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CDO pricing using single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model with stochastic correlation and random factor loading

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ABSTRACT

We consider the valuation of CDO tranches with single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, where the involved distributions are mixtures of Gaussian distribution and $\mathcal{NIG}$ distribution. In addition, we consider two cases for stochastic correlation and random factor loadings instead of constant factor loadings. We analyze the unconditional characteristic function of accumulated loss of the reference portfolio, and derive the loss distribution through the fast Fourier transform. Moreover, using the loss distribution and semi-analytic approach, we can get the CDO tranches spreads.

1. Introduction

In recent years, collateralized debt obligations (CDOs) were probably the most important type of multi-name credit derivatives. A CDO consists of a portfolio of reference entities (e.g. bonds, loans, residential and commercial mortgages) whose credit risk is sold to investors who, in return for an agreed payment (usually a periodic fee), will bear the losses of the portfolio derived from the default of the reference entities. Through a securitization technique, CDOs repackage a portfolio credit risk into tranches according to credit risk. During the life of the transaction the resulting losses affect first the so-called equity piece and then, after the equity tranche has been exhausted, the mezzanine tranches. Further losses, due to credit events on a large number of reference entities, are supported by senior and super senior tranches. The credit risk of the portfolio underlying the CDO is sold in these tranches. Generally, a tranche is defined by a lower attachment point K_L and an upper detachment point K_U. The buyers of the tranche $[K_L, K_U]$ will bear all losses of the portfolio value in excess of K_L and up to K_U percent of the initial value of the portfolio. CDO tranching allows the holders of each tranche to limit their loss exposure to $K_U - K_L$ percent of the initial portfolio value.

The base for pricing CDO tranches is to model the default losses in portfolio of all reference entities. Since it is a well-known fact that defaults appear in all entities and so cannot be treated as independent random events, then, we must find an efficient approach to solve this. A common technique for the description of codependence of defaults is to specify a copula that governs the joint distribution of default times, and many cases can be found in [1,2]. More precisely, the copula methodology will help us model the dependency between the default-correlation entities, and many works have proved the factor copula approach is a powerful tool for pricing CDOs within a semi-analytical framework, see [3,4]. However, there still exists a correlation smile behavior in calculating the correlations that are implied by the market prices of tranches. The main explanation of this phenomenon is the lack of tail dependence of the Gaussian copula. Then, various authors have proposed

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different approaches to bring more tail dependence into the model. One approach is the introducing stochastic correlation (see [5,6]) or stochastic risk exposure (see [7]). The risk exposures may or not be associated with a factor structure and may or may not be factor dependent. When the correlation is stochastic and independent of the factor, we will consider a stochastic correlation model. When the correlation depends upon the factor, we will discuss a local correlation model with random factor loadings (see [7]). Meanwhile, many other authors proposed different approaches to use a copula that exhibits more tail dependence: Examples are the α-stable copula in Claudio Ferrarese [8], Marshall-Olkin copula in Andersen and Sidenius [7], the double t distribution in Hull and White [2], and normal inverse Gaussian distribution in [9]. Recently, Geng Xu [10] used the mixture copula model of multi-Gaussian distributions and Dezhong Wang [11] used double mixture of t and Gaussian copula to price the CDO tranches, and they found these mixture copula models fitted better than the previous models, however, they did not consider the stochastic correlation and random factor loadings. Motivated by these mixture copula models, we combine the normal inverse Gaussian ($\mathcal{NIG}$) distribution with Gaussian distribution, construct an $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, in addition, for solving the dynamic correlation, we consider stochastic correlation and random factor loadings.

The paper is organized as follows. Section 2 gives a semi-analytic approach for pricing CDO tranche. Section 3 presents $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model. Sections 4 and 5 discuss the $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model where that factor loadings are the stochastic correlation and random factor loadings, respectively.

