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Note

On the expressive power of temporal logic for infinite words

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Abstract

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In this paper, we give an algebraic proof of the equivalence between temporal logic and star-free languages on infinite words. The proof is based on a result of Schützenberger that characterizes the star-free languages of finite words by using particular prefix codes.

Introduction

In this paper, we study the languages definable in propositional temporal logic (PTL) in relation with ω-star-free languages by using algebraic techniques.

It is known that the equivalence of the PTL definable languages and the ω-star-free languages follows from the combination of two independent results. First, Kamp [8] proved that first-order logic and PTL are equivalent. Secondly, the equivalence between star-free languages and first-order definable languages was obtained along the works of Thomas [31] and Ladner [11]. This suggests that it may be possible to obtain a direct proof. Such a proof has been proposed by Peikert [17] and Zuck [39]. In this paper, we give a self-contained algebraic proof.

In the case of finite words, Perrin and Pin and the present author [3] gave such an algebraic proof using the wreath product and the well known theorem of Schützenberger which characterizes the star-free languages by their syntactic aperiodic monoids. Unfortunately, we were not able to extend this proof to the case of infinite words, mainly because of the definition of the wreath product.

Our proof now is therefore rather different but is also based on a theorem of Schützenberger which asserts that the star-free languages can be generated from letters using union, concatenation product, and star operations restricted to prefix languages with a bounded synchronization delay. In the first part of our proof, we give an algorithm to pass from a PTL formula to ω-languages, from which we deduce that the PTL definable ω-languages are star-free. In the second part, we prove that the class of PTL definable languages is a Boolean algebra closed under left concatenation product with star-free languages of finite words; it follows that the class of star-free languages of infinite words is included in the class of PTL definable languages.

1. Preliminaries

1.1. Propositional temporal logic

Let A be a finite alphabet. In propositional temporal logic, formulas are built up on the following vocabulary:

- a set P of atomic propositions $\{P_a \mid a \in A\}$,
- two constants T and F,
- Boolean connectives $\vee$, $\wedge$, $\neg$,
- temporal operators $\bigcirc$ (Next), W (Always), $\Diamond$ (Eventually) and U (Until).

The formation rules are:

T, F and P_a are formulas for any $a \in A$; if φ and ψ are formulas then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\bigcirc\varphi$, $\Diamond\varphi$, $(\mathrm{W}\varphi)$ and $\varphi \mathrm{~U~} \psi$ are formulas.

Semantics are now defined by induction on the formation rules as follows:

let $u \in A^\omega$, $i \geq 1$; $\langle u, i\rangle \models \varphi$ will denote "u satisfies the formula φ at the instant i"

$\langle u, i\rangle \models P_a$	if the ith letter of u is an a,
$\langle u, i\rangle \models \bigcirc\varphi$	if $\langle u, i+1\rangle \models \varphi$,
$\langle u, i\rangle \models \Diamond\varphi$	if there exists j, $i \leq j$, $\langle u, j\rangle \models \varphi$,
$\langle u, i\rangle \models \mathrm{W}\varphi$	if for every j, $i \leq j$, $\langle u, j\rangle \models \varphi$,
$\langle u, i\rangle \models \varphi \mathrm{~U~} \psi$	if there exists k, $i \leq k$, such that $\langle u, k\rangle \models \psi$ and for every j, $i \leq j < k$, $\langle u, j\rangle \models \varphi$.

For example if $A = \{a, b, c\}$ then $\langle ababab(bc)^\omega, 1\rangle \models (P_a \vee P_b) \mathrm{~U~} P_c$.

If φ is a temporal formula, u is said to satisfy φ if $\langle u, 1\rangle \models \varphi$ and $L(\varphi)$ will denote the set of all the words satisfying φ. For example, if $A = \{a, b\}$,

$$L(P_a) = aA^\omega, \qquad L(\mathrm{W}[(P_a \supset \bigcirc P_b) \wedge (P_b \supset \bigcirc P_a)] \wedge P_a) = (ab)^\omega,$$

$$L(\Diamond(P_a \wedge \bigcirc P_b)) = A^* abA^\omega.$$

There exists a link between temporal operators and operations on sets:

$$L(P_a) = aA^\omega, \qquad L(\bigcirc\varphi) = AL(\varphi), \qquad L(\Diamond\varphi) = A^* L(\varphi);$$

for the Until operator U, we define the following operation (denoted $\mathcal{U}$): given two sets L and K both of finite words or both of infinite words,

$$L\,\mathcal{U}\,K = \{x \in A^* \mid x = uv \text{ where } v \in K \text{ and } u'v \in L \text{ for every right factor of } u\}.$$

The following relation stands: $L(\varphi \text{ U } \psi) = L(\varphi)\ \mathcal{U}\ L(\psi)$ for all temporal formulas φ and ψ.

