon > 0$ and any $B \in Bor(K)$ there is $D \in \mathcal{D}$ such that $\mu(D \triangle B) < \varepsilon$.

We consider here nonempty compact and convex subsets $M \subseteq P(K)$. The dichotomy announced by the title of this note states, in particular, that every such a space $M$ either has a $G_\delta$ point or contains a measure $\mu$ of uncountable Maharam type. Roughly speaking, this means that to build a complicated topological space like $M$ one needs large measures.

Subsequent sections illustrate how our dichotomy works. The condition that $\mu \in P(K)$ is a $G_\delta$ point in $P(K)$ is equivalent to some measure-theoretic property of $\mu$, namely to $\mu$ being strongly countably determined. It follows that every compact

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space $K$ either carries a strongly countably determined measure (topologically simple, behaving like measures on metric spaces) or a Radon measure of uncountable type.

For a given compact space $K$, the existence of Radon measures on $K$ of uncountable Maharam types is connected with the existence of continuous surjections from $K$ or $P(K)$ onto Tichonov cubes. This enables us to derive from our main theorem some purely topological results on compact spaces of measures.

Finally, we give some applications of our dichotomy to Grothendieck-like properties of Banach spaces of the form $C(K)$.

Our main theorem arose as a generalization of a result due to Borodulin-Nadzieja [3, Theorem 4.3] and was further inspired by a theorem due to Haydon, Levy and Odell [7, Corollary 3C]. Although our dichotomy is a generalization of the latter, its proof seems to be more straightforward.

## 2. Dichotomy

If $X$ is a topological space and $x \in X$ then $\chi(x, X)$ denotes the local character of $X$ at $x$, i.e. the minimal cardinality of a local base at $x$. Recall that in a compact space $X$ the local character at $x \in X$ agrees with the minimal cardinality of a family of open sets intersecting to $\{x\}$.

In this section we shall present the main result of this paper stated as Theorem 2.2 below. We first recall the following useful fact due to Douglas (see [4], Theorem 1), which is reproduced here together with its short proof.

**Lemma 2.1.** *Let $K$ be a compact space and let $\mathcal{F} \subseteq C(K)$ be any family of functions. Suppose that measures $\mu_1, \mu_2 \in P(K)$ satisfy $\mu_1 \neq \mu_2$ and $\mu_1(f) = \mu_2(f)$ for every $f \in \mathcal{F}$.*
*If $\mu = \frac{\mu_1+\mu_2}{2}$ then $\mathcal{F}$ is not dense in $L_1(\mu)$.*

**Proof.** We have $\mu = \frac{\mu_1+\mu_2}{2}$ so $\mu_1$ is absolutely continuous with respect to $\mu$, in fact $\mu_1 \leqslant 2 \cdot \mu$. By the Radon–Nikodym theorem there exists $h \in L_\infty(\mu)$ such that $\mathrm{d}\mu_1 = h\,\mathrm{d}\mu$. Note that $1 - h \neq 0$ since $\mu_1 \neq \mu_2$.

For any function $f \in \mathcal{F}$ we have

$$\int f(1-h)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu - \int fh\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\mu_1 = 0,$$

which proves that $\mathcal{F}$ lies in the kernel of a nonzero continuous functional on $L_1(\mu)$ so is not dense in $L_1(\mu)$. $\square$

**Theorem 2.2.** *Let $K$ be a compact space and $\kappa$ be any cardinal number of uncountable cofinality. If $M$ is a nonempty compact and convex subspace of $P(K)$ then either*

(D1) *there exists $\mu \in M$, such that $\chi(\mu, M) < \kappa$, or*
(D2) *there exists $\mu \in M$, such that $\mathrm{mt}(\mu) \geqslant \kappa$.*

**Proof.** Assuming that for any $\mu \in M$ we have that $\chi(\mu, M) \geqslant \kappa$ we shall prove that there is a measure $\mu \in M$ with $\mathrm{mt}(\mu) \geqslant \kappa$.