2. Semi-analytic approach for pricing CDO tranches

For pricing a CDO tranche, that is, finding the fair spreads of the tranche, we must analyze the values of two legs, default leg (DL) and premium leg (PL) using semi-analytic approach. The value of DL presents the value of tranche losses triggered by credit events during the CDO lifetime, and the value of PL is the premium payments weighted by the outstanding asset (original tranche amount minus accumulated losses). Under the risk-neutral measure, the expected value of both legs should be equal, applying this we can derive the fair spread S of tranche $[K_L, K_U]$, i.e.

$$S = \frac{E[DL_{[K_L,K_U]}]}{E[PL_{[K_L,K_U]}]}. \tag{1}$$

Now let us describe the method to calculate DL and PL for CDO tranche $[K_L, K_U]$ in detail. For convenience, we assume the reference portfolio consists of n entities, and introduce the following notations:

- R_i—the recovery rate of the ith reference entity;
- M_i—the notional of the ith reference entity;
- V_i—the asset value of the ith reference entity;
- A_i—the lower default barrier of the ith reference entity;
- τ_i—the default time of the ith reference entity, i.e., $\tau_i := \inf\{t \geqslant 0\colon\ V_i \leqslant A_i\}$;
- T—the maturity time;
- r—the risk-free discount rate ($r > 0$).

Letting $l_i := M_i(1 - R_i)$, then accumulated loss at time t is given by:

$$L(t) = \sum_{i=1}^{n} M_i(1 - R_i)\mathbf{1}_{\{\tau_i \leqslant t\}} = \sum_{i=1}^{n} l_i \mathbf{1}_{\{\tau_i \leqslant t\}}, \tag{2}$$

and the aggregate tranche loss of $[K_L, K_U]$ is:

$$L_{[K_l,K_U]}(t) = \min\bigl\{L(t), K_U\bigr\} - \min\bigl\{L(t), K_L\bigr\}. \tag{3}$$

For the default leg, let us assume that

$$0 \leqslant t_0 < t_1 < \cdots < t_{M-1} < t_M = T \tag{4}$$

denote the spread payment dates. Now we derive the following results:

Theorem 2.1. *Under the risk-neutral probability measure, the price S of CDO tranche $[K_L, K_U]$ is given by*

$$S = \frac{E[\sum_{m=1}^{M} \mathrm{e}^{-rt_m}(L_{[K_L,K_U]}(t_m) - L_{[K_L,K_U]}(t_{m-1}))]}{E[\sum_{m=1}^{M} \Delta t_m \mathrm{e}^{-rt_m} \min\{\max[K_U - L_{[K_L,K_U]}(t_m), 0], K_U - K_L\}]}. \tag{5}$$

Proof. First, the expected value of the default leg of the tranche $[K_L, K_U]$ with respect to the risk neutral probability measure can be written as follows:

$$E[DL_{[K_L,K_U]}] = \int_{t_0}^{t_M} \mathrm{e}^{-rt_u}\, dE\, L_{[K_L,K_U]}(u) = E\left[\sum_{m=1}^{M} \mathrm{e}^{-rt_m}\big(L_{[K_L,K_U]}(t_m) - L_{[K_L,K_U]}(t_{m-1})\big)\right]. \tag{6}$$

Second, the expected value of the premium leg of tranche $[K_L, K_U]$ is the present value of all expected spread payments, and it is calculated by

$$E[PL_{[K_L,K_U]}] = E\left[\sum_{m=1}^{M} S\Delta t_m \mathrm{e}^{-rt_m} \min\big\{\max\big[K_U - L_{[K_L,K_U]}(t_m), 0\big], K_U - K_L\big\}\right], \tag{7}$$

where $\Delta t_m = t_m - t_{m-1}$.

Substituting (6) and (7) into (1) yields the conclusion (5). □

Remark 2.1. Eq. (5) shows, the key issue for pricing the CDO tranche $[K_L, K_U]$ is to find the distribution of the accumulated loss $L_{[K_L,K_U]}(t_m)$ of a CDO portfolio. In the following sections we will present the $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, and analyze the probability distribution of $L(t)$.

3. Single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model

Before giving the single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, let us consider the $\mathcal{NIG}$ distribution (see [8,9]) and $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ distribution.

