

Modules in monoidal model categories

L. Gaunce Lewis Jr.^a, Michael A. Mandell^{b,*}

^a *Department of Mathematics, Syracuse University, Syracuse, NY 13244-1150, United States*

^b *Department of Mathematics, Indiana University, Bloomington, IN 47405, United States*

Received 12 June 2006; received in revised form 21 September 2006

Available online 28 November 2006

Communicated by C.A. Weibel

Abstract

This paper studies the existence of and compatibility between derived change of ring, balanced product, and function module derived functors on module categories in monoidal model categories.

© 2006 Elsevier B.V. All rights reserved.

MSC: Primary: 18G55; secondary: 55P99

1. Introduction

A “monoidal model category” \mathcal{M} is a category having a symmetric monoidal closed structure and a closed model category structure which satisfy certain compatibility conditions (reviewed in Section 2 below). These conditions ensure that the homotopy category $\text{Ho}\mathcal{M}$ inherits a symmetric monoidal structure and that the localization functor is (lax) symmetric monoidal. Schwede and Shipley began the study of monoidal model categories in [12]. There, they provide good criteria for the categories of modules and algebras over a monoid A in \mathcal{M} to inherit closed model structures from \mathcal{M} . The purpose of this paper is to study the existence and behavior of the derived functors of certain commonly used functors relating various categories of modules over a monoid in a monoidal model category. These functors are all variants of the “function object” and “balanced product” constructions.

Let \wedge denote the symmetric monoidal product, I denote the unit, and $[-,-]$ denote the internal function object for \mathcal{M} . A left A -module is an object M of \mathcal{M} with an associative and unital left action map $A \wedge M \rightarrow M$. For left A -modules L and M , the left A -module function object ${}^A[L, M]$ is the equalizer in \mathcal{M}

$${}^A[L, M] \longrightarrow [L, M] \rightrightarrows [A \wedge L, M],$$

where one of the righthand arrows is induced by the A -action $A \wedge L \rightarrow L$ and the other the composite of the map $[L, M] \rightarrow [A \wedge L, A \wedge M]$ and the A -action $A \wedge M \rightarrow M$. Similarly, for a right A -module M and a left A -module N , the balanced product $M \wedge_A N$ is the coequalizer in \mathcal{M}

$$M \wedge A \wedge N \rightrightarrows M \wedge N \longrightarrow M \wedge_A N,$$

* Corresponding author.

E-mail addresses: lglewis@syr.edu (L.G. Lewis Jr.), mmandell@indiana.edu (M.A. Mandell).

where one of the lefthand arrows is induced by the right action of A on M and the other by the left action of A on N .

It is clear from the definitions that the set ${}_A\mathcal{M}(L, M)$ of left A -module maps from L to M is naturally in bijective correspondence with the set $\mathcal{M}(I, {}^A[L, M])$ of maps in \mathcal{M} from the unit I to ${}^A[L, M]$. Thus, the left A -function object construction enriches the category ${}_A\mathcal{M}$ of left A -modules over the category \mathcal{M} . For objects X in \mathcal{M} and L, M in ${}_A\mathcal{M}$, the objects $L \wedge X$ and $[X, M]$ inherit left A -actions from L and the enriched parametrized adjunctions

$${}^A[L \wedge X, M] \cong [X, {}^A[L, M]] \cong {}^A[L, [X, M]]$$

indicate that the constructions $L \wedge X$ and $[X, M]$ provide tensors and cotensors for the enrichment of ${}_A\mathcal{M}$ over \mathcal{M} . The forgetful functor ${}_A\mathcal{M} \rightarrow \mathcal{M}$ has enriched left and right adjoints, called the free and cofree functors, sending an object X of \mathcal{M} to $A \wedge X$ and $[A, X]$, respectively. These functors have a rich structure of interrelations and coherences that the enriched category theory language concisely encodes and which would be tedious to list in terms of individual natural isomorphisms.

Our first objective is to describe conditions under which all of this structure passes over to the homotopy categories. Much of it passes over with no restrictions other than a very standard one on the model structure inherited by ${}_A\mathcal{M}$ from \mathcal{M} . A closed model structure on the module category ${}_A\mathcal{M}$ is said to have fibrations and weak equivalences created in \mathcal{M} if a map f in ${}_A\mathcal{M}$ is a fibration or weak equivalence in the model structure for ${}_A\mathcal{M}$ if and only if it is one in the model structure for \mathcal{M} . The following is the most basic theorem in this direction.

Theorem 1.1. *Let \mathcal{M} be a monoidal model category, and let A be a monoid in \mathcal{M} . If the category of left A -modules is a closed model category with fibrations and weak equivalences created in \mathcal{M} , then:*

- (a) *The right derived functor ${}^A\llbracket -, - \rrbracket$ of ${}^A[-, -]$ exists and enriches $\text{Ho } {}_A\mathcal{M}$ over $\text{Ho } \mathcal{M}$.*
- (b) *The right derived functor $\llbracket -, - \rrbracket$ of the cotensor functor $[-, -]: \mathcal{M}^{\text{op}} \times {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$ exists and provides cotensors for $\text{Ho } {}_A\mathcal{M}$ over $\text{Ho } \mathcal{M}$.*
- (c) *The left derived functor $\widehat{\wedge}$ of the tensor functor $\wedge: {}_A\mathcal{M} \times \mathcal{M} \rightarrow {}_A\mathcal{M}$ exists and provides tensors for $\text{Ho } {}_A\mathcal{M}$ over $\text{Ho } \mathcal{M}$.*

The enrichment concisely encodes many relations and coherences that are less obvious for these derived functors than for the corresponding functors on \mathcal{M} and ${}_A\mathcal{M}$. For example, the interpretation of $\widehat{\wedge}$ as a tensor encodes coherent associativity natural isomorphisms as well as various adjunctions. This theorem is a special case of a general theorem for closed model categories enriched over monoidal model categories, discussed in Section 3.

The condition that fibrations and weak equivalences are created in \mathcal{M} obviously implies that the forgetful functor from ${}_A\mathcal{M}$ to \mathcal{M} and its left adjoint free functor form a Quillen adjunction, and so induce a derived adjunction on the homotopy categories. Likewise, since the cotensor in $\text{Ho } {}_A\mathcal{M}$ is the right derived functor $\llbracket -, - \rrbracket$ of $[-, -]: \mathcal{M}^{\text{op}} \times {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$, it follows that its composition with the derived forgetful functor to $\text{Ho } \mathcal{M}$ is naturally isomorphic to the right derived functor of $[-, -]: \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$. The corresponding assertions about the existence of a right adjoint for the derived forgetful functor and that the derived forgetful functor preserves tensors need not hold in general, but require an additional hypothesis on A . This hypothesis depends only on A viewed as an object of \mathcal{M} , and it is convenient to state in a general context for further use in the statements below.

Definition 1.2. Let \mathcal{C} be a closed model category that is also enriched over a monoidal model category \mathcal{M} by function objects ${}^{\mathcal{C}}[-, -]$. An object C is said to be *semicofibrant* in \mathcal{C} when the functor ${}^{\mathcal{C}}[C, -]$ preserves fibrations and acyclic fibrations.

This property is explored further in Section 6. The following result proved in that section provides enough information about this notion for our present purposes.

Proposition 1.3. *Let A be a monoid in a monoidal model category \mathcal{M} for which the module category ${}_A\mathcal{M}$ is a closed model category with fibrations and weak equivalences created in \mathcal{M} .*

- (a) *M is semifibrant in ${}_A\mathcal{M}$ if and only if the functor $M \wedge (-): \mathcal{M} \rightarrow {}_A\mathcal{M}$ preserves cofibrations and acyclic cofibrations.*

- (b) If M is cofibrant in ${}_A\mathcal{M}$ then it is semifibrant in ${}_A\mathcal{M}$. Moreover, if the unit I is cofibrant in \mathcal{M} , then an object M of ${}_A\mathcal{M}$ is semifibrant in ${}_A\mathcal{M}$ if and only if it is cofibrant in ${}_A\mathcal{M}$.
- (c) If M is semifibrant in ${}_A\mathcal{M}$ and $M \rightarrow N$ is a cofibration in ${}_A\mathcal{M}$, then N is semifibrant in ${}_A\mathcal{M}$.
- (d) A is semifibrant in ${}_A\mathcal{M}$. In particular, I is semifibrant in \mathcal{M} .

If the monoid A is semifibrant when considered as an object of \mathcal{M} , then all of the enriched structure of ${}_A\mathcal{M}$ over \mathcal{M} discussed above passes to the homotopy categories.

Theorem 1.4. *Let \mathcal{M} be a monoidal model category, and let A be a monoid in \mathcal{M} . If the category of left A -modules is a closed model category with fibrations and weak equivalences created in \mathcal{M} , then:*

- (a) *The left derived functor of the free functor $\mathcal{M} \rightarrow {}_A\mathcal{M}$ exists and is a $\text{Ho}\mathcal{M}$ -enriched left adjoint to the forgetful functor from $\text{Ho}{}_A\mathcal{M}$ to $\text{Ho}\mathcal{M}$.*
- (b) *The forgetful functor $\text{Ho}{}_A\mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ preserves cotensors. Moreover, if A is semifibrant as an object of \mathcal{M} , then:*
- (c) *The right derived functor of the cofree functor $\mathcal{M} \rightarrow {}_A\mathcal{M}$ exists and is a $\text{Ho}\mathcal{M}$ -enriched right adjoint to the forgetful functor from $\text{Ho}{}_A\mathcal{M}$ to $\text{Ho}\mathcal{M}$.*
- (d) *The forgetful functor $\text{Ho}{}_A\mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ preserves tensors.*

The hypothesis above that A is semifibrant as an object of \mathcal{M} seems to hold quite generally: Often the category of monoids in \mathcal{M} forms a closed model category where the unit map $I \rightarrow A$ for a cofibrant monoid A is a cofibration in \mathcal{M} . In that case parts (c) and (d) of Proposition 1.3 imply that cofibrant monoids are semifibrant objects of \mathcal{M} . This applies in particular when the hypotheses of the main theorem of Schwede–Shipley [12, 4.1] hold.

The results above are special cases of more general results about bimodule categories and functors between such categories. If A and B are monoids in \mathcal{M} , then an (A, B) -bimodule M in \mathcal{M} is an object of \mathcal{M} with commuting left A -module and right B -module structures. Equivalently, it may be described as a left $A \wedge B^{\text{op}}$ -module. The category of (A, B) -bimodules is denoted ${}_A\mathcal{M}_B$, and the $A \wedge B^{\text{op}}$ -module function object ${}^{A \wedge B^{\text{op}}}[-, -]$ is denoted ${}^A[-, -]^B$.

Any object X of \mathcal{M} carries a canonical (I, I) -bimodule structure. Moreover, any right A -module N carries a canonical (I, A) -bimodule structure. Analogously, any left B -module M is canonically a (B, I) -bimodule. Thus, we can identify the category \mathcal{M} with the categories ${}_I\mathcal{M}$, \mathcal{M}_I and ${}_I\mathcal{M}_I$. Similarly, ${}_A\mathcal{M}$ and \mathcal{M}_B can be identified with ${}_A\mathcal{M}_I$ and ${}_I\mathcal{M}_B$, respectively. With this perspective all results below for categories of bimodules specialize to corresponding results for categories of left and/or right modules.

For any (A, B) -bimodule M and any left B -module N , the balanced product $M \wedge_B N$ is naturally a left A -module. More generally, for monoids A, B , and C , we can consider $-\wedge_B-$ to be a functor

$$-\wedge_B -: {}_A\mathcal{M}_B \times {}_B\mathcal{M}_C \rightarrow {}_A\mathcal{M}_C.$$

Similarly, if M is an (A, B) -bimodule and P is a right A -module, then the function objects ${}^A[M, P]$ and ${}^A[P, M]$ inherit B -actions from M . These actions are left and right, respectively. Generalizing this, we can think of the left A -module function object construction ${}^A[-, -]$ as a functor

$${}^A[-, -]: {}_A\mathcal{M}_B^{\text{op}} \times {}_A\mathcal{M}_C \rightarrow {}_B\mathcal{M}_C.$$

Analogously, we can think of the right C -module function object construction $[-, -]^C$ as a functor

$$[-, -]^C: {}_A\mathcal{M}_C^{\text{op}} \times {}_B\mathcal{M}_C \rightarrow {}_B\mathcal{M}_A.$$

The three constructions ${}^A[-, -]$, $-\wedge_B-$, and $[-, -]^C$ are related by natural isomorphisms

$${}^A[M, [N, P]^C]^B \cong {}^A[M \wedge_B N, P]^C \cong {}^B[N, {}^A[M, P]]^C, \tag{1.5}$$

for M in ${}_A\mathcal{M}_B$, N in ${}_B\mathcal{M}_C$, and P in ${}_A\mathcal{M}_C$. Note that the second isomorphism is precisely the first isomorphism for the opposite monoids under the isomorphisms of categories

$${}_B\mathcal{M}_C \cong {}_{C^{\text{op}}}\mathcal{M}_{B^{\text{op}}} \quad {}_A\mathcal{M}_B \cong {}_{B^{\text{op}}}\mathcal{M}_{A^{\text{op}}} \quad {}_A\mathcal{M}_C \cong {}_{C^{\text{op}}}\mathcal{M}_{A^{\text{op}}}. \tag{1.6}$$

All of our module categories are enriched over \mathcal{M} and each of the constructions ${}^A[-, -]$, $- \wedge_B -$, and $[-, -]^C$ gives a \mathcal{M} -enriched bifunctor. With this viewpoint, the two isomorphisms above become two enriched parametrized adjunctions. The following theorem extends these adjunctions to the homotopy categories.