We construct inductively a family $\{\mu_\alpha \in M\colon\ \alpha \leqslant \kappa\}$ of measures from $M$ and a family $\{f_\alpha\colon\ \alpha < \kappa\}$ of functions from $C(K)$ such that, writing for any $\alpha < \kappa$,

$$\mathcal{F}_\alpha = \{f_\xi\colon\ \xi \leqslant \alpha\} \cup \big\{|f_\xi - f_\zeta|\colon\ \zeta < \xi \leqslant \alpha\big\},$$

the following are satisfied

(i) $\mu_\beta(g) = \mu_\alpha(g)$ whenever $\alpha < \beta \leqslant \kappa$ and $g \in \mathcal{F}_\alpha$;
(ii) for every $\beta < \kappa$

$$\varepsilon_\beta = \inf_{\alpha<\beta} \int |f_\alpha - f_\beta|\,\mathrm{d}\mu_\beta > 0.$$

Let us note first that once the construction is done, the measure $\mu = \mu_\kappa$ will have the desired property. Indeed, as $\kappa$ has uncountable cofinality, there is $\varepsilon > 0$ such that $\varepsilon_\beta \geqslant \varepsilon$ for $\beta$ from some set $Y \subseteq \kappa$ of size $\kappa$. Using (i), for any $\beta, \beta' \in Y$, $\beta' < \beta < \kappa$ we have

$$\int |f_\beta - f_{\beta'}|\,\mathrm{d}\mu_\kappa = \int |f_\beta - f_{\beta'}|\,\mathrm{d}\mu_\beta > \varepsilon_\beta \geqslant \varepsilon,$$

so the family $\{f_\beta\colon\ \beta \in Y\}$ witnesses that $\mathrm{mt}(\mu_\kappa) \geqslant \kappa$.

We shall now describe the inductive step: let us assume that we have already constructed measures $\{\mu_\alpha \in M\colon\ \alpha < \beta\}$ and continuous functions $\{f_\alpha\colon\ \alpha < \beta\}$ so that conditions (i)–(ii) are satisfied.

Let us denote $\mathcal{F}^* = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$ and for every $\alpha < \beta$ write

$$M_\alpha = \big\{\nu \in M\colon\ \nu(f) = \mu_\alpha(f) \text{ for all } f \in \mathcal{F}_\alpha\big\}.$$

It follows from (i) that $\{M_\alpha\colon\ \alpha < \beta\}$ is a decreasing family of nonempty and closed sets in $M$ so, by compactness, $M_\beta = \bigcap_{\alpha<\beta} M_\alpha$ is nonempty, too.

For fixed $f \in \mathcal{F}^*$ and $\alpha < \beta$, the set $\{\nu \in M\colon\ \nu(f) = \mu_\alpha(f)\}$ is a $G_\delta$ subset of $M$. It follows that $M_\beta$ is an intersection of less than $\kappa$ open sets in $M$ (since $|\mathcal{F}^*| < \kappa$), so $M_\beta$ cannot consist of a single point (by our assumption that (D1) does not hold).

Take $\nu', \nu'' \in M_\beta$, $\nu' \neq \nu''$ and put $\mu_\beta = \frac{\nu'+\nu''}{2}$. By Lemma 2.1 $\mathcal{F}^*$ is not dense in $L_1(\mu_\beta)$ so there is a continuous function $f_\beta \in C(K)$ such that

$$\inf_{\xi<\beta} \int |f_\xi - f_\beta| \,\mathrm{d}\mu_\beta > 0,$$

and we are done. □

Let us note that the alternatives of Theorem 2.2 have rather different content: (D1) is a topological statement on the space $M$ while (D2) names a measure-theoretic property of elements of $M$. We shall see below, however, that in fact (D1) is equivalent to another purely measure-theoretic property. Likewise, we shall mention instances when (D2) has natural topological consequences.