Definition 3.1 *($\mathcal{NIG}$ distribution).* The $\mathcal{NIG}$ distribution (normal inverse Gaussian distribution) is a mixture of normal and inverse Gaussian distribution. A random variable U follows a $\mathcal{NIG}$ distribution with parameters α, β, μ and δ if its density function is of the form

$$f_{\mathcal{NIG}}(x;\alpha,\beta,\mu,\delta) = \frac{\delta\alpha\exp(\delta\gamma + \beta(x-\mu))}{\pi\sqrt{\delta^2 + (x-\mu)^2}} K_1\big(\alpha\sqrt{\delta^2+(x-\mu)^2}\big), \tag{8}$$

where $K_1(w) := \frac{1}{2}\int_0^\infty \exp(-\frac{1}{2}w(t+t^{-1}))\,dt$, $0 \leqslant |\beta| < \alpha$ and $\delta > 0$.

We denote the $\mathcal{NIG}$ distribution by $\mathcal{NIG}(\alpha,\beta,\mu,\delta)$. If a random variable $U \sim \mathcal{NIG}(\alpha,\beta,\mu,\delta)$, then,

$$E[U] = \mu + \delta\frac{\beta}{\gamma}, \qquad \mathrm{Var}[U] = \delta\frac{\alpha^2}{\gamma^3}, \tag{9}$$

where $\gamma := \sqrt{\alpha^2 - \beta^2}$.

Proposition 3.1. *The main properties of the $\mathcal{NIG}$ distribution class are the scaling property*

$$U \sim \mathcal{NIG}(\alpha,\beta,\mu,\delta) \quad\Rightarrow\quad cU \sim \mathcal{NIG}\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta\right) \quad \text{for } c \in R, \tag{10}$$

and the stability under convolution for independent random variables U_1 and U_2

$$U_1 \sim \mathcal{NIG}(\alpha,\beta,\mu_1,\delta_1) \quad \text{and} \quad U_2 \sim \mathcal{NIG}(\alpha,\beta,\mu_2,\delta_2) \quad\Rightarrow\quad U_1+U_2 \sim \mathcal{NIG}(\alpha,\beta,\mu_1+\mu_2,\delta_1+\delta_2). \tag{11}$$

Proof. See [9]. □

Definition 3.2 *($\mathcal{M}_{G\text{-}\mathcal{NIG}}$ distribution).* The $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ distribution is a mixture distribution of Gaussian distribution and $\mathcal{NIG}$ distribution, and a random variable $X \sim \mathcal{M}_{G\text{-}\mathcal{NIG}}(0,1;\alpha,\beta,\mu,\delta;p)$ is given by

$$X = \begin{cases} U, & \text{with probability } 1-p, \\ V, & \text{with probability } p, \end{cases} \tag{12}$$

where $U \sim \mathcal{NIG}(x;\alpha,\beta,\mu,\delta)$, V is a random variable that followed a standard Gaussian distribution, and $p \in (0,1)$ is the proportion of the Gaussian component in the mixture distribution X, α, β, μ and δ is defined as in Definition 3.1. Denote the probability density function of X by $f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x;0,1;\alpha,\beta,\mu,\delta;p)$, then

$$f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x;0,1;\alpha,\beta,\mu,\delta;p) = \frac{p}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) + (1-p) f_{\mathcal{NIG}}(x;\alpha,\beta,\mu,\delta). \tag{13}$$

Now we give the single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model.

Definition 3.3 *(Single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model).* Suppose that the ith asset value V_i of the reference entities is given by

$$V_i = \rho_i Y + \sqrt{1-\rho_i^2}\epsilon_i, \quad i = 1, \dots, n, \tag{14}$$

where the factor loadings ρ_i, $i = 1, \dots, n$, is a constant and takes values in $[0, 1]$, Y is the common factor of the market, ϵ_i, $i = 1, \dots, n$, is the idiosyncratic factor of the ith reference entity, and we assume that Y and ϵ_i are independent with random variables with

$$Y \sim \mathcal{M}_{G\text{-}\mathcal{NIG}}\left(0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\right), \tag{15}$$

$$\epsilon_i \sim \mathcal{M}_{G\text{-}\mathcal{NIG}}\left(0, 1; \frac{\sqrt{1-\rho_i^2}}{\rho_i}\alpha, \frac{\sqrt{1-\rho_i^2}}{\rho_i}\beta, -\frac{\sqrt{1-\rho_i^2}}{\rho_i}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho_i^2}}{\rho_i}\frac{\gamma^3}{\alpha^2}; p\right), \quad i = 1, \dots, n. \tag{16}$$

