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Ω-Invariance in Control Systems with Bounded Controls

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The concepts of weakly Ω-invariant sets and strictly weakly Ω-invariant sets in control systems with bounded controls are defined and analyzed. Computable conditions for weak Ω-invariance are derived, and the question of existence of rest points and stationary points in weakly Ω-invariant sets is considered. For linear dynamics, properties of weakly Ω-invariant sets are studied, and questions of constrained reachability are investigated.

1. INTRODUCTION

Consider an autonomous control system described by n ordinary differential equations

$$\dot{x} = g(x, w), \tag{1.1}$$

where $x \in R^n$ is the state vector, $w \in R^m$ is the control vector, and for a subset $\Gamma \subset R^m$, the set of admissible controls W_Γ is the set of all Lebesgue measurable functions w: $[0, \infty) \to \Gamma$.

Consider a control law $w = h(x, u)$ so that the "closed loop" equations are given by

$$\dot{x} = f(x, u), \tag{1.2}$$

where $f(x, u) = g(x, h(x, u))$ and where $u \in R^m$ is a new control vector. For a subset $\Omega \subset R^m$ the new set of controls U_Ω is then the set of Lebesgue measurable functions u: $[0, \infty) \to \Omega$, and given a subset $X \subset R^n$, the pair X, Ω is called admissible [1] if for each $x \in X$ and each $u \in \Omega$, $h(x, u) \in \Gamma$. The following question is then of considerable interest to the synthesis of feedback systems. Given an (admissible) pair of subsets X, Ω; under what conditions does there exist for each $x_0 \in X$ a control $u \in U_\Omega$ such that the corresponding (unique)

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solution $x(t) = \phi(t, 0, x_0, u)$ to (1.2) on $[0, \infty)$ satisfying $x(0) = x_0$ remains in X for all $t > 0$?

Roxin [2], in his study of "general control systems" (which are defined through their so called "attainability functions"), called sets X possessing the aforementioned property *positively weakly invariant*. He considered positively weakly invariant sets as a generalization of invariant sets of ordinary stability theory of dynamical systems (see, e.g., [3]) and studied problems of weak and strong stability in such sets via generalized Liapunov functions.

Weak invariance also received attention for ordinary differential equations (without uniqueness) by Yorke [4] (see also [3]) and for systems described by contingent equations by Yorke [5] (see also [6]) and by Bebernes and Schuur [7]. In [4], where in the main, problems of weak invariance and stability of ordinary differential equations were investigated, Yorke also gave a sufficient (but not necessary) condition for positive weak invariance of a closed set $X \subset R^n$ relative to control system (1.2). Bebernes and Schuur [7] gave a necessary and sufficient condition for positive weak invariance of contingent systems based on a theorem of Nagumo [8], which was also a principal tool in [4].

In the present paper we will be concerned with control system (1.2). We assume that $\Omega \subset R^m$ is compact and wall positively weakly invariant sets *weakly Ω-invariant* to emphasize the dependence of this property on the restraint set Ω. We focus our primary attention on weak Ω-invariance of compact convex sets $X \subset R^n$.

The paper is organized as follows: In Section 2 we define the concepts of *weak Ω-subtangentiality* and *strict weak Ω-subtangentiality* of a map f to a compact convex subset $X \subset R^n$ and characterize some properties of such maps. In Section 3 we prove a necessary and sufficient condition for a compact convex subset $X \subset R^n$ to be weakly Ω-invariant (Theorem 3.2). It is also shown that under certain conditions a weakly Ω-invariant set contains an Ω-rest point, i.e., a point $x_0 \in X$ such that $f(x_0, u_0) = 0$ for some $u_0 \in \Omega$ (Theorem 3.3). The paper is concluded in Section 4, where some properties of weakly Ω-invariant sets for systems with linear dynamics are investigated.

2. Preliminaries

For a nonempty subset $X \subset R^n$ we denote by ∂X, $\text{int}(X)$, $\bar{X}$, and $H(X)$ its boundary, interior, closure, and convex hull, respectively. If X is compact and convex, we also denote, respectively, by $ri(X)$ and $rb(X)$ its relative interior and relative boundary. We let $\varnothing$ denote the empty set. We denote the Euclidean norm and inner product in R^n by $\| \; \|$ and $\langle \cdot, \cdot \rangle$, respectively, and let $B \triangleq \{v \in R^n \mid \| v \| = 1\}$.