Theorem 1.7. *Let \mathcal{M} be a monoidal model category, and let A , B , and C be monoids in \mathcal{M} . Assume that each category of modules in each statement below is a closed model category with fibrations and weak equivalences created in \mathcal{M} . If B is semifibrant when considered as an object in \mathcal{M} , then:*

- (a) *The right derived functor $\text{Ext}_A(-_B, -_C)$ of*

$${}^A[-, -]: {}_A\mathcal{M}_B^{\text{op}} \times {}_A\mathcal{M}_C \rightarrow {}_B\mathcal{M}_C$$

exists and is enriched over $\text{Ho}\mathcal{M}$.

- (b) *The left derived functor $\text{Tor}_B({}_A-, -_C)$ of*

$$\wedge_B: {}_A\mathcal{M}_B \times {}_B\mathcal{M}_C \rightarrow {}_A\mathcal{M}_C$$

exists, is enriched over $\text{Ho}\mathcal{M}$, and forms an enriched parametrized adjunction with $\text{Ext}_A(-_B, -_C)$:

$${}^A\llbracket \text{Tor}_B({}_A M, N_C), P \rrbracket^C \cong {}^B\llbracket N, \text{Ext}_A(M_B, P_C) \rrbracket^C. \tag{1.8}$$

- (c) *Let M be a (A, B) -bimodule. If the underlying left A -module of M is semifibrant in ${}_A\mathcal{M}$, then $\text{Tor}_B({}_A M, -_C)$ is the left derived functor of $M \wedge_B -$, $\text{Ext}_A(M_B, -_C)$ is the right derived functor of ${}^A[M, -]$, and the adjunction*

$$M \wedge_B (-): {}_B\mathcal{M}_C \rightleftarrows {}_A\mathcal{M}_C : {}^A[M, -]$$

is a Quillen adjunction.

See Definition 5.5 in Section 5 for a precise definition of the enrichment of the derived functor of a bifunctor.

Part (c) above applies in particular when M is a cofibrant (A, B) -bimodule, because then the underlying A -module of M is cofibrant in ${}_A\mathcal{M}$ and therefore semifibrant in ${}_A\mathcal{M}$. To see this, note that the right adjoint of the forgetful functor ${}_A\mathcal{M}_B \rightarrow {}_A\mathcal{M}$ is the functor $[B, -]: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}_B$, which preserves fibrations and acyclic fibrations since by hypothesis B is semifibrant in \mathcal{M} . It follows that the forgetful functor ${}_A\mathcal{M}_B \rightarrow {}_A\mathcal{M}$ preserves cofibrations and acyclic cofibrations.

Another interesting case of part (c) occurs when $A = B$ and $M = B$. Then the statement is that the functors $\text{Tor}_B({}_B B, -_C)$ and $\text{Ext}_B(B_B, -_C)$ are naturally isomorphic to the identity functor on $\text{Ho}{}_B\mathcal{M}_C$.

Isomorphism (1.8) of Theorem 1.7 can be coupled with the isomorphisms (1.6) of categories to obtain the pair of enriched isomorphisms

$${}^A\llbracket M, \text{Ext}_{C^{\text{op}}}({}_B N, {}_A P) \rrbracket^B \cong {}^A\llbracket \text{Tor}_B({}_A M, N_C), P \rrbracket^C \cong {}^B\llbracket N, \text{Ext}_A(M_B, P_C) \rrbracket^C \tag{1.9}$$

which are the derived versions of the isomorphisms (1.5).

The universal property of derived functors implies that functors in the previous theorem are appropriately natural in the monoids A, B, C . In fact, the natural transformations so obtained are enriched.

Theorem 1.10. *Let \mathcal{M} be a monoidal model category, and assume that each category of modules in each statement below is a closed model category with fibrations and weak equivalences created in \mathcal{M} .*

- (a) *When the underlying objects of B and B' are semifibrant in \mathcal{M} , maps of monoids $A' \rightarrow A$, $B' \rightarrow B$, and $C' \rightarrow C$, induce an enriched natural transformation of bifunctors (from $\text{Ho}{}_A\mathcal{M}_B^{\text{op}}, \text{Ho}{}_A\mathcal{M}_C$ to $\text{Ho}{}_{B'}\mathcal{M}_{C'}$)*

$$\text{Ext}_A(-_B, -_C) \rightarrow \text{Ext}_{A'}(-_{B'}, -_{C'})$$

making Ext appropriately functorial in the monoid variables. In particular, this transformation is compatible with the natural transformation ${}^A[-, -] \rightarrow {}^{A'}[-, -]$,

- (b) *When the underlying objects of B and B' are semifibrant in \mathcal{M} , maps of monoids $A' \rightarrow A$, $B' \rightarrow B$, and $C' \rightarrow C$, induce an enriched natural transformation of bifunctors (from $\text{Ho}{}_A\mathcal{M}_B, \text{Ho}{}_B\mathcal{M}_C$ to $\text{Ho}{}_{A'}\mathcal{M}_{C'}$)*

$$\text{Tor}_{B'}({}_{A'}-, -_{C'}) \rightarrow \text{Tor}_B({}_A-, -_C),$$

making Tor appropriately functorial in the monoid variables. In particular, this transformation is compatible with the natural transformation $\wedge_{B'} \rightarrow \wedge_B$,

(c) The Ext and Tor natural transformations above are appropriately compatible with the adjunction isomorphism of Theorem 1.7(b).

In favorable situations, the underlying object in $\text{Ho}\mathcal{M}$ of $\text{Tor}_B(A-, -C)$ should only depend on B and not on A and C . Similarly, the underlying object in $\text{Ho}\mathcal{M}$ of $\text{Ext}_A(-B, -C)$ should only depend on A and not on B and C . The natural transformations of Theorem 1.10 allow us to convert this intuition into the following precise statement. In it, we drop the notation for any monoid variable when it is the unit I .

Theorem 1.11. *Let \mathcal{M} be a monoidal model category, and let $A, B,$ and C be monoids in \mathcal{M} . Assume that each category of modules in each statement below is a closed model category with fibrations and weak equivalences created in \mathcal{M} . If B is semifibrant when considered as an object in \mathcal{M} , then:*

(a) The natural transformation

$$\text{Ext}_A(-B, -C) \rightarrow \text{Ext}_A(-, -) = {}^A\llbracket -, - \rrbracket$$

in $\text{Ho}\mathcal{M}$ induced by the unit maps $I \rightarrow B$ and $I \rightarrow C$ is an isomorphism.

(b) If the underlying object of A is semifibrant in \mathcal{M} , then the natural transformation

$$\text{Tor}_B(-, -C) \rightarrow \text{Tor}_B(A-, -C)$$

in $\text{Ho}\mathcal{M}_C$ induced by the unit map $I \rightarrow A$ is an isomorphism. Similarly, if the underlying object of C is semifibrant in \mathcal{M} , then the natural transformation

$$\text{Tor}_B(A-, -) \rightarrow \text{Tor}_B(A-, -C)$$

in $\text{Ho}\mathcal{M}_A$ induced by the unit map $I \rightarrow C$ is an isomorphism.

Since a map in $\text{Ho}_{B'}\mathcal{M}_{C'}$ is an isomorphism if and only if it is sent to an isomorphism in $\text{Ho}\mathcal{M}$, it follows from part (a) that for any maps of monoids $B' \rightarrow B$ and $C \rightarrow C'$, when B and B' have underlying objects that are semifibrant in \mathcal{M} , the induced map $\text{Ext}_A(-B, -C) \rightarrow \text{Ext}_A(-B', -C')$ is an isomorphism in $\text{Ho}_{B'}\mathcal{M}_{C'}$. Likewise, it follows from part (b) that for any map of monoids $A' \rightarrow A$ whose underlying objects are semifibrant in \mathcal{M} , the induced map $\text{Tor}_B(A'-, -C) \rightarrow \text{Tor}_B(A-, -C)$ is an isomorphism in $\text{Ho}_{A'}\mathcal{M}_C$. Similarly, for any map of monoids $C' \rightarrow C$ whose underlying objects are semifibrant in \mathcal{M} , the induced map $\text{Tor}_B(A-, -C') \rightarrow \text{Tor}_B(A-, -C)$ is an isomorphism in $\text{Ho}_A\mathcal{M}_{C'}$.

Since for Quillen adjunctions the left derived functor of the composite of the left adjoints is the composite of the left derived functors, the last part of Theorem 1.7 gives an ‘‘associativity’’ isomorphism for the derived functors.

Theorem 1.12. *Let \mathcal{M} be a monoidal model category, and let $A, B, C,$ and D be monoids in \mathcal{M} . Assume that each category of modules in the statement below is a closed model category with fibrations and weak equivalences created in \mathcal{M} . If the underlying objects of B and C are semifibrant in \mathcal{M} , then there is a canonical enriched natural isomorphism of trifunctors*

$$\text{Tor}_B({}_A L, (\text{Tor}_C({}_B M, N_D))_D) \cong \text{Tor}_C({}_A (\text{Tor}_B({}_A L, M_C)), N_D),$$

compatible with the associativity isomorphism for the symmetric monoidal product in $\text{Ho}\mathcal{M}$ and satisfying the evident analogue of the pentagon law. Adjointly, there is a canonical enriched natural isomorphism of trifunctors

$$\text{Ext}_A((\text{Tor}_B({}_A M, N_C))_C, P_D) \cong \text{Ext}_B(N_C, (\text{Ext}_A(M_B, P_D))_D).$$

Several special cases of the results presented above are of particular interest. These include:

Tensors and cotensors. Although treated explicitly in Theorem 1.1, the existence and interpretation of tensors and cotensors as derived functors also follows from the general bimodule theorems above. Tensors and cotensors in ${}_A\mathcal{M}$ comprise the special case of the isomorphisms (1.5) in which $B = C = I$. Likewise, tensors and cotensors in $\text{Ho}_A\mathcal{M}$ comprise the special case of the isomorphisms (1.9) in which $B = C = I$. This indicates that $\text{Tor}({}_A M, X)$ provides the tensor $M \widehat{\otimes} X$ for $\text{Ho}_A\mathcal{M}$. The last part of the Theorem 1.11 indicates that the tensor $M \widehat{\otimes} X$ in $\text{Ho}_A\mathcal{M}$ agrees with the derived monoidal product $M \widehat{\wedge} X$ in $\text{Ho}\mathcal{M}$ when the underlying object in \mathcal{M} of A is semifibrant. Moreover, it follows that for a map of monoids $A \rightarrow B$ whose underlying objects are semifibrant, the derived forgetful (or

“pullback”) functor $\text{Ho}_B \mathcal{M} \rightarrow \text{Ho}_A \mathcal{M}$ preserves tensors. This special case of isomorphisms (1.9) also implies that $\text{Ext}(X, {}_A M)$ provides the cotensors for $\text{Ho}_A \mathcal{M}$, and that these are preserved by the derived forgetful functor to $\text{Ho} \mathcal{M}$. Note that tensors are preserved by all enriched left adjoints and cotensors are preserved by all enriched right adjoints, and so the remarks on preservation of tensors and cotensors also follow from the observations on extension of scalars and coextension of scalars below.

Extension of scalars. Let $B \rightarrow A$ be a map of monoids in \mathcal{M} , and assume that the categories ${}_A \mathcal{M}$ and ${}_B \mathcal{M}$ are closed model categories with fibrations and weak equivalences created in \mathcal{M} . It then follows formally that the extension of scalars functor $A \wedge_B (-): {}_B \mathcal{M} \rightarrow {}_A \mathcal{M}$ and the forgetful (or pullback) functor ${}_A \mathcal{M} \rightarrow {}_B \mathcal{M}$ form a Quillen adjunction. When A and B are semifibrant in \mathcal{M} , Theorem 1.7 implies that the left derived extension of scalars functor is given by $\text{Tor}_B({}_A A, -)$ and Theorem 1.11 implies that it is naturally isomorphic to $\text{Tor}_B(A, -)$ in $\text{Ho} \mathcal{M}$. In particular, in this case, when $B \rightarrow A$ is a weak equivalence, the extension of scalars adjunction is a Quillen equivalence.

Coextension of scalars. Let $A \rightarrow B$ be a map of monoids in \mathcal{M} , and assume that the categories ${}_A \mathcal{M}$ and ${}_B \mathcal{M}$ are closed model categories with fibrations and weak equivalences created in \mathcal{M} . The forgetful functor ${}_B \mathcal{M} \rightarrow {}_A \mathcal{M}$ has a right adjoint given by ${}^A[B, -]$. If B is semifibrant in ${}_A \mathcal{M}$, then Theorem 1.7 implies that this is a Quillen adjunction and it identifies the right derived coextension of scalars functor as $\text{Ext}_A(B_B, -)$. Moreover, Theorem 1.11 implies that this functor is naturally isomorphic to ${}^A[[B, -]]$ in $\text{Ho} \mathcal{M}$.