1.2. Semigroups and infinite words

In this section, we briefly recall the algebraic tools we need to study ω-languages. More details can be found in [19, 20].

A semigroup S is a set equipped with an associative multiplication. A finite semigroup S is *aperiodic* if there exists an integer n such that, for every $x \in S$, $x^n = x^{n+1}$. Given two elements s and t of S, we set $st^{-1} = \{x \in S \mid xt = s\}$.

Definition 1.1. Let S be a finite semigroup. $(s, e) \in S \times S$ is a *linked pair* of S if $se = e$ and $e^2 = e$. Let $L(S)$ denote the set of all the linked pairs of S. Let S be a finite semigroup and $\varphi : A^+ \to S$ be a morphism. A *φ-simple* subset is a nonempty subset of A^ω of the form $s\varphi^{-1}[e\varphi^{-1}]^\omega$ where $(s, e) \in S \times S$ is a linked pair of S.

Definition 1.2. Let $L \subset A^\omega$. L is *recognized* by a finite semigroup S if there exists a morphism $\varphi : A^+ \to S$ such that L is a finite union of φ-simple subsets. L is also said to be recognized by φ.

There is a sharper definition.

Definition 1.3. Let S be a finite semigroup, $\varphi : A^+ \to S$ a morphism and $L \subset A^\omega$. L is *saturated* by φ (or S) if, for every $(s, t) \in S \times S$,

$$L \cap s\varphi^{-1}[t\varphi^{-1}]^\omega = \emptyset \quad \text{or} \quad s\varphi^{-1}[t\varphi^{-1}]^\omega \subset L.$$

(One can deal with linked pairs only, i.e. use only φ-simple subsets).

It is easy to see that if L is saturated by S then L is recognized by S, but the converse is not true.

The following result is a consequence of Ramsey's Theorem and will often be used in the sequel.

Lemma 1.4 (Perrin [19]). *Let A be a possible infinite alphabet. Let S be a finite semigroup and $\varphi : A^+ \to S$ be a morphism. For each word $x \in A^\omega$, there exists a linked pair (s, e) such that $x \in s\varphi^{-1}[e\varphi^{-1}]^\omega$.*

1.3. ω-star-free languages

The notion of star-free language can be extended to infinite words in various ways, summarized in the following theorem.

Theorem 1.5 (Thomas [31], Perrin [20]). *Let $L \subset A^\omega$. The following conditions are equivalent*:

(1) *L is a finite union of languages of type XY^ω where X and Y^+ are star-free languages of A^*.*

(2) *There exists an aperiodic monoid which recognizes L.*

(3) *There exists an aperiodic monoid which saturates L.*

(4) *L belongs to the smallest Boolean algebra in A^ω that is closed under left concatenation with star-free languages of A^*.*

Any ω-language which satisfies one of the previous assertions is said to be star-free. The first and third characterizations will be used in the second section to prove that any PTL definable language is star-free, while the fourth one will be used when proving the reverse assertion in the third section.

1.4. A theorem of Schützenberger

We first recall the definition of a set with a bounded synchronization delay, which comes from the theory of codes [1].

A language $L \subset A^*$ has a *bounded synchronization delay* (b.s.d.) if there exists a positive integer d such that for every $u \in L^d$, for every $m_1, m_2 \in A^*$,

$$m_1 u m_2 \in L^* \Rightarrow m_1 u, \; u m_2 \in L^*.$$

The smallest integer d verifying this condition is called the synchronization delay of L. A language L has synchronization delay 0 if, for every $m_1, m_2 \in A^*$, $m_1 m_2 \in L^*$ implies $m_1, m_2 \in L^*$.