## 3. Strongly determined measures

A measure $\mu \in P(K)$ is said to be *strongly countably determined* if there exists a continuous map $f : K \to [0,1]^\omega$, such that for any compact set $F \subseteq K$ we have $\mu(F) = \mu(f^{-1} f[F])$. Strongly countably determined measures were introduced by Babiker [1] who gave them the name *uniformly regular measures*, motivated by considerations related to uniform spaces. A measure $\mu$ is strongly countably determined if and only if there is a countable family $\mathcal{Z}$ of closed $G_\delta$ subsets of $K$ such that

$$\mu(U) = \sup\big\{\mu(Z)\colon\ Z \in \mathcal{Z}, Z \subseteq U\big\}, \tag{$*$}$$

for every open set $U \subseteq K$, see [1]. The latter condition seems to justify the name strongly countably determined; note that if we relax the property to saying that $\mathcal{Z}$ is a countable family of closed sets then a measure $\mu$ satisfying $(*)$ is called *countably determined*. (Strongly) countably determined measures were considered by Pol [14], Mercourakis [11], Plebanek [12] and Marciszewski and Plebanek [10].

We shall consider the following more general notion (see [9]).

**Definition 3.1.** Let us say that a measure $\mu \in P(K)$ is strongly $\kappa$-determined if there exists continuous map $f : K \to [0,1]^\kappa$ such that for any compact set $F \subseteq K$ we have $\mu(F) = \mu(f^{-1} f[F])$. We shall write $\mathrm{sd}(\mu) = \kappa$ to denote the least cardinal $\kappa$ for which $\mu$ is strongly $\kappa$-determined.

It can be checked that, as in the case $\kappa = \omega$, $\mathrm{sd}(\mu)$ is the minimal cardinality of an infinite family $\mathcal{Z}$ of closed $G_\delta$ sets such that for any open set $U \subseteq K$ and any $\varepsilon > 0$, there is $Z \in \mathcal{Z}$, $Z \subseteq U$ and $\mu(U \setminus Z) < \varepsilon$.

Clearly $\mathrm{mt}(\mu) \leqslant \mathrm{sd}(\mu)$, since the family approximating all open sets as in (*) needs to $\triangle$-approximate all Borel sets by regularity of a measure. Note that the Dirac measure $\mu = \delta_x$, where $x \in K$, is strongly countably determined if and only if $x$ is a $G_\delta$ point in $K$; in fact we always have $\mathrm{sd}(\delta_x) = \chi(x, K)$.

In [14] Pol proved that if a measure $\mu \in P(K)$ is strongly countably determined then $\chi(\mu, P(K)) = \omega$. A straightforward modification of the argument used in [14, Proposition 2] yields the following.

**Proposition 3.2.** *For any compact space $K$ and a measure $\mu \in P(K)$ we have $\chi(\mu, P(K)) \leqslant \mathrm{sd}(\mu)$.*

We shall check now that in fact the equality $\chi(\mu, P(K)) = \mathrm{sd}(\mu)$ always holds.

For the sake of the next lemma we recall some standard fact concerning finitely additive measures on algebras of sets. Suppose that $\mathcal{A}$ is an algebra of subsets of some space $X$. If $\mathcal{L} \subseteq \mathcal{A}$ is a lattice (that is $\mathcal{L}$ is closed under finite unions and intersections) then a finitely additive measure $\mu$ on $\mathcal{A}$ is said to be $\mathcal{L}$-regular if

$$\mu(A) = \sup\big\{\mu(L)\colon\ L \subseteq A, L \in \mathcal{L}\big\},$$

for every $A \in \mathcal{A}$. The following result can be found in Bachman and Sultan [2], Theorem 2.1.

**Theorem 3.3.** *Let $\mathcal{A} \subseteq P(X)$ be an algebra of sets. Let $\mathcal{L} \subseteq \mathcal{A}$ be a lattice and let $\mu$ be an $\mathcal{L}$-regular, finitely additive measure on $\mathcal{A}$. Assume $\mathcal{K} \supseteq \mathcal{L}$ is also a lattice. Then $\mu$ extends to a $\mathcal{K}$-regular, finitely additive measure $\nu$ on alg$(\mathcal{A} \cup \mathcal{K})$.*

If $f : K \to L$ is a surjective map between compact spaces and $\mu \in P(K)$ is a measure on $K$, then by $f[\mu]$ we denote the image measure $f[\mu] \in P(K)$ which is for $A \in Bor(K)$ defined as $f[\mu](A) = \mu(f^{-1}[A])$.