We call the model given by (14)–(16) *single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model.*

4. Stochastic correlation

Now we introduce the stochastic correlation into the above $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, and present the ith asset value as follows:

$$V_i = \widetilde{\rho}_i Y + \sqrt{1-\widetilde{\rho}_i^2}\epsilon_i, \quad i = 1, \dots, n, \tag{17}$$

where $\epsilon_i \sim \mathcal{M}_{G\text{-}\mathcal{NIG}}(0, 1; \frac{\sqrt{1-\widetilde{\rho}_i^2}}{\widetilde{\rho}_i}\alpha, \frac{\sqrt{1-\widetilde{\rho}_i^2}}{\widetilde{\rho}_i}\beta, -\frac{\sqrt{1-\widetilde{\rho}_i^2}}{\widetilde{\rho}_i}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1-\widetilde{\rho}_i^2}}{\widetilde{\rho}_i}\frac{\gamma^3}{\alpha^2}; p)$, $i = 1, \dots, n$, Y is given as in Section 3, all these being jointly independent, $\widetilde{\rho}_i$ are some random variables taking values in $[0, 1]$ and independent from the Y and ϵ_i. Consequently, conditioning on Y, V_i, $i = 1, \dots, n$, remain independent.

Base on the $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model, we will consider two stochastic correlations below proposed by X. Burtschell et al. (see [12]).

4.1. Case for binary structure

In this case the correlation random variable follows a binary structure:

$$\widetilde{\rho}_i = (1 - B_i)\rho_1 + B_i\rho_2, \tag{18}$$

where ρ_1, ρ_2 are constants and take values in $[0, 1]$, B_i, $i = 1, \dots, n$, are independent Bernoulli random variables such that

$$B_i = \begin{cases} 1, & \text{with probability } q, \\ 0, & \text{with probability } 1-q. \end{cases} \tag{19}$$

Substituting (18) into (17) yields

$$V_i = \big((1 - B_i)\rho_1 + B_i\rho_2\big)Y + \sqrt{1-\big((1 - B_i)\rho_1 + B_i\rho_2\big)^2}\epsilon_i, \quad i = 1, \dots, n. \tag{20}$$

Then, conditioning on Y, we have the conditional probability of default time for the ith reference entity:

$$\begin{aligned}
p_t^{i|Y} &= \mathbb{P}(\tau_i \leqslant t \mid Y = y) \\
&= \sum_{l=0}^{1}\mathbb{P}(\tau_i \leqslant t \mid Y = y, B_i = l)\mathbb{P}(B_i = l) \\
&= \sum_{l=0}^{1}\mathbb{P}(V_i \leqslant A_i \mid Y = y, B_i = l)\mathbb{P}(B_i = l) \\
&= (1-q)\int_{-\infty}^{\frac{A_i-\rho_1 y}{\sqrt{1-\rho_1^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(x; 0, 1; \frac{\sqrt{1-\rho_1^2}}{\rho_1}\alpha, \frac{\sqrt{1-\rho_1^2}}{\rho_1}\beta, -\frac{\sqrt{1-\rho_1^2}}{\rho_1}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho_1^2}}{\rho_1}\frac{\gamma^3}{\alpha^2}; p\right)dx \\
&\quad + q\int_{-\infty}^{\frac{A_i-\rho_2 y}{\sqrt{1-\rho_2^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(x; 0, 1; \frac{\sqrt{1-\rho_2^2}}{\rho_2}\alpha, \frac{\sqrt{1-\rho_2^2}}{\rho_2}\beta, -\frac{\sqrt{1-\rho_2^2}}{\rho_2}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1-\rho_2^2}}{\rho_2}\frac{\gamma^3}{\alpha^2}; p\right)dx.
\end{aligned} \tag{21}$$