For example, any subset B of the alphabet A has synchronization delay 0, $B^*(A \backslash B)$ has synchronization delay 1. But $(a^2)^*$ has no b.s.d. for any letter $a \in A$.

The previous definition is simplified when we restrict ourselves to prefix codes: a set $L \subset A^*$ is a prefix code if L does not contain any proper left factor of its words. Then, L has a b.s.d. if there exists a positive integer d such that for every $u \in L^d$, for every $m_1, m_2 \in A^*$, $m_1 u m_2 \in L^* \Rightarrow m_1 u, \; m_2 \in L^*$.

The theorem of Schützenberger can now be given.

Theorem 1.6 (Schützenberger [28]). *The class of star-free languages of finite words is the smallest class of languages that contains $\emptyset$, $\{1\}$, $\{a\}$ for any letter a and that is closed under union, concatenation product and star operations restricted to prefix codes with a bounded synchronization delay.*

2. PTL definable languages are star-free

The aim of this section is to prove the following result.

Theorem 2.1. *$L(\varphi)$ is star-free for any temporal formula φ.*

Proof. This result is proved by induction on the structure of temporal formulas. If $\varphi = \mathrm{T}$ then $L(\varphi) = A^\omega$ is star-free; if $\varphi = P_a$ then $L(\varphi) = aA^\omega$ is star-free.

Let φ and ψ be temporal formulas such that $L(\varphi)$ et $L(\psi)$ are star-free. One has

$$L(\varphi \vee \psi) = L(\varphi) \cup L(\psi), \qquad L(\neg\varphi) = A^\omega \setminus L(\varphi),$$
$$L(\bigcirc\varphi) = AL(\varphi), \qquad L(\Diamond\varphi) = A^* L(\varphi),$$

and these languages are star-free according to Theorem 1.5(4).

It remains to prove that the language $L(\varphi \mathrm{U} \psi)$ is star-free. In the following result $L(\varphi \mathrm{U} \psi)$ is described using the two languages $L(\varphi)$ and $L(\psi)$. Let φ and ψ be temporal formulas such that $L(\varphi)$ et $L(\psi)$ are star-free. Then there exists an aperiodic semigroup S, a morphism $g : A^+ \to S$ which saturates $L(\varphi)$ and a subset P_φ of $S \times S$ such that

$$L(\varphi) = \bigcup_{(s,e) \in P_\varphi \cap L(S)} sg^{-1}[eg^{-1}]^\omega.$$

Proposition 2.2. *With the above notations, one has*

$$L(\varphi \mathrm{U} \psi) = L(\psi) \cup \bigcup_{(n,f) \in L(S)} \{A^* \setminus A^*[A^* \setminus X_{n,f}]\} . \{L(\psi) \cap ng^{-1}[fg^{-1}]^\omega\},$$

with $X_{n,f} = \bigcup_{s \in S; (s,f) \in P_\varphi} (sn^{-1})g^{-1}$.

Proof. Let K denote the right-hand term of the equality: we first prove that $L(\varphi \mathrm{U} \psi) \subset K$. Let $x \in L(\varphi \mathrm{U} \psi)$. Then either $x \in L(\psi)$ and $x \in K$, either $x \notin L(\psi)$ and x can be written $x = a_1 a_2 \ldots$ with the following conditions: there exists a positive integer k such that $u = a_{k+1} a_{k+2} \ldots \in L(\psi)$ and, for $1 \leqslant i \leqslant k$, $a_i \ldots a_k u \in L(\varphi)$. According to Lemma 1.4, there exists a linked pair $(n,f) \in S \times S$ such that $u \in ng^{-1}[fg^{-1}]^\omega$. It follows that $u \in L(\psi) \cap ng^{-1}[fg^{-1}]^\omega$, and it suffices to prove that

$$a_1 \ldots a_k \in \{A^* \setminus A^*[A^* \setminus X_{n,f}]\}.$$

The following lemma will enlighten the meaning of the previous formula.