Consider a compact and convex set $X \subset R^n$. For $x \in \partial X$, let

$$V_x = \{v \in B \mid \langle v, y - x \rangle \leqslant 0; y \in X\}$$

denote the set of unit outward normals to X at x. A point $x \in \partial X$ is called *regular* if V_x is a single vector (denoted v_x), and a compact convex set X is called *smooth* if all its boundary points are regular. For X compact and convex and any $y \in R^n$ we denote by $P_X(y)$ the *projection of y on X*; i.e., $P_X(y)$ is the unique closest point of X to y.

For $x \in R^n$ and a number $\epsilon > 0$ we let $S_\epsilon(x) \triangleq \{y \in R^n \mid \|y - x\| < \epsilon\}$. Similarly, for a subset $X \subset R^n$ we let $S_\epsilon(X) \triangleq \bigcup_{x \in X} S_\epsilon(x)$ and denote by X_ϵ its closure.

It is readily verified (see also [9, 10]) that if $X \subset R^n$ is compact and convex, then X_ϵ is compact, convex, and smooth for each $\epsilon > 0$. Moreover, given $y \in \partial X_\epsilon$, then $y - P_X(y) = \epsilon v_y$ and $v_y \in V_{P_X(y)}$.

We will later also need the following

LEMMA 2.1. *Let $X \subset R^n$ be compact and convex. Then the map V of ∂X into closed subsets of B defined by $V(x) = V_x$ is upper-semicontinuous.*

Proof. Assume the lemma is false and that V is not upper-semicontinuous at some $x_0 \in \partial X$. Then for some $\epsilon_0 > 0$ (see, e.g., [11]) there is a sequence $\{\delta_i\}$, $\delta_i > 0$, $\delta_i \to 0$ and associated sequences $\{x_i\}$ and $\{v_i\}$ such that $x_i \in S_{\delta_i}(x_0) \cap \partial X$, $v_i \in V_{x_i}$, $v_i \notin S_{\epsilon_0}(V_{x_0})$, and $v_i \to v_0 \in B$ (the latter convergence being due to the compactness of B). Since $S_{\epsilon_0}(V_{x_0})$ is open, it follows that $v_0 \notin S_{\epsilon_0}(V_{x_0})$. Hence in view of the convexity of X and the continuity of the inner product function, the set $W_k = \{x \in X \mid \langle v_k, x - x_k \rangle > 0\}$ is nonempty for sufficiently large k, a contradiction since $v_k \in V_{x_k}$. ∎

Let $X \subset R^n$ be compact and convex. We call a vector $z \in R^n$ *subtangential* (*strictly subtangential*) to X at $x \in \partial X$ if $\langle v, z \rangle \leqslant 0$ (< 0) for all $v \in V_x$. Similarly, a map $f: R^n \times R^m \to R^n$ is called *weakly* (*strictly weakly*) *Ω-subtangential to X at $x \in \partial X$* (relative to a given subset $\Omega \subset R^m$) if there exists $u \in \Omega$ such that $f(x, u)$ is subtangential (strictly subtangential) to X at x. f is called *weakly* (*strictly weakly*) *Ω-subtangential to X* provided the condition holds for each $x \in \partial X$.

In view of Lemma 2.1 we can also characterize strict weak Ω-subtangentiality as follows.

LEMMA 2.2. *Assume $X \subset R^n$ is compact and convex and that the following conditions hold.*

(A_1) *$f(x, u)$ is continuous in both arguments and is continuously differentiable in x.*

(A_2) *$\Omega \subset R^m$ is nonempty and compact.*

Then f is strictly weakly Ω-subtangential to X if and only if for some $\alpha_0 < 0$ there exists for each $x \in \partial X$ a $u \in \Omega$ satisfying $\langle v, f(x, u)\rangle \leqslant \alpha_0$ for all $v \in V_x$.

