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A variational model for stress analysis in cracked laminates with arbitrary symmetric lay-up under general in-plane loading

M. Hajikazemi *, M.H. Sadr

Aerospace Engineering Department and Centre of Excellence in Computational Aerospace Engineering, Amirkabir University of Technology, Tehran, Iran

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ABSTRACT

The present research work presents a variational approach for stress analysis in a general symmetric laminate, having a uniform distribution of ply cracks in a single orientation, subject to general in-plane loading. Using the principle of minimum complementary energy, an optimal admissible stress field is derived that satisfies equilibrium, boundary and traction continuity conditions. Natural boundary conditions have been derived from the variational principle to overcome the limitations of the existing methodology on the analysis of general symmetric laminates. Thus, a systematic way to formulate boundary value problem for general symmetric laminates containing many cracked and un-cracked plies has been derived, and appropriate mathematical tools can then be employed to solve them. The obtained results are in excellent agreement with the available results in the literature. In the field of matrix cracks analysis for symmetric laminates, the present formulation is the most complete variational model developed so far.

1. Introduction

In many branches of engineering, like aerospace and civil engineering, composite laminates are increasingly used as structural components. The observed damage process of laminated composites during operation is rather complex consisting of matrix cracks, delaminations, fiber-matrix debondings, fiber breakage, etc. However, matrix cracking parallel to the fiber direction on off-axis plies is usually the first damage mode observed for in-plane loading. Although matrix cracking is not critical from a final fracture point of view, its presence triggers the initiation of other damage modes like delamination and fiber breakage or provides pathways for entry of corrosive liquids that may subsequently lead to fracture. In addition, matrix cracks lead to stiffness reduction and stress redistribution to adjacent plies, which are needed for computation of fiber-dominated failure modes and laminate strength. Moreover, the first step in any matrix cracking analysis is to obtain stress state for laminate containing matrix cracks. Therefore, the capability in analyzing stress field of a cracked laminate has played an important role in the development of damage mechanics for laminated composites. Such capability has evolved over decades to build up in terms of accuracy and versatility, from ply-discount method in early days to shear-lag model (Garrett and Bailey, 1977), stress-based variational approach (Hashin, 1985), finite element (Herakovich et al., 1988), stress transfer model (McCartney, 1992), finite strip method (Li et al., 1994), displacement-based variational approach (Berthelot et al., 1996), etc. The outcome of such micromechanical stress analyses offer supports to other approaches in damage mechanics, such as self-consistent approach (Laws et al., 1983), continuum damage mechanics (Lundmark and Varna, 2005; Singh and Talreja, 2008) and discrete damage mechanic (Barbero et al., 2011). The reader is referred to the following publications for a detailed review of the developments in the area of matrix cracking analysis (Nairn, 2000a; Berthelot, 2003).

Among the approximate analytical models, the stress-based variational approach (Hashin, 1985, 1986, 1987; Nairn, 1989; Nairn and Hu, 1992; Kuriakose and Talreja, 2004; Vinogradov and Hashin, 2010) and stress transfer model (McCartney, 1992, 2000; McCartney and Pierse, 1997; Katerelos et al., 2006) have shown to be more accurate from the micromechanical point of view (Berthelot, 1997; Nairn, 2000a,b) in comparison to other methods (e.g. shear-lag).

Stress transfer model of McCartney (1992) is a 2D analysis, which considers the stress and displacement components based on the generalized plane strain assumptions. The analysis satisfies the equilibrium equations, the interface continuities and the boundary conditions. However, some of the stress-strain relations and boundary conditions are satisfied in an average sense. McCartney (2000) has extended the stress transfer model to analyze general symmetric laminates with arbitrary stacking sequence, having a uniform distribution of ply cracks in a single orientation, under general in-plane loading. The technique is basically analytical,

* Corresponding author. Address: 424 Hafez Ave, Tehran 15875-4413, Iran. Tel.: +98 9121826367.

E-mail address: hajikazemi@aut.ac.ir (M. Hajikazemi).

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but because of the resulting complexity, the analysis must be handled numerically in some steps while making predictions of the behavior of laminate. The stress transfer model can use the ply refinement technique (Takeda et al., 2000; McCartney and Pierse, 1997) where each layer of the laminate is subdivided into plies having the same properties in order that important through the thickness variations of the stress and displacement components could be taken into account. Today, stress transfer model is considered as one of the most efficient, versatile and accurate methods to analyze stress field for the laminate containing matrix cracks.

Hashin (1985) has analyzed the stress distributions in cross ply cracked laminates under tension or shear using a stress-based variational approach. He has presented an approximate 2D stress representation, which automatically satisfies the equilibrium equations, interface and boundary conditions that must be satisfied by stress components. The so-called admissible stress field was then used in conjunction with variational techniques to minimize the complementary energy and thus to provide an optimal solution for the stress field. It has been revealed that the stress field obtained by the variational approach does an excellent job of predicting stiffness reduction and crack grow experiments (Vinogradov and Hashin, 2010; Nairn, 2000a,b). Nevertheless, the variational approach has mostly been used for treatment of either cross-ply laminates (Hashin, 1985, 1986, 1987; Nairn, 1989; Varna and Berglund, 1992; Nairn and Hu, 1992; Rebiere et al., 2001; Kuriakose and Talreja, 2004) or other symmetric laminates that by averaging out the off-axis plies are reduced to cross-ply (Joffe and Varna, 1999; Li and Lim, 2005). Recently, Vinogradov and Hashin (2010) have extended the capability of the variational approach to analyze stress field and consequently stiffness reduction of angle ply laminates. It should be noted that the mathematical model for all mentioned variational works involves effectively only two layers, one cracked and one un-cracked, representing a three-layered laminate after applying symmetry considerations. Therefore, these models do not have the capability of analyzing stress field for the laminates with multiple cracked and un-cracked layers, which cannot be simplified to a two-layer model using symmetry, due to the lack of boundary conditions for un-cracked layers. More recently, Li and Hafeez (2009) have overcome this drawback by introducing some boundary conditions as an outcome of variational procedure and translational symmetry (Li et al., 2009), called natural boundary conditions, in the terminology of variational calculus. As a result, the applicability of the variational approach has been extended fundamentally for considering multiple layers laminates. However, their model (Li and Hafeez, 2009) has only considered cross ply laminates under axial loading due to the assumed admissible stress field.

In the current work, an attempt has been made to extend the applicability of the variational approach for analyzing cracked symmetric laminates with arbitrary stacking sequence under general in-plane loading. An admissible stress field that satisfies equilibrium and all the boundary and continuity conditions is constructed and it is used in conjunction with the principle of minimum complementary energy to achieve the optimal stress state of a general symmetric cracked laminate with multiple cracked and un-cracked layers. A systematic way of evaluating governing equations is developed, which is completely analytical, and consequently, the model could enjoy the advantages of ply refinement technique. To donduct the analysis for considering general symmetric lay-ups, a set of natural boundary conditions is considered as an outcome of variational procedure and translational symmetry. Stress state of cracked laminate estimated by the suggested approach is in excellent agreement with the results obtained from the formulations of Hashin (1985) and Li and Hafeez (2009), which are special cases of the current formulation.

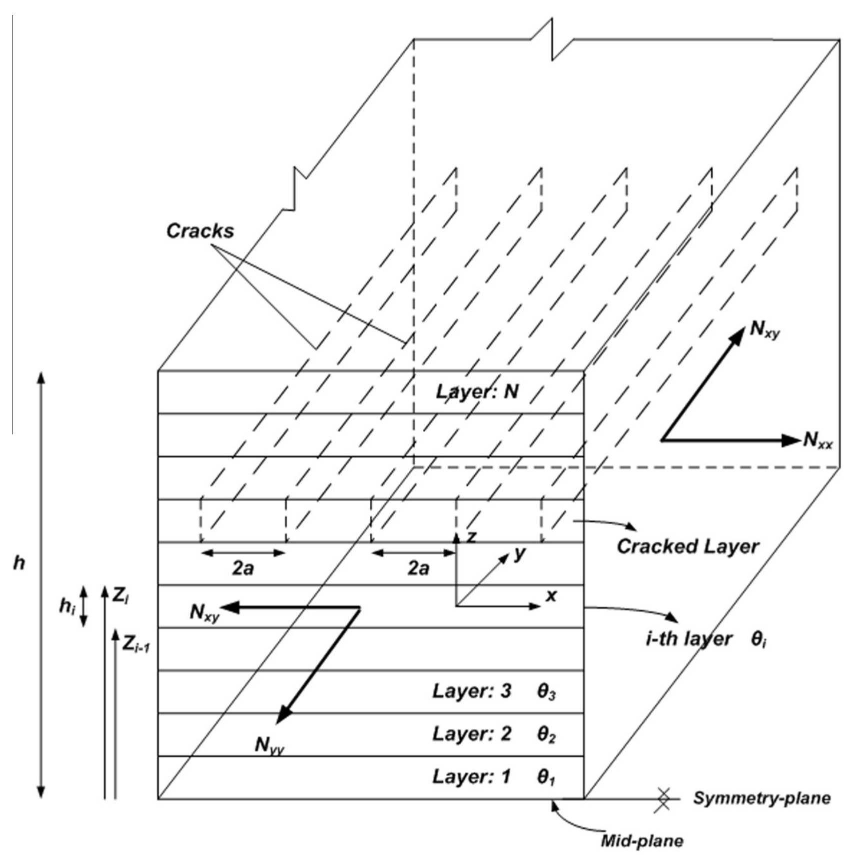

Fig. 1. Geometry of an arbitrary symmetric laminate containing cracks in one 90° layer (only the upper set of N layers ($z > 0$) is shown).

2. Theoretical formulation

Consider a symmetric multilayered laminate including 2N perfectly bonded layers, which can have any combination of orientations while the symmetry about the mid-plane of the laminate is preserved. As laminate symmetry is assumed, it is better to consider only the upper set of N layers as shown in Fig. 1. A global set of rectangular Cartesian coordinates is chosen having the origin at the center of the laminate as shown in Fig. 1. The x-direction defines the longitudinal or axial direction, the y-direction defines the in-plane transverse direction and the z-direction defines the direction through the thickness. The locations of the $N-1$ interfaces in one half of the laminate ($z>0$) are specified by $z=z_i$; $i=1,2,\ldots,N-1$. The mid-plane of the laminate is specified by $z=z_0=0$ and the external surface is demonstrated by $z=z_N=\text{h}$, where 2h is the total thickness of the laminate. The thickness of the ith layer is denoted by $h_i=z_i-z_{i-1}$. The orientation of the ith layer is specified by the angle θ_i (measured clockwise) between the x-axis and the fiber direction of this layer. The laminate must be such that the orientation of fibers in at least one set of plies is aligned in y-direction. This assumption is not a limitation as general in-plane loading conditions are considered so that if cracks form in another single orientation the laminate can rotate so that the crack planes are parallel to the y-axis and the applied stresses transform to values appropriate for the new orientation. The stress and strain components and also material properties associated with the ith layer are denoted by a superscript or subscript i. Some layers might have similar properties, so that through the thickness variations in the stress fields can adequately be modeled. We assume that the laminate can be infinitely extended in both x and y directions (see Fig. 1), so that the effect of edges is neglected.

The laminate is subjected to external uniform membrane loads of N_{xx}, N_{yy} and N_{xy} in the coordinate system of xyz associated with cracks. In an un-cracked laminate, the only nonzero components of the stress tensor defined in the coordinate system associated with cracks are $\sigma_{xx}^{0(i)}, \sigma_{yy}^{0(i)}, \sigma_{xy}^{0(i)}$, where the superscript 0 denotes the undamaged state and the superscript (i), $i=1,2,\ldots,N$, denotes the number of the layer. The stresses are spatially uniform within each layer and linear functions of the applied load of N_{xx}, N_{yy} and N_{xy}.