Lemma 2.3. *Let X be a language of A^*. Then $(A^* \setminus A^*(A^* \setminus X))$ is the set of words all of whose right factors are in X.*

Proof. It is easy to see that $A^*(A^* \setminus X)$ is the set of words which have no right factor in X. $\square$

Therefore, it remains to prove that, for any $1 \leqslant i \leqslant k$,

$$(*) \qquad a_i \ldots a_k \in \bigcup_{s \in S; (s,f) \in P_\varphi} (sn^{-1})g^{-1}.$$

Let B_i denote a possibly infinite alphabet in bijection with the set $\{a_i, \ldots, a_k\} \cup \{n, f\}g^{-1}$ and let $g_i : B_i \to S$ be the morphism defined by

$$a_j g_i = a_j g \quad \text{for } i \leqslant j \leqslant k, \qquad v g_i = \begin{cases} n & \text{if } v \in ng^{-1}, \\ f & \text{if } v \in fg^{-1}. \end{cases}$$

Notice that $vg_i = vg$ for every $v \in B_i$.

Since the word u belongs to $ng^{-1}[fg^{-1}]^\omega$, the word $a_i \ldots a_k u$ can be factorized on the alphabet B_i as

$$a_i \ldots a_k u = (a_i)(a_{i+1}) \ldots (a_k)(v_0)(v_1) \ldots (v_n) \ldots$$

with $u = v_0 v_1 \ldots v_n \ldots$, $v_0 \in ng^{-1}$ and $v_r \in fg^{-1}$ for $r > 0$.

It follows from Lemma 1.4 that there exists a linked pair $(s_i, e_i) \in S \times S$ such that

$$(a_i)(a_{i+1}) \ldots (a_k)(v_0)(v_1) \ldots (v_n) \ldots \in s_i g_i^{-1}[e_i g_i^{-1}]^\omega.$$

Using the definition of g_i one deduces that $a_i \ldots a_k u \in s_i g^{-1}[e_i g^{-1}]^\omega$. As g saturates $L(\varphi)$, $s_i g^{-1}[e_i g^{-1}]^\omega$ is included in $L(\varphi)$ and $(s_i, e_i) \in P_\varphi$. The word $a_i \ldots a_k u$ can be factorized using only letters of the alphabet B_i as

$$a_i \ldots a_k u = (a_i)(a_{i+1}) \ldots (a_k) v_0 v_1 \ldots v_n \ldots \quad \text{with } v_0 \in ng^{-1} \text{ and } v_j \in fg^{-1} \text{ for } j > 0,$$

and "overfactorized" by putting together some factors of the previous factorization as $a_i \ldots a_k u = t_0 \ldots t_n \ldots$ where $t_0 \in s_i g^{-1}$, $t_j \in e_i g^{-1}$ for $j > 0$ and $t_j \in B_i^*$ for $j \geqslant 0$ (see Fig. 1). By choosing j large enough, the word t_j can be written as the product of words $v_k \in fg^{-1}$, as $t_j = v_{j,1} \ldots v_{j,r_j}$.

Therefore, $e_i = t_j g = (v_{j,1})g \ldots (v_{j,r_j})g = f$, whence $e_i = f$ and $(s_i, f) \in P_\varphi$.

Moreover, the word $a_i \ldots a_k$ is a left factor of a word $t_0 \ldots t_q$ and therefore

$$a_i \ldots a_k v_0 v_1 \ldots v_l = t_0 t_1 \ldots t_q \quad \text{for some } l \geqslant 0.$$

Thus $(a_i \ldots a_k)ghf^l = s_i f^q$. Since $f^2 = f$, $nf = f$ and $s_i f = s_i$, the following equality holds: $(a_i \ldots a_k)gn = s_i$. This implies that $(a_i \ldots a_k)g \in s_i n^{-1}$ and so $a_i \ldots a_k \in (s_i n^{-1})g^{-1}$, which proves (∗) and the inclusion $L(\varphi \text{ U } \psi) \subset K$.

It remains to prove the opposite inclusion: $K \subset L(\varphi \text{ U } \psi)$. Let $x \in K$. If $x \in L(\psi)$, $x \in L(\varphi \text{ U } \psi)$. Otherwise there exists a linked pair $(n, f) \in S \times S$ such that $x = vw$, where

$$w \in L(\psi) \cap ng^{-1}[fg^{-1}]^\omega \quad \text{and} \quad v \notin A^*[A^* \backslash X_{n,f}].$$