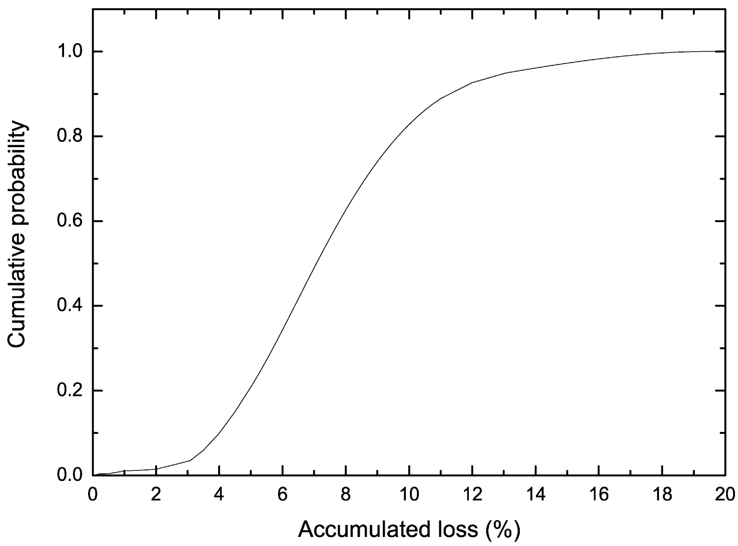

Fig. 1. Cumulative distribution function of accumulated loss.

Theorem 4.1. *The unconditional characteristic function of $L(t)$ is expressed by*

$$E\big[\mathrm{e}^{juL(t)}\big] = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \big[1 + \big(\mathrm{e}^{jul_i} - 1\big) p_t^{i|Y}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\bigg(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\bigg)\, dy. \tag{22}$$

Furthermore, using the fast Fourier transform, we can derive the probability distribution function of the accumulated loss $L(t)$.

Proof. Since V_i, $i = 1, \dots, n$, conditioned on the common factor Y are independent, then, the conditional characteristic function of $L(t)$ given Y is expressed by

$$E\big[\mathrm{e}^{juL(t)} \mid Y\big] = E\big[\mathrm{e}^{ju\sum_{i=1}^{n} l_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \mid Y\big] = \prod_{i=1}^{n} E\big[\mathrm{e}^{jul_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \mid Y\big], \tag{23}$$

where j is an imaginary unit such that $j^2 = -1$.

The characteristic function for the default loss of the ith reference entity can be written as:

$$E\big[\mathrm{e}^{jul_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \mid Y\big] = \mathrm{e}^{jul_i} p_t^{i|Y} + \big(1 - p_t^{i|Y}\big) = 1 + \big(\mathrm{e}^{jul_i} - 1\big) p_t^{i|Y}. \tag{24}$$

So, from (21), (23) and (24) we conclude

$$E\big[\mathrm{e}^{juL(t)} \mid Y\big] = \prod_{i=1}^{n} \big[1 + \big(\mathrm{e}^{jul_i} - 1\big) p_t^{i|Y}\big]. \tag{25}$$

By integrating out the common factor Y we get the unconditional characteristic function as follows:

$$E\big[\mathrm{e}^{juL(t)}\big] = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \big[1 + \big(\mathrm{e}^{jul_i} - 1\big) p_t^{i|Y}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\bigg(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\bigg)\, dy. \tag{26}$$

Using the fast Fourier transform (see [13]), we can derive the probability distribution function of the accumulated loss $L(t)$. $\square$

Now we give a numerical example to get the loss distribution of $L(t)$ by using the fast Fourier transform. For convenience, we assume $A_i = A = 0.4$ for $i = 1, 2, \dots, n$, and let $n = 100$, $\alpha = 1.2$, $\beta = -0.5$, $\gamma = 0.5$, $q = 0.3$; $p = 0.2$, $\rho_1 = 0.1$, $\rho_2 = 0.2$, $l = 0.80$. We illustrate the cumulative probability of accumulated loss in Fig. 1.