Free and cofree functors bimodule structure. For the map of monoids $I \rightarrow A$, the extension of scalars functor and coextension of scalars functor are called the free functor and the cofree functor. These functors have the extra structure that they factor through the forgetful functor ${}_A \mathcal{M}_A \rightarrow {}_A \mathcal{M}$. Assume that the categories of ${}_A \mathcal{M}$ and ${}_A \mathcal{M}_A$ are closed model categories with fibrations and weak equivalences created in \mathcal{M} and that A is semifibrant in \mathcal{M} . Then $A \wedge A^{\text{op}}$ is also semifibrant in \mathcal{M} and Theorem 1.11 implies that the functors

$$\mathbb{F}(-) = \text{Tor}_{(A \wedge A^{\text{op}})A}(-), \quad \text{and} \quad \mathbb{F}^\sharp(-) = \text{Ext}(A_{A \wedge A^{\text{op}}}, -),$$

provide factorizations of the derived free and cofree functors through $\text{Ho}_A \mathcal{M}_A$. The adjunctions also identify $\mathbb{F}X$ as $A \widehat{\otimes} X$, the tensor (in $\text{Ho}_A \mathcal{M}_A$) of A with the object X of $\text{Ho} \mathcal{M}$. Since tensors commute with enriched left adjoints, we obtain natural isomorphisms in $\text{Ho}_A \mathcal{M}$ (or $(\text{Ho}_A \mathcal{M})^{\text{op}}$)

$$M \widehat{\otimes} X \cong \text{Tor}_A({}_A \mathbb{F}X, M), \quad \text{and} \quad [[X, M]] \cong \text{Ext}_A(\mathbb{F}X_A, M).$$

The isomorphism ${}^A[[M, \mathbb{F}^\sharp X]] \cong [[M, X]]$ from the enriched adjunction also refines to an isomorphism in $\text{Ho} \mathcal{M}_A$

$$\text{Ext}_A(M, \mathbb{F}^\sharp X_A) \cong \text{Ext}({}_A M, X)$$

as an instance of the universal map of enriched derived functors. Although not a direct result of the results listed above, this last enriched natural transformation is an immediate consequence of the more general Theorem 5.4 in Section 5 below.

In practice, many monoidal model categories have additional properties that make the semifibrant hypotheses in the results above unnecessary in certain cases. The process of eliminating these hypotheses is discussed in Section 8.

2. Monoidal model categories

This section reviews the terminology and basic theory of monoidal model categories from [12] (and [5]). The definition of a monoidal model category involves constraints on the interaction of the model structure with the closed symmetric monoidal structure. The imposed conditions suffice to ensure that the homotopy category inherits a closed symmetric monoidal category structure and that the localization functor is lax symmetric monoidal. The conditions are stated in terms of the following two standard maps. Let $f: A \rightarrow B$, $g: K \rightarrow L$, and $h: X \rightarrow Y$ be maps in a symmetric monoidal closed category \mathcal{M} . Then the maps

$$\square[g, h]: [L, X] \rightarrow [K, X] \times_{[K, Y]} [L, Y]$$

and

$$f \square g: (A \wedge L) \cup_{A \wedge K} (B \wedge K) \rightarrow B \wedge L$$

are defined by the diagrams

$$\begin{array}{ccc}
 [L, X] & \xrightarrow{\square[g, h]} & [L, Y] \\
 \downarrow & \searrow & \downarrow g^* \\
 [K, X] \times_{[K, Y]} [L, Y] & \longrightarrow & [L, Y] \\
 \downarrow & & \downarrow \\
 [K, X] & \xrightarrow{h_*} & [K, Y]
 \end{array} \tag{2.1}$$

and

$$\begin{array}{ccc}
 A \wedge K & \xrightarrow{f \wedge \text{id}} & B \wedge K \\
 \text{id} \wedge g \downarrow & & \downarrow \\
 A \wedge L & \longrightarrow & (A \wedge L) \cup_{A \wedge K} (B \wedge K) \\
 & \searrow & \downarrow f \square g \\
 & & B \wedge L
 \end{array} \tag{2.2}$$

in which the squares are a pullback and a pushout, respectively.

Definition 2.3. A monoidal model category \mathcal{M} is a closed model category with a closed symmetric monoidal structure satisfying the following two axioms:

- (Enr): If $g: K \rightarrow L$ is a cofibration and $h: X \rightarrow Y$ is a fibration, then $\square[g, h]$ is a fibration. Moreover, if either g or h is also a weak equivalence, then so is $\square[g, h]$.
- (Unit): There exist a cofibrant object I_c and a weak equivalence $\omega: I_c \rightarrow I$ such that the composite $\tilde{\ell}_c$ of the adjoint of the unit isomorphism and ω^*

$$\tilde{\ell}_c: Z \rightarrow [I, Z] \rightarrow [I_c, Z]$$

is a weak equivalence for every fibrant object Z .

The first axiom, the *Enrichment Axiom*, is the internal version of Quillen’s axiom (SM7). We have given it in a form that easily generalizes to the context of enriched categories in the next section. Each of the above axioms may be reformulated adjointly in terms of \wedge , and these reformulations seem to be easier to work with in practice. The adjoint form of the Enrichment Axiom is called the *Pushout Product Axiom* [12, 3.1]

Proposition 2.4. Let \mathcal{M} be a closed model category with a closed symmetric monoidal structure. Then \mathcal{M} satisfies the Enrichment Axiom (Enr) if and only if it satisfies the Pushout Product Axiom:

- (PP): If $f: A \rightarrow B$ and $g: K \rightarrow L$ are cofibrations, then so is $f \square g$. Moreover, if either f or g is also a weak equivalence, then so is $f \square g$.

When \mathcal{M} satisfies these axioms, it satisfies the Unit Axiom (Unit) if and only if it satisfies the following axiom:

- (Unit’): There exists a cofibrant object I_c and a weak equivalence $\omega: I_c \rightarrow I$ such that the composite ℓ_c of $\text{id} \wedge \omega$ and the unit isomorphism

$$\ell_c: X \wedge I_c \rightarrow X \wedge I \cong X$$

is a weak equivalence for every cofibrant object X .

The equivalence of the axioms (Enr) and (PP) follows from the characterization of (acyclic) cofibrations and (acyclic) fibrations in terms of lifting properties, using the $(-\wedge-, [-, -])$ -adjunction applied to f, g , and h as above (see, for example, [5, 4.2.2]). The equivalence of the two unit axioms is closely related to the construction of the derived product and function functors and our discussion of it is postponed until after Proposition 2.6 below.

To describe the implications of (PP) for the functors \wedge and $[-, -]$, we must first recall the standard model structures on the opposite of a closed model category and the product of two closed model categories. If \mathcal{M} and \mathcal{M}' are closed model categories, then $\mathcal{M}' \times \mathcal{M}$ is a closed model category whose cofibrations, fibrations, and weak equivalences are the maps that are cofibrations, fibrations, and weak equivalences, respectively, in each coordinate. Also, \mathcal{M}^{op} is a closed model category whose cofibrations, fibrations, and weak equivalences are the maps that are the opposites of fibrations, cofibrations, and weak equivalences, respectively. In particular, the fibrant objects in \mathcal{M}^{op} are the cofibrant objects in \mathcal{M} .

Proposition 2.5. *Let \mathcal{M} be a monoidal model category, and X, Z objects of \mathcal{M} that are cofibrant and fibrant, respectively.*

- (a) *The functors $X \wedge (-)$ and $(-) \wedge X$ preserve cofibrations and acyclic cofibrations.*
- (b) *The functor $[X, -]$ preserves fibrations and acyclic fibrations.*
- (c) *The functor $[-, Z]$ converts cofibrations and acyclic cofibrations in \mathcal{M} into fibrations and acyclic fibrations, respectively.*
- (d) *The functor $\wedge : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ preserves weak equivalences between cofibrant objects.*
- (e) *The functor $[-, -] : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$ preserves weak equivalences between fibrant objects.*

Proof. Applying the Pushout Product Axiom to the map from the initial object to a cofibrant object X gives that \wedge preserves cofibrations and acyclic cofibrations in either variable when the other variable is cofibrant. A second application of the Pushout Product Axiom then indicates that \wedge preserves acyclic cofibrations between cofibrant objects. Coupled with this observation, Brown’s lemma [1, 9.9] implies that \wedge preserves weak equivalences between cofibrant objects. A similar argument, using the Enrichment Axiom (Enr), proves the claim about $[-, -]$. \square

Parts (d) and (e) of this proposition indicate that \wedge and $[-, -]$ satisfy Quillen’s criterion (Proposition 1 in [11, p. I.4.2] or Proposition 9.3 in [1]) for the existence of a left derived functor $\widehat{\wedge}$ and a right derived functor $\llbracket -, - \rrbracket$, respectively. Moreover, these derived functors can be constructed so that

$$X \widehat{\wedge} Y = X \wedge Y \quad \text{and} \quad \llbracket Y, Z \rrbracket = [Y, Z]$$

whenever X and Y are cofibrant objects of \mathcal{M} and Z is a fibrant object of \mathcal{M} . In particular, if Y is cofibrant, then $(-) \widehat{\wedge} Y$ and $\llbracket Y, - \rrbracket$ are the left and right derived functors of $(-) \wedge Y$ and $[Y, -]$ respectively. Quillen’s criterion for adjoint derived functors (Theorem 3 in [11, pp. I.4.5ff] or Theorem 9.7 in [1]) then implies that $(-) \widehat{\wedge} Y$ and $\llbracket Y, - \rrbracket$ are adjoint. These observations are summarized in the following proposition.

Proposition 2.6. *If \mathcal{M} is a monoidal model category, then the left derived functor $\widehat{\wedge}$ of \wedge and the right derived functor $\llbracket -, - \rrbracket$ of $[-, -]$ exist and give a parametrized adjunction*

$$\text{Ho.}\mathcal{M}(X \widehat{\wedge} Y, Z) \cong \text{Ho.}\mathcal{M}(X, \llbracket Y, Z \rrbracket).$$

This parametrized adjunction is the source of an easy proof of the equivalence of the Unit Axioms (Unit) and (Unit’). The functors $(-) \wedge I_c$ and $[I_c, -]$ are Quillen adjoints, and induce the adjoint functors $(-) \widehat{\wedge} I_c$ and $\llbracket I_c, - \rrbracket$ on the homotopy category $\text{Ho.}\mathcal{M}$. The map in axiom (Unit) is a natural transformation in $\text{Ho.}\mathcal{M}$ from the identity functor to $\llbracket I_c, - \rrbracket$. Axiom (Unit) asserts that this natural transformation is a natural isomorphism. The map in axiom (Unit’) is the adjoint natural transformation from $(-) \widehat{\wedge} I_c$ to the identity. That axiom asserts that this adjoint natural transformation is a natural isomorphism. Since each natural transformation is a natural isomorphism if and only if its adjoint is, the two axioms are equivalent.

From this it follows that the unit isomorphism for \wedge induces a unit isomorphism for $\widehat{\wedge}$. Using the description of the derived functor $\widehat{\wedge}$ in terms of cofibrant approximations, it is straightforward to check that the associativity isomorphism for \wedge induces an associativity isomorphism for $\widehat{\wedge}$. Combined with Proposition 2.6, these observations prove most of the following result. For a more complete discussion, see [5, 4.3.2].

Proposition 2.7. *Let \mathcal{M} be a monoidal model category. The derived product $\widehat{\wedge}$ and the derived function objects $\llbracket -, - \rrbracket$ provide the homotopy category $\text{Ho.}\mathcal{M}$ with a closed symmetric monoidal structure. Moreover, the localization functor $\mathcal{M} \rightarrow \text{Ho.}\mathcal{M}$ is lax symmetric monoidal.*

3. Enriched model categories

Although most of the main results stated in the introduction only make sense for module categories, the most basic result, [Theorem 1.1](#), applies more generally to closed model categories enriched over a monoidal model category. Moreover, since the enrichment of the derived balanced product and function functors described in our main results concisely encodes much of the coherence among the derived functors we discuss, it is particularly convenient to work in the context of enriched categories as much as possible. Our first objective in this section is to introduce axioms for the interaction of a model category structure with an enriched category structure which imply the “expected” relationship between the homotopy category, the enrichment, and the homotopy category of the enriching monoidal model category. The discussion of this relationship then begins with [Theorem 3.10](#), the generalization of [Theorem 1.1](#) to enriched model categories. The discussion then continues in the next section with [Theorem 4.2](#), which states the universal property of the enrichment of the homotopy category, and with a study of enrichments of derived functors. Finally, we conclude the discussion in [Section 5](#) with a study of enriched derived bifunctors and enriched parametrized adjunctions.

Recall that a category \mathcal{C} enriched over a closed symmetric monoidal category \mathcal{M} consists of:

- (i) A class $\text{Ob}(\mathcal{C})$ of objects of \mathcal{C} ,
- (ii) For each C, D in $\text{Ob}(\mathcal{C})$ a mapping object ${}^{\mathcal{C}}[C, D]$ in \mathcal{M} ,
- (iii) A composition law given by maps

$$\circ: {}^{\mathcal{C}}[D, E] \wedge {}^{\mathcal{C}}[C, D] \rightarrow {}^{\mathcal{C}}[C, E]$$

in \mathcal{M} for each C, D, E in $\text{Ob}(\mathcal{C})$, and

- (iv) Identity morphisms, which are maps $\text{id}_C: I \rightarrow {}^{\mathcal{C}}[C, C]$ in \mathcal{M} for each C in $\text{Ob}(\mathcal{C})$.

These morphisms are required to satisfy the appropriate associativity and unit conditions (see, for example, [\[6, 1.2\]](#)).

The ordinary category underlying \mathcal{C} has the same objects as \mathcal{C} and morphism sets given by

$$\mathcal{C}(C, D) = \mathcal{M}(I, {}^{\mathcal{C}}[C, D]).$$

The composition law and identity morphisms for this underlying category are derived from the composition law and identity morphisms in \mathcal{M} above.

More informally, an enrichment over \mathcal{M} of an ordinary category \mathcal{C} is an isomorphism (or merely equivalence) between \mathcal{C} and the underlying category of a \mathcal{M} -enriched category. For example, \mathcal{M} is enriched over itself by the isomorphism

$$\mathcal{M}(I, [X, Y]) \cong \mathcal{M}(I \wedge X, Y) \cong \mathcal{M}(X, Y).$$

The following definition describes the standard procedure for pushing enrichments forward along a monoidal functor.