**Lemma 3.4.** *Let $g : K \to L$ be a continuous surjection between compact spaces $K$ and $L$ and let $F \subseteq K$ be a closed set. Given $\lambda \in P(L)$, there exists a measure $\mu \in P(K)$ such that $g[\mu] = \lambda$ and $\mu(F) = \lambda(g[F])$.*

**Proof.** Let

$$\mathcal{A} = \{g^{-1}[B]:\ B \in Bor(L)\}, \qquad \mathcal{L} = \{g^{-1}(H):\ H = \bar{H} \subseteq L\}.$$

Then $\mathcal{A}$ is an algebra of sets, $\mathcal{L} \subseteq \mathcal{A}$ is a lattice and putting $\mu_0(g^{-1}[B]) = \lambda(B)$ we have an $\mathcal{L}$-regular measure $\mu_0$ defined on $\mathcal{A}$.

Let $\mathcal{L}'$ be the lattice generated by $\mathcal{L} \cup \{F\}$ and $\mathcal{A}' = alg(\mathcal{A} \cup \{F\})$. Using Theorem 3.3 we can extend $\mu_0$ to a finitely additive measure $\mu'$ on $\mathcal{A}'$ which is $\mathcal{L}'$-regular. Note that we then have

$$\mu'(F) = \mu_0^*(F) = \inf\{\lambda(B):\ F \subseteq g^{-1}[B]\} = \lambda(g[F]).$$

Finally, $\mu'$ can be extended to a finitely additive closed-regular measure $\mu$ on an algebra of sets containing $\mathcal{A}'$ and all closed subsets of $K$. Such $\mu$ is then countably additive and extends to a Radon measure by a standard measure extension theorem, see [2] for details. □

**Theorem 3.5.** *If $K$ is a compact space and $\mu \in P(K)$ then $\chi(\mu, P(K)) = \mathrm{sd}(\mu)$.*

**Proof.** By Proposition 3.2, we only need to show $\chi(\mu, P(K)) \geqslant \mathrm{sd}(\mu)$.

Fix $\mu \in P(K)$ and denote $\chi(\mu, P(K)) = \kappa$. We can find a family $\{f_\xi:\ \xi < \kappa\}$ of continuous functions $f_\xi : K \to [0,1]$ such that whenever $\nu \in P(K)$ and $\nu(f_\xi) = \mu(f_\xi)$ for all $\xi < \kappa$ then $\nu = \mu$.

Consider now the diagonal mapping

$$f = \Delta_{\xi<\kappa} : K \to [0,1]^\kappa, \qquad f(x)(\xi) = f_\xi(x).$$

If we suppose that $\mathrm{sd}(\mu) > \kappa$ then there is a closed set $F \subseteq K$ such that $\mu(f^{-1}f[F]) > \mu(F)$. Let $\lambda = f[\mu] \in P([0,1]^\kappa)$; by Lemma 3.4 there is a measure $\nu \in P(K)$, such that $f[\nu] = \lambda$ and $\nu(F) = \lambda(f[F]) = \mu(f^{-1}f[F]) > \mu(F)$, which in particular means that $\nu \neq \mu$.

On the other hand, $\lambda = f[\mu] = f[\nu]$ so for every $\xi < \kappa$

$$f_\xi[\mu] = \pi_\xi \circ f[\mu] = \pi_\xi \circ f[\nu] = f_\xi[\nu],$$

where $\pi_\xi : [0,1]^\kappa \to [0,1]$ is a projection, and therefore

$$\int f_\xi \, \mathrm{d}\mu = \int_0^1 t \, \mathrm{d} f_\xi[\mu](t) = \int_0^1 t \, \mathrm{d} f_\xi[\nu](t) = \int f_\xi \, \mathrm{d}\nu,$$

which is a contradiction. □

Recall that a topological space $K$ is scattered of every subset of $K$ has an isolated point. Nonscatteredness of the compact space $K$ is equivalent to the existence of a continuous surjection $K \to [0,1]$ and to the existence of a nonatomic measure on $K$.