4.2. Case for symmetric dependence structure

Another case for modelling stochastic correlation, takes a more sophisticated way to consider the systemic risk, i.e., the correlation random variable is assumed to follow a symmetric dependence structure

$$\widetilde{\rho}_i = (1 - B_s)(1 - B_i)\rho + B_s, \quad i = 1, \dots, n, \tag{27}$$

where B_s, B_i, $i = 1, \dots, n$, are independent Bernoulli random variables and constant $\rho \in [0, 1]$. We still adopt the expression in (19) for random variable B_i, and denote B_s as follows:

$$B_s = \begin{cases} 1, & \text{with probability } \hat{q}, \\ 0, & \text{with probability } 1 - \hat{q}. \end{cases} \tag{28}$$

The general model of (17) can be written as:

$$V_i = \big((1 - B_s)(1 - B_i)\rho + B_s\big)Y + (1 - B_s)\big(\sqrt{1 - \rho^2}(1 - B_i) + B_i\big)\epsilon_i, \quad i = 1, \dots, n, \tag{29}$$

then, conditioning on Y and B_s, we have

$$\begin{aligned} p_t^{i|Y, B_s=1} &= \mathbb{P}(\tau_i \leqslant t \mid Y = y, B_s = 1) \\ &= \mathbf{1}_{\{y \leqslant A_i\}}, \end{aligned} \tag{30}$$

$$\begin{aligned} p_t^{i|Y, B_s=0} &= \mathbb{P}(\tau_i \leqslant t \mid Y = y, B_s = 0) \\ &= \mathbb{P}\big((1 - B_i)\rho Y + \big(\sqrt{1 - \rho^2}(1 - B_i) + B_i\big)\epsilon_i \leqslant A_i \bigm| Y = y, B_s = 0\big) \\ &= \mathbb{P}\big(\rho Y + \sqrt{1 - \rho^2}\epsilon_i \leqslant A_i \bigm| Y = y, B_s = 0, B_i = 0\big)\mathbb{P}(B_i = 0) \\ &\quad + \mathbb{P}(\epsilon_i \leqslant A_i \mid Y = y, B_s = 0, B_i = 1)\mathbb{P}(B_i = 1) \\ &= (1 - q) \int_{-\infty}^{\frac{A_i - \rho y}{\sqrt{1-\rho^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(x; 0, 1; \frac{\sqrt{1 - \rho^2}}{\rho}\alpha, \frac{\sqrt{1 - \rho^2}}{\rho}\beta, -\frac{\sqrt{1 - \rho^2}}{\rho}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1 - \rho^2}}{\rho}\frac{\gamma^3}{\alpha^2}; p\right) dx \\ &\quad + q \int_{-\infty}^{A_i} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(x; 0, 1; \frac{\sqrt{1 - \rho^2}}{\rho}\alpha, \frac{\sqrt{1 - \rho^2}}{\rho}\beta, -\frac{\sqrt{1 - \rho^2}}{\rho}\beta\frac{\gamma^2}{\alpha^2}, \frac{\sqrt{1 - \rho^2}}{\rho}\frac{\gamma^3}{\alpha^2}; p\right) dx. \end{aligned} \tag{31}$$

Now we consider the computation of probability distribution function of the accumulated loss $L(t)$. Similar to Section 4.1, we analyze the characteristic function $E[\mathrm{e}^{juL(t)}]$ for $L(t)$. Using the tower property to integrate out B_s yields

$$E\big[\mathrm{e}^{juL(t)}\big] = \hat{q}E\big[\mathrm{e}^{juL(t)} \bigm| B_s = 1\big] + (1 - \hat{q})E\big[\mathrm{e}^{juL(t)} \bigm| B_s = 0\big], \tag{32}$$

where j is defined as in Section 4.1.

Conditioning upon $B_s = 1$ ($B_s = 0$), next we will give the conditional characteristic functions $E[\mathrm{e}^{juL(t)} \mid B_s = 1]$ ($E[\mathrm{e}^{juL(t)} \mid B_s = 0]$, respectively) of $L(t)$ as follows:

$$\begin{aligned} E\big[\mathrm{e}^{juL(t)} \bigm| B_s = 1\big] &= E\big[E\big[\mathrm{e}^{juL(t)} \bigm| Y = y, B_s = 1\big]\big] \\ &= \int_{-\infty}^{\infty} E\big[\mathrm{e}^{ju(\sum_{i=1}^n l_i \mathbf{1}_{\{\tau_i \leqslant t\}})} \bigm| Y = y, B_s = 1\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\right) dy. \end{aligned} \tag{33}$$