It is assumed that the ply crack distribution in damaged 90 plies is uniform, having a separation 2a, and that the cracks in each damaged 90 ply of the laminate are in the same plane. The cracked laminate can be seen as a sequence of laminate fragments, bounded by pairs of adjacent cracks (see Fig. 1). Further, it will be shown that each fragment has the same 'admissible' traction boundary conditions in the crack planes and hence can be treated separately. In Fig. 2 a fragment of length 2a is indicated and it will serve as an elementary cell for the construction of admissible stress field. The origin of the coordinate system is located in the mid length of the fragment. The geometry of the fragment is then symmetrical with respect to the xy and yz planes.

Following the approach developed by Hashin (1985), the stresses in the cracked material are represented as a superposition of the stresses in the un-cracked material and some yet unknown perturbation stresses due to the presence of the cracks.

$$\tilde{\sigma}_{mn}^{(i)}(\mathrm{X}) = \sigma_{mn}^{0(i)} + \sigma_{mn}^{(i)}(\mathrm{X}) \tag{1}$$

where $m, n = x, y, z$. The second term in Eq. (1) is the stress in the ith ply of undamaged laminate, which can be obtained from a simple analysis using classic laminate theory. The last term in Eq. (1) is the perturbation stress in ith ply, which in contrast to the stresses in the undamaged laminate, is a function of location. For a general symmetric laminate, all the components of the perturbation stress tensor are expected to be nonzero, even when some of the external membrane forces are not applied. It is necessary to find an admissible stress field that satisfies the equilibrium equations $\tilde{\sigma}_{mn,n}^{(i)}(\mathrm{X})=0$, traction boundary and continuity conditions, namely:

1. Zero traction condition on the external surfaces $z=\pm h$: $\tilde{\sigma}_{xz}^{(N)} = \tilde{\sigma}_{yz}^{(N)} = \tilde{\sigma}_{zz}^{(N)} = 0$.
2. Continuity condition at the interface between the plies at $z=z_i$, $i=1,2,\ldots,N-1$: $\tilde{\sigma}_{xz}^{(i)} = \tilde{\sigma}_{xz}^{(i+1)}$, $\tilde{\sigma}_{yz}^{(i)} = \tilde{\sigma}_{yz}^{(i+1)}$, $\tilde{\sigma}_{zz}^{(i)} = \tilde{\sigma}_{zz}^{(i+1)}$.
3. Zero traction condition on the crack surfaces $x=\pm a$: $\tilde{\sigma}_{xx}^{(k)} = \tilde{\sigma}_{xz}^{(k)} = \tilde{\sigma}_{xy}^{(k)} = 0$, where superscript ($k$) represents cracked plies. In addition, the stress field should balance the membrane forces applied to the laminate.

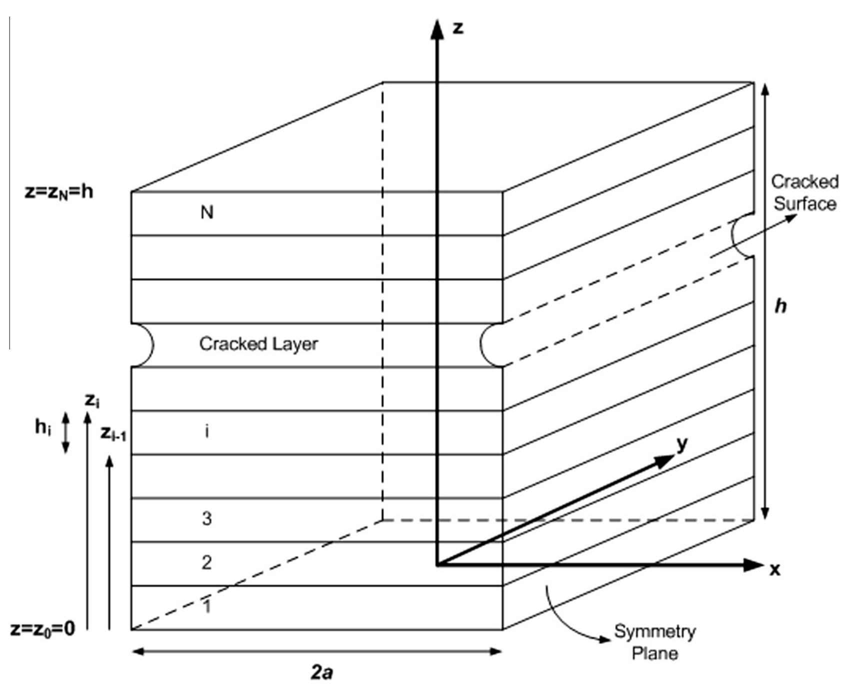

Fig. 2. Schematic of an elementary cell containing a cracked layer for the construction of admissible stress field.

It should be mentioned that in the coordinate system where the crack planes are parallel to the y-axis, the stress fields are independent of the y coordinate due to this fact that cracks extend across the entire width. Moreover, following the approach developed by Hashin (1985), we assume that the membrane perturbation stresses of each layer vary merely along the x direction, normal to the crack surfaces, and are denoted as:

$$\sigma_{xx}^{(i)}(\mathrm{X}) = -\varphi_i(x)/h_i \tag{2}$$

$$\sigma_{xy}^{(i)}(\mathrm{X}) = -\psi_i(x)/h_i \tag{3}$$

$$\sigma_{yy}^{(i)}(\mathrm{X}) = -\eta_i(x)/h_i \tag{4}$$

where $\varphi_i(x)$, $\psi_i(x)$ and $\eta_i(x)$ are unknown functions of x coordinate. The other components of the stress tensor are allowed to depend on both x and z coordinates.

Using Eqs. (2)–(4) the equilibrium equations $\tilde{\sigma}_{mn,n}^{(i)}(\mathrm{X}) = 0$ reduce to:

$$-\varphi_i'(x)/h_i + \sigma_{xz,z}^{(i)} = 0 \tag{5}$$

$$-\psi_i'(x)/h_i + \sigma_{yz,z}^{(i)} = 0 \tag{6}$$

$$\sigma_{xz,x}^{(i)} + \sigma_{zz,z}^{(i)} = 0 \tag{7}$$

The solution of the equilibrium equations can be written in the form of:

$$\sigma_{xz}^{(i)}(x,z) = \varphi_i'(x)(z - z_i)/h_i + f_i(x) \tag{8}$$

$$\sigma_{yz}^{(i)}(x,z) = \psi_i'(x)(z - z_i)/h_i + g_i(x) \tag{9}$$

$$\sigma_{zz}^{(i)}(x,z) = -\frac{1}{2h_i}\varphi_i''(x)(z - z_i)^2 - zf_i'(x) + j_i(x) \tag{10}$$

where $f_i(x)$, $g_i(x)$ and $j_i(x)$, $i = 1,2,\ldots,N$ are unknown functions, which will be determined later and primes denote derivatives with respect to x.

The external membrane forces applied to the laminate should be balanced. The load is applied so that N_{xx}, N_{yy} and N_{xy} remain constant as cracks appear in the laminate. Therefore, for an undamaged laminate:

$$N_{xx} = \int_{-h}^{h} \tilde{\sigma}_{xx}\,dz = 2\sum_{i=1}^{N}\sigma_{xx}^{0(i)}h_i \tag{11}$$

$$N_{xy} = \int_{-h}^{h} \tilde{\sigma}_{xy}\,dz = 2\sum_{i=1}^{N}\sigma_{xy}^{0(i)}h_i \tag{12}$$

$$N_{yy} = \int_{-h}^{h} \tilde{\sigma}_{yy}\,dz = 2\sum_{i=1}^{N}\sigma_{yy}^{0(i)}h_i \tag{13}$$

And for a cracked laminates, we have:

$$N_{xx} = \int_{-h}^{h} \tilde{\sigma}_{xx}\,dz = 2\sum_{i=1}^{N}(\sigma_{xx}^{0(i)} - \varphi_i(x)/h_i)h_i \tag{14}$$

$$N_{xy} = \int_{-h}^{h} \tilde{\sigma}_{xy}\,dz = 2\sum_{i=1}^{N}(\sigma_{xy}^{0(i)} - \psi_i(x)/h_i)h_i \tag{15}$$

$$N_{yy} = \int_{-h}^{h} \tilde{\sigma}_{yy}\,dz = 2\sum_{i=1}^{N}(\sigma_{yy}^{0(i)} - \eta_i(x)/h_i)h_i \tag{16}$$

Therefore, the following relations are valid:

$$\sum_{i=1}^{N}\varphi_i(x) = 0 \tag{17}$$

$$\sum_{i=1}^{N}\psi_i(x) = 0 \tag{18}$$

$$\sum_{i=1}^{N}\eta_i(x) = 0 \tag{19}$$

It can be seen that equilibrium of external forces provides three relationships between the perturbation functions as shown in Eqs. (17)–(19).

The boundary conditions for each cracked ply (k) on the crack surfaces $x = \pm a$ become:

$$\tilde{\sigma}_{xx}^{(k)}(x = \pm a) = 0 \Rightarrow \sigma_{xx}^{(k)}(\pm a, z) = -\sigma_{xx}^{0(k)} \tag{20}$$

$$\tilde{\sigma}_{xy}^{(k)}(x = \pm a) = 0 \Rightarrow \sigma_{xy}^{(k)}(\pm a, z) = -\sigma_{xy}^{0(k)} \tag{21}$$

$$\tilde{\sigma}_{xz}^{(k)}(x = \pm a) = 0 \Rightarrow \sigma_{xz}^{(k)}(\pm a, z) = 0 \tag{22}$$

The boundary conditions on the external surface of $z = z_N = h$ are:

$$\tilde{\sigma}_{xz}^{(N)}(x, z = h) = 0 \Rightarrow \sigma_{xz}^{(N)}(x, z = h) = 0 \tag{23}$$

$$\tilde{\sigma}_{yz}^{(N)}(x, z = h) = 0 \Rightarrow \sigma_{yz}^{(N)}(x, z = h) = 0 \tag{24}$$

$$\tilde{\sigma}_{zz}^{(N)}(x, z = h) = 0 \Rightarrow \sigma_{zz}^{(N)}(x, z = h) = 0 \tag{25}$$

Symmetry with respect to xy plane requires:

$$\tilde{\sigma}_{xz}^{(1)}(x, z = 0) = 0 \Rightarrow \sigma_{xz}^{(1)}(x, z = 0) = 0 \tag{26}$$

$$\tilde{\sigma}_{yz}^{(1)}(x, z = 0) = 0 \Rightarrow \sigma_{yz}^{(1)}(x, z = 0) = 0 \tag{27}$$

Traction continuity at N−1 interfaces ($z = z_i$, $i = 1,2,\ldots,N-1$) requires:

$$\tilde{\sigma}_{xz}^{(i)}(x, z = z_i) = \tilde{\sigma}_{xz}^{(i+1)}(x, z = z_i) \Rightarrow \sigma_{xz}^{(i)}(x, z = z_i) = \sigma_{xz}^{(i+1)}(x, z = z_i) \tag{28}$$

$$\tilde{\sigma}_{yz}^{(i)}(x, z = z_i) = \tilde{\sigma}_{yz}^{(i+1)}(x, z = z_i) \Rightarrow \sigma_{yz}^{(i)}(x, z = z_i) = \sigma_{yz}^{(i+1)}(x, z = z_i) \tag{29}$$

$$\tilde{\sigma}_{zz}^{(i)}(x, z = z_i) = \tilde{\sigma}_{zz}^{(i+1)}(x, z = z_i) \Rightarrow \sigma_{zz}^{(i)}(x, z = z_i) = \sigma_{zz}^{(i+1)}(x, z = z_i) \tag{30}$$

Substituting Eq. (26) into Eq. (8) and Eq. (27) into Eq. (9), it could be concluded that $f_1(x) = \varphi_1'(x)$ and $g_1(x) = \psi_1'(x)$. Moreover, Eqs. (28) and (29) provide enough recursive relations to find $f_i(x)$, $g_i(x)$, $i = 2,\ldots,N$, respectively. Therefore, substituting Eqs. (28) and (29) into Eqs. (8) and (9), it can be concluded that:

$$f_i(x) = \sum_{j=1}^{i}\varphi_j'(x) \tag{31}$$

$$g_i(x) = \sum_{j=1}^{i}\psi_j'(x) \tag{32}$$

Correspondingly, we have:

$$\sigma_{xz}^{(i)}(x,z) = \varphi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \varphi_j'(x) \tag{33}$$

$$\sigma_{yz}^{(i)}(x,z) = \psi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \psi_j'(x) \tag{34}$$

It can be easily verified that the traction boundary conditions in Eqs. (23) and (24) are automatically satisfied considering Eqs. (17) and (18), respectively. Li and Hafeez (2009) have shown that this is the nature of any models in which $\sigma_{xx}^{(i)}, \sigma_{xy}^{(i)}$ are assumed to remain constant through-the-thickness of each ply.