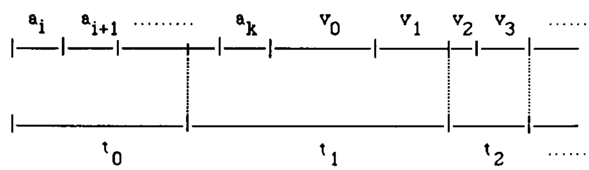

Fig. 1.

w can be factorized as $w = w_0 w_1 \dots w_n \dots$ with $w_0 g = n$ and $w_i g = f$ for $i > 0$. On the other hand, $v = a_1 \dots a_k$ and, by Lemma 2.3, for $1 \leq i \leq k$, $a_i \dots a_k \in X_{n,f}$. According to the definition of $X_{n,f}$, there exists $s_i \in S$ such that $(s_i, f) \in P_\varphi$ and $a_i \dots a_k \in (s_i n^{-1}) g^{-1}$. It follows in particular that $(a_i \dots a_k w_0) g = (a_i \dots a_k) g n = s_i$ and hence $a_i \dots a_k w \in s_i g^{-1} [f g^{-1}]^\omega$. Since $(s_i, f) \in P_\varphi$, $a_i \dots a_k w \in L(\varphi)$. In particular, since $x = a_i \dots a_k w$, $x \in L(\varphi \text{ U } \psi)$. □

3. ω-star-free languages are PTL definable

This section is devoted to star-free languages of infinite words. It is proved that such languages can be defined by temporal formulas.

Theorem 3.1. *Let $\mathscr{C}$ be the class of languages of A^ω that can be defined by a temporal formula. Then the class of star-free languages of A^ω is included in $\mathscr{C}$.*

Proof. The characterization of star-free languages of infinite words given by Theorem 1.5(4) will be used to prove this result. Consequently, it suffices to prove that $\mathscr{C}$ is a Boolean algebra closed under left concatenation with star-free languages of A^*. It is easy to see that $\mathscr{C}$ a Boolean algebra. Indeed, A^ω is equal to $L(\text{T})$, whence $A^\omega \in \mathscr{C}$. Moreover, if L and K are two languages of $\mathscr{C}$ then L and K can be defined by two temporal formulas respectively, say, φ and ψ. Hence $A^\omega \setminus L = L(\neg\varphi)$ and $L \cup K = L(\varphi \vee \psi)$.

It remains to prove that if S is a star-free language and if L is a language of $\mathscr{C}$, then SL belongs to $\mathscr{C}$. For this purpose, we construct a temporal formula defining SL as a function of the formula φ defining L, using the following definitions adapted from [17].

Definition 3.2. Let ψ be a temporal formula, let L_1, L_2 be two star-free languages of A^* and let L be a prefix language with a bounded synchronization delay, say d. The following temporal formulas are defined:

(1) $\varphi(\emptyset, \psi) = \text{F}$, $\varphi(1, \psi) = \psi$,

(2) $\varphi(a, \psi) = P_a \wedge \bigcirc\psi$,

(3) $\varphi(L_1 \cup L_2, \psi) = \varphi(L_1, \psi) \vee \varphi(L_2, \psi)$,

(4) $\varphi(L_1 L_2, \psi) = \varphi(L_1, \varphi(L_2, \psi))$,

(5) if $d = 0$,

$$\varphi(L^*, \psi) = \psi \vee [\varphi(L, \text{T}) \text{ U } \varphi(L, \psi)],$$

and if $d > 0$,

$$\varphi(L^*, \psi) = \psi \vee \varphi(L, \psi) \vee \cdots \vee \varphi(L^{2d-1}, \psi)$$
$$\vee [\varphi(L^d, \text{T}) \wedge \{(\varphi(L^{d+1}, \text{T}) \vee \neg\varphi(L^d, \text{T})) \text{ U } \varphi(L^{2d}, \psi)\}].$$

Let $\mathcal{S}$ denote the class of languages S of finite words such that, for every temporal formula ψ, $L(\varphi(S, \psi)) = SL(\psi)$. $\mathcal{S}$ satisfies the following property.

Proposition 3.3. *$\mathcal{S}$ contains the star-free languages of finite words.*

Proof. According to Theorem 1.6, we only have to prove that $A^*\mathcal{S}$ contains $\emptyset$, $\{1\}$, $\{a\}$ for every letter a of the alphabet, and that $\mathcal{S}$ is closed under union, concatenation, and star operations restricted to prefix subsets with a bounded synchronisation delay.