Definition 3.1. Let $\lambda: \mathcal{M} \rightarrow \mathcal{N}$ be a lax symmetric monoidal functor between two symmetric monoidal closed categories \mathcal{M} and \mathcal{N} . Let \mathcal{C} be a category enriched over \mathcal{M} . The induced category $\lambda_*\mathcal{C}$ enriched over \mathcal{N} has the same object set as \mathcal{C} and morphism objects in \mathcal{N} given by

$$\lambda_*\mathcal{C}[C, D] = \lambda({}^{\mathcal{C}}[C, D]).$$

The composition and identity maps in \mathcal{N} for $\lambda_*\mathcal{C}$ are obtained by applying λ to the analogous maps for \mathcal{C} in \mathcal{M} and composing with the appropriate morphisms giving λ its lax monoidal structure. There is a canonical functor from the underlying category of \mathcal{C} to that of $\lambda_*\mathcal{C}$ which is the identity on objects and on morphisms is

$$\begin{aligned} \mathcal{C}(C, D) = \mathcal{M}(I_{\mathcal{M}}, {}^{\mathcal{C}}[C, D]) &\xrightarrow{\lambda} \mathcal{N}(\lambda I_{\mathcal{M}}, \lambda({}^{\mathcal{C}}[C, D])) \\ &\rightarrow \mathcal{N}(I_{\mathcal{N}}, \lambda({}^{\mathcal{C}}[C, D])) = \lambda_*\mathcal{C}(C, D), \end{aligned}$$

where the second map comes from the unit map for λ .

The monoidal functor of interest to us is the localization functor $\lambda: \mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ associated to a monoidal model category \mathcal{M} . If \mathcal{C} is enriched over \mathcal{M} , then $\lambda_*\mathcal{C}$ is a sort of homotopy category. For example, when \mathcal{M} is the monoidal model category of spaces, the enrichment of \mathcal{C} over \mathcal{M} is given by function spaces ${}^{\mathcal{C}}[C, D]$. The morphism sets of $\lambda_*\mathcal{C}$ are then the path components of these function spaces. Thus, when λ is the localization functor for a monoidal model category, $\lambda_*\mathcal{C}$ is a natural generalization of the traditional notion of a homotopy category. If \mathcal{C} also carries a closed model structure, then it is natural to inquire about the relationship between $\text{Ho}\mathcal{C}$ and $\lambda_*\mathcal{C}$. Without some restrictions on the model structure on \mathcal{C} , there need not even be a functor comparing $\text{Ho}\mathcal{C}$ and $\lambda_*\mathcal{C}$. However, there is an obvious generalization of the Enrichment Axiom for monoidal model categories to the context of closed model categories enriched over a monoidal model category. For the statement of this axiom, we need the following generalization of the map $\square[g, h]$ from Section 2. Let $f: A \rightarrow B, g: K \rightarrow L$, and $h: X \rightarrow Y$ be maps in a category \mathcal{C} enriched over a monoidal model category \mathcal{M} . Then

$$\square[g, h]: {}^{\mathcal{C}}[L, X] \rightarrow {}^{\mathcal{C}}[K, X] \times_{{}^{\mathcal{C}}[K, Y]} {}^{\mathcal{C}}[L, Y]$$

is the map defined by the pullback analogous to (2.1) with ${}^{\mathcal{C}}[-, -]$ in place of $[-, -]$. The generalization of the Enrichment Axiom for monoidal model categories to the context of enriched categories is

(Enr): If $g: K \rightarrow L$ is a cofibration in \mathcal{C} and $h: X \rightarrow Y$ is a fibration in \mathcal{C} , then $\square[g, h]$ is a fibration in \mathcal{M} .
 Moreover, if either g or h is also a weak equivalence, then so is $\square[g, h]$.

The following analog of Proposition 2.5 describes the implications of this Enrichment Axiom for the functor ${}^{\mathcal{C}}[-, -]$. (Recall for part (c) that a fibrant object in \mathcal{M}^{op} is a cofibrant object in \mathcal{M} .)

Proposition 3.2. *Let \mathcal{C} be a closed model category enriched over a monoidal model category \mathcal{M} such that the Enrichment Axiom is satisfied.*

- (a) *If C is cofibrant, then the functor ${}^{\mathcal{C}}[C, -]$ preserves fibrations and acyclic fibrations.*
- (b) *If D is fibrant, then the functor ${}^{\mathcal{C}}[-, D]$ converts cofibrations and acyclic cofibrations in \mathcal{C} into fibrations and acyclic fibrations in \mathcal{M} , respectively.*
- (c) *The functor ${}^{\mathcal{C}}[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{M}$ preserves weak equivalences between fibrant objects.*

Proposition 3.2(c) indicates that the restriction of the composite functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \xrightarrow{{}^{\mathcal{C}}[-, -]} \mathcal{M} \xrightarrow{\lambda} \text{Ho}\mathcal{M}$$

to the full subcategory $\mathcal{C}_{\text{cf}}^{\text{op}} \times \mathcal{C}_{\text{cf}}$ (consisting of the objects that are both cofibrant and fibrant) converts weak equivalences into isomorphisms. It follows from the universal property of localization that the functor $\mathcal{C}_{\text{cf}} \rightarrow \lambda_*\mathcal{C}_{\text{cf}}$ factors through the category $\text{Ho}\mathcal{C}_{\text{cf}}$ to give a comparison functor

$$\Upsilon: \text{Ho}\mathcal{C}_{\text{cf}} \rightarrow \lambda_*\mathcal{C}_{\text{cf}}.$$

This comparison functor is the subject of the following *Homotopy/Unit Axiom*:

(HoUnit). The functor $\Upsilon: \text{Ho}\mathcal{C}_{\text{cf}} \rightarrow \lambda_*\mathcal{C}_{\text{cf}}$ is an isomorphism of categories.

In other words, the Homotopy/Unit Axiom requires that whenever C and D are cofibrant–fibrant objects of \mathcal{C} , the map $\text{Ho}\mathcal{C}(C, D) \rightarrow \lambda_*\mathcal{C}(C, D)$ is a bijection.

This axiom turns out to generalize the Unit Axiom in the definition of a monoidal model category. It is shown below that it is equivalent to both of the more obvious generalizations of the Unit Axiom that become available when \mathcal{C} is tensored or cotensored over \mathcal{M} . The Enrichment Axiom and Homotopy/Unit Axiom suffice to ensure the appropriate relationship between the enriched structure and the homotopy category.

Definition 3.3. Let \mathcal{M} be a monoidal model category and let \mathcal{C} be a closed model category that is enriched over \mathcal{M} . Then \mathcal{C} is an *enriched model category* if it satisfies the Enrichment Axiom (Enr) and the Homotopy/Unit Axiom (HoUnit).

When \mathcal{C} is tensored or cotensored over \mathcal{M} , both the Enrichment Axiom (Enr) and the Homotopy/Unit Axiom (HoUnit) have alternate forms that are easier to verify in practice. For the statements of these alternative forms, we need the following generalizations of the maps $\square[g, h]$ and $f\square g$ defined in Section 2. Let $f: A \rightarrow B$ and $h: X \rightarrow Y$ be maps in \mathcal{C} and $g: K \rightarrow L$ be a map in \mathcal{M} . Then the maps

$$\square[g, h]: [L, X] \rightarrow [K, X] \times_{[K, Y]} [L, Y]$$

and

$$f \boxtimes g: (A \otimes L) \cup_{A \otimes K} (B \otimes K) \rightarrow B \otimes L$$

are defined as the pullback analogous to (2.1) and as the pushout analogous to (2.2) (with \otimes replacing \wedge), respectively.

The following proposition provides the alternative forms of the Enrichment Axiom. It follows easily from the characterization of (acyclic) cofibrations and (acyclic) fibrations in \mathcal{M} in terms of lifting properties, using the tensor or cotensor adjunction.

Proposition 3.4. *Let \mathcal{M} be a monoidal model category and let \mathcal{C} be a closed model category that is also enriched over \mathcal{M} .*

- (a) *If \mathcal{C} has tensors, then the Enrichment Axiom (Enr) is equivalent to the following Pushout Tensor Product Axiom:*
 (PTP): *If $f: A \rightarrow B$ is a cofibration in \mathcal{C} and $g: K \rightarrow L$ is a cofibration in \mathcal{M} , then $f \boxtimes g$ is a cofibration in \mathcal{C} . Moreover, if either f or g is also a weak equivalence, then so is $f \boxtimes g$.*
- (b) *If \mathcal{C} has cotensors, then the Enrichment Axiom (Enr) is equivalent to the following Cotensor Axiom:*
 (Cot): *If $g: K \rightarrow L$ is a cofibration in \mathcal{M} and $h: X \rightarrow Y$ is a fibration in \mathcal{C} , then $\square[g, h]$ is a fibration in \mathcal{C} . Moreover, if either g or h is also a weak equivalence, then so is $\square[g, h]$.*

The following proposition provides the alternative forms of the Homotopy/Unit Axiom (HoUnit):

Proposition 3.5. *Let \mathcal{M} be a monoidal model category and let \mathcal{C} be a closed model category that is enriched over \mathcal{M} and satisfies the Enrichment Axiom (Enr).*

- (a) *If \mathcal{C} has tensors, then the Homotopy/Unit Axiom is equivalent to the following Tensor Unit Axiom*
 (TUnit): *When C is cofibrant in \mathcal{C} , the map $\ell_c: C \otimes I_c \rightarrow C \otimes I \cong C$ is a weak equivalence.*
- (b) *If \mathcal{C} has cotensors, then the Homotopy/Unit Axiom is equivalent to the following Cotensor Unit Axiom*
 (CUnit): *When D is fibrant in \mathcal{C} , the map $\tilde{\ell}_c: D \cong [I, D] \rightarrow [I_c, D]$ is a weak equivalence.*

The proof of this result is similar to the proof of the equivalence of the two unit axioms (Unit) and (Unit') given in Section 2. It makes use of the adjunction relating the derived functors of the tensor and cotensor functors, and this proof is delayed until after our discussion of the existence of these derived functors. The following extension of Proposition 2.5 to the context of tensors and cotensors is needed in the discussion. Its proof again follows by adjunction from the characterization of (acyclic) cofibrations and (acyclic) fibrations in terms of lifting. (Recall for part (b) that a fibrant object in \mathcal{M}^{op} is a cofibrant object of \mathcal{M} .)

Proposition 3.6. *Let \mathcal{M} be a monoidal model category and let \mathcal{C} be a closed model category that is enriched over \mathcal{M} and satisfies the Enrichment Axiom (Enr).*

- (a) *Assume \mathcal{C} has tensors. If C and X are cofibrant in \mathcal{C} and \mathcal{M} , respectively, then $C \otimes (-)$ and $(-) \otimes X$ preserve cofibrations and acyclic cofibrations. Also, the functor $\otimes: \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{C}$ preserves weak equivalences between cofibrant objects.*
- (b) *Assume \mathcal{C} has cotensors. If X is cofibrant in \mathcal{M} , then $[X, -]$ preserves fibrations and acyclic fibrations in \mathcal{C} . If D is fibrant in \mathcal{C} , then $[-, D]$ converts cofibrations and acyclic cofibrations in \mathcal{M} into fibrations and acyclic fibrations in \mathcal{C} , respectively. Also, the functor $[-, -]: \mathcal{M}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ preserves weak equivalences between fibrant objects.*

Proposition 3.6 implies that the tensor and cotensor adjunctions for \mathcal{C} are Quillen adjunctions in each variable. This observation and an argument analogous to the proof of Proposition 2.6 proves the following proposition.

Proposition 3.7. *Let \mathcal{M} be a monoidal model category and let \mathcal{C} be a closed model category that is enriched over \mathcal{M} and satisfies the Enrichment Axiom (Enr).*

(a) *If \mathcal{C} has tensors then the left derived functor $\widehat{\otimes}$ of \otimes exists and with ${}^{\mathcal{C}}\llbracket -, - \rrbracket$ gives a parametrized adjunction*

$$\mathrm{Ho}\mathcal{C}(C \widehat{\otimes} X, D) \cong \mathrm{Ho}\mathcal{M}(X, {}^{\mathcal{C}}\llbracket C, D \rrbracket).$$

(b) *If \mathcal{C} has cotensors then the right derived functor $\llbracket -, - \rrbracket$ of $[-, -]$ exists and with ${}^{\mathcal{C}}\llbracket -, - \rrbracket$ gives a parametrized adjunction*

$$\mathrm{Ho}\mathcal{C}(C, \llbracket X, D \rrbracket) \cong \mathrm{Ho}\mathcal{M}(X, {}^{\mathcal{C}}\llbracket C, D \rrbracket).$$

A cofibrant approximation $I_c \rightarrow I$ to the unit I for \mathcal{M} yields natural transformations

$$\ell_c : C \otimes I_c \rightarrow C \otimes I \cong C$$

and

$$\tilde{\ell}_c : D \cong [I, D] \rightarrow [I_c, D]$$

relating the inclusion functor on $\mathrm{Ho}\mathcal{C}_{cf} \rightarrow \mathrm{Ho}\mathcal{C}$ to the functors $- \otimes I_c$ and $[I_c, -]$. The axioms (Unit') and (Unit) assert that these natural transformations are natural isomorphisms. The adjunction of Proposition 3.7 allows us to relate these natural transformations to the comparison functor $\Upsilon : \mathrm{Ho}\mathcal{C}_{cf} \rightarrow \lambda_*\mathcal{C}_{cf}$ via the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Ho}\mathcal{C}_{cf}(C, D) & \xrightarrow{(\tilde{\ell}_c)_*} & \mathrm{Ho}\mathcal{C}_{cf}(C, [I_c, D]) \\ \downarrow \ell_c^* & \searrow \Upsilon & \downarrow \cong \\ \mathrm{Ho}\mathcal{C}(C \otimes I_c, D) & \xrightarrow{\lambda_*\mathcal{C}(C, D)} & \mathrm{Ho}\mathcal{M}(I_c, {}^{\mathcal{C}}\llbracket C, D \rrbracket) \end{array} \quad (3.8)$$

(Only the relevant part of this diagram exists when \mathcal{C} has tensors but not cotensors or vice-versa.) Clearly each of the maps $(\tilde{\ell}_c)_*$, Υ , and ℓ_c^* in this diagram is an isomorphism if and only if the either of the other maps is an isomorphism. This implies that (HoUnit) is equivalent to (CUnit) and (TUnit) whenever either axiom makes sense.