Substituting Eq. (25) into Eq. (10) and using Eqs. (31) and (32), one concludes that

$$j_N(x) = h\sum_{j=1}^{N} \varphi_j'' \tag{35}$$

Moreover, Eq. (30) provides enough recursive relations to find $j_i(x),\ i = 1, 2, \ldots, N-1$. Therefore, substituting Eqs. (30) into Eq. (10), it is concluded that

$$j_i(x) = \sum_{j=1}^{N} h\varphi_j''(x) - \frac{1}{2}\sum_{j=i+1}^{N} (z_j + z_{j-1})\varphi_j''(x) \tag{36}$$

Finally, the admissible stress field that satisfies all equilibrium equations, tractions and continuity boundary conditions can be summarized as:

$$\tilde{\sigma}_{xx}^{(i)}(x) = \sigma_{xx}^{0(i)} - \varphi_i(x)/h_i \tag{37}$$

$$\tilde{\sigma}_{yy}^{(i)}(x) = \sigma_{yy}^{0(i)} - \eta_i(x)/h_i \tag{38}$$

$$\tilde{\sigma}_{xy}^{(i)}(x) = \sigma_{xy}^{0(i)} - \psi_i(x)/h_i \tag{39}$$

$$\tilde{\sigma}_{xz}^{(i)}(x,z) = \varphi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \varphi_j'(x) \tag{40}$$

$$\tilde{\sigma}_{yz}^{(i)}(x,z) = \psi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \psi_j'(x) \tag{41}$$

$$\tilde{\sigma}_{zz}^{(i)}(x,z) = -\frac{1}{2h_i}\varphi_i''(x)(z - z_i)^2 - z\sum_{j=1}^{i}\varphi_j''(x) + h\sum_{j=1}^{N}\varphi_j''(x) - \frac{1}{2}\sum_{j=i+1}^{N}(z_j + z_{j-1})\varphi_j''(x) \tag{42}$$

where $\varphi_i(x)$, $\psi_i(x)$ and $\eta_i(x)$, $i = 1,2,\ldots,N$ are unknown perturbation functions yet to be determined. It should be noted that Eqs. (17)–(19) represent three relations between unknown perturbation functions, thus, the number of unknown perturbation functions which should be determined is actually $3N - 3$.

Notice that the expressions for the admissible stress field are valid for any in-plane loading conditions. The influence of the actual loads under consideration comes through the constant field in the boundary conditions at cracked surfaces of $x = \pm a$ (Eqs. (20)–(22)) for the unknown functions that will be discussed in detail.As discussed above, the general expressions for the perturbation stresses contain $3N - 3$ unknown functions. We will be evaluating the optimal functions that minimize the complementary energy of the cracked laminate. The total complementary energy $\tilde{U}_C$ associated with the admissible stresses in a laminate subject to traction boundary conditions is defined as follows:

$$\begin{gathered}\tilde{U}_C = \tfrac{1}{2}\int_V \tilde{\sigma} S \tilde{\sigma}\, dV = \tfrac{1}{2}\int_V (\sigma + \sigma^0) S (\sigma + \sigma^0)\, dV = U_C^0 + U_C + 2U_m, \\ U_C^0 = \tfrac{1}{2}\int_V \sigma^0 S \sigma^0 dV, \quad U_C = \tfrac{1}{2}\int_V \sigma S \sigma\, dV, \quad U_m = \tfrac{1}{2}\int_V \sigma S \sigma^0\, dV\end{gathered} \tag{43}$$

where S is the local compliance matrix and V is the volume of the cracked laminate. Hashin (1985) went through a lengthy proof to demonstrate that U_m vanishes. This is in fact a direct consequence of the virtual work principle. Thus, we have:

$$\tilde{U}_C = \frac{1}{2}\int_V \tilde{\sigma} S \tilde{\sigma}\, dV = U_C^0 + U_C \tag{44}$$

where U_C^0 is the total complementary potential energy before cracking, which does not contribute to the variation. For a cracked laminate, we would minimize the functional over the volume of a fragment of length $2a$ bounded by two adjacent transverse cracks, such that $|x| \leqslant a$ and $|z| \leqslant h$. Due to the symmetry with respect to mid-plane $z = 0$, only the region $0 \leqslant z \leqslant h$ can be considered. Consequently:

$$\begin{gathered}U_C = \sum_{i=1}^{N}\left(\int_{-a}^{a}\int_{z_{i-1}}^{z_i} W^{(i)} dz\, dx\right), \\ W^{(i)} = \{\sigma^{(i)}\}^T [S^{(i)}] \{\sigma^{(i)}\}, \\ \{\sigma^{(i)}\}^T = \{\sigma_{xx}^{(i)}, \sigma_{yy}^{(i)}, \sigma_{zz}^{(i)}, \sigma_{yz}^{(i)}, \sigma_{xz}^{(i)}, \sigma_{xy}^{(i)}\}\end{gathered} \tag{45}$$

where $W^{(i)}$ is the perturbation stress energy density of ply (i) and $[S^{(i)}]$ is the compliance matrix of ply (i) in the coordinate system associated with cracks. In order to compute $[S^{(i)}]$, the compliance matrix of the unidirectional fiber composite material should be rotated to the corresponding angles of the plies θ_i, e.g. 90° for any cracked plies. Substituting the expressions for the perturbation stresses (Eqs. (2), (3), (4), (40), (41), (42)) and inserting the rotated compliance matrices into Eq. (45), we can perform the integration over z. The result of integration can be written in the following form:

$$U_C = \int_{-a}^{a} F(x, \{\overline{\varphi}\}, \{\overline{\varphi'}\}, \{\overline{\varphi''}\}, \{\overline{\psi}\}, \{\overline{\psi'}\}, \{\overline{\eta}\})\, dx \tag{46}$$

where

$$\begin{aligned}&F(x, \{\overline{\varphi}\}, \{\overline{\varphi'}\}, \{\overline{\varphi''}\}, \{\overline{\psi}\}, \{\overline{\psi'}\}, \{\overline{\eta}\}) \\ &\quad = \{\overline{\varphi}\}^T[C_{11}^{00}]\{\overline{\varphi}\} + \{\overline{\psi}\}^T[C_{22}^{00}]\{\overline{\psi}\} + \{\overline{\eta}\}^T[C_{33}^{00}]\{\overline{\eta}\} \\ &\quad + \{\overline{\varphi}\}^T[C_{12}^{00}]\{\overline{\psi}\} + \{\overline{\varphi}\}^T[C_{13}^{00}]\{\overline{\eta}\} + \{\overline{\psi}\}^T[C_{23}^{00}]\{\overline{\eta}\} \\ &\quad + \{\overline{\varphi'}\}^T[C_{11}^{11}]\{\overline{\varphi'}\} + \{\overline{\psi'}\}^T[C_{22}^{11}]\{\overline{\psi'}\} + \{\overline{\varphi'}\}^T[C_{12}^{11}]\{\overline{\psi'}\} \\ &\quad + \{\overline{\varphi''}\}^T[C_{11}^{20}]\{\overline{\varphi}\} + \{\overline{\varphi''}\}^T[C_{12}^{20}]\{\overline{\psi}\} + \{\overline{\varphi''}\}^T[C_{13}^{20}]\{\overline{\eta}\} \\ &\quad + \{\overline{\varphi''}\}^T[C_{11}^{22}]\{\overline{\varphi''}\}\end{aligned} \tag{47}$$

where:

$$\begin{gathered}\{\overline{\varphi}\}^T = \{\varphi_1, \varphi_2, \ldots, \varphi_N\}_{1\times N}, \quad \{\overline{\varphi'}\}^T = \{\varphi_1', \varphi_2', \ldots, \varphi_N'\}_{1\times N}, \\ \{\overline{\varphi''}\}^T = \{\varphi_1'', \varphi_2'', \ldots, \varphi_N''\}_{1\times N}, \quad \{\overline{\psi}\}^T = \{\psi_1, \psi_2, \ldots, \psi_N\}_{1\times N}, \\ \{\overline{\psi'}\}^T = \{\psi_1', \psi_2', \ldots, \psi_N'\}_{1\times N}, \quad \{\overline{\eta}\}^T = \{\eta_1, \eta_2, \ldots, \eta_N\}_{1\times N}\end{gathered} \tag{48}$$

In addition, the coefficient matrices of $[C_{11}^{00}]_{N\times N}$, etc., with the superscripts corresponding to the order of derivatives and subscripts corresponding to independent unknown functions involved (e.g. $[C_{13}^{20}]$ is associated with $\{\overline{\varphi''}\}$ and $\{\overline{\eta}\}$ or $[C_{12}^{11}]$ is associated with $\{\overline{\varphi'}\}$ and $\{\overline{\psi'}\}$), were evaluated systematically as given in Appendix. It should be noted that all the unknown functions are not independent and one out of any n unknowns of $\varphi_i, \psi_i, \eta_i$, $i = 1, 2, \ldots, N$ can be eliminated using Eqs. (17)–(19), respectively. In order to have a systematic way for developing the formulation, we choose the nearest un-cracked ply to the upper surface of the laminate (assigned its ply number to the integer variable m) and eliminate its perturbation functions based on Eqs. (17)–(19). Thus, it is necessary to rewrite Eqs. (46)–(48) based on the independent unknown functions as follows:

$$U_C = \int_{-a}^{a} F(x, \{\varphi\}, \{\varphi'\}, \{\varphi''\}, \{\psi\}, \{\psi'\}, \{\eta\})\, dx \tag{49}$$

where:

$$\begin{aligned} &F(x, \{\varphi\}, \{\varphi'\}, \{\varphi''\}, \{\psi\}, \{\psi'\}, \{\eta\}) \\ &\quad = \{\varphi\}^T[A_{11}^{00}]\{\varphi\} + \{\psi\}^T[A_{22}^{00}]\{\psi\} + \{\eta\}^T[A_{33}^{00}]\{\eta\} \\ &\qquad + \{\varphi\}^T[A_{12}^{00}]\{\psi\} + \{\varphi\}^T[A_{13}^{00}]\{\eta\} + \{\psi\}^T[A_{23}^{00}]\{\eta\} \\ &\qquad + \{\varphi'\}^T[A_{11}^{11}]\{\varphi'\} + \{\psi'\}^T[A_{22}^{11}]\{\psi'\} + \{\varphi'\}^T[A_{12}^{11}]\{\psi'\} \\ &\qquad + \{\varphi''\}^T[A_{11}^{20}]\{\varphi\} + \{\varphi''\}^T[A_{12}^{20}]\{\psi\} + \{\varphi''\}^T[A_{13}^{20}]\{\eta\} \\ &\qquad + \{\varphi''\}^T[A_{11}^{22}]\{\varphi''\} \end{aligned} \tag{50}$$