Proof. Sufficiency is obvious. To prove necessity, define the real function g on ∂X by

$$g(x) = \min_{u\in\Omega} \max_{v\in V_x} \langle v, f(x, u)\rangle.$$

By hypothesis, $g(x) < 0$ for each $x \in \partial X$. In view of Lemma 2.1 it is readily verified [11, pp. 113–116] that g is an upper-semicontinuous function and hence attains a maximum $\alpha_0 < 0$ on the compact set ∂X. ∎

We conclude this section with the following approximation result.

LEMMA 2.3. *Assume* (A_1) *and* (A_2) *hold and that f is weakly Ω-subtangential to a compact convex subset $X \subset R^n$. Then, given any $\alpha > 0$, there exists $\epsilon(\alpha) > 0$ such that for any ϵ, $0 < \epsilon < \epsilon(\alpha)$, and each $x \in \partial X_\epsilon$, there exists a $u \in \Omega$ satisfying $\langle v_x, f(x, u)\rangle \leqslant \alpha$.*

Proof. If the lemma is false then for some $\alpha_0 > 0$ there exists a sequence $\{\epsilon_i\}$, $\epsilon_i > 0$, $\epsilon_i \to 0$ and an associated sequence $\{x_i\}$, $x_i \in \partial X_{\epsilon_i}$ such that $\langle v_{x_i}, f(x_i, u)\rangle > \alpha_0$ for all $u \in \Omega$. Denoting $y_i = P_X(x_i) \in \partial X$, it is clear that $\| x_i - y_i \| = \epsilon_i$ and for some subsequence $\{i'\}$ of $\{i\}$, $x_{i'} \to x^* \in \partial X$ and $v_{x_{i'}} \to v^* \in V_{x^*}$, the latter convergence being due to the upper-semicontinuity of V and the fact that $v_{x_i} \in V_{y_i}$ for all i. Hence, since f is continuous, we conclude that $\langle v^*, f(x^*, u)\rangle \geqslant \alpha_0 > 0$ for all $u \in U_\Omega$, violating the weak Ω-subtangentiality of f to X. ∎

3. WEAK Ω-INVARIANCE

We consider system (1.2) and will assume throughout that conditions (A_1) and (A_2) (of Lemma 2.2) hold. We will denote both vectors in Ω and functions in U_Ω by u, the meaning always being clear from the context. Conditions (A_1) and (A_2) guarantee that for any compact set $X \subset R^n$ and any $\epsilon > 0$ there exists $T > 0$ such that the following holds: Given any $x_0 \in X$ and any $u \in U_\Omega$, there exists a unique solution $x(t) = \phi(t, 0, x_0, u)$ on $[0, T]$ to (1.2) (satisfying $x(0) = x_0$), and $x(t) \in X_\epsilon$ for all $t \in [0, T]$.

We will also need the following condition.

(A_3) *For each $x \in R^n$ the velocity set $f(x, \Omega) \triangleq \{f(x, u) \mid u \in \Omega\}$ is convex.*

For each $x \in R^n$ and $t \geqslant 0$ we denote by $F_\Omega(x, t)$ the *reachable set* from x

in time t, that is $F_\Omega(x, t) = \cup \{\phi(t, 0, x, u) \mid u \in U_\Omega\}$. For a compact subset $X \subset R^n$ and $\epsilon > 0$ we define

$$T(X, \epsilon) = \sup\{T \mid F_\Omega(x, t) \in X_\epsilon;\ x \in X,\ 0 \leqslant t \leqslant T\}.$$

We then have the following variant of a well-known theorem due to Fillipov [12] (see also [13, 14]), which will be required in the sequel.