**Corollary 3.6.** *Let $\kappa$ be a cardinal number of uncountable cofinality. If $K$ is any compact space which is not scattered then either there a nonatomic measure $\mu \in P(K)$ such that $\mathrm{sd}(\mu) < \kappa$ or there is a (nonatomic) measure $\mu \in P(K)$ with $\mathrm{mt}(\mu) \geqslant \kappa$.*

**Proof.** If $K$ is not scattered then there is a continuous surjection $g : K \to [0,1]$. Writing $\lambda$ for the Lebesgue measure on $[0,1]$ we consider the set

$$M = \{\mu \in P(K):\ g[\mu] = \lambda\}.$$

Then $M$ is a compact and convex set which is $G_\delta$ in $P(K)$. Applying Theorem 2.2 we get the result. □

For $\kappa = \omega_1$ we get the following generalization of a result due to Borodulin-Nadzieja [3, Theorem 4.6], which was stated in the context of Boolean algebras or zero-dimensional spaces.

**Corollary 3.7.** *Every compact nonscattered space either carries a nonatomic strongly countably determined measure or a Radon measure of uncountable type.*

Of course a compact space can carry both strongly countably determined measures and measures of uncountable type. Note that infinite products such as $K = 2^\kappa$, $\kappa > \omega$, are examples of spaces not admitting strongly countably determined measures. Other examples of $K$ with such a property are mentioned below in connection with the Grothendieck property.

There are several classes of spaces carrying only measures of countable type, which necessarily admit strongly countably determined measures; this list of such classes includes

(1) Eberlein compacta (i.e. spaces homeomorphic to weakly compact subsets of Banach spaces);
(2) ordered compact spaces;
(3) Corson compacta under $\mathrm{MA}(\omega_1)$;
(4) consistently, all first-countable compact spaces, see Plebanek [12];
(5) Rosenthal compacta, see Todorčević [16], cf. Marciszewski and Plebanek [10].

The reader may consult Mercourakis [11] for basic facts and further references concerning (1)–(3). In connection with (4) note that it is relatively consistent that every Radon measure on a first-countable compactum is strongly countably determined [12] but there is an open problem posed by D.H. Fremlin if this is a consequence of $\mathrm{MA}(\omega_1)$.

## 4. Mappings onto cubes

In [5] Fremlin proved assuming $\mathrm{MA}(\omega_1)$ that if a compact space $K$ carries a Radon measure of type $\geqslant \omega_1$, then there is a continuous surjection $f : K \to [0,1]^{\omega_1}$; cf. Plebanek [12] for further discussion. As we have mentioned uncountable products are typical examples of spaces on which there are no strongly countably determined measures; combining Fremlin's result with Theorem 2.2 and Theorem 3.5 we get the following partial converse.

**Corollary 4.1.** *Assuming* $\mathrm{MA}(\omega_1)$, *if a compact space $K$ admits no strongly countably determined measure then there is a continuous surjection $f : K \to [0,1]^{\omega_1}$.*

If there is a continuous surjection from a compact space $K$ onto $[0,1]^\kappa$ for some $\kappa$ then it can be extended to a continuous mapping $P(K) \to [0,1]^\kappa$. The existence of the latter surjection is also closely related to types of measures on $K$; the following is due to Talagrand [15] and requires no additional set-theoretic assumptions.

**Theorem 4.2.** *If $\kappa \geqslant \omega_2$ is a cardinal of uncountable cofinality and $K$ is a compact space then there is a continuous surjection $P(K) \to [0,1]^\kappa$ if (and only if) $K$ carries a Radon measure of type $\geqslant \kappa$.*

The above result and Theorem 2.2 yield immediately the following.