Where in the new notation with the assumption that mth ply represents an un-cracked ply, we have:

$$\begin{aligned} \{\varphi\}^T &= \{\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_{m+1}, \ldots, \varphi_N\}_{1\times N-1}, \\ \{\varphi'\}^T &= \{\varphi'_1, \varphi'_2, \ldots, \varphi'_{m-1}, \varphi'_{m+1}, \ldots, \varphi'_N\}_{1\times N-1}, \\ \{\varphi''\}^T &= \{\varphi''_1, \varphi''_2, \ldots, \varphi''_{m-1}, \varphi''_{m+1}, \ldots, \varphi''_N\}_{1\times N-1}, \\ \{\psi\}^T &= \{\psi_1, \psi_2, \ldots, \psi_{m-1}, \psi_{m+1}, \ldots, \psi_N\}_{1\times N-1}, \\ \{\psi'\}^T &= \{\psi'_1, \psi'_2, \ldots, \psi'_{m-1}, \psi'_{m+1}, \ldots, \psi'_N\}_{1\times N-1}, \\ \{\eta\}^T &= \{\eta_1, \eta_2, \ldots, \eta_{m-1}, \eta_{m+1}, \ldots, \eta_N\}_{1\times N-1} \end{aligned} \tag{51}$$

Again, the remaining unknowns will be denoted as $\{\varphi\}$, $\{\psi\}$, $\{\eta\}$ hereafter in this paper and they are the independent unknown functions for the mathematical formulation of the problem. A ply corresponding to an independent unknown function will be referred to as an 'independent ply'. In addition, in order to evaluate new coefficient matrices of $[A_{11}^{00}]_{(N-1)\times(N-1)}$, etc, we have:

$$\begin{aligned} &[A_{11}^{00}]_{(N-1)\times(N-1)} = [M]^T[C_{11}^{00}][M], \quad [A_{22}^{00}]_{(N-1)\times(N-1)} = [M]^T[C_{22}^{00}][M], \\ &\textit{similarly} \quad \ldots \quad [A_{11}^{22}]_{(N-1)\times(N-1)} = [M]^T[C_{11}^{22}][M] \end{aligned} \tag{52}$$

where $[M]_{N\times(N-1)}$ is a matrix to consider the Eqs. (17)–(19), in order to remove dependent unknown functions effects in the coefficient matrices, which can be defined as below:

$$[M]_{N\times(N-1)} = \left\{ \begin{array}{ll} M_{i,j} = -1, & i = m \\ M_{i,i} = 1, & i > m \\ M_{i,i-1} = 1, & i < m \\ M_{i,j} = 0, & else \end{array} \right\} \tag{53}$$

where m is one of the un-cracked plies (the nearest un-cracked ply to the upper surface of the laminate), which is assumed as the dependent ply.

The functions $\{\varphi\}_{N-1,1}$, $\{\psi\}_{N-1,1}$ and $\{\eta\}_{N-1,1}$ are now to be determined from minimization of the complementary energy functional. It is well known that when a variation is taken for the total complementary potential energy, the stationary value condition, $d\tilde{U}_C = 0$, leads to Euler-Lagrange equations as the governing equations for the problem as follows:

$$\begin{aligned} &\frac{\partial F}{\partial\{\varphi\}} - \frac{d}{dx}\left(\frac{\partial F}{\partial\{\varphi'\}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial\{\varphi''\}}\right) = 0, \\ &or \quad [B_1]\{\varphi''''\} + [B_2]\{\varphi''\} + [B_3]\{\varphi\} + [B_4]\{\psi''\} \\ &+[B_5]\{\psi\} + [B_6]\{\eta''\} + [B_7]\{\eta\} = 0 \end{aligned} \tag{54}$$

$$\begin{aligned} &\frac{\partial F}{\partial\{\psi\}} - \frac{d}{dx}\left(\frac{\partial F}{\partial\{\psi'\}}\right) = 0, \\ &or \quad [B_4]^T\{\varphi''\} + [B_5]^T\{\varphi\} + [B_8]\{\psi''\} + [B_9]\{\psi\} + [B_{10}]\{\eta\} = 0 \end{aligned} \tag{55}$$

$$\begin{aligned} &\frac{\partial F}{\partial\{\eta\}} = 0, \\ &or \quad [B_6]^T\{\varphi''\} + [B_7]^T\{\varphi\} + [B_{10}]^T\{\psi\} + [B_{11}]\{\eta\} = 0 \end{aligned} \tag{56}$$

where

$$\begin{aligned} &[B_1] = [A_{11}^{22}] + [A_{11}^{22}]^T, \quad [B_2] = [A_{11}^{20}] + [A_{11}^{20}]^T - [A_{11}^{11}] - [A_{11}^{11}]^T, \\ &[B_3] = [A_{11}^{00}] + [A_{11}^{00}]^T, \quad [B_4] = -[A_{12}^{11}] + [A_{12}^{20}], \quad [B_5] = [A_{12}^{00}], \\ &[B_6] = [A_{13}^{20}], \quad [B_7] = [A_{13}^{00}], \quad [B_8] = -[A_{22}^{11}] - [A_{22}^{11}]^T, \\ &[B_9] = [A_{22}^{00}] + [A_{22}^{00}]^T, \quad [B_{10}] = [A_{23}^{00}], \quad [B_{11}] = [A_{33}^{00}] + [A_{33}^{00}]^T \end{aligned} \tag{57}$$

The Eqs. (56) define $\{\eta\}$ in terms of the other two unknown functions:

$$\{\eta\} = -[B_{11}]^{-1}([B_6]^T\{\varphi''\} + [B_7]^T\{\varphi\} + [B_{10}]^T\{\psi\}) \tag{58}$$

Substituting above expression back to the functional in Eq. (50), we can rewrite it in terms of two sets of independent unknown functions of $\{\varphi\}$ and $\{\psi\}$:

$$\begin{aligned} &F(x, \{\varphi\}, \{\varphi'\}, \{\varphi''\}, \{\psi\}, \{\psi'\}) \\ &\quad = \{\varphi\}^T[P_{11}^{00}]\{\varphi\} + \{\psi\}^T[P_{22}^{00}]\{\psi\} + \{\varphi\}^T[P_{12}^{00}]\{\psi\} \\ &\qquad + \{\varphi'\}^T[P_{11}^{11}]\{\varphi'\} + \{\psi'\}^T[P_{22}^{11}]\{\psi'\} + \{\varphi'\}^T[P_{12}^{11}]\{\psi'\} \\ &\qquad + \{\varphi''\}^T[P_{11}^{20}]\{\varphi\} + \{\varphi''\}^T[P_{12}^{20}]\{\psi\} + \{\varphi''\}^T[P_{11}^{22}]\{\varphi''\} \end{aligned} \tag{59}$$

where

$$\begin{aligned} &[P_{11}^{00}] = [A_{11}^{00}] - \frac{1}{4}[A_{13}^{00}][A_{33}^{00}]^{-1}[A_{13}^{00}]^T, \quad [P_{22}^{00}] = [A_{22}^{00}] - \frac{1}{4}[A_{23}^{00}][A_{33}^{00}]^{-1}[A_{23}^{00}]^T, \\ &[P_{12}^{00}] = [A_{12}^{00}] - \frac{1}{2}[A_{13}^{00}][A_{33}^{00}]^{-1}[A_{23}^{00}]^T, \quad [P_{11}^{11}] = [A_{11}^{11}], \quad [P_{22}^{11}] = [A_{22}^{11}], \\ &[P_{12}^{11}] = [A_{12}^{11}], \quad [P_{11}^{20}] = [A_{11}^{20}] - \frac{1}{2}[A_{13}^{20}][A_{33}^{00}]^{-1}[A_{13}^{00}]^T, \\ &[P_{12}^{20}] = [A_{12}^{20}] - \frac{1}{2}[A_{13}^{20}][A_{33}^{00}]^{-1}[A_{23}^{00}]^T, \quad [P_{11}^{22}] = [A_{11}^{22}] - \frac{1}{4}[A_{13}^{20}][A_{33}^{00}]^{-1}[A_{13}^{20}]^T \end{aligned} \tag{60}$$

Again, the Euler-Lagrange equations as the governing equations for the problem based on functional in Eq. (59) are as follows:

$$[T_1]\{\varphi''''\} + [T_2]\{\varphi''\} + [T_3]\{\varphi\} + [T_4]\{\psi''\} + [T_5]\{\psi\} = 0 \tag{61}$$

$$[T_4]^T\{\varphi''\} + [T_5]^T\{\varphi\} + [T_6]\{\psi''\} + [T_7]\{\psi\} = 0 \tag{62}$$

where

$$\begin{aligned} &[T_1] = [P_{11}^{22}] + [P_{11}^{22}]^T, \quad [T_2] = [P_{11}^{20}] + [P_{11}^{20}]^T - [P_{11}^{11}] - [P_{11}^{11}]^T, \\ &[T_3] = [P_{11}^{00}] + [P_{11}^{00}]^T, \quad [T_4] = -[P_{12}^{11}] + [P_{12}^{20}], \quad [T_5] = [P_{12}^{00}], \\ &[T_6] = -[P_{22}^{11}] - [P_{22}^{11}]^T, \quad [T_7] = [P_{22}^{00}] + [P_{22}^{00}]^T \end{aligned} \tag{63}$$

The Eqs. (61) and (62) are an extension of work done by Vinogradov and Hashin (2010) for two-layer angle ply laminates into the more applicable case of general symmetric laminates with multiple-layer cracked and un-cracked plies. Note here that for a cross-ply laminates $[T_4] = [T_5] = 0$ and the Eqs. (61) and (62) become uncoupled.

$$[T_1]\{\varphi''''\} + [T_2]\{\varphi''\} + [T_3]\{\varphi\} = 0 \tag{64}$$

$$[T_6]\{\psi''\} + [T_7]\{\psi\} = 0 \tag{65}$$

which is the case considered by Hashin (1985) for a two-layer cross ply and (Li and Hafeez, 2009) for a multiple-layer cross ply laminates; with only one (usually negligible for a cross-ply) difference that here the perturbation of σ_{yy} has not been neglected.