THEOREM 3.1. *Let $X \subset R^n$ be compact and assume* (A_1)–(A_3) *hold. Then*

(i) *For each $\epsilon > 0$ and $T < T(X, \epsilon)$, the map $(x, t) \mapsto F_\Omega(x, t)$ is a Housdorff continuous map of $X \times [0, T]$ into closed subsets of X_ϵ.*

(ii) *For any $\epsilon > 0$ and $T < T(X, \epsilon)$ let $\{x_i(t)\}$, $x_i(t) = \phi(t, 0, x_i, u_i)$, $x_i \in X$, $u_i \in U_\Omega$ be a sequence of solutions to* (1.2) *on $[0, T]$. Then there is a uniformly convergent subsequence $\{x_k(t)\}$ (on $[0, T]$) of $\{x_i(t)\}$ with limit $x(t)$, where $x(t)$ is a solution to* (1.2) *for some $u \in U_\Omega$.*

DEFINITION. A subset $X \subset R^n$ is called *weakly Ω-invariant* (*strictly weakly Ω-invariant*) relative to system (1.2) if for each $x_0 \in X$ there exists a control $u \in U_\Omega$ such that

$$x(t) = \phi(t, 0, x_0, u) \in X(\mathrm{int}(X)) \qquad \text{for all } t > 0.$$

The following sufficient condition for strict weak Ω-invariance will be used below to prove the main necessary and sufficient condition for weak Ω-invariance (Theorem 3.2).

LEMMA 3.1. *Assume* (A_1) *and* (A_2) *hold, and let $X \subset R^n$ be compact and convex. If f is strictly weakly Ω-subtangential to X, then X is strictly weakly Ω-invariant.*

Proof. First note that X has nonempty interior. Indeed, if $\mathrm{int}(X) = \varnothing$, then $X \subseteq \mathscr{L}$ for some hyperplane $\mathscr{L} \subset R^n$. But then for some $v \in B$, both v and $-v$ are in V_x for each $x \in X$ and strict weak Ω-subtangentiality is violated.

For $x_0 \in \partial X$ choose (as in Lemma 2.2) $u_0 \in \Omega$ such that

$$\langle v, f(x_0, u_0)\rangle \leqslant \alpha_0 < 0 \qquad \text{for all } v \in V_{x_0}.$$

Applying as control the *constant* function $u(t) = u_0$, it is readily verified that $\phi(t, 0, x_0, u_0) \in \mathrm{int}(X)$ for $t > 0$ sufficiently small. Hence, if X is not strictly weakly Ω-invariant there is some $x_0 \in \mathrm{int}(X)$ such that for each $u \in U_\Omega$ there exists $t(u) > 0$ such that $x_\partial(u) = \phi(t(u), 0, x_0, u) \in \partial X$, $\phi(t, 0, x_0, u) \in \mathrm{int}(X)$ for $0 \leqslant t < t(u)$ and $\sup\{t(u) \mid u \in U_\Omega\} = t^* < \infty$. Consider any sequence

$\{u_i\}$, $u_i \in U_\Omega$, $t(u_i) \to t^*$. By compactness of ∂X we may assume that $x_\partial(u_i) \to x^* \in \partial X$. For any $0 < h < t(u_i)$ let

$$\psi(u_i, h) \triangleq [x_\partial(u_i) - \phi(t(u_i) - h, 0, x_0, u_i)]/h.$$

By convexity of X, $\langle v, \psi(u_i, h)\rangle > 0$ for all $v \in V_{x_\partial(u_i)}$. Letting $i \to \infty$ and $h \to 0$, it is easily verified that f is not strictly weakly Ω-subtangential to X at x^*. ▮

THEOREM 3.2. *Let $X \subset R^n$ be compact and convex and assume* (A_1) *and* (A_2) *hold. A necessary condition for weak Ω-invariance of X is that f be weakly Ω-subtangential to X. If* (A_3) *also holds, the condition is also sufficient.*

Proof. Necessity. Choose any $x_0 \in \partial X$ and let $u_0 \in U_\Omega$ satisfy $\phi(t, 0, x_0, u_0) \in X$ for all $t \geqslant 0$. Clearly, for any $t > 0$ we have

$$\langle v, [\phi(t, 0, x_0, u_0) - x_0]/t\rangle \leqslant 0 \qquad \text{for all } v \in V_{x_0}.$$

Letting $t \to 0$, we readily conclude the existence of $u^* \in \Omega$ such that $\langle v, f(x_0, u^*)\rangle \leqslant 0$ for all $v \in V_{x_0}$. Hence f is weakly Ω-subtangential to X.