Since default times are independent conditioning upon Y and B_s, then

$$E\big[\mathrm{e}^{ju\sum_{i=1}^n l_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \bigm| Y = y, B_s = 1\big] = \prod_{i=1}^n E\big[\mathrm{e}^{jul_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \bigm| Y = y, B_s = 1\big], \tag{34}$$

where the conditional characteristic function of the default loss of the ith reference entity can be computed by:

$$E\big[\mathrm{e}^{jul_i \mathbf{1}_{\{\tau_i \leqslant t\}}} \bigm| Y = y, B_s = 1\big] = \mathrm{e}^{jul_i} p_t^{i|Y, B_s=1} + \big(1 - p_t^{i|Y, B_s=1}\big) = 1 + \big(\mathrm{e}^{jul_i} - 1\big)p_t^{i|Y, B_s=1}. \tag{35}$$

Substituting (34) and (35) into (33) yields

$$E\big[\mathrm{e}^{juL(t)} \bigm| B_s = 1\big] = \int_{-\infty}^{\infty} \prod_{i=1}^n \big[1 + \big(\mathrm{e}^{jul_i} - 1\big)p_t^{i|Y, B_s=1}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\right) dy. \tag{36}$$

Similarly, we can obtain

$$\begin{aligned}
E\big[\mathrm{e}^{juL(t)} \mid B_s=0\big] &= E\big[E\big[\mathrm{e}^{juL(t)} \mid B_s=0\big] \mid Y=y\big] \\
&= \int_{-\infty}^{\infty} E\big[\mathrm{e}^{juL(t)} \mid Y=y, B_s=0\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y;0,1;\alpha,\beta,-\beta\frac{\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2};p\right)dy \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{n} E\big[\mathrm{e}^{jul_i \mathbf{1}_{\{\tau_i\leqslant t\}}} \mid Y=y, B_s=0\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y;0,1;\alpha,\beta,-\beta\frac{\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2};p\right)dy \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \big[1+\big(\mathrm{e}^{jul_i}-1\big)p_t^{i|Y,B_s=0}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y;0,1;\alpha,\beta,-\beta\frac{\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2};p\right)dy. \qquad (37)
\end{aligned}$$

Combining (36), (37) with (32), similar to Theorem 4.1, we obtain the following conclusions:

Theorem 4.2. *The unconditional characteristic function of $L(t)$ is expressed by*

$$\begin{aligned}
E\big[\mathrm{e}^{juL(t)}\big] &= \hat{q}\int_{-\infty}^{\infty} \prod_{i=1}^{n} \big[1+\big(\mathrm{e}^{jul_i}-1\big)p_t^{i|Y,B_s=1}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y;0,1;\alpha,\beta,-\beta\frac{\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2};p\right)dy \\
&\quad + (1-\hat{q})\int_{-\infty}^{\infty} \prod_{i=1}^{n} \big[1+\big(\mathrm{e}^{jul_i}-1\big)p_t^{i|Y,B_s=0}\big] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y;0,1;\alpha,\beta,-\beta\frac{\gamma^2}{\alpha^2},\frac{\gamma^3}{\alpha^2};p\right)dy, \qquad (38)
\end{aligned}$$

and the loss distribution $L(t)$ can be derived by the fast Fourier transform.

5. Random factor loadings

As shown by Andersen and Sidenius (see [7]) that the random factor loadings seem to fit market data well. It was therefore a natural extension to see if a combination of the above $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model and the random factor loadings in one model should provide an even better fit. In the following we are going to consider the single factor $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model where the factor loadings depend deterministically on the common factor Y.

Following closely the lines of discussion presented by Andersen and Sidenius [7], we express the $\mathcal{M}_{G\text{-}\mathcal{NIG}}$ copula model with random loadings as follows. In this case, the ith asset value is modelled by

$$V_i = \rho_i(Y)Y + \sqrt{1-\mathrm{Var}\big[\rho_i(Y)Y\big]}\epsilon_i + b_i, \qquad (39)$$

where $b_i = -E[\rho_i(Y)Y]$.