The Eqs. (61) and (62) are a pair of coupled systems of simultaneous linear ordinary differential equations with constant coefficients arising from the variational analysis of general symmetric multiple-ply laminate containing cracks in some layers. Using differential operator notation, the system of Eqs. (61) and (62) may be expressed as

$$[T_1]D^4\{\varphi\} + [T_2]D^2\{\varphi\} + [T_3]\{\varphi\} + [T_4]D^2\{\psi\} + [T_5]\{\psi\} = 0 \tag{66}$$

$$[T_4]^T D^2\{\varphi\} + [T_5]^T\{\varphi\} + [T_6]D^2\{\psi\} + [T_7]\{\psi\} = 0 \tag{67}$$

where D is the differential operator defined as

$$D = \frac{d}{dx} \quad \text{and if generalized} \quad D^n = \frac{d^n}{dx^n} \tag{68}$$

Combining these systems, we obtain the coupled system, expressed as follows:

$$\begin{pmatrix} [T_1]D^4 + [T_2]D^2 + [T_3] & [T_4]D^2 + [T_5] \\ [T_4]^T D^2 + [T_5]^T & [T_6]D^2 + [T_7] \end{pmatrix} \begin{Bmatrix} \{\varphi\} \\ \{\psi\} \end{Bmatrix} = 0 \tag{69}$$

Let us denote the differential operator matrix on the left hand side of the above equation by $MM(D)$:

$$MM(D) = \begin{pmatrix} [T_1]D^4 + [T_2]D^2 + [T_3] & [T_4]D^2 + [T_5] \\ [T_4]^T D^2 + [T_5]^T & [T_6]D^2 + [T_7] \end{pmatrix} \tag{70}$$

Then, the characteristic equation for the coupled system would be

$$|MM(\lambda)| = \det\left\{ \begin{pmatrix} [T_1]\lambda^4 + [T_2]\lambda^2 + [T_3] & [T_4]\lambda^2 + [T_5] \\ [T_4]^T \lambda^2 + [T_5]^T & [T_6]\lambda^2 + [T_7] \end{pmatrix} \right\} = 0 \tag{71}$$

The characteristic equation (71) for the coupled system is a polynomial of degree $6(N-1)$ in λ.

If the lay-up is cross-ply or the system is uncoupled, i.e. $[T_4] = [T_5] = 0$, the Eqs. (64) and (65) should be solved separately. In that case, there are two characteristic equations as follows:

$$|MM(\lambda)| = \det\{([T_1]\lambda^4 + [T_2]\lambda^2 + [T_3])\} = 0 \tag{72}$$

$$|MM(\lambda)| = \det\{([T_6]\lambda^2 + [T_7])\} = 0 \tag{73}$$

where the first characteristic equation is a polynomial of degree $4(N-1)$ and the second one is a polynomial of degree $2(N-1)$ in λ.

Since the derivatives in both coupled and uncoupled systems are all even, the parameters $\lambda_i,\ i = 1, 2, \ldots, 6(N-1)$ appear in both positive and negative valued pairs of square roots which may not always be real. In the case of a complex root, it is always accompanied by its conjugate due to the fact that all coefficients of the equation are real. Moreover, it is feasible to obtain all roots numerically for Eq. (71) or Eqs. (72) and (73) using Maple software, Matlab software or any other mathematical software. Once the characteristic equations are solved and the all roots are found for a laminate, it is possible to write the functions $\{\varphi\}$ and $\{\psi\}$ as series expansions of exponential functions as long as the values λ_i are all distinct. In other words, for a coupled system, each of these solution functions is of the following form

$$\begin{aligned} \varphi_k(x) &= \sum_{i=1}^{6(N-1)} q_{i,k} e^{\lambda_i x}, \\ \psi_k(x) &= \sum_{i=1}^{6(N-1)} q_{i,k+N-1} e^{\lambda_i x}, \quad k = 1, 2, \ldots, m-1, m+1, \ldots, N \end{aligned} \tag{74}$$

where $q_{i,k}$ are arbitrary constants that should be determined using boundary conditions. It should be mentioned that all values of $q_{i,k}$ are not independent. It is due to this fact that they are members of eigenvectors concerning characteristic equation roots of λ_i.

In addition, for the uncoupled systems, each of these solution functions is of the following form

$$\varphi_k(x) = \sum_{i=1}^{4(N-1)} q_{i,k} e^{\lambda_i x}, \quad k = 1, 2, \ldots, m-1,\ m+1, \ldots, N \tag{75}$$

$$\psi_k(x) = \sum_{i=1}^{2(N-1)} q_{i,k+N-1} e^{\lambda_i x}, \quad k = 1, 2, \ldots, m-1,\ m+1, \ldots, N \tag{76}$$

This is because the system is one of linear ordinary differential equations with constant coefficients. For complex roots, the terms associated with a conjugating pair can be expressed in term of products of exponential and harmonic functions to avoid the appearance of imaginary numbers in the expression. It is necessary that all λ_i values are distinct for expressions (74)–(76) to hold. This uniqueness condition is checked after solving Eq. (71) for coupled systems and Eqs. (72), (73) for uncoupled systems, and it is found to be satisfied by the input data in all considered cases. When repeated roots are present, the form of the solution needs to be modified a little to accommodate them. These are all standard treatments in ordinary differential equations and hence not described in details here. Moreover, it is noted that the coupled system in Eqs. (61) and (62) is very similar to the equations which has been derived by McCartney (2000) using a stress transfer model. The method of numerically solving the system of differential equations (61) and (62) is
described by Hannaby (1997) for coupled case (i.e. $[T_4] \neq 0$ *or* $[T_5] \neq 0$) as well as the uncoupled case (i.e. $[T_4] = [T_5] = 0$) (Hannaby, 1993). The reader can also refer to these publications to find details about solving of the governing equations. Once the governing equations are solved, the last step is finding arbitrary constants using boundary conditions.

It could be easily seen that one system (Eq. (61)) is fourth order in terms of the variables φ and second order in terms of the variables ψ and the other system (Eq. (62)) is second order both in terms of φ and the ψ. Thus, the first system requires $4(N-1)$ boundary conditions and the second needs $2(N-1)$ boundary conditions, making $6(N-1)$ in total. Suppose that there are N_c cracked and N_u un-cracked plies. Then clearly, we have:

$$N_c + N_u = N \tag{77}$$

Eqs. (20)–(22) should be satisfied For each of the cracked plies, and these equations can be written in terms of unknown functions as comes in the following:

$$\varphi_i(a) = \varphi_i(-a) = \sigma_{xx}^{0(i)} h_i \tag{78}$$

$$\psi_i(a) = \psi_i(-a) = \sigma_{xy}^{0(i)} h_i \tag{79}$$

$$\varphi_i'(a) = \varphi_i'(-a) = 0 \tag{80}$$

There are clearly $6N_c$ of these boundary conditions.

For an un-cracked ply, the physical conditions available in the problem can only produce four boundary conditions. Take the in-plane shear stress σ_{xy}, axial stress σ_{xx} and transverse shear stress σ_{xz} from the free body diagram as shown in Fig. 3, to start with. The continuity consideration leads to

$$\begin{aligned}&\sigma_{xy1} = \sigma_{xy2} \quad \text{and} \quad \sigma_{xy3} = \sigma_{xy4},\\ &\sigma_{xx1} = \sigma_{xx2} \quad \text{and} \quad \sigma_{xx3} = \sigma_{xx4},\\ &\sigma_{xz1} = \sigma_{xz2} \quad \text{and} \quad \sigma_{xz3} = \sigma_{xz4},\end{aligned} \tag{81}$$

While the periodic condition or translational symmetry requires

$$\begin{aligned}&\sigma_{xy1} = \sigma_{xy3} \quad \text{and} \quad \sigma_{xy2} = \sigma_{xy4}\\ &\sigma_{xx1} = \sigma_{xx3} \quad \text{and} \quad \sigma_{xx2} = \sigma_{xx4}\\ &\sigma_{xz1} = \sigma_{xz3} \quad \text{and} \quad \sigma_{xz2} = \sigma_{xz4}\end{aligned} \tag{82}$$

Consequently:

$$\sigma_{xy2} = \sigma_{xy3} \rightarrow \sigma_{xy}(a,z) = \sigma_{xy}(-a,z) \Rightarrow \psi_i(a) = \psi_i(-a) \tag{83}$$

$$\sigma_{xx2} = \sigma_{xx3} \rightarrow \sigma_{xx}(a,z) = \sigma_{xx}(-a,z) \Rightarrow \varphi_i(a) = \varphi_i(-a) \tag{84}$$

$$\sigma_{xz2} = \sigma_{xz3} \rightarrow \sigma_{xz}(a,z) = \sigma_{xz}(-a,z) \Rightarrow \varphi_i'(a) = \varphi_i'(-a) \tag{85}$$

The rotational symmetry about the vertical central axis can also serve another boundary condition (Li and Hafeez, 2009). This symmetry is always present in the laminate, cracked or not, in general (Li and Reid, 1992; Li et al., 2009). It is also seen by the loading condition as long as the loads the laminate is subjected to can be expressed in terms of generalized stresses (membrane forces) as defined in the classic laminate theory. The rotational symmetry on σ_{xy} and σ_{xx} yields the same condition as in Eqs. (83) and (84), respectively. However, for σ_{xz}, as it is anti-symmetric under this particular symmetry transformation, this symmetry requires

$$\sigma_{xz2} = -\sigma_{xz3} \rightarrow \sigma_{xz}(a,z) = -\sigma_{xz}(-a,z) \Rightarrow \varphi_i'(a) = -\varphi_i'(-a) \tag{86}$$

where together with Eq. (85), the boundary conditions associated with transverse shear can be given as

$$\sigma_{xz}(a,z) = \sigma_{xz}(-a,z) = 0 \Rightarrow \varphi_i'(a) = \varphi_i'(-a) = 0 \tag{87}$$

With Eqs. (22) and (80) for cracked plies and (87) for un-cracked plies, it can be seen that the transverse shear stress at $x = \pm a$ vanishes in all plies of the laminate, both cracked and un-cracked. This leads to the fact that the first order derivative of φ_i vanishes at $x = \pm a$ for all plies. Finally, the Eqs. (83), (84), and (87), which belong to independent un-cracked plies, clearly provide $4(N_u - 1)$ boundary conditions.

The physical construction of the problem itself does not offer any more boundary conditions without resorting to displacements. As a result, there will not be sufficient boundary conditions directly from the physical conditions. McCartney (1992, 2000) has introduced a displacement boundary condition. In a stress-based approach, as is the case here, displacements are not involved, and one has to find extra boundary conditions in terms of stresses for each independent un-cracked lamina before the solution can be determined.

Li and Hafeez (2009) have shown that for any laminate having more than two un-cracked plies, an extension is required in terms of boundary conditions. There is a shortfall of two boundary conditions for each independent un-cracked ply ($2(N_u - 1)$ in total). In the next paragraphs, it will be shown that the fifth and sixth boundary conditions for each independent un-cracked ply can be obtained mathematically from the variational calculus itself in terms of natural boundary conditions.