Sufficiency. Consider a convergent sequence $\{\alpha_i\}$, $\alpha_i > 0$, $\alpha_i \to 0$. By Lemma 2.3 there exists an associated sequence $\{\epsilon_i\}$, $\epsilon_i > 0$, $\epsilon_i \to 0$ such that for each $i = 1, 2,\ldots$ and $0 < \epsilon < \epsilon_i$ there exists for $x \in \partial X_\epsilon$ a vector $u \in \Omega$ satisfying $\langle v_x, f(x, u)\rangle \leqslant \alpha_i$. Let $\Lambda_i \triangleq \{w \in R^n \mid \|w\| \leqslant 2\alpha_i\}$ and consider the system

$$\dot{x} = g(x, u, w) \triangleq f(x, u) + w, \qquad (S^i)$$

where $u \in U_\Omega$ and $w \in W_{\Lambda_i}$ (W_{Λ_i} being the set of measurable functions w: $[0, \infty) \to \Lambda_i$). Clearly, for each i, g is strictly weakly $\Omega \times \Lambda_i$-subtangential to X_{ϵ_i}, and hence (by Lemma 3.1) X_{ϵ_i} is strictly weakly $\Omega \times \Lambda_i$-invariant.

Let x_0 be any point of X and for each $i = 1, 2,\ldots$ let $(u_i, w_i) \in U_\Omega \times W_{\Lambda_i}$ be a control pair such that the corresponding solution $x_i(t)$ to (S^i) satisfies $x_i(0) = x_0$ and $x_i(t) \in X_{\epsilon_i}$ for all $t \geqslant 0$. Choose $0 < T < T(X, \epsilon_1)$ arbitrarily. Since $X_{\epsilon_j} \subset X_{\epsilon_k}$ and $W_{\Lambda_j} \subset W_{\Lambda_k}$ for all $j \geqslant k$, conditions (A_1)–(A_3) imply (in view of Theorem 3.1) that there is a uniformly convergent subsequence $\{x_{i'}(t)\}$ on $[0, T]$ to a function $x(t)$, which is clearly a solution to (1.2) for some $u \in U_\Omega$ and which satisfies $x(t) \in X$ for all $0 \leqslant t \leqslant T$. Using the above convergence argument repeatedly, weak Ω-invariance follows. ▮

Remark 3.1. In [4] Yorke gave a sufficient condition for weak invariance (in a control system) that in the present setting can be stated as follows.

Consider system (1.2), *assume* (A_1) *holds, and let $X \subset R^n$ be compact and convex. If there is a continuous map w: $X \to \Omega$ such that $f(x, w(x))$ is subtangential to X at each $x \in \partial X$, then X is weakly Ω-invariant.*

Although this condition is not easily verifiable, we mention it here since it is independent of Theorem 3.2 in that the convexity of $f(x, \Omega)$ (and even the compactness of Ω) is not required. It is worth noting, however, that this sufficient condition is not necessary in general, as the following simple example illustrates: Let $f(x, u) = x + u$, let $X = [0, \frac{1}{2}]$ and $\Omega = \{-1, 0\}$. X is clearly weakly Ω-invariant, but since Ω consists of only two points, subtangentiality cannot be satisfied by a continuous w.

Remark 3.2. Although in the present paper we are concerned with weak Ω-invariance of convex sets, it should be noted that Theorem 3.2 could be generalized to sets that are not necessarily convex. Such a generalization could be obtained by using the same approximation approach as was used to prove Theorem 3.2 after suitable modification of the concepts of subtangentiality and strict subtangentiality. The generalization could also be obtained by applying Fillipov's theorem on measurable selection [12] to Theorem 2 of [7] (see also [5]) or by combining Roxin's results on generalized dynamical systems [2, 15] with Fillipov's theorem.

An important question that arises in connection with weakly Ω-invariant sets X is that of existence of points $x \in X$ at, or near, which it is possible to maintain the state by means of controls in U_Ω.