Our motivating examples of enriched model categories are provided by the following result.

Proposition 3.9. *Let \mathcal{M} be a monoidal model category, let A be a monoid in \mathcal{M} , and assume the category ${}_A\mathcal{M}$ of left A modules is a closed model category with fibrations and weak equivalences created in \mathcal{M} . Then ${}_A\mathcal{M}$ is an enriched model category.*

Proof. ${}_A\mathcal{M}$ has tensors induced by \wedge and cotensors induced by $[-, -]$. Moreover, the Enrichment Axiom (Enr) for \mathcal{M} implies the Cotensor Axiom (Cot) for ${}_A\mathcal{M}$ and the Unit Axiom (Unit) for \mathcal{M} implies the Cotensor Unit Axiom (CUnit) for ${}_A\mathcal{M}$. \square

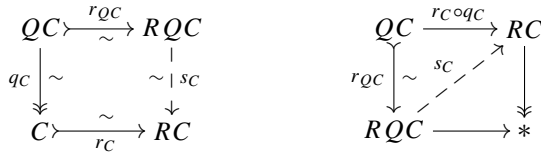
The following is our fundamental result about enriched model categories. Coupled with the previous proposition, it implies Theorem 1.1 of the introduction.

Theorem 3.10. *Let \mathcal{C} be an enriched model category over a monoidal model category \mathcal{M} . Then the right derived functor ${}^{\mathcal{C}}\llbracket -, - \rrbracket$ of ${}^{\mathcal{C}}[-, -]$ exists and enriches $\mathrm{Ho}\mathcal{C}$ over $\mathrm{Ho}\mathcal{M}$. Further, if \mathcal{C} is tensored or cotensored over \mathcal{M} , then $\mathrm{Ho}\mathcal{C}$ is likewise tensored or cotensored over $\mathrm{Ho}\mathcal{M}$.*

Proof. It follows from Proposition 3.2(c) that the right derived functor ${}^{\mathcal{C}}\llbracket -, - \rrbracket$ of ${}^{\mathcal{C}}[-, -]$ exists and ${}^{\mathcal{C}}\llbracket X, Y \rrbracket$ may be computed as ${}^{\mathcal{C}}\llbracket QX, RY \rrbracket$ where $QX \rightarrow X$ is a cofibrant approximation and $Y \rightarrow RY$ is a fibrant approximation. Every object C of $\mathrm{Ho}\mathcal{C}$ is isomorphic in $\mathrm{Ho}\mathcal{C}$ to an object RQC of $\mathrm{Ho}\mathcal{C}_{cf}$; conjugating by these isomorphisms gives both an isomorphism ${}^{\mathcal{C}}\llbracket RQC, RQD \rrbracket \cong {}^{\mathcal{C}}\llbracket C, D \rrbracket$ and Quillen's equivalence of $\mathrm{Ho}\mathcal{C}_{cf}$ with $\mathrm{Ho}\mathcal{C}$. The Homotopy/Unit Axiom requires that $\Upsilon : \mathrm{Ho}\mathcal{C}_{cf} \rightarrow \lambda_*\mathcal{C}_{cf}$ is an isomorphism, and viewing $\lambda_*\mathcal{C}_{cf}$ as a full enriched

subcategory of $\lambda_*\mathcal{C}$, it follows that $\text{Ho}\mathcal{C}_{cf}$ and therefore $\text{Ho}\mathcal{C}$ is enriched by $\mathcal{C}[[-, -]]$. This completes the proof of the first statement of the theorem. **Proposition 3.7** is a first step toward proving that $\text{Ho}\mathcal{C}$ is tensored and/or cotensored over $\text{Ho}\mathcal{M}$ when \mathcal{C} is. However, the adjunctions provided by that proposition are ordinary, rather than enriched adjunctions. To complete the proof of the theorem, we prove a stronger version of **Proposition 3.7** with enriched adjunctions as **Corollary 4.11** in our discussion of enriched functors and adjunctions in the next section. \square

For concreteness and to introduce notation used in the next section, we describe in more detail the composition law constructed in the previous proof. For each object C of \mathcal{C} , choose and fix an acyclic fibration $q_C: QC \rightarrow C$ with QC cofibrant (with q_C the identity if C is cofibrant), an acyclic cofibration $r_C: C \rightarrow RC$ with RC fibrant (with r_C the identity if C is fibrant), and a factorization $s_C: RQC \rightarrow RC$ of the composite $QC \rightarrow RC$, i.e., a map s_C making the diagram on the left commute.



Such a factorization exists by the lifting property of cofibrations with respect to acyclic fibrations illustrated on the diagram on the right above. Note that s_C is a weak equivalence by the two-out-of-three property. We choose s_C to be the identity when C is cofibrant. Then $s_C^{-1} \circ r_C = r_{QC} \circ q_C^{-1}$ is an isomorphism in $\text{Ho}\mathcal{C}$ from C to the cofibrant–fibrant object RQC .

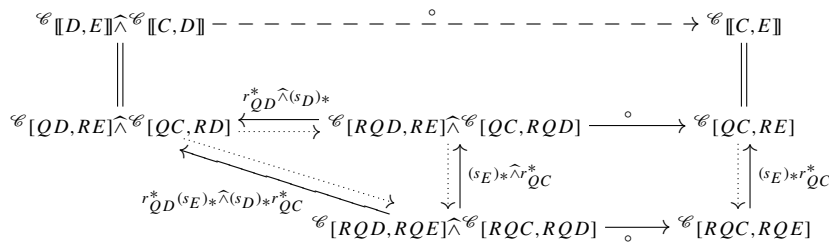
The purpose of the choice of the maps s is that it allows us to identify the isomorphism

$$(r_{QC} \circ q_C^{-1})_* (r_{QD} \circ q_D^{-1})^{-1}: \mathcal{C}[[RQC, RQD]] \rightarrow \mathcal{C}[[C, D]]$$

in the proof of **Theorem 3.10** above with the map

$$\mathcal{C}[[RQC, RQD]] = \mathcal{C}[[RQC, RQD]] \xrightarrow{(s_D)_* r_{QC}^*} \mathcal{C}[[QC, RD]] = \mathcal{C}[[C, D]].$$

The composition in $\text{Ho}\mathcal{C}$ therefore fits into the following commutative diagram in $\text{Ho}\mathcal{M}$, where the dotted arrows are the inverses of the isomorphisms indicated by the corresponding backward solid arrows.



The horizontal arrows in the bottom right square are the composition in $\lambda_*\mathcal{C}$. We can regard the middle row as a definition of the composition in the enrichment of $\text{Ho}\mathcal{C}$.

4. Enriched functors and enriched derived functors

This section continues the study of enriched model categories with a discussion of enriched functors. We characterize the enrichment of the homotopy category of an enriched model category in terms of a universal property with respect to enriched functors. This leads to a generalization to enriched functors of Quillen’s criterion for the existence of derived functors and a corresponding theory of enriched Quillen adjunctions.

Recall that, for categories \mathcal{C} and \mathcal{D} enriched over \mathcal{M} , an *enriched functor* $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ consists of a function $\Phi: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$ together with maps

$$\Phi_{C,C'}: \mathcal{C}[[C, C']] \rightarrow \mathcal{D}[\Phi C, \Phi C']$$

in \mathcal{M} consistent with the identity morphisms and composition law. We also write Φ for the functor on the underlying categories; this underlying functor is given by $\mathcal{M}(I, \Phi_{C,C'})$. More generally, when \mathcal{C} is enriched over \mathcal{M} , \mathcal{D} is enriched over \mathcal{N} , and $\lambda: \mathcal{M} \rightarrow \mathcal{N}$ is a lax symmetric monoidal functor, a λ -enriched functor $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ (or \mathcal{N} -enriched, when λ is understood) consists of a function Φ on objects and maps in \mathcal{N}

$$\Phi_{C,C'}: \lambda(\mathcal{C}[C, C']) \rightarrow \mathcal{D}[\Phi C, \Phi C']$$

consistent with the identity morphisms and composition law. The following well-known proposition essentially provides an equivalent alternate definition of a λ -enriched functor in terms of the \mathcal{N} -enriched category $\lambda_*\mathcal{C}$ of the previous section.

Proposition 4.1. *For any lax symmetric monoidal functor $\lambda: \mathcal{M} \rightarrow \mathcal{N}$ and any \mathcal{M} -enriched category \mathcal{C} , λ induces a λ -enriched functor $\mathcal{C} \rightarrow \lambda_*\mathcal{C}$, and this λ -enriched functor is initial. In other words, for any \mathcal{N} -enriched category \mathcal{D} , any λ -enriched functor $\mathcal{C} \rightarrow \mathcal{D}$ factors uniquely through an \mathcal{N} -enriched functor $\lambda_*\mathcal{C} \rightarrow \mathcal{D}$.*

We are mainly concerned with the case where λ is the localization functor $\mathcal{M} \rightarrow \text{Ho}\mathcal{M}$. Using this special case of a λ -enriched functor, we can identify the homotopy category of an enriched model category by a universal property. To avoid confusion with the localization functor $\lambda: \mathcal{M} \rightarrow \text{Ho}\mathcal{M}$, we denote the localization functor $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ as γ .

Theorem 4.2. *Let \mathcal{C} be an enriched model category over a monoidal model category \mathcal{M} . The localization functor $\gamma: \mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ is λ -enriched and is the initial λ -enriched functor that sends weak equivalences to isomorphisms. In other words, for any $\text{Ho}\mathcal{M}$ -enriched category \mathcal{H} , any λ -enriched functor $\mathcal{C} \rightarrow \mathcal{H}$ that sends weak equivalences to isomorphisms factors uniquely through a $\text{Ho}\mathcal{M}$ -enriched functor $\text{Ho}\mathcal{C} \rightarrow \mathcal{H}$.*

Proof. The enriched localization functor is given by the universal maps $\mathcal{C}[C, D] \rightarrow \mathcal{C}[[C, D]]$ of the right derived functor; we need to check that these maps assemble into an enriched functor. The fact that they preserve the identity morphisms is clear, and so it suffices to check that they preserve composition. Consider the following diagram in $\text{Ho}\mathcal{M}$ written in the notation introduced at the end of the previous section.

$$\begin{array}{ccc}
 \mathcal{C}[D, E] \hat{\wedge}^{\mathcal{C}} [C, D] & \xrightarrow{\circ} & \mathcal{C}[C, E] \\
 \downarrow (r_E)_* \hat{\wedge} q_C^* & \searrow \circ & \downarrow (r_E)_* q_C^* \\
 \mathcal{C}[D, RE] \hat{\wedge}^{\mathcal{C}} [QC, D] & \xleftarrow{r_D^* \hat{\wedge} (q_D)_*} \mathcal{C}[RD, RE] \hat{\wedge}^{\mathcal{C}} [QC, QD] & \\
 \downarrow q_D^* \hat{\wedge} (r_D)_* & \downarrow s_D^* \hat{\wedge} (r_{QD})_* & \\
 \mathcal{C}[QD, RE] \hat{\wedge}^{\mathcal{C}} [QC, RD] & \xleftarrow{r_{QD}^* \hat{\wedge} (s_D)_*} \mathcal{C}[RQD, RE] \hat{\wedge}^{\mathcal{C}} [QC, RQD] & \xrightarrow{\circ} \mathcal{C}[QC, RE]
 \end{array} \tag{4.3}$$

The top row is the composition in $\lambda_*\mathcal{C}$ and the bottom row is essentially the composition in $\text{Ho}\mathcal{C}$. The square

$$\begin{array}{ccc}
 \mathcal{C}[RD, RE] \hat{\wedge}^{\mathcal{C}} [QC, QD] & \xrightarrow{r_D^* \hat{\wedge} (q_D)_*} & \mathcal{C}[D, RE] \hat{\wedge}^{\mathcal{C}} [QC, D] \\
 \downarrow s_D^* \hat{\wedge} (r_{QD})_* & & \downarrow \circ \\
 \mathcal{C}[RQD, RE] \hat{\wedge}^{\mathcal{C}} [QC, RQD] & \xrightarrow{\circ} & \mathcal{C}[QC, RE]
 \end{array}$$

(where the right vertical arrow is the curved arrow in diagram (4.3)) commutes by dinaturality since $r_D \circ q_D = s_D \circ r_{QD}$, and the remaining squares commute by naturality. It follows that $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ is $\text{Ho}\mathcal{M}$ -enriched.