As Euler-Lagrange's equations are derived using variational calculus, terms also emerge which take their values at boundaries resulting from steps of integration by parts. For the variation of the functional to vanish so that the functional takes its stationary value, these terms must also vanish. This leads to boundary conditions, called natural boundary conditions in variational principles. For the current problem with the functional given in Eq. (49), the natural boundary conditions obtained directly from the variational calculus are as following:

$$\left[\left(\frac{\partial F}{\partial\{\varphi'\}} - \frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)^T \{\partial\varphi\}\right]_{x=-a}^{x=a} = 0 \tag{88}$$

$$\left[\left(\frac{\partial F}{\partial\{\psi'\}}\right)^T \{\partial\psi\}\right]_{x=-a}^{x=a} = 0 \tag{89}$$

The left-hand side of the above equations, shortly denoted as $[XY]_{x=-a}^{x=+a}$, can be manipulated as (Li and Hafeez, 2009), (Li, 2008) and (Li et al., 2009):

$$\begin{aligned}[XY]_{x=-a}^{x=+a} &= X_{+a}Y_{+a} - X_{-a}Y_{-a} = \frac{1}{2}(X_{+a} - X_{-a})(Y_{+a} + Y_{-a})\\ &\quad + \frac{1}{2}(X_{+a} + X_{-a})(Y_{+a} - Y_{-a})\end{aligned} \tag{90}$$

$(Y_{+a} - Y_{-a}) = \{\partial\varphi\}_{+a} - \{\partial\varphi\}_{-a} = 0$ *or* $(Y_{+a} - Y_{-a}) = \{\partial\psi\}_{+a} - \{\partial\psi\}_{-a} = 0$, are based on (84) and (83), respectively. Therefore

$$[XY]_{x=-a}^{x=+a} = \frac{1}{2}(X_{+a} - X_{-a})(Y_{+a} + Y_{-a}) \tag{91}$$

And Eqs. (88) and (89) reduce to

$$\begin{aligned}&\left[\left(\frac{\partial F}{\partial\{\varphi'\}} - \frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)^T \{\partial\varphi\}\right]_{x=-a}^{x=a}\\ &= \frac{1}{2}\left(\left(\frac{\partial F}{\partial\{\varphi'\}} - \frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)_{+a} - \left(\frac{\partial F}{\partial\{\varphi'\}} - \frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)_{-a}\right)^T\\ &\quad (\{\partial\varphi\}_{+a} + \{\partial\varphi\}_{-a}) = 0\end{aligned} \tag{92}$$

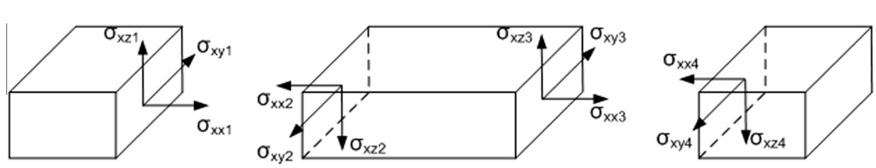

Fig. 3. Axial stress, in-plane shear stress, and transverse shear stress at the boundary of an un-cracked lamina.

$$\left[\left(\frac{\partial F}{\partial\{\psi'\}}\right)^T\{\partial\psi\}\right]_{x=-a}^{x=a}=\frac{1}{2}\left(\left(\frac{\partial F}{\partial\{\psi'\}}\right)_{+a}-\left(\frac{\partial F}{\partial\{\psi'\}}\right)_{-a}\right)^T \left(\{\partial\psi\}_{+a}+\{\partial\psi\}_{-a}\right)=0 \tag{93}$$

As $(\{\partial\varphi\}_{+a}+\{\partial\varphi\}_{-a})$ represents the variations of $(\{\varphi\}_{+a}+\{\varphi\}_{-a})$ and also $(\{\partial\psi\}_{+a}+\{\partial\psi\}_{-a})$ represents the variations of $(\{\psi\}_{+a}+\{\psi\}_{-a})$, which are arbitrary for un-cracked plies but vanish for cracked ones, for the above expression to vanish, one must have the following relationship as the natural boundary conditions for the un-cracked plies

$$\left(\frac{\partial F}{\partial\{\varphi'\}}-\frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)_{+a}-\left(\frac{\partial F}{\partial\{\varphi'\}}-\frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)_{-a}=0 \tag{94}$$

$$\left(\frac{\partial F}{\partial\{\psi'\}}\right)_{+a}-\left(\frac{\partial F}{\partial\{\psi'\}}\right)_{-a}=0 \tag{95}$$

Given F as expressed in Eq. (59), the natural boundary conditions in Eqs. (94) and (95) can be obtained as

$$[T_1](\{\varphi'''(-a)\}-\{\varphi'''(+a)\})+[T_4](\{\psi'(-a)\}-\{\psi'(+a)\})=0 \tag{96}$$

$$[T_6](\{\psi'(-a)\}-\{\psi'(+a)\})=0 \tag{97}$$

In obtaining Eqs. (96) and (97), physical boundary conditions in Eq. (87) have been considered. $[T_4]$ is associated with off-axis plies and it disappears for cross ply laminates. Apparently, Eqs. (96) and (97) apply only to independent un-cracked plies and clearly provide $2(N_u-1)$ boundary conditions.

As mentioned above, Eqs. (78)–(80) prepare $6N_c$, Eqs. (83), (84), and (87) prepare $4(N_u-1)$ and Eqs. (96) and (97) prepare $2(N_u-1)$ boundary conditions which provide

$$6N_c+4(N_u-1)+2(N_u-1)=6(N_c+N_u)-6=6(N-1) \tag{98}$$

boundary conditions in total, as required.

3. Results and discussions

The numerical results for stress field will be given for graphite/epoxy and E-glass/epoxy laminates. The properties are summarized as follows:

For graphite/epoxy (Hashin, 1985; Li and Hafeez, 2009)

$$\begin{aligned}&E_1=208.3\ \text{GPa},\quad E_2=E_3=6.5\ \text{GPa},\quad G_{12}=1.65\ \text{GPa}\\&G_{23}=2.3\ \text{GPa},\quad \upsilon_{12}=\upsilon_{13}=0.255,\quad \upsilon_{23}=0.413,\\&\text{Ply thickness}=0.203\ \text{mm}\end{aligned} \tag{99}$$

For E-glass/epoxy (Varna et al., 1999)

$$\begin{aligned}&E_1=44.7\ \text{GPa},\quad E_2=E_3=12.7\ \text{GPa},\quad G_{12}=5.8\ \text{GPa},\\&G_{23}=4.5035\ \text{GPa},\quad \upsilon_{12}=\upsilon_{13}=0.297,\quad \upsilon_{23}=0.41,\\&\text{Ply thickness}=0.144\ \text{mm}\end{aligned} \tag{100}$$

For comparison, the case of $[0/90]_s$ graphite/epoxy laminate ($h_1=h_2=0.203$ mm) is considered, which has been presented in Hashin (1985). It is noted that for the mentioned lay-up (i.e. $[0/90]_s$) $[T_4]=[T_5]=0$ and the Eqs. (61) and (62) become uncoupled, therefore, they should be solved separately. Moreover, the lay-up is containing only two layers, one cracked and one un-cracked representing a three-layered laminate after the application of symmetry considerations; thus, there is no need to use the natural boundary conditions presented in Eqs. (96) and (97) because there is no independent un-cracked ply. The stresses are plotted in Figs. 4 and 5 at several typical locations for the crack spacing $2a=4\times0.203$ mm under uniaxial tension and in-plane shear, respectively. It is noted that the stresses in Fig. 4 have all been normalized with respect to the direct stress in the x-direction in the 90-lamina before cracking, and stresses in Fig. 5 have all been normalized with respect to the in-plane shear stress in the 90-lamina before cracking, which have been obtained from the classic laminate theory. In addition to the results of the present analysis, the results obtained from the variational method by Hashin (1985) are also presented for validation; his method is based on plane stress assumptions, which neglects the perturbation of σ_{yy}. It is worth mentioning that the current method can predict the exact results of Hashin's (1985); it is enough to put the perturbation of σ_{yy} in Eq. (58) equal to zero. Therefore, it can be concluded that the Hashin's formulation is the special cases of the current formulation for cross ply laminate with only two layers, one cracked and one un-cracked, which do not need more boundary conditions other than those provided by the physical conditions available in the problem. The excellent agreement between two sets of results is observed. All of the features and trends as obtained from

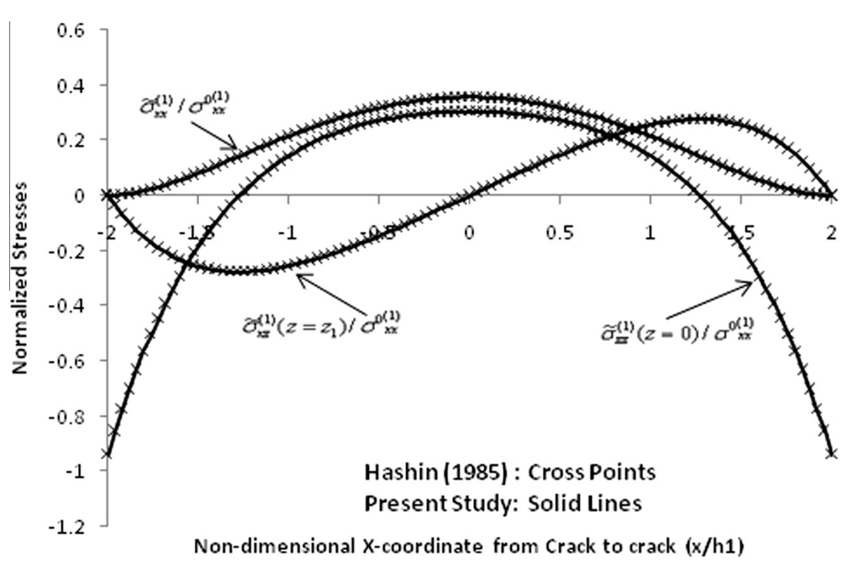

Fig. 4. $[0/90]_s$ graphite/epoxy laminate under uniaxial tension: distributions of stresses between two cracks with crack spacing $2a=4\times0.203$ mm.

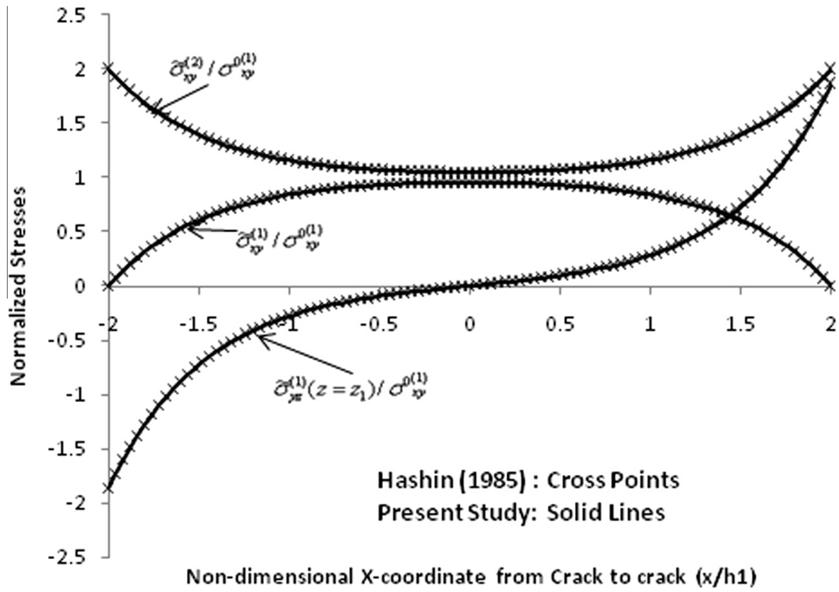

Fig. 5. $[0/90]_s$ graphite/epoxy laminate under shear: distributions of stresses between two cracks with crack spacing $2a = 4 \times 0.203$ mm.

Hashin's (1985) analysis are observed here. This agreement shows that the perturbation of σ_{yy} is actually negligible for the assumed cross ply. It is worth mentioning that both approaches are approximate, however, in the current approach all equilibrium equations, the interface continuities and the boundary conditions are satisfied exactly. Moreover, the obtained stress field minimizes the complementary energy. Therefore, for symmetric cracked laminates, with a uniform distribution of ply cracks in a single orientation, the present stress field is the optimal admissible stress field that can be developed based upon the single fundamental assumption that the in-plane stresses in each ply element are independent from the through-thickness direction.