(1) Obviously $\emptyset, \{1\} \in A^*\mathcal{S}$.

(2) Let $a \in A$; then

$$L(\varphi(a, \psi)) = L(P_a \wedge \bigcirc\psi) = L(P_a) \cap AL(\psi) = aA^\omega \cap AL(\psi) = aL(\psi)$$

and therefore $\{a\} \in A^*\mathcal{S}$ for every $a \in A$.

(3) Let $S_1, S_2 \in S^*\mathcal{S}$. Then

$$\begin{aligned} L(\varphi(S_1 \cup S_2, \psi)) &= L(\varphi(S_1, \psi) \vee \varphi(S_2, \psi)) \\ &= L(\varphi(S_1, \psi)) \cup L(\varphi(S_2, \psi)) \\ &= S_1L(\psi) \cup S_2L(\psi) = (S_1 \cup S_2)L(\psi) \end{aligned}$$

and $S_1 \cup S_2 \in A^*\mathcal{S}$.

(4) Moreover, $L(\varphi(S_1S_2, \psi)) = L(\varphi(S_1, \varphi(S_2, \psi))) = S_1L(\varphi(S_2, \psi))$ because $S_1 \in A^*\mathcal{S}$ and $L(\varphi(S_1S_2, \psi)) = S_1S_2L(\psi)$ because $S_2 \in A^*\mathcal{S}$. Consequently, $S_1S_2 \in A^*\mathcal{S}$.

Now let us consider a prefix language S of $A^*\mathcal{S}$ with a synchronization delay d. If $d = 0$, we have, since $S \in A^*\mathcal{S}$,

$$L(\varphi(S^*, \psi)) = L(\psi) \cup [SA^\omega \,\mathcal{U}\, SL(\psi)].$$

Now if $d \neq 0$, $S^k \in A^*\mathcal{S}$ for all $k > 0$, and thus

$$\begin{aligned} L(\varphi(S^*, \psi)) = {} & L(\psi) \cup SL(\psi) \cup \cdots \cup S^{2d-1}L(\psi) \\ & \cup [S^dA^\omega \cap \{(S^{d+1}A^\omega \cup (A^\omega \backslash S^dA^\omega)) \,\mathcal{U}\, S^{2d}L(\psi)\}]. \end{aligned}$$

The following lemma concludes the proof.

Lemma 3.4. *For every prefix code S with a b.s.d., for every language L in A^ω, if $d = 0$ then $S^*L = L \cup [SA^\omega \,\mathcal{U}\, SL]$, and if $d \neq 0$ then*

$$\begin{aligned} S^*L = {} & L \cup SL \cup \cdots \cup S^{2d-1}L \\ & \cup [s^dA^\omega \cap \{(S^{d+1}A^\omega \cup (A^\omega \backslash S^dA^\omega)) \,\mathcal{U}\, S^{2d}L\}]. \end{aligned}$$

Proof. We first treat the case $d = 0$. Let $u \in S^*L$. Then $u \in L$ or u can be written as $u = u_1u_2 \ldots u_nvw$ with $n \geq 0$, $u_i \in S$ for $1 \leq i \leq n$, $v \in S$ and $w \in L$. Since $vw \in SL$, it suffices to prove that, for every right factor m of the word $u_1u_2 \ldots u_n$, $mvw \in SA^\omega$. Since m is a right factor of $u_1u_2 \ldots u_n$, there exists a $j \in \{1, \ldots, n\}$, and there exists a possibly empty word m_1 such that $m_1m = u_j \ldots u_n$. then $m_1mv \in S^*$. Since S is of delay 0, it follows $m_1, mv \in S^*$ and moreover, as $v \neq 1$, $mv \in S^+$. Finally, $mvw \in SA^\omega$, and $u \in SA^\omega \,\mathcal{U}\, SL$.

Conversely, let $u \in L \cup [SA^\omega \,\mathcal{U}\, SL]$. If $u \in L$ then $u \in S^*L$. Otherwise, $u \in SA^\omega \,\mathcal{U}\, SL$ and either $u \in SL$ and then $u \in S^*L$, or $u = a_1 \dots a_n vw$ with $v \in S$, $w \in L$, and for every i, $1 \leqq i \leqq n$, $a_i \in A$ and $a_i \dots a_n vw \in SA^\omega$. Since, for every i, $a_i \dots a_n vw \in SA^\omega$, there exists a (possibly empty) word m_i such that $a_i m_i \in S$. Since S is of delay 0, $a_i \in S$ for every integer i, $1 \leqslant i \leqslant n$. Then $u \in S^*L$.