Given any λ -enriched functor $\Phi: \mathcal{C} \rightarrow \mathcal{H}$ that sends weak equivalences to isomorphisms, the natural maps $\Phi((r_E)_* \hat{\wedge} q_C^*)$, $\Phi(q_D^* \hat{\wedge} (r_D)_*)$, and $\Phi((r_E)_* q_C^*)$ are isomorphisms in $\text{Ho}\mathcal{M}$, and it follows from diagram (4.3) that Φ factors uniquely through a $\text{Ho}\mathcal{M}$ -enriched functor $\text{Ho}\mathcal{C} \rightarrow \mathcal{H}$. \square

Next we discuss derived functors in the enriched model category context. We concentrate our discussion on left derived functors to avoid tedious repetition. Recall that for a functor $\Phi: \mathcal{C} \rightarrow \mathcal{H}$, the left derived functor $L\Phi: \text{Ho}\mathcal{C} \rightarrow \mathcal{H}$ (if it exists) is defined to be the right Kan extension of Φ along the localization functor $\gamma: \mathcal{C} \rightarrow \text{Ho}\mathcal{C}$. In other words, the left derived functor (if it exists) as part of its structure comes with a natural

transformation $\phi: L\Phi \circ \gamma \rightarrow \Phi$ which is final among natural transformations $F \circ \gamma \rightarrow \Phi$. The definition of enriched derived functors therefore first requires review of the definition of enriched natural transformations.

Definition 4.4. An *enriched natural transformation* α between enriched functors $\Phi, \Phi': \mathcal{C} \rightarrow \mathcal{D}$ is a natural transformation between the underlying functors that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{C}[C, C'] & \xrightarrow{\Phi_{C,C'}} & \mathcal{D}[\Phi C, \Phi C'] \\ \Phi'_{C,C'} \downarrow & & \downarrow (\alpha_{C'})_* \\ \mathcal{D}[\Phi' C, \Phi' C'] & \xrightarrow{(\alpha_C)_*} & \mathcal{D}[\Phi C, \Phi' C'] \end{array}$$

If, instead, Φ and Φ' are λ -enriched functors, then a λ -enriched natural transformation is a \mathcal{N} -enriched natural transformation from Φ to Φ' , considered as \mathcal{N} -enriched functors out of $\lambda_*\mathcal{C}$.

We offer the following definition in analogy with the definition of left derived functor.

Definition 4.5. Let \mathcal{C} be an enriched model category over a monoidal model category \mathcal{M} . Let \mathcal{H} be a $\text{Ho}\mathcal{M}$ -enriched category and let $\Phi: \mathcal{C} \rightarrow \mathcal{H}$ be a λ -enriched functor. We say that a $\text{Ho}\mathcal{M}$ -enriched functor $L^{\mathcal{M}}\Phi: \text{Ho}\mathcal{C} \rightarrow \mathcal{H}$ and λ -enriched natural transformation $\phi^{\mathcal{M}}: L^{\mathcal{M}}\Phi \circ \gamma \rightarrow \Phi$ forms the *enriched left derived functor* of Φ when $\phi^{\mathcal{M}}$ is final among λ -enriched natural transformations $F \circ \gamma \rightarrow \Phi$. In other words, given any $\text{Ho}\mathcal{M}$ -enriched functor $F: \text{Ho}\mathcal{C} \rightarrow \mathcal{H}$ and λ -natural transformation $\alpha: F \circ \gamma \rightarrow \Phi$, there exists a unique $\text{Ho}\mathcal{M}$ -enriched natural transformation $\theta: F \rightarrow L^{\mathcal{M}}\Phi$ such that $\phi^{\mathcal{M}} \circ \theta = \alpha$.

The enriched right derived functor is defined analogously, or equivalently, as $R^{\mathcal{M}}\Phi = (L^{\mathcal{M}}\Phi^{\text{op}})^{\text{op}}$, for $\bar{\Phi}^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{H}^{\text{op}}$. Note that without further hypotheses on Φ , the underlying functor and natural transformation of the enriched left derived functor need not agree with the left derived functor of Φ when both exist. In the case when they do agree, we say that $L^{\mathcal{M}}\Phi, \phi^{\mathcal{M}}$ provide an *enrichment* of the left derived functor. Next we extend Quillen’s criterion for the existence of left derived functors to the enriched context, and show that under its hypotheses, the enriched left derived functor exists and provides an enrichment for the derived functor.

Quillen’s criterion for the existence of a left derived functor asserts that when $\Phi: \mathcal{C} \rightarrow \mathcal{H}$ preserves weak equivalences between cofibrant objects, the left derived functor exists and can be computed using the cofibrant approximations QC . In detail, for each map $f: C \rightarrow D$ in \mathcal{C} , we choose $Qf: QC \rightarrow QD$ to be a lift of $f \circ q_C$, i.e., choose a function $Q_{C,D}$ making the following diagram commute.

$$\begin{array}{ccc} \mathcal{C}(C, D) & \overset{Q_{C,D}}{\dashrightarrow} & \mathcal{C}(QC, QD) \\ & \searrow q_C^* & \swarrow (q_D)_* \\ & \mathcal{C}(QC, D) & \end{array}$$

Although Q is not a functor, implicit in the statement and explicit in the proof of Quillen’s criterion is that when Φ preserves weak equivalences between cofibrant objects, the composite $\Phi \circ Q$ becomes a functor and $\Phi(q)$ a natural transformation. In the enriched context, the map $(q_D)_*: \mathcal{C}[QC, QD] \rightarrow \mathcal{C}[QC, D]$ is an acyclic fibration, and so is an isomorphism in $\text{Ho}\mathcal{M}$. Thus, there exists a unique map $Q_{C,D}$ in $\text{Ho}\mathcal{M}$ making the following diagram in $\text{Ho}\mathcal{M}$ commute.

$$\begin{array}{ccc} \mathcal{C}[C, D] & \overset{Q_{C,D}}{\dashrightarrow} & \mathcal{C}[QC, QD] \\ & \searrow q_C^* & \swarrow \simeq (q_D)_* \\ & \mathcal{C}[QC, D] & \end{array}$$

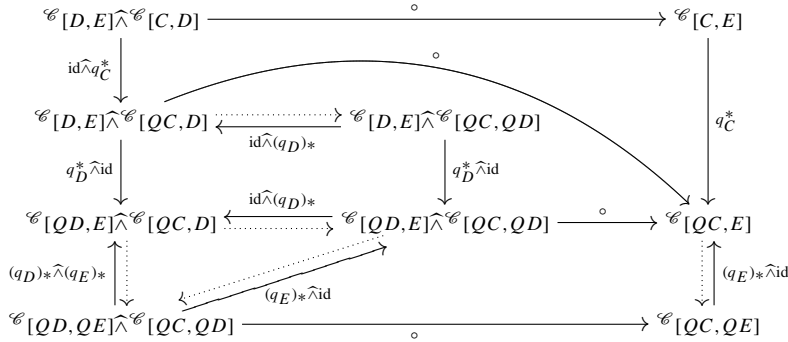
This leads to the following observation.

Lemma 4.6. *There is an enriched functor $Q: \lambda_*\mathcal{C} \rightarrow \lambda_*\mathcal{C}_c$ extending the function Q on objects. The maps q assemble to an enriched natural transformation from Q to the identity in $\lambda_*\mathcal{C}$.*

Proof. As indicated above, the enriched functor Q is defined as the map in $\text{Ho}\mathcal{M}$

$$Q_{C,D}: \mathcal{C}[C, D] \xrightarrow{q_C^*} \mathcal{C}[QC, D] \xleftarrow{(q_D)_*} \mathcal{C}[QC, QD]$$

and it is clear from this definition that q is an enriched natural transformation provided that Q is an enriched functor. To see that Q is an enriched functor, consider the following diagram in $\text{Ho}\mathcal{M}$.



This diagram, like diagram (4.3), commutes by naturality and dinaturality. \square

The following theorem now extends Quillen’s criterion to λ -enriched functors. The corresponding criterion for right derived functors also holds (and follows by considering $\Phi^{\text{op}}: \mathcal{C}^{\text{op}} \rightarrow \mathcal{H}^{\text{op}}$).

Theorem 4.7 (Enriched Quillen Criterion). *Let \mathcal{C} be an enriched model category over a monoidal model category \mathcal{M} , \mathcal{H} be a category enriched over $\text{Ho}\mathcal{M}$, and $\Phi: \mathcal{C} \rightarrow \mathcal{H}$ be a $\text{Ho}\mathcal{M}$ -enriched functor. If Φ takes weak equivalences between cofibrant objects to isomorphisms in \mathcal{H} , then the enriched left derived functor exists and provides an enrichment for the left derived functor.*

Proof. The composite enriched functor $\Phi \circ Q: \lambda_*\mathcal{C} \rightarrow \mathcal{H}$ sends weak equivalences to isomorphisms, and so factors through an enriched functor $L^{\mathcal{M}}\Phi: \text{Ho}\mathcal{C} \rightarrow \mathcal{H}$, and $\phi^{\mathcal{M}} = \Phi(q)$ gives a natural transformation from $L^{\mathcal{M}}\Phi \circ \gamma = \Phi \circ Q$ to Φ . It is easy to see from the diagrams preceding Lemma 4.6 that the underlying functor and natural transformation are the left derived functor and universal natural transformation constructed by Quillen. Given any enriched functor $F: \text{Ho}\mathcal{C} \rightarrow \mathcal{H}$ and any enriched natural transformation $\alpha: F \circ \gamma \rightarrow \Phi$, the maps $\theta_C = \alpha_{QC} \circ F(q_C^{-1})$ assemble to a natural transformation $\theta: F \rightarrow L^{\mathcal{M}}\Phi$, and this is the unique natural transformation θ such that $\phi^{\mathcal{M}} \circ \theta = \alpha$. \square

Next we discuss Quillen adjunctions. Recall that given an adjunction between closed model categories \mathcal{C} and \mathcal{D} , the following are equivalent:

- (i) The left adjoint preserves cofibrations and acyclic cofibrations.
- (ii) The right adjoint preserves fibrations and acyclic fibrations.
- (iii) The left adjoint preserves cofibrations and the right adjoint preserves fibrations.

Such an adjunction is called a *Quillen adjunction*. The left adjoint $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ then preserves weak equivalences between cofibrant objects, and the right adjoint $\Theta: \mathcal{D} \rightarrow \mathcal{C}$ preserves weak equivalences between fibrant objects. Quillen’s criterion for the existence of left derived functors then applies to $\gamma_{\mathcal{D}} \circ \Phi$ and Quillen’s criterion for the existence of right derived functors to $\gamma_{\mathcal{C}} \circ \Theta$. We write $\mathbf{L}\Phi$ and $\mathbf{R}\Theta$ for the corresponding (unenriched and, if applicable, enriched) derived functors obtained from $\gamma_{\mathcal{D}} \circ \Phi$ and $\gamma_{\mathcal{C}} \circ \Theta$. The fundamental theorem of model category theory is that $\mathbf{L}\Phi$ and $\mathbf{R}\Theta$ remain adjoints. To extend this to the enriched context we first must recall the definition of an enriched adjunction.

Definition 4.8. An adjunction between enriched functors $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ and $\Theta: \mathcal{D} \rightarrow \mathcal{C}$ is said to be *enriched* if the unit and counit of the adjunction are both enriched natural. This condition is equivalent to the requirement that the adjunction isomorphism

$$\mathcal{C}(C, \Theta D) \cong \mathcal{D}(\Phi C, D)$$

lifts to an isomorphism

$$\mathcal{C}[C, \Theta D] \cong \mathcal{D}[\Phi C, D]$$

that is enriched natural in each variable.

Theorem 4.9 (Enriched Quillen Adjunction). Let \mathcal{C} and \mathcal{D} be enriched model categories over the monoidal model category \mathcal{M} . If an enriched adjunction (Φ, Θ) between \mathcal{C} and \mathcal{D} is a Quillen adjunction, then the derived adjunction $(\mathbf{L}\Phi, \mathbf{R}\Theta)$ is also enriched.

Proof. If we write $\eta: \text{Id} \rightarrow \Theta\Phi$ for the unit of the (Φ, Θ) adjunction, then the unit of the $(\mathbf{L}\Phi, \mathbf{R}\Theta)$ adjunction is $(\Theta r) \circ \eta \circ q^{-1}$, and this is clearly enriched when η is, and likewise for the counit. \square

As promised in the last section, we now complete the proof of [Theorem 3.10](#). This amounts to recalling the notions of tensors and cotensors and applying the result above.

Definition 4.10. For an object C of \mathcal{C} and an object X of \mathcal{M} , the associated *tensor* $C \otimes X$ and *cotensor* $[X, C]$ are objects of \mathcal{C} , unique up to an enriched natural isomorphism when they exist, for which there are enriched natural isomorphisms

$$\mathcal{C}[C \otimes X, -] \cong [X, \mathcal{C}[C, -]] \quad \text{and} \quad \mathcal{C}[-, [X, C]] \cong [X, \mathcal{C}[-, C]].$$

If $C \otimes X$ exists for all X , then for formal reasons $C \otimes (-)$ is an enriched functor, and we can interpret the natural isomorphism above as an enriched adjunction. Analogous observations hold for $[-, C]$.

Applying the previous theorem to these enriched adjunctions gives the following corollary and thereby completes the proof of [Theorem 3.10](#).

Corollary 4.11. Let \mathcal{C} be an enriched model category over a monoidal model category \mathcal{M} .

- (a) If \mathcal{C} has tensors then so does $\text{Ho}\mathcal{C}$ and the tensor in $\text{Ho}\mathcal{C}$ is the left derived functor $\widehat{\otimes}$ of the tensor \otimes in \mathcal{C} .
- (b) If \mathcal{C} has cotensors then so does $\text{Ho}\mathcal{C}$ and the cotensor in $\text{Ho}\mathcal{C}$ is the right derived functor $\llbracket -, - \rrbracket$ of the cotensor $[-, -]$ in \mathcal{C} .