As the second step, the case of $[0/90_2/0]_s$ laminate ($h_1 = h_3 = 0.203$ mm & $h_2 = 0.406$ mm) presented in Li and Hafeez (2009) is considered. Again, for the mentioned cross ply lay-up (i.e. $[0/90_2/0]_s$), $[T_4] = [T_5] = 0$ and the Eqs. (61) and (62) become uncoupled, therefore, they should be solved separately. However, the lay-up contains three layers in the upper part (considering upper part of symmetric laminate), one cracked and two uncracked layers; thus, there is one independent un-cracked layer and the Eqs. (96) and (97) should be implemented to achieve enough boundary conditions. The stresses at several typical locations for the crack spacing $2a = 4 \times 0.203$ mm are plotted in Fig. 6. It is noted that the stresses in Fig. 6 have all been normalized with respect to the direct stress in the x-direction in the 90-lamina before cracking. Beside the presentation of the results obtained from the present analysis, the results obtained from variational method by Li and Hafeez (2009) based on plane strain assumptions are also presented for validation, which also neglects the perturbation of σ_{yy} (usually negligible for a cross-ply). It is noted that Li and Hafeez

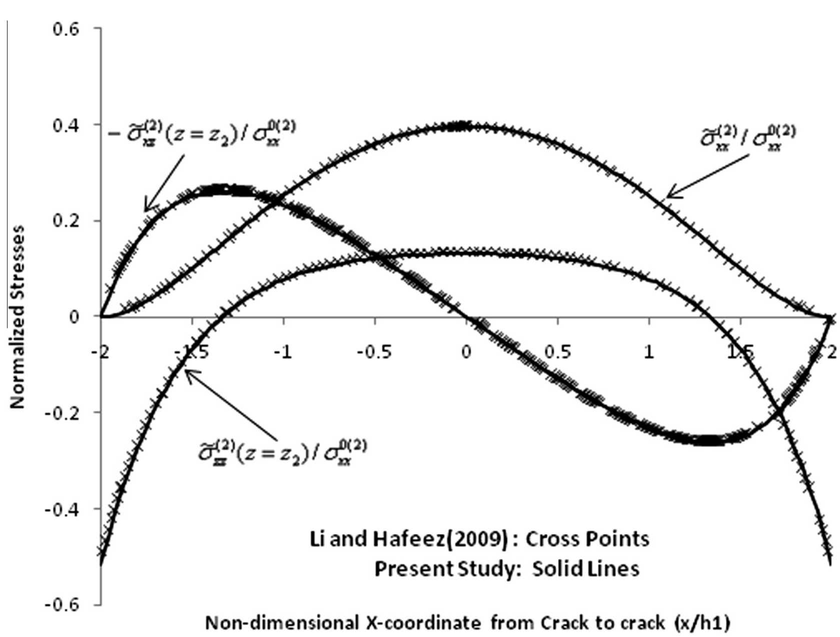

Fig. 6. $[0/90_2/0]_s$ graphite/epoxy laminate under uniaxial tension: distributions of stresses between two cracks with crack spacing $2a = 4 \times 0.203$ mm.

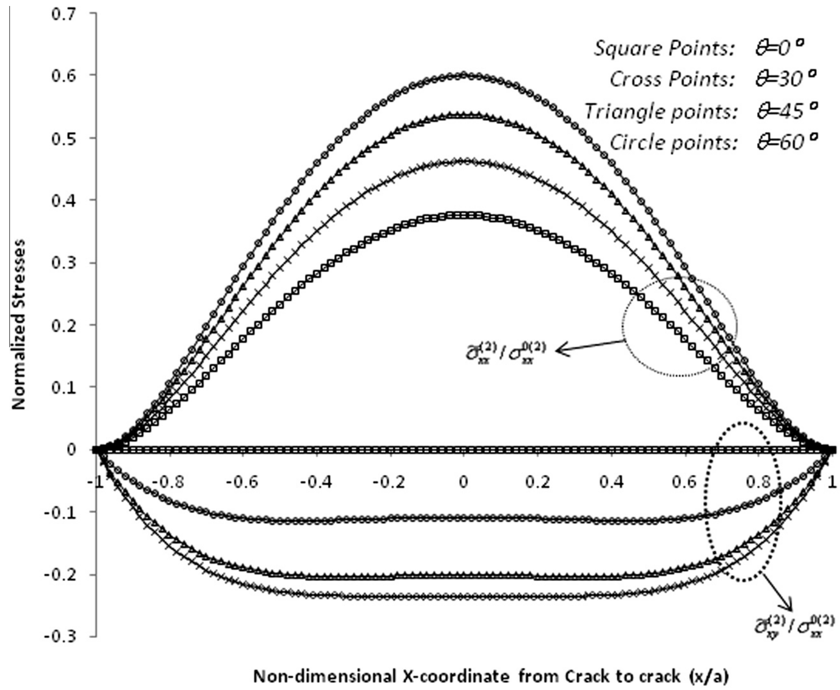

Fig. 7. $[\theta/90_8/\theta_{0.5}]_s$ E-glass/epoxy laminate under uniaxial tension: distributions of stresses between two cracks with crack spacing $2a = 8 \times 0.144$ mm.

(2009) have considered the lower part of the symmetric laminates, therefore, the out-of-plane shear stresses have been presented in Fig. 6 applying a negative sign so that the results are compatible with those of Li and Hafeez (2009). The excellent agreement between two sets of results is observed. This agreement shows that the Li and Hafeez's results (2009) can be easily obtained by the present formulation, which means that their formulation is an especial case of the current formulation. Moreover, it can be concluded that the perturbation of σ_{yy} is actually negligible for the assumed cross ply.

In order to show the capability of the method in handling stress analysis of cracked laminates with general symmetric lay-up, a set of $[\theta/90_8/\theta_{0.5}]_s$ E-glass/epoxy laminates with different θ angles ($\theta = 0^\circ, 30^\circ\,45^\circ$ and 60°) is considered and the stresses are plotted at several typical locations for the crack spacing $2a = 8 \times 0.144$ mm under uniaxial tension in Figs. 7–10. It should be noted that for $\theta = 0^\circ$ the Eqs. (61) and (62) are uncoupled, however, for the remaining cases under consideration the equations are coupled and should be solved together. Moreover, all lay-ups under consideration need more boundary conditions in addition to those provided by the physical conditions available in the problem. It is due to this fact that there is one independent un-cracked ply in the laminates and consequently the Eqs. (96) and (97) should be implemented to provide enough boundary

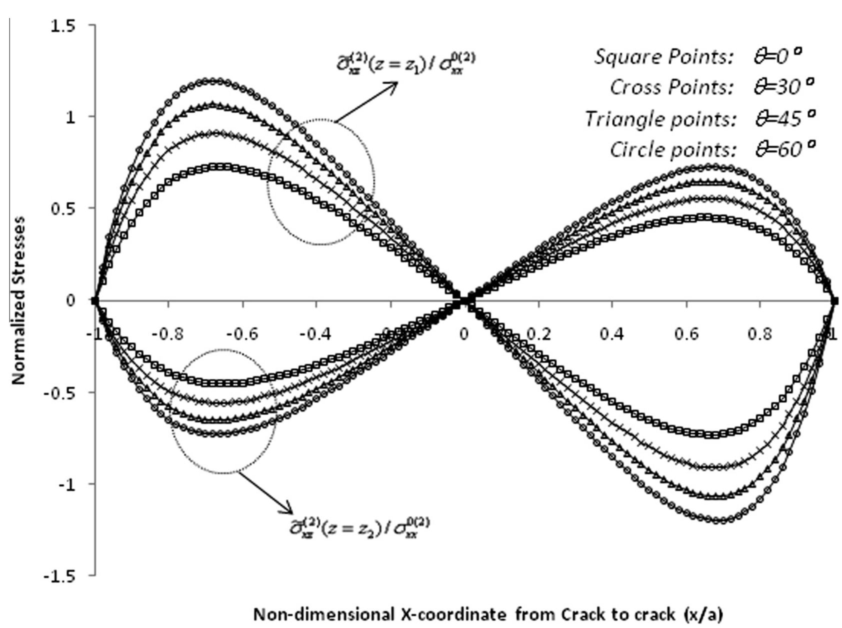

Fig. 8. $[\theta/90_8/\theta_{0.5}]_s$ E-glass/epoxy laminate under uniaxial tension: distributions of out of plane shear stresses between two cracks with crack spacing $2a = 8 \times 0.144$ mm.

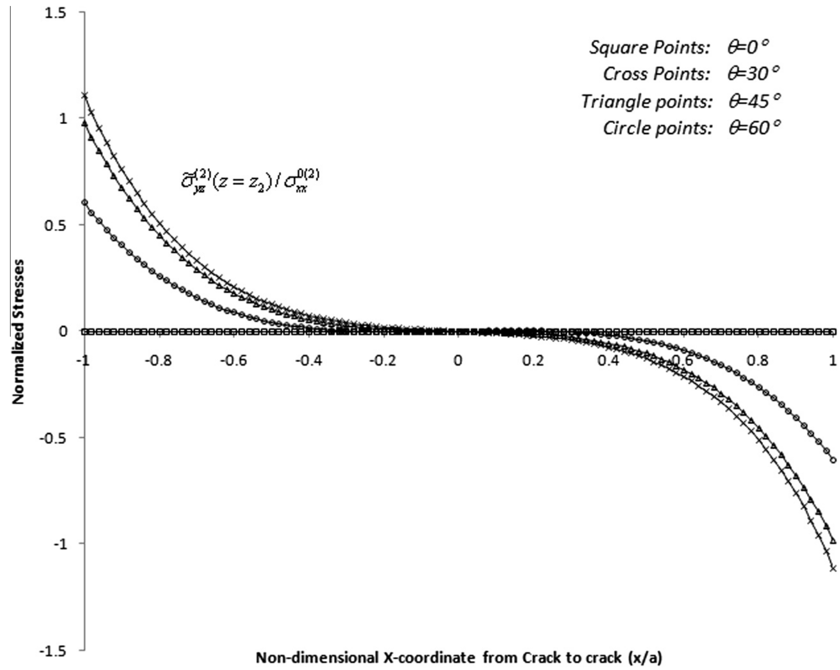

Fig. 9. $[\theta/90_8/\theta_{0.5}]_s$ E-glass/epoxy laminate under uniaxial tension: distributions of out of plane shear stresses between two cracks with crack spacing $2a = 8 \times 0.144$ mm.

conditions. It is also noted that the stresses in these figures have all been normalized with respect to the axial stress in the x-direction of the 90-lamina before cracking.

Fig. 7 shows the variations of non-dimensional axial and in-plane shear stresses versus non-dimensional x-coordinate in the cracked 90-lamina. It is clearly seen in Fig. 7 that the axial stress in cracked ply $\tilde{\sigma}_{xx}^{(2)}$ has its maximum value at the mid-point between the cracks. This maximum value is smaller than $\sigma_{xx}^{0(2)}$ and increases with increasing the θ angle. It is also seen that under uniaxial tension, the in-plane shear stresses are zero for cross ply laminate (i.e. $\theta = 0°$).

The variations of non-dimensional through-the-thickness shear stresses $\tilde{\sigma}_{xz}^{(2)}$ and $\tilde{\sigma}_{yz}^{(2)}$ versus non-dimensional x-coordinate are depicted in Figs. 8 and 9, respectively. It is clearly seen in Fig. 9 that under uniaxial tension, $\tilde{\sigma}_{yz}^{(2)}$ is zero for cross ply laminate (i.e. $\theta = 0^{\circ}$).