**Corollary 4.3.** *If $\kappa \geqslant \omega_2$ is a cardinal of uncountable cofinality then for a compact space $K$ either*

(i) *$P(K)$ has points of character $< \kappa$, or*
(ii) *$P(K)$ can be continuously mapped onto $[0,1]^\kappa$.*

The reader is referred to a survey paper [13] for further discussion. We shall only mention here a problem related to Corollary 4.3. Let us recall that the tightness of a topological space $X$ is the minimal cardinal number $\tau(X)$ such that whenever $A \subseteq X$ and $x \in \bar{A}$ then there is $I \subseteq A$ such that $|I| \leqslant \tau(X)$ and $x \in \bar{I}$.

If $K$ is such a compact space that $P(K)$ has tightness $\leqslant \omega_1$ then $P(K)$ cannot be continuously mapped onto $[0,1]^{\omega_2}$ because the tightness of the latter space is $\omega_2$ and tightness is not increased by continuous surjections between compact spaces; hence Theorem 4.2 implies in particular that then every measure $\mu \in P(K)$ has the Maharam type at most $\omega_1$. We do not know if the following holds true.

**Question 4.4.** Assume that $K$ is such a compact space that $P(K)$ has countable tightness. Does this imply that every measure $\mu \in P(K)$ has countable Maharam type?

It seems that the only result in that direction, requiring no additional set-theoretic assumptions, is a theorem stating that if $K$ is Rosenthal compact then $K$ admits only measures of countable type; the result was obtained by Jean Bourgain but the first published proof is due to Todorčević [16]; see also [10].

## 5. On Haydon–Levy–Odell result

If $(v_n)_n$ is a sequence in some vector space then a sequence $(w_n)_n$ is said to be a *convex block subsequence* of $(v_n)_n$ if for some sequence of natural numbers $k_1 < k_2 < \cdots$, every $w_n$ is a convex combination of vectors $v_i$ for $k_n \leqslant i < k_{n+1}$. The following result was proved by Haydon, Levy and Odell (see [7], Corollary 3C).

**Theorem 5.1.** *If $K$ is compact and in $P(K)$ there is a sequence $(\lambda_n)$, with no weak* convergent convex block subsequence, then there is $\mu \in P(K)$ such that $\mathrm{mt}(\mu) \geqslant \mathfrak{p}$.*

Recall that the cardinal number $\mathfrak{p}$ mentioned here is the largest cardinal having the property:

*If $\kappa < \mathfrak{p}$ and $(M_\alpha)_{\alpha<\kappa}$ is a family of subsets of $\omega$ with $\bigcap_{\alpha\in F} M_\alpha$ infinite for all finite $F \subseteq \kappa$, then there exists an infinite $M \subseteq \omega$ with $M \setminus M_\alpha$ finite for all $\alpha < \kappa$.*

Theorem 5.1 has several interesting consequences, for instance it is one of the main ingredients of the proof that assuming $\mathfrak{p} = \mathfrak{c} > \omega_1$, for every infinite compact space $K$, the Banach space $C(K)$ has either $l_\infty$ or $c_0$ as a quotient, see [7], cf. Koszmider [8] for more information.

We note here that one can easily derive Theorem 5.1 from our dichotomy as follows.

**Proof of Theorem 5.1.** Set

$$M = \bigcap_{n=1}^{\infty} \overline{\mathrm{conv}}\{\lambda_k\colon k \geqslant n\}.$$

By Theorem 2.2 applied to such a set $M$ and $\kappa = \mathfrak{p}$ it is sufficient to check that $\chi(\mu, M) \geqslant \mathfrak{p}$ for every $\mu \in M$. But if we suppose that $\chi(\mu, M) < \mathfrak{p}$ for some $\mu \in M$ then $\mu$ is a limit of some convex block subsequence of $(\lambda_n)_n$, which can be derived from the definition of $\mathfrak{p}$, as indicated in Lemma 5.2 below. $\square$

**Lemma 5.2.** *Let $(M_n)_n$ be a decreasing sequence of separable topological spaces and let $M = \bigcap_{n=1}^{\infty} M_n$. If $x$ is a nonisolated point in $M$ and $\chi(x, M) < \mathfrak{p}$, then there is a sequence $(x_n)_n$ in $M_1$, convergent to $x$ and such that for any $k \in \omega$, $x_n \in M_k$ for all but finitely many $n$.*