5. Enriched bifunctors and their derived functors

In the context of ordinary category theory, bifunctors such as the functor taking a pair of objects C and D in a category \mathcal{C} to their product $C \times D$ or to the morphism set $\mathcal{C}(C, D)$ have as their domains categories of the form $\mathcal{C} \times \mathcal{C}$, $\mathcal{C}^{\text{op}} \times \mathcal{C}$, or $\mathcal{C} \times \mathcal{D}$. However, the domains of the analogous enriched bifunctors, such as the tensor product functor, enriched hom functors, and the tensor and cotensor functors are not product categories like $\mathcal{C} \times \mathcal{D}$, but more complex enriched categories of the form $\mathcal{C} \wedge \mathcal{D}$, as can be seen in the familiar examples of additive categories. In the context of enriched model categories, this problem with domain categories for bifunctors is compounded by the fact that the morphism sets of the ordinary category underlying an enriched category like $\mathcal{C} \wedge \mathcal{D}$ typically have no tractable description. As a result, we cannot expect to be able to impose a useful model structure on these categories. The purpose of this section is to propose a definition of enriched derived functors in this context and to study when they exist and fit into (parametrized) enriched adjunctions.

We begin by reviewing the definition of enriched bifunctor. For categories \mathcal{C} and \mathcal{D} enriched over \mathcal{M} , the enriched category $\mathcal{C} \wedge \mathcal{D}$ is defined to have objects $\text{Ob}(\mathcal{C} \wedge \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$ and for objects $(C, D), (C', D')$ in $\text{Ob}(\mathcal{C} \wedge \mathcal{D})$, the morphism object $\mathcal{C} \wedge \mathcal{D}[(C, D), (C', D')]$ in \mathcal{M} is defined to be

$$\mathcal{C}[C, C'] \wedge \mathcal{D}[D, D'].$$

An enriched bifunctor from \mathcal{C}, \mathcal{D} to an enriched category \mathcal{E} is defined to be an enriched functor $\mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{E}$. The ordinary bifunctor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ underlying an enriched bifunctor $\mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{E}$ is obtained by precomposing with the functor from $\mathcal{C} \times \mathcal{D}$ to the underlying category of $\mathcal{C} \wedge \mathcal{D}$ that takes

$$f \in \mathcal{M}(I, {}^{\mathcal{C}}[C, C']) = \mathcal{C}(C, C'), \quad g \in \mathcal{M}(I, {}^{\mathcal{D}}[D, D']) = \mathcal{D}(D, D')$$

to

$$f \wedge g \in \mathcal{M}(I, {}^{\mathcal{C}}[C, C'] \wedge {}^{\mathcal{D}}[D, D']) = (\mathcal{C} \wedge \mathcal{D})((C, D), (C', D')).$$

An enriched natural transformation of bifunctors is an enriched natural transformation of functors $\mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{E}$, or equivalently, a natural transformation that is enriched in each variable separately.

In the context of a monoidal model category \mathcal{M} , since $\lambda: \mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ is lax symmetric monoidal, we have a canonical $\text{Ho}\mathcal{M}$ -enriched functor

$$\lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \lambda_*(\mathcal{C} \wedge \mathcal{D}), \tag{5.1}$$

and so a λ -enriched bifunctor $\mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{H}$ induces a $\text{Ho}\mathcal{M}$ -enriched bifunctor $\lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \mathcal{H}$. However, the functor in (5.1) is typically not an equivalence; for example, when \mathcal{C} and \mathcal{D} are enriched model categories, the $\text{Ho}\mathcal{M}$ -enriched localization functors $\gamma_{\mathcal{C}}: \lambda_*\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ and $\gamma_{\mathcal{D}}: \lambda_*\mathcal{D} \rightarrow \text{Ho}\mathcal{D}$ induce a $\text{Ho}\mathcal{M}$ -enriched functor

$$\gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}}: \lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D},$$

but we do not expect an enriched functor from $\lambda_*(\mathcal{C} \wedge \mathcal{D})$ to $\text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D}$. The following theorem describing the universal property of $\gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}}$ is the bifunctor analog of Theorem 4.2; its proof is a straightforward application of diagram (4.3).

Theorem 5.2. *Let \mathcal{C} and \mathcal{D} be enriched model categories over the monoidal model category \mathcal{M} . The $\text{Ho}\mathcal{M}$ -enriched bifunctor $\gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}}: \lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D}$ is initial among $\text{Ho}\mathcal{M}$ -enriched bifunctors that take weak equivalences in each variable to isomorphisms; in other words, if \mathcal{H} is a $\text{Ho}\mathcal{M}$ -enriched category and $\Phi: \lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \mathcal{H}$ is a $\text{Ho}\mathcal{M}$ -enriched functor that sends weak equivalences to isomorphisms, then Φ factors uniquely through a $\text{Ho}\mathcal{M}$ -enriched functor*

$$\text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D} \rightarrow \mathcal{H}.$$

Next we discuss the enriched left derived functors of enriched bifunctors. We offer the following definition.

Definition 5.3. Let \mathcal{C} and \mathcal{D} be enriched model categories over a monoidal model category \mathcal{M} , let \mathcal{H} be a category enriched over $\text{Ho}\mathcal{M}$, and let $\Phi: \lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \mathcal{H}$ be a $\text{Ho}\mathcal{M}$ -enriched bifunctor. An enriched bifunctor $L^{\mathcal{M}}\Phi: \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D} \rightarrow \mathcal{H}$ with an enriched natural transformation $\phi^{\mathcal{M}}: L^{\mathcal{M}}\Phi \circ (\gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}}) \rightarrow \Phi$ forms the enriched left derived bifunctor of Φ when $\phi^{\mathcal{M}}$ is final among enriched natural transformations $F \circ (\gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}}) \rightarrow \Phi$. We say that the enriched left derived bifunctor enriches the left derived functor of Φ if the left derived functor $L\Phi, \phi$ of Φ exists and is the restriction to $\text{Ho}\mathcal{C} \times \text{Ho}\mathcal{D}$ of the underlying functor and natural transformation of $L^{\mathcal{M}}\Phi, \phi^{\mathcal{M}}$.

Enriched right derived bifunctors are defined analogously. The following theorem is the bifunctor equivalent of Theorem 4.7. The corresponding result for right derived functors also holds (and follows by considering the appropriate enriched opposite categories).

Theorem 5.4. *Let \mathcal{C} and \mathcal{D} be enriched model categories over a monoidal model category \mathcal{M} , let \mathcal{H} be a category enriched over $\text{Ho}\mathcal{M}$, and let $\Phi: \lambda_*\mathcal{C} \widehat{\wedge} \lambda_*\mathcal{D} \rightarrow \mathcal{H}$ be a $\text{Ho}\mathcal{M}$ -enriched bifunctor. If Φ takes weak equivalences between cofibrant objects to isomorphisms in \mathcal{H} , then the enriched left derived bifunctor exists and enriches the left derived functor.*

Proof. We obtain $L^{\mathcal{M}}\Phi$ by factoring $\Phi \circ (Q \widehat{\wedge} Q)$ using Theorem 5.2. Given an enriched functor $F: \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D} \rightarrow \mathcal{H}$ and an enriched natural transformation $\alpha: F \circ \gamma_{\mathcal{C}} \widehat{\wedge} \gamma_{\mathcal{D}} \rightarrow \Phi$, then $F(q^{-1} \widehat{\wedge} q^{-1})$ is the unique enriched natural transformation factoring α . \square

The definition and theorems above have obvious generalizations to trifunctors and functors of any number of variables. In general for $\mathcal{C}_0, \dots, \mathcal{C}_m$ enriched model categories over a monoidal model category \mathcal{M} , and an enriched functor of m -variables

$$\Phi: \mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_m \rightarrow \mathcal{C}_0,$$

we write

$$\mathbf{L}\Phi: \mathbf{Ho}\mathcal{C}_1 \widehat{\wedge} \cdots \widehat{\wedge} \mathbf{Ho}\mathcal{C}_m \rightarrow \mathbf{Ho}\mathcal{C}_0$$

for the enriched left derived functor of

$$\gamma_{\mathcal{C}_0} \circ \lambda_* \Phi: \lambda_* \mathcal{C}_1 \widehat{\wedge} \cdots \widehat{\wedge} \lambda_* \mathcal{C}_m \rightarrow \lambda_* \mathcal{C}_0 \rightarrow \mathbf{Ho}\mathcal{C}_0$$

when it exists and extends the left derived functor, and call it the *enriched total left derived functor*. The enriched total right derived functor is defined analogously and denoted $\mathbf{R}\Phi$. The following terminology is also convenient.

Definition 5.5. For Φ as above, we say that the left derived functor of Φ is *enriched* when the enriched left derived functor of $\gamma \circ \lambda_* \Phi$ exists and extends the left derived functor. Likewise, we say that the right derived functor of Φ is enriched when the enriched right derived functor of $\gamma \circ \lambda_* \Phi$ exists and extends the right derived functor.

Functors of many variables admit many sorts of compositions, and the same kind of results as usual for the composition of derived functors of a single variable apply to all the possible compositions of total derived functors of many variables. The following proposition suffices for our purposes in Section 7. We phrase the proposition for the enriched total derived functors but it is really an assertion about the unenriched total derived functors.

Proposition 5.6. Let $\mathcal{C}_0, \dots, \mathcal{C}_m, \mathcal{D}_0, \dots, \mathcal{D}_n$, and \mathcal{E} be enriched model categories over a monoidal model category \mathcal{M} . If

$$\Phi: \mathcal{C}_0 \wedge \mathcal{D}_0 \rightarrow \mathcal{E}, \quad \Psi: \mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_m \rightarrow \mathcal{C}_0, \quad \Xi: \mathcal{D}_1 \wedge \cdots \wedge \mathcal{D}_n \rightarrow \mathcal{D}_0,$$

are enriched functors that send tuples of cofibrant objects to cofibrant objects and preserve weak equivalences between tuples of cofibrant objects, then the universal map

$$(\mathbf{L}\Phi) \circ (\mathbf{L}\Psi \widehat{\wedge} \mathbf{L}\Xi) \rightarrow \mathbf{L}(\Phi \circ (\Psi \wedge \Xi))$$

is an isomorphism.

All of the enriched bifunctors of interest to us appear in enriched parametrized adjunctions, and it is important that these adjunctions pass to homotopy categories. The general context we study is when have a pair of bifunctors

$$\Phi: \mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{E} \quad \text{and} \quad \Theta: \mathcal{D}^{\text{op}} \wedge \mathcal{E} \rightarrow \mathcal{C}$$

that form an enriched parametrized adjunction. This means that we have isomorphisms

$${}^{\mathcal{E}}[\Phi(C, D), E] \cong {}^{\mathcal{C}}[C, \Theta(D, E)]$$

that are enriched natural in all three variables. The following proposition describes the two pairs of equivalent conditions that together suffice to ensure that such a parametrized adjunction passes properly to homotopy categories.

Proposition 5.7. Let \mathcal{C}, \mathcal{D} , and \mathcal{E} be enriched model categories over a monoidal model category \mathcal{M} , and let $\Phi: \mathcal{C} \wedge \mathcal{D} \rightarrow \mathcal{E}$ and $\Theta: \mathcal{D}^{\text{op}} \wedge \mathcal{E} \rightarrow \mathcal{C}$ be a pair of \mathcal{M} -enriched bifunctors forming an enriched parametrized adjunction. The following two conditions are equivalent:

- (i) $\Phi(-, D)$ preserves cofibrations and acyclic cofibrations for all cofibrant D in \mathcal{D} .
- (ii) $\Theta(D, -)$ preserves fibrations and acyclic fibrations for all cofibrant D in \mathcal{D} .

If Φ and Θ satisfy these conditions, then the following two conditions are equivalent:

- (a) $\Phi(C, -)$ preserves weak equivalences between cofibrant objects for all cofibrant C in \mathcal{C} .
- (b) $\Theta(-, E)$ preserves weak equivalences between cofibrant objects for all fibrant E in \mathcal{E} .

Proof. The equivalence of (i) and (ii) is just a special case of the standard result about Quillen adjunctions. Assuming (i) and (ii), let $f: D \rightarrow D'$ be a weak equivalence between two cofibrant objects in \mathcal{D} , C be a cofibrant object of \mathcal{C} , and E be a fibrant object of \mathcal{E} . Then $\Phi(C, D)$ and $\Phi(C, D')$ are cofibrant and $\Theta(D, E)$ and $\Theta(D', E)$ are fibrant by (i) and (ii). It follows that we can identify the commuting diagram on the left below with the commuting diagram on the right below.

$$\begin{array}{ccc}
 \mathcal{E}[\Phi(C, D'), E] \xrightarrow{\cong} \mathcal{C}[C, \Theta(D', E)] & & \mathcal{E}[\llbracket \Phi(C, D'), E \rrbracket] \xrightarrow{\cong} \mathcal{C}[\llbracket C, \Theta(D', E) \rrbracket] \\
 \Phi(\text{id}_C, f)^* \downarrow & & \Phi(\text{id}_C, f)^* \downarrow \\
 \mathcal{E}[\Phi(C, D), E] \xrightarrow{\cong} \mathcal{C}[C, \Theta(D, E)] & & \mathcal{E}[\llbracket \Phi(C, D), E \rrbracket] \xrightarrow{\cong} \mathcal{C}[\llbracket C, \Theta(D, E) \rrbracket] \\
 \Theta(f, \text{id}_E)_* \downarrow & & \Theta(f, \text{id}_E)_* \downarrow
 \end{array}$$

Then $\Phi(\text{id}_C, f)^*$ is an isomorphism for every cofibrant C in \mathcal{C} and every fibrant E in \mathcal{E} if and only if $\Theta(f, \text{id}_E)_*$ is an isomorphism for every cofibrant C in \mathcal{C} and every fibrant E in \mathcal{E} . Now by the enriched Yoneda lemma in $\text{Ho}\mathcal{C}$ and $\text{Ho}\mathcal{E}$, we see that $\Phi(\text{id}_C, f)$ is a weak equivalence for every cofibrant C if and only if $\Theta(f, \text{id}_E)$ is a weak equivalence for every fibrant E in \mathcal{E} . \square

We can now state our main result on the passage of parametrized adjunctions to homotopy categories.