The variations of non-dimensional through-the-thickness axial stresses $\tilde{\sigma}_{zz}$ versus non-dimensional x-coordinate are depicted in Fig. 10. It is seen that in the plane of symmetry (i.e. $z = 0$), the maximum tensile $\tilde{\sigma}_{zz}$ occurs at the center point between any two cracks and the maximum compressive $\tilde{\sigma}_{zz}$ occurs at the crack tips.

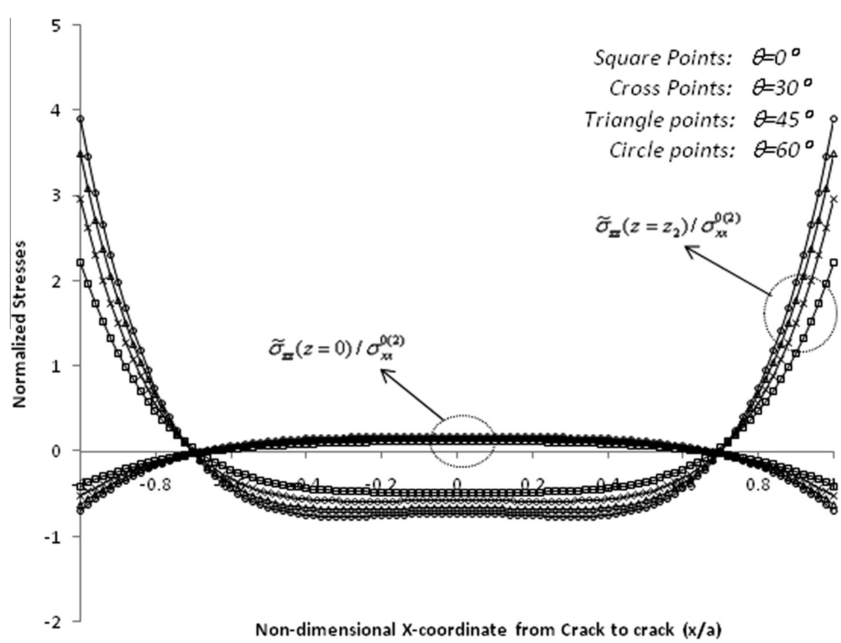

Fig. 10. $[\theta/90_8/\theta_{0.5}]_s$ E-glass/epoxy laminate under uniaxial tension: distributions of out of plane axial stresses between two cracks with crack spacing $2a = 8 \times 0.144$ mm.

4. Conclusion

A variational model for analyzing stress field in general symmetric cracked laminates under general in-plane loading is developed. To do that, an admissible stress field is constructed that satisfies equilibrium and all the boundary and continuity conditions. We used these equations in conjunction with the principle of minimum complementary energy to get the optimal stress state of a cracked laminate with multiple cracked and uncracked layers. The formulations have been derived in general terms in this paper and to the best knowledge of the authors, the present model is the most complete variational model developed so far.

The formulation can be a little enhanced to accurately estimate stiffness reduction and crack evolution of cracked laminates with general symmetric lay-up. This certainly necessitates a future detailed research work. It should be noted that similar analysis can be performed for a symmetric laminate with the cracks in the outer plies taking into account the possibility of formation of non-symmetric or staggered crack arrays, as in similar research works (Nairn and Hu, 1992). However, a more complex geometry compared to those considered here, which is not symmetric, is needed as an elementary cell to consider staggered crack arrays.

Appendix A

$$[C_{11}^{00}]_{N\times N} \Rightarrow (C_{11}^{00})_{i,j} = \begin{cases} \frac{S_{11}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-1}$$

$$[C_{22}^{00}]_{N\times N} \Rightarrow (C_{22}^{00})_{i,j} = \begin{cases} \frac{S_{66}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-2}$$

$$[C_{33}^{00}]_{N\times N} \Rightarrow (C_{33}^{00})_{i,j} = \begin{cases} \frac{S_{22}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-3}$$

$$[C_{12}^{00}]_{N\times N} \Rightarrow (C_{12}^{00})_{i,j} = \begin{cases} \frac{2S_{16}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-4}$$

$$[C_{13}^{00}]_{N\times N} \Rightarrow (C_{13}^{00})_{i,j} = \begin{cases} \frac{2S_{12}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-5}$$

$$[C_{23}^{00}]_{N\times N} \Rightarrow (C_{23}^{00})_{i,j} = \begin{cases} \frac{2S_{26}^{(j)}}{h_j} & i=j \\ 0 & i\neq j \end{cases} \tag{A-6}$$

$$[C_{11}^{11}]_{N\times N} \Rightarrow (C_{11}^{11})_{i,j} = \begin{cases} -\frac{2h_jS_{55}^{(j)}}{3} + \sum_{k=j}^{N} h_k S_{55}^{(k)} & i=j \\ \sum_{k=i}^{N} h_k S_{55}^{(k)} & i>j \\ -h_jS_{55}^{(j)} + \sum_{k=j}^{N} h_k S_{55}^{(k)} & i<j \end{cases} \tag{A-7}$$

$$[C_{22}^{11}]_{N\times N} \Rightarrow (C_{22}^{11})_{i,j} = \begin{cases} -\frac{2h_jS_{44}^{(j)}}{3} + \sum_{k=j}^{N} h_k S_{44}^{(k)} & i=j \\ \sum_{k=i}^{N} h_k S_{44}^{(k)} & i>j \\ -h_jS_{44}^{(j)} + \sum_{k=j}^{N} h_k S_{44}^{(k)} & i<j \end{cases} \tag{A-8}$$

$$[C_{12}^{11}]_{N\times N} \Rightarrow (C_{12}^{11})_{i,j} = \begin{cases} -\frac{4h_jS_{45}^{(j)}}{3} + \sum_{k=j}^{N} 2h_k S_{45}^{(k)} & i=j \\ -h_iS_{45}^{(i)} + \sum_{k=i}^{N} 2h_k S_{45}^{(k)} & i>j \\ -h_jS_{45}^{(j)} + \sum_{k=j}^{N} 2h_k S_{45}^{(k)} & i<j \end{cases} \tag{A-9}$$

$$[C_{11}^{20}]_{N\times N} \Rightarrow (C_{11}^{20})_{i,j} = \begin{cases} S_{13}^{(j)}(\frac{h_j}{3} - 2h + z_j + z_{j-1}) & i=j \\ S_{13}^{(j)}(-2h + z_i + z_{i-1}) & i>j \\ S_{13}^{(j)}(-2h + z_j + z_{j-1}) & i<j \end{cases} \tag{A-10}$$

$$[C_{12}^{20}]_{N\times N} \Rightarrow (C_{12}^{20})_{i,j} = \begin{cases} S_{36}^{(j)}(\frac{h_j}{3} - 2h + z_j + z_{j-1}) & i=j \\ S_{36}^{(j)}(-2h + z_i + z_{i-1}) & i>j \\ S_{36}^{(j)}(-2h + z_j + z_{j-1}) & i<j \end{cases} \tag{A-11}$$

$$[C_{13}^{20}]_{N\times N} \Rightarrow (C_{13}^{20})_{i,j} = \begin{cases} S_{23}^{(j)}\left(\frac{h_j}{3} - 2h + z_j + z_{j-1}\right) & i=j \\ S_{23}^{(j)}(-2h + z_i + z_{i-1}) & i>j \\ S_{23}^{(j)}(-2h + z_j + z_{j-1}) & i<j \end{cases} \tag{A-12}$$

$$[C_{11}^{22}]_{N\times N} \Rightarrow (C_{11}^{22})_{i,j} = \begin{cases} \begin{aligned} & S_{33}^{(j)}\frac{(z_j-z_{j-1})^3}{20} + \sum_{k=j}^{N}\left(\tfrac{1}{3}S_{33}^{(k)}(z_k^3 - z_{k-1}^3)\right) \\ & + S_{33}^{(j)}\frac{(15z_{j-1}^3 - 25z_jz_{j-1}^2 + 5z_{j-1}z_j^2 + 5z_j^3)}{60} \\ & + \frac{(z_j+z_{j-1})^2}{4}\sum_{k=1}^{j-1}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) \\ & + h^2\sum_{k=1}^{N}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) - h(z_j+z_{j-1})\sum_{k=1}^{i-1}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) \\ & -\tfrac{h}{3}S_{33}^{(j)}h_j^2 - h\sum_{k=j}^{N}(S_{33}^{(k)}(z_k^2 - z_{k-1}^2)) \end{aligned} & i=j \\ \begin{aligned} & \sum_{k=i}^{N}\left(\tfrac{2}{3}S_{33}^{(k)}(z_k^3 - z_{k-1}^3)\right) \\ & + \frac{(z_j+z_{j-1})(z_i+z_{i-1})}{4}\sum_{k=1}^{j-1}\left(2S_{33}^{(k)}(z_k - z_{k-1})\right) \\ & + 2h^2\sum_{k=1}^{N}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) - h(z_i+z_{i-1})\sum_{k=1}^{i-1}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) \\ & -\tfrac{h}{3}S_{33}^{(j)}h_j^2 + \tfrac{h_j^2}{6}S_{33}^{(j)}(z_i+z_{i-1}) - h\sum_{k=j}^{N}\left(S_{33}^{(k)}(z_k^2 - z_{k-1}^2)\right) \\ & (\tfrac{z_i+z_{i-1}}{2})\sum_{k=j}^{i-1}\left(S_{33}^{(k)}(z_k^2 - z_{k-1}^2)\right) \end{aligned} & i>j \\ \begin{aligned} & S_{33}^{(j)}\frac{\left(15z_{j-1}^3 - 25z_jz_{j-1}^2 + 5z_{j-1}z_j^2 + 5z_j^3\right)}{60} \\ & - h(z_i+z_{i-1})\sum_{k=1}^{i-1}\left(S_{33}^{(k)}(z_k - z_{k-1})\right) \\ & -\tfrac{h}{3}S_{33}^{(j)}h_j^2 - h\sum_{k=j}^{N}\left(S_{33}^{(k)}(z_k^2 - z_{k-1}^2)\right) \end{aligned} & i<j \end{cases} \tag{A-13}$$

where $S^{(i)}$ is the compliance matrix of the ith ply in the laminate coordinate system (the coordinate system associated with cracks), which can be stated in terms of engineering constants and ply angle θ_i as follows:

$$[S^{(i)}] = [R^{(i)}]^T \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_1} & \frac{-\nu_{13}}{E_1} & 0 & 0 & 0 \\ \frac{-\nu_{21}}{E_2} & \frac{1}{E_2} & \frac{-\nu_{23}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{31}}{E_3} & \frac{-\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} [R^{(i)}] \tag{A-14}$$

$$[R^{(i)}] = \begin{bmatrix} \cos^2(\theta_i) & \sin^2(\theta_i) & 0 & 0 & 0 & \sin(2\theta_i) \\ \sin^2(\theta_i) & \cos^2(\theta_i) & 0 & 0 & 0 & -\sin(2\theta_i) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta_i) & -\sin(\theta_i) & 0 \\ 0 & 0 & 0 & \sin(\theta_i) & \cos(\theta_i) & 0 \\ -\frac{1}{2}\sin(2\theta_i) & \frac{1}{2}\sin(2\theta_i) & 0 & 0 & 0 & -\frac{1}{2}\cos(2\theta_i) \end{bmatrix} \quad \text{(A-15)}$$

In the equations above the elastic properties E, v, G denote Young's modulus, Poisson's ratio and shear modulus, respectively. Subscripts 1, 2 attached to axial and transverse elastic constants appearing in Eq. (A-14) refer to in-plane stresses and deformation while the corresponding subscripts 3 denotes elastic constants that involve out-of-plane stresses and deformations.

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