DEFINITION. Consider system (1.2) with restraint set $\Omega \subset R^m$. A point $x_0 \in X$ is called an *Ω-rest point* if there exist $u_0 \in \Omega$ such that $f(x_0, u_0) = 0$. A point $x_0 \in X$ is called an *Ω-stationary point* if for any $\epsilon > 0$ there exists an open subset $C(\epsilon)$ satisfying $x_0 \in C(\epsilon) \subset S_\epsilon(x_0)$ and $X \cap C(\epsilon)$ is weakly Ω-invariant.

Remark 3.3. One should note that in general an Ω-stationary point need not be an Ω-rest point nor need an Ω-rest point be Ω-stationary.

The following theorem provides a sufficient condition for the existence of rest points in a weakly Ω-invariant set.

THEOREM 3.3. *Consider system* (1.2) *and assume* (A_1)–(A_3) *hold. If a compact convex set $X \subset R^n$ is weakly Ω-invariant, it contains an Ω-rest point.*

Proof. For each $t \geqslant 0$ and $x \in X$, weak Ω-invariance implies that $F_\Omega(x, t) \cap X \neq \varnothing$. With the aid of Theorem 3.1, it can be seen that for $t > 0$ sufficiently small, the map $x \mapsto G_t(x) \triangleq H(F_\Omega(x, t)) \cap X$ is an uppersemicontinuous map of X into the set of nonempty closed convex subsets of itself. Hence, by Kakutani's fixed point theorem [16] there exists $x_t \in X$ such that $x_t \in G_t(x_t)$. For any sequence $\{t_i\}$, $t_i > 0$, $t_i \to 0$ let $\{x_i\}$ be an associated sequence such that $x_i \in G_{t_i}(x_i)$. By compactness of X we can assume that $\{x_i\}$ converges to a limit $x^* \in X$ that we claim to be an Ω-rest point. Indeed, if $0 \notin f(x^*, \Omega)$ then in view of (A_1)–(A_3) there exist $v \in B$ and $\alpha < 0$ such that $\langle v, f(x^*, u)\rangle \leqslant \alpha$ for all $u \in \Omega$. Hence, for some $\epsilon > 0$ and each $x \in S_\epsilon(x^*)$, $\langle v, f(x, u)\rangle < 0$ for all $u \in \Omega$. But then it is readily verified that there exists $t > 0$ such that for $x \in S_{\epsilon/2}(x^*)$, $x \notin H(F_\Omega(x, \tau))$ for all $0 < \tau < t$, a contradiction. ∎

4. Linear Dynamics and Constrained Reachability

We conclude the paper with an investigation of some properties of weakly Ω-invariant sets for systems with linear dynamics, i.e.,

$$\dot{x} = f(x, u) = Ax + Bu, \tag{4.1}$$

where A and B are constant real matrices. In this case assumption (A_1) is always satisfied and assumption (A_3) holds whenever Ω is convex. Moreover, we also have the following well-known lemma (see, e.g., [13]).

Lemma 4.1. *Consider system* (4.1) *and assume $\Omega \subset R^m$ is compact. Then for each $x \in R^n$ and each $t \geqslant 0$, $F_\Omega(x, t) = F_{H(\Omega)}(x, t)$.*

Definition. A subset $X \subset R^n$ is called *approximately weakly Ω-invariant* relative to system (1.2) if given any $\epsilon > 0$ there exists for each $x \in X$ a control $u \in U_\Omega$ such that $\phi(t, 0, x, u) \in X_\epsilon$ for all $t \geqslant 0$.

In view of Lemma 4.1 we then have the following.

Theorem 4.1. *Consider system* (4.1) *and let $X \subset R^n$ and $\Omega \subset R^m$ both be compact. Then X is approximately weakly Ω-invariant if and only if X is weakly $H(\Omega)$-invariant.*