We now turn to the case $d > 0$. Put

$$K = L \cup SL \cup \dots \cup S^{2d-1}L$$
$$\cup [S^d A^\omega \cap \{(S^{d+1}A^\omega \cup (A^\omega \backslash S^d A^\omega)) \,\mathcal{U}\, S^{2d}L\}].$$

We first prove the inclusion $K \subset S^*L$. Let $u \in K$. If $u \in L \cup SL \cup \dots \cup S^{2d-1}L$, then $u \in S^*L$. Otherwise, either $u \in S^{2d}L$ and $u \in S^*L$, or $u \in S^d A^\omega$ and u can be written as $a_1 \dots a_n v_1 v_2 w$ where $v_1, v_2 \in S^d$, $w \in L$ and, for $1 \leqslant j \leqslant n$, $n > 0$,

$$(*) \qquad a_j \dots a_n v_1 v_2 w \in S^{d+1}A^\omega \cup (A^\omega \backslash S^d A^\omega).$$

The result is a consequence of the following lemma.

Lemma 3.5. *Let $x = a_1 \dots a_n$, v_1, $v_2 \in S^d$, $w \in L$ be words such that*
(1) $a_1 \dots a_n v_1 v_2 w \in S^d A^\omega$ *and*
(2) *for* $1 \leqslant j \leqslant n$, $a_j \dots a_n v_1 v_2 w \in S^{d+1}A^\omega \cup (A^\omega \backslash S^d A^\omega)$.
Then $a_1 \dots a_n v_1 v_2 w \in S^*L$.

Proof. By induction on n. If $n = 1$, then $u = a_1 v_1 v_2 w$. Since $u \in S^d A^\omega$, condition (2) with $j = 1$ turns to $u \in S^{d+1}A^\omega$. Therefore u can be written as $u = t_1 \dots t_{d+1} x$ where each t_i is a word in S and x is a word in A^ω. Several possibilities occur:

(a) If $t_1 = a_1$ then $u \in S^*L$, because $u = a_1 v_1 v_2 w$ with a_1, v_1, $v_2 \in S^*$ and $w \in L$. Otherwise there are still two possibilities:

(b1) Either there exists an integer j, $1 \leqslant j \leqslant d+1$, such that $a_1 v_1 v_2$ is a left factor of $t_1 \dots t_j$; then there exists a word m such that $a_1 v_1 v_2 m = t_1 \dots t_j \in S^*$. As $v_2 \in S^d$ and as S is prefix b.s.d. of positive delay d, it follows $a_1 v_1 v_2 \in S^d$ and $u \in S^*L$.

(b2) Or there exist two words m_1, m_2 such that $m_1 t_2 \dots t_{d+1} m_2 = v_1 v_2 \in S^*$. Then $t_2 \dots t_{d+1} \in S^d$ so $m_2 \in S^*$ and $u = t_1 \dots t_{d+1} m_2 w \in S^*L$.

For the general case, condition (2) with $j = 1$ is again applied and u can be written as $u = t_1 \dots t_{d+1} x$ where each t_i is a word in S. Similar cases occur:

(a) If there exists an integer j, $1 \leqslant j \leqslant n-1$, such that $t_1 = a_1 \dots a_j$, then the induction hypothesis is applied to the word $y = a_{j+1} \dots a_n v_1 v_2 w$; indeed, since $t_2 \dots t_{d+1}$ is a left factor of y, y belongs to $S^d A^\omega$ and (1) is satisfied. (2) is clearly satisfied. Then $y \in S^*L$ and $u = t_1 y \in S^*L$. Otherwise there are still two cases:

(b1) Either there exists an integer i, $1 \leqslant i \leqslant d+1$, such that $a_1 \dots a_n v_1 v_2$ is a left factor of $t_1 \dots t_i$; then there exists a word m such that $a_1 \dots a_n v_1 v_2 m = t_1 \dots t_i \in S^*$. As $v_2 \in S^d$, it follows that $a_1 \dots a_n v_1 v_2 \in S^*$ and $u \in S^*L$. This case can be illustrated by Fig. 2.