Now we give a simplest form associated with (39). The factor $\rho_i(Y)$ is an $R \to R$ function to which, following the original model (see [7]), is given by:

$$\rho_i(Y) = h_1\mathbf{1}_{\{Y<\varrho\}} + h_2\mathbf{1}_{\{Y\geqslant\varrho\}}, \qquad (40)$$

where ϱ is a positive constant, h_1, h_2 are some input parameters with $h_1, h_2 > 0$.

Thus, the conditional default probability given Y of the ith asset can be expressed by:

$$\begin{aligned}
p_t^{i|Y} &= \mathbb{P}(\tau_i \leqslant t \mid Y=y) = \mathbb{P}(V_i \leqslant A_i \mid Y=y) \\
&= \mathbf{1}_{\{y<\varrho\}} \int_{-\infty}^{\frac{A_i-b_i-h_1y}{\sqrt{1-h_1^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x;0,1;\alpha_1,\beta_1,\mu_1,\sigma_1;p)\,dx \\
&\quad + \mathbf{1}_{\{y\geqslant\varrho\}} \int_{-\infty}^{\frac{A_i-b_i-h_2y}{\sqrt{1-h_2^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x;0,1;\alpha_2,\beta_2,\mu_2,\sigma_2;p)\,dx, \quad i=1,\dots,n, \qquad (41)
\end{aligned}$$

where

$$b_i = -h_1\int_{-\infty}^{\varrho} y f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(y;0,1;\alpha_1,\beta_1,\mu_1,\sigma_1;p)\,dy - h_2\int_{\varrho}^{\infty} y f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(y;0,1;\alpha_2,\beta_2,\mu_2,\sigma_2;p)\,dy, \qquad (42)$$

$$\alpha_k = \frac{\sqrt{1-h_k^2}}{h_k}\alpha, \qquad \beta_k = \frac{\sqrt{1-h_k^2}}{h_k}\beta, \qquad \mu_k = -\frac{\sqrt{1-h_k^2}}{h_k}\beta\frac{\gamma^2}{\alpha^2}, \qquad \sigma_k = \frac{\sqrt{1-h_k^2}}{h_k}\frac{\gamma^3}{\alpha^2}, \quad k=1,2. \qquad (43)$$

Denote

$$F^1_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(\frac{A_i - b_i - h_1 y}{\sqrt{1-h_1^2}}\right) = \int_{-\infty}^{\frac{A_i - b_i - h_1 y}{\sqrt{1-h_1^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x; 0, 1; \alpha_1, \beta_1, \mu_1, \sigma_1; p)\,dx, \quad i = 1, \ldots, n, \tag{44}$$

$$F^2_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(\frac{A_i - b_i - h_2 y}{\sqrt{1-h_2^2}}\right) = \int_{-\infty}^{\frac{A_i - b_i - h_2 y}{\sqrt{1-h_2^2}}} f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}(x; 0, 1; \alpha_2, \beta_2, \mu_2, \sigma_2; p)\,dx, \quad i = 1, \ldots, n, \tag{45}$$

then, analogous to the method described in Section 4.1, we arrive at the following results:

Theorem 5.1. *The characteristic function $E[\mathrm{e}^{juL(t)}]$ of the accumulated loss $L(t)$ of the reference portfolio is given by*

$$\begin{aligned} E\big[\mathrm{e}^{juL(t)}\big] = {} & \int_{-\infty}^{\varrho} \prod_{i=1}^{N}\left[1 + \left(\mathrm{e}^{jul_i} - 1\right) F^1_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(\frac{A_i - b_i - h_1 y}{\sqrt{1-h_1^2}}\right)\right] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\right) dy \\ & + \int_{\varrho}^{\infty} \prod_{i=1}^{N}\left[1 + \left(\mathrm{e}^{jul_i} - 1\right) F^2_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(\frac{A_i - b_i - h_2 y}{\sqrt{1-h_2^2}}\right)\right] f_{\mathcal{M}_{G\text{-}\mathcal{NIG}}}\left(y; 0, 1; \alpha, \beta, -\beta\frac{\gamma^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}; p\right) dy, \end{aligned} \tag{46}$$

and the distribution of $L(t)$ can be derived from the fast Fourier transform.

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