Proof. That approximate weak Ω-invariance of X implies weak $H(\Omega)$-invariance is immediate from Theorem 3.1. Conversely, assume that X is weakly $H(\Omega)$-invariant. Then by Lemma 4.1, for any $x \in X$ and $t \geqslant 0$, $F_\Omega(x, t) \cap X \neq \varnothing$. Hence given any $x \in X$ and any strictly increasing sequence of positive numbers $\{t_i\}$, there exists a control $u \in U_\Omega$ such that $\phi(t_i, 0, x, u) \in X$ for all $i = 1, 2,\ldots$. Choosing $\epsilon > 0$ arbitrarily and setting $t_1 = t_i - t_{i-1} = T$ $(i = 2, 3,\ldots)$, where $0 < T < T(X, \epsilon)$, then implies that $\phi(t, 0, x, u) \in X_\epsilon$ for all $t \geqslant 0$, and the proof is complete. ∎

Remark 4.1. One should observe that while weak $H(\Omega)$-invariance implies approximate weak Ω-invariance, it does not imply weak Ω-invariance.

Below we will need the following notation. Let $C \subset R^n$ be a nonempty compact and convex subset and let c_0 be a given point of C. For any given $\lambda \geqslant 0$ we denote $\lambda C(c_0) \triangleq \{(1 - \lambda)\, c_0 + \lambda c \mid c \in C\}$ and for any point $c \in C$

we let $\gamma_{c_0}(c \mid C) \triangleq \inf\{\lambda \geqslant 0 \mid c \in \lambda C(c_0)\}$. Clearly, for $0 \leqslant \lambda \leqslant 1$, $\lambda C(c_0) \subset C$, and $0 \leqslant \gamma_{c_0}(c \mid C) \leqslant 1$ for all $c \in C$. If $c_0 \in ri(C)$, then for $0 < \lambda < 1$, $ri[\lambda C(c_0)] \neq \varnothing$, and for any $0 < \lambda_1 < \lambda_2 < 1$,

$$c_0 \in ri(\lambda_1 C(c_0)) \subset ri(\lambda_2 C(c_0) \subset ri(C).$$

Moreover, given $c \in C$ and setting $\lambda_c = \gamma_{c_0}(c \mid C)$, it follows that $c \in rb(\lambda_c C(c_0))$.

Consider now system (4.1) and assume that both $X \subset R^n$ and $\Omega \subset R^m$ are compact and convex. Assume $f = Ax + Bu$ is weakly Ω-subtangential to X. Then X is weakly Ω-invariant and hence contains an Ω-rest point x_0. It is readily verified that for $0 < \lambda < 1$, f is then also weakly $\lambda\Omega(u_0)$-subtangential to $\lambda X(x_0)$, where $u_0 \in \Omega$ satisfies $Ax_0 + Bu_0 = 0$. Since $\lambda\Omega(u_0) \subset \Omega$, it follows that f is also weakly Ω-subtangential to $\lambda X(x_0)$. A similar conclusion holds for strict weak Ω-subtangentiality relative to Ω-rest points $x_0 \in \text{int}(X)$. In view of the above we can thus state the following.

THEOREM 4.2. *Consider system* (4.1), *let* $X \subset R^n$ *be compact and convex, and let* $\Omega \subset R^m$ *be compact. If* X *is approximately weakly* Ω*-invariant and* $x_0 \in X$ *is any* $H(\Omega)$*-rest point, then for any* $0 \leqslant \lambda < 1$ *the set* $\lambda X(x_0)$ *is also approximately weakly* Ω*-invariant.*

Proof. If X is approximately weakly Ω-invariant, then it is weakly $H(\Omega)$-invariant, so that f is weakly $H(\Omega)$-subtangential to X and hence also to $\lambda X(x_0)$. Consequently $\lambda X(x_0)$ is weakly $H(\Omega)$-invariant and by Theorem 4.1 it is approximately weakly Ω-invariant. ▮

COROLLARY 4.1. *Consider system* (4.1), *assume* $X \subset R^n$ *is compact and convex and* $\Omega \subset R^m$ *is compact. Assume in addition that* $\text{int}(X) \neq \varnothing$ *and that some* $x_0 \in \text{int}(X)$ *is a* $H(\Omega)$*-rest point of* X. *If* X *is approximately weakly* Ω*-invariant, then* $\text{int}(X)$ *is weakly* Ω*-invariant and* x_0 *is an* Ω*-stationary point.*