(b2) Or there exist two words m_1, m_2 such that $m_1 t_2 \dots t_{d+1} m_2 = v_1 v_2 \in S^*$, as shown in Fig. 3. Yet $t_2 \dots t_{d+1} \in S^d$; then $m_2 \in S^*$ and $u = t_1 \dots t_{d+1} m_2 w \in S^*L$. This concludes that proof of Lemma 3.5. $\square$

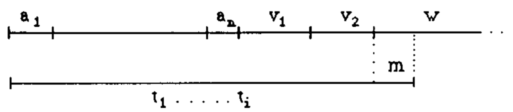

Fig. 2.

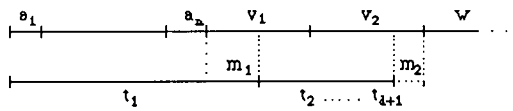

Fig. 3.

Proof of Lemma 3.4 (*continued*). It remains to prove the inclusion $S^*L \subset K$. Let $u \in S^*L$. If $u \in L \cup SL \cup \cdots \cup S^{2d-1}L$, then $u \in K$. If $u \in S^{2d}L$ then $u \in S^{d+1}A^\omega$ and $u \in K$. Otherwise, u can be written as $u = u'v_1v_2w$ where $u' \in S^*$, v_1, $v_2 \in S^d$ and $w \in L$. It suffices to prove that

$$u \in S^dA^\omega \cap \{(S^{d+1}A^\omega \cup (A^\omega \backslash S^dA^\omega))\ \mathcal{U}\ S^{2d}L\}.$$

Clearly, $u \in S^dL$, and it remains to prove that, for every suffix z of u',

$$f = zv_1v_2w \in S^{d+1}A^\omega \cup (A^\omega \backslash S^dA^\omega)$$

which is equivalent to showing that $zv_1v_2w \in S^dA^\omega$ implies $zv_1v_2w \in S^{d+1}A^\omega$. Suppose $f \in S^dA^\omega$. Then $f = t_1 \ldots t_dx \in S$ and $x \in A^\omega$. Set $t = t_1 \ldots t_d$. there are two cases:

(i) *t is a proper prefix of zv_1v_2*. Set $u' = pz$. Then

$$u = pzv_1v_2w = ptx$$

and there exists a $m_1 \in A^+$ such that $ptm_1 = pzv_1v_2 = u'v_1v_2 \in S^*$. Since $t \in S^d$, it follows that $m_1 \in S^+$. Thus $zv_1v_2 = tm_1S^{d+1}A^*$ and $f = zv_1v_2w \in S^{d+1}A^\omega$.

(ii) *zv_1v_2 is a prefix of t*. Then $t = zv_1v_2m'$ where $m' \in A^*$, and since $t \in S^+$ and $v_2 \in S^d$, it follows that $zv_1v_2 \in S^+$. Now since $v_1 \in S^d$, it follows that $zv_1 \in S^+$. Therefore

$$f = zv_1v_2w \in S^{d+1}A^\omega.$$

This concludes the proof of Lemma 3.4. □

Proof of Theorem 3.1 (*conclusion*). Lemma 3.4 shows that if S is a prefix set of $A^*\mathcal{S}$ with b.s.d., then S^* belongs to $A^*\mathcal{S}$. This proves Proposition 3.3. Now, by Theorem 1.6, $A^*\mathcal{S}$ contains the star-free sets of A^*. It follows that for every star-free language S, for every language L in the class $\mathcal{C}$, the language SL is defined by the formula $\varphi(S, \psi)$ where ψ is a temporal formula that defines L. The class $\mathcal{C}$ is then closed under left concatenation with a star-free language in A^*. By Theorem 1.5, it follows that the class of star-free ω-languages of A^ω is contained in $\mathcal{C}$ (Theorem 3.1). □

Conclusion

The algebraic proof of the equivalence between PTL and star-free languages of infinite words is therefore complete. It is probably possible to extend this result to larger classes of languages by using on the one hand some more powerful temporal operators (see Wolper [37], for instance) and on the other hand, some other results of Schützenberger. This will be a subject of a future paper.

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