**Proof.** Since each $M_n$ is separable, there is a countable set $D \subseteq M_1$, such that for any $n \in \omega$, $D \cap M_n$ is dense in $M_n$. Let $\{U_\alpha\colon \alpha < \chi(x, M)\}$ a family of open subsets of $M_1$ such that $\{U_\alpha \cap M\colon \alpha < \chi(x, M)\}$ is a local base at $x \in M$. Consider

$$\mathcal{C} = \bigl\{U_\alpha \cap D\colon \alpha < \chi(x, M)\bigr\} \cup \{M_n \cap D\colon n < \omega\}.$$

Since $\chi(x, M) < \mathfrak{p}$, there exists an infinite $A \subseteq D$ with $A \setminus C$ finite for any $C \in \mathcal{C}$. Now elements of $A$ form a desired sequence converging to $x$. $\square$

Recall that a Banach space $X$ is called a *Grothendieck space* if every weak* convergent sequence in the dual space $X^*$ is also weakly convergent. Note that if $K$ is an infinite compact space and the Banach space $C(K)$ is Grothendieck then for every sequence $(\mu_n)_n$ from $C(K)^*$ which is *weak** convergent to $\mu \in C(K)^*$ one has $\mu(B) = \lim_n \mu_n(B)$ for all $B \in Bor(K)$.

It follows from Theorem 5.1 that $\mathrm{mt}(\mu) \geqslant \mathfrak{p}$ for some $\mu \in P(K)$: indeed, if $(x_n)_n$ is a sequence of distinct elements of $K$ then it is not difficult to check that the sequence of measures $(\delta_{x_n})_n$ has no convex block subsequence which converges weakly. Hence, by the Grothendieck property and Theorem 5.1, $K$ must carry a Radon measure of type $\geqslant \mathfrak{p}$; this fact is a particular case of Haydon's result from [6].

**Proposition 5.3.** *If $K$ is a compact space without isolated points and $C(K)$ is a Grothendieck space then*

(a) $\mathrm{sd}(\mu) \geqslant \omega_1$ *for every $\mu \in P(K)$; and*
(b) $\mathrm{sd}(\mu) \geqslant \mathfrak{p}$ *for every $\mu$ which is supported by a separable subspace of $K$.*

**Proof.** We shall discuss (b); one gets (a) by a simple modification of an argument below. Suppose that $\kappa = \mathrm{sd}(\mu) < \mathfrak{p}$ for some $\mu \in P(K)$.

Let $\mu$ have an atom, say $c = \mu(\{x_0\}) > 0$ for $x_0 \in K$. If $U$ is an open set containing $x_0$ then there is an open set $V$ with $x_0 \in V \subseteq U$ and $\mu(V \setminus \{x_0\}) < c/2$. It follows that if $\mathcal{Z}$ is a family of closed $G_\delta$ sets witnessing that $\mathrm{sd}(\mu) = \kappa$ then there is $Z \in \mathcal{Z}$ such that $Z \subseteq V$ and $\nu(V \setminus Z) < c/2$ which implies $x_0 \in Z \subseteq V$. Using this remark we conclude that $\chi(x_0, K) \leqslant \kappa$; as $\kappa < \mathfrak{p}$ and $x_0$ is not isolated, there is a sequence $x_n \in K \setminus \{x_0\}$ converging to $x_0$, a contradiction with the Grothendieck property.

Let $\mu$ be nonatomic and $\mu(K_0) = 1$ for some separable $K_0 \subseteq K$; take a countable dense subset $A$ of $K_0$. Then $\mu$ is in the closure of a countable set

$$\left\{ \sum_{i=1}^{n} r_i \delta_{a_i} \colon r_i \in \mathbb{Q}, a_i \in A \right\},$$

which in view of $\kappa = \mathrm{sd}(\mu) = \chi(\mu, P(K))$ implies that $\mu$ is a *weak** limit of a sequence of purely atomic measures; a contradiction since such a sequence cannot converge weakly. □

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