Proof. For each $x \in \text{int}(X)$, $\lambda_x \triangleq \gamma_{x_0}(x \mid X) < 1$ so that $\lambda_x X(x_0) \subset \text{int}(X)$. Hence for some $\epsilon > 0$ we have $[\lambda_x X(x_0)]_\epsilon \subset \text{int}(X)$, and since $x \in \partial(\lambda_x X(x_0))$, the approximate weak Ω-invariance of $\lambda_x X(x_0)$ establishes that $\text{int}(X)$ is weakly Ω-invariant. By the same argument we conclude that for each $0 < \lambda < 1$ the set $\text{int}(\lambda X(x_0))$ is weakly Ω-invariant so that x_0 is an Ω-stationary point. ▮

We established certain conditions for weakly Ω-invariant sets X to contain Ω-rest points and Ω-stationary points. It is interesting to find conditions under which such points can be reached or even approached in finite time from arbitrary points in X via trajectories that are wholly contained in X. Below in Theorem 4.3 we give a partial answer to these questions.

DEFINITION. Let $X \subset R^n$ be weakly Ω-invariant. An Ω-rest point (or an Ω-stationary point) $\hat{x} \in X$ is called *weakly reachable in* X if given any $x \in X$ and $\epsilon > 0$ there exists $u \in U_\Omega$ and $0 \leqslant t_1 < \infty$ such that

$$x(t) = \phi(t, 0, x, u) \in X \qquad \text{for all } t \geqslant 0$$

and $x(t) \in S_\epsilon(\hat{x})$ for all $t > t_1$. $\hat{x}$ is said to be *reachable in* X if given any $x \in X$ there exists $0 \leqslant t_1 < \infty$ and a control $u \in U_\Omega$ such that

$$x(t) = \phi(t, 0, x, u) \in X \qquad \text{for all } t \geqslant 0 \quad \text{and} \quad x(t_1) = \hat{x}.$$

THEOREM 4.3. *Consider system* (4.1), *let* $\Omega \subset R^m$ *be compact, let* $X \subset R^n$ *be compact, convex, and weakly Ω-invariant. Assume that* $\text{int}(H(\Omega)) \neq \varnothing$, $\text{int}(X) \neq \varnothing$, *and* $f = Ax + Bu$ *is strictly weakly Ω-subtangential to* X. *Then*

(i) *Every Ω-stationary point* $\hat{x} \in \text{int}(X)$ *is weakly reachable in* X.

(ii) *If* $\text{rank}[B, AB,..., A^{n-1}B] = n$, *then every Ω-stationary point* $\hat{x} \in \text{int}(X)$ *such that* $A\hat{x} + B\hat{u} = 0$ *for some* $\hat{u} \in \text{int}(H(\Omega))$ *is reachable in* X.

Proof. First note that strict weak Ω-subtangentiality of f implies that X is strictly weakly Ω-invariant. Upon combining this fact with Lemma 4.1 (which implies trajectory approximation), we conclude the theorem holds for controls in U_Ω whenever it holds for controls in $U_{H(\Omega)}$. Hence we will assume that Ω is convex.

(i) For each $x \in X$ let $\lambda_x = \gamma_{x_0}(x \mid X)$. If $x \neq x_0$, then $0 < \lambda_x \leqslant 1$ and $\lambda_x X(x_0)$ is also strictly weakly Ω-invariant. Let $U_\Omega(x)$ denote the subset of U_Ω for which $u \in U_\Omega(x)$ implies that $\lambda_{\phi(t_2,0,x,u)} \leqslant \lambda_{\phi(t_1,0,x,u)}$ whenever $t_2 \geqslant t_1$. Clearly, $U_\Omega(x) \neq \varnothing$ for all $x \in X$. If x_0 is not weakly reachable, then for some $\hat{x} \in X$ $\inf_{u \in U_\Omega(\hat{x})}[\lim_{t\to\infty} \lambda_{\phi(t,0,\hat{x},u)}] = \hat{\lambda} > 0$. But then one readily concludes that $\hat{\lambda}X(x_0)$ is not strictly weakly Ω-invariant, a contradiction.

(ii) Follows immediately upon combining (i) with the well-known result on local controllability [13, Corollary 2, p. 84]. ∎

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