Theorem 5.8. *Let (Φ, Θ) be an enriched parametrized adjunction satisfying both pairs of equivalent conditions of Proposition 5.7. Then the enriched total left derived bifunctor*

$$\mathbf{L}\Phi: \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{E}$$

of Φ and the enriched total right derived bifunctor

$$\mathbf{R}\Theta: \text{Ho}\mathcal{D}^{\text{op}} \widehat{\wedge} \text{Ho}\mathcal{E} \rightarrow \text{Ho}\mathcal{C}$$

of Θ exist and form an enriched parametrized adjunction on the homotopy categories.

Proof. The enriched total derived bifunctors exist by Theorem 5.4 and its analogue for right derived functors. For fixed D in \mathcal{D} , write Φ_D for the enriched functor $\Phi(-, QD): \mathcal{C} \rightarrow \mathcal{E}$ and Θ_D for the enriched functor $\Theta(QD, -): \mathcal{E} \rightarrow \mathcal{C}$. Then by Theorem 4.9, the total derived functors $\mathbf{L}\Phi_D: \text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{E}$ and $\mathbf{R}\Theta_D: \text{Ho}\mathcal{E} \rightarrow \text{Ho}\mathcal{C}$ are adjoint. By construction, $\mathbf{L}\Phi_D$ coincides with $\mathbf{L}\Phi(-, D)$ viewed as an enriched functor $\text{Ho}\mathcal{C} \rightarrow \text{Ho}\mathcal{E}$, and $\mathbf{R}\Theta_D$ coincides with $\mathbf{R}\Theta(D, -)$ viewed as an enriched functor $\text{Ho}\mathcal{E} \rightarrow \text{Ho}\mathcal{C}$. Viewing $\mathbf{L}\Phi_{(-)}(-)$ as the bifunctor $\mathbf{L}\Phi: \text{Ho}\mathcal{C} \widehat{\wedge} \text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{E}$, then formally there exists precisely one way to make $\mathbf{R}\Theta_{(-)}$ an enriched bifunctor $\text{Ho}\mathcal{D}^{\text{op}} \widehat{\wedge} \text{Ho}\mathcal{E} \rightarrow \text{Ho}\mathcal{C}$ that is a parametrized right adjoint to $\mathbf{L}\Phi$; we have to show that for fixed E in \mathcal{E} , the functor $\mathbf{R}\Theta_{(-)}(E): \text{Ho}\mathcal{D}^{\text{op}} \rightarrow \text{Ho}\mathcal{C}$ so obtained coincides with $\mathbf{R}\Theta(-, E)$. Denote the counit of the $\mathbf{L}\Phi_{D'}, \mathbf{R}\Theta_{D'}$ adjunction as ϵ' and the natural isomorphism $\mathcal{E}[\llbracket \mathbf{L}\Phi_D(-), - \rrbracket] \cong \mathcal{E}[\llbracket -, \mathbf{R}\Theta_D(-) \rrbracket]$ as α' . Then as a contravariant $\text{Ho}\mathcal{M}$ -enriched functor on $\text{Ho}\mathcal{D}$, $\mathbf{R}\Theta_{(-)}(E)$ is the map

$$\begin{aligned}
 \mathcal{D}[\llbracket D, D' \rrbracket] &\xrightarrow{\mathbf{L}\Phi(\mathbf{R}\Theta_{D'}(E), -)} \mathcal{E}[\llbracket \mathbf{L}\Phi(\mathbf{R}\Theta_{D'}(E), D), \mathbf{L}\Phi(\mathbf{R}\Theta_{D'}(E), D') \rrbracket] \\
 &\xrightarrow{\epsilon'_*} \mathcal{E}[\llbracket \mathbf{L}\Phi(\mathbf{R}\Theta_{D'}(E), D), E \rrbracket] \xrightarrow{\alpha'} \mathcal{E}[\llbracket \mathbf{R}\Theta_{D'}(E), \mathbf{R}\Theta_D(E) \rrbracket].
 \end{aligned}$$

Unwinding the definition of $\mathbf{L}\Phi, \mathbf{R}\Theta$, and using naturality in \mathcal{D} of the Φ, Θ adjunction, a little bit of work identifies this map as the composite in $\text{Ho}\mathcal{M}$

$$\begin{aligned}
 \mathcal{D}[\llbracket D, D' \rrbracket] &= \mathcal{D}[QD, RD'] \simeq \mathcal{D}[QD, RQD] \xrightarrow{\Phi(Q\Theta(RQD', RE), Q(-))} \\
 &\quad \mathcal{C}[\Phi(Q\Theta(RQD', RE), QD), \Phi(Q\Theta(RQD', RE), RQD')] \xrightarrow{\epsilon_*} \\
 &\quad \mathcal{C}[\Phi(Q\Theta(RQD', RE), QD), RE] \xrightarrow{\alpha} \mathcal{E}[Q\Theta(RQD', RE), \Theta(QD, RE)] \\
 &\quad \simeq \mathcal{E}[Q\Theta(QD', RE), \Theta(QD, RE)] = \mathcal{E}[\llbracket \mathbf{R}\Theta(D', E), \mathbf{R}\Theta(D, E) \rrbracket],
 \end{aligned}$$

where ϵ is the counit and α the isomorphism for the Φ, Θ adjunction. Unwinding the Φ, Θ adjunction identifies this composite as the functor $\mathbf{R}\Theta$. \square

6. Semicofibrant objects

As explained in the introduction, semifibrant objects in \mathcal{M} are of intrinsic interest because in practice monoids in \mathcal{M} can often be approximated by weakly equivalent monoids whose underlying objects are semifibrant, but when the unit I is not cofibrant, monoids typically cannot have underlying objects that are cofibrant. We need some further observations on the properties of semifibrant objects for the proofs of the main results of the introduction that are phrased in terms of semifibrant objects. In this section, we collect these observations and some additional facts about semifibrant objects that seem potentially useful.

Recall that an object C in a closed model category \mathcal{C} enriched over the monoidal model category \mathcal{M} is semifibrant when the functor ${}^{\mathcal{C}}[C, -]: \mathcal{C} \rightarrow \mathcal{M}$ preserves fibrations and acyclic fibrations. Clearly, this notion is most useful when \mathcal{C} is an enriched model category, and we have the following proposition that generalizes parts of Proposition 1.3.

Proposition 6.1. *Let \mathcal{C} be an enriched model category.*

- (a) *If C is cofibrant in \mathcal{C} , then it is semifibrant in \mathcal{C} . Moreover, if the unit I is cofibrant in \mathcal{M} , then an object C of \mathcal{C} is semifibrant in \mathcal{C} if and only if it is cofibrant in \mathcal{C} .*
- (b) *If C is semifibrant in \mathcal{C} and $C \rightarrow D$ is a cofibration in \mathcal{C} , then D is semifibrant in \mathcal{C} .*

Proof. The first part of part (a) is a special case of part (b), which follows immediately from the Enrichment Axiom. For the second part of part (a), suppose I is cofibrant in \mathcal{M} and let C be a semifibrant object in \mathcal{C} ; then to see that C is cofibrant, we just need to see that for any acyclic fibration $X \rightarrow Y$ and any map $C \rightarrow Y$, there exists a lift $C \rightarrow X$. A map $C \rightarrow Y$ specifies a map in \mathcal{M} from I to ${}^{\mathcal{C}}[C, Y]$. Since C is semifibrant, ${}^{\mathcal{C}}[C, X] \rightarrow {}^{\mathcal{C}}[C, Y]$ is an acyclic fibration in \mathcal{M} . Since I is cofibrant in \mathcal{M} , we can lift $I \rightarrow {}^{\mathcal{C}}[C, Y]$ to $I \rightarrow {}^{\mathcal{C}}[C, X]$, and this specifies the lift $C \rightarrow X$ in \mathcal{C} of $C \rightarrow Y$. \square

We also need the following general theorem about semifibrant objects. It is proved at the end of the section.

Theorem 6.2. *Let \mathcal{C} be an enriched model category and D a fibrant object of \mathcal{C} . Then ${}^{\mathcal{C}}[-, D]$ preserves weak equivalences between semifibrant objects.*

Applying the theorem to the cofibrant approximation $QC \rightarrow C$, we obtain the following corollary.

Corollary 6.3. *Under the hypotheses of Theorem 6.2, if C is a semifibrant object of \mathcal{C} , then the canonical map ${}^{\mathcal{C}}[C, D] \rightarrow {}^{\mathcal{C}}[[C, D]]$ is an isomorphism in $\text{Ho}\mathcal{M}$.*

The following proposition explains many of the properties of semifibrant objects.

Proposition 6.4. *Let \mathcal{C} be an enriched model category that has tensors.*

- (a) *C is semifibrant if and only if $C \otimes -: \mathcal{M} \rightarrow \mathcal{C}$ preserves cofibrations and acyclic cofibrations.*
- (b) *If C is semifibrant, then $C \otimes I_c$ is cofibrant, and $\ell_c: C \otimes I_c \rightarrow C$ is a weak equivalence.*

Proof. Part (a) is the usual result on Quillen adjunctions applied to the adjoint pair $C \otimes -: \mathcal{M} \rightarrow \mathcal{C}$ and ${}^{\mathcal{C}}[C, -]: \mathcal{C} \rightarrow \mathcal{M}$. The first statement of part (b) follows from applying part (a) to the cofibration $0 \rightarrow I_c$ where 0 is the initial object. For the second statement in part (b), consider the map

$${}^{\mathcal{C}}[C, D] \rightarrow [I_c, {}^{\mathcal{C}}[C, D]] \cong {}^{\mathcal{C}}[C \otimes I_c, D].$$

When D is fibrant in \mathcal{C} , ${}^{\mathcal{C}}[C, D]$ is fibrant in \mathcal{M} , and so this map is a weak equivalence by the Unit Axiom in \mathcal{M} . Since the composite is induced by the map $\ell_c: C \otimes I_c \rightarrow C$, applying Corollary 6.3 and the enriched Yoneda lemma, we see that ℓ_c is a weak equivalence. \square

Finally, we need the following two propositions which are specific to the case of module categories. The first proposition is clear from the definition of semifibrant. Together with it, Propositions 6.1 and 6.4 subsume Proposition 1.3 from the introduction.

Proposition 6.5. Assume that ${}_A\mathcal{M}$ is a closed model category with fibrations and weak equivalences created in \mathcal{M} . Then A , considered as an object of ${}_A\mathcal{M}$, is semifibrant in ${}_A\mathcal{M}$.

The following proposition is a formal statement of the observation following Theorem 1.7 and plays a key role in the arguments in the next section.

Proposition 6.6. Assume that ${}_A\mathcal{M}$ and ${}_A\mathcal{M}_B$ are closed model categories with fibrations and weak equivalences created in \mathcal{M} . If the monoid B is semifibrant as an object of \mathcal{M} , then the forgetful functor ${}_A\mathcal{M}_B \rightarrow {}_A\mathcal{M}$ preserves cofibrations and takes semifibrant objects in ${}_A\mathcal{M}_B$ to semifibrant objects in ${}_A\mathcal{M}$.

Proof. The functor $[B, -]: {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}_B$ is right adjoint to the forgetful functor from ${}_A\mathcal{M}_B$ to ${}_A\mathcal{M}$. Since B is semifibrant in \mathcal{M} , this right adjoint preserves fibrations and acyclic fibrations. Thus, the forgetful functor from ${}_A\mathcal{M}_B$ to ${}_A\mathcal{M}$ preserves cofibrations (and by hypothesis on the model structures, all weak equivalences). If M is semifibrant in ${}_A\mathcal{M}_B$, then we see from the natural isomorphism

$${}^A[M, -] \cong {}^A[M, [B, -]]^B: {}_A\mathcal{M} \rightarrow \mathcal{M}$$

that ${}^A[M, -]$ preserves fibrations and acyclic fibrations, and so M is semifibrant in ${}_A\mathcal{M}$. \square

We close the section with the proof of Theorem 6.2. Let \mathcal{C} be an enriched model category, let D be a fibrant object, and let $f: C' \rightarrow C$ be a weak equivalence between semifibrant objects; we need to see that $f^*: \mathcal{C}[C, D] \rightarrow \mathcal{C}[C', D]$ is a weak equivalence. Since $\mathcal{C}[-, D]$ converts acyclic cofibrations to acyclic fibrations, by factoring the map from C to the final object, it suffices to consider the case when C is fibrant. Likewise, by factoring the map f , it suffices to consider the case when f is an acyclic fibration.

The idea for the proof is to construct some kind of “map” $g: C \rightarrow C'$ such that the composite $f \circ g: C \rightarrow C$ is the identity and the composite $g \circ f: C' \rightarrow C'$ is (left) homotopic to the identity. The induced composite $g^* \circ f^*$ then would be the identity and $f^* \circ g^*$ would be (right) homotopic to the identity, and so still a weak equivalence. We can actually do this in the case when I is cofibrant using a version of the argument of Proposition 6.1 (or the proposition itself). We generalize this argument and make this idea rigorous as follows:

Since C is semifibrant and f is an acyclic fibration, the map $f_*: \mathcal{C}[C, C'] \rightarrow \mathcal{C}[C, C]$ is an acyclic fibration. Let $\tilde{\text{id}}_C: I_c \rightarrow \mathcal{C}[C, C]$ be the composite of $I_c \rightarrow I$ and the map $I \rightarrow \mathcal{C}[C, C]$ representing the identity of C . Then since I_c is cofibrant, we can lift $\tilde{\text{id}}_C$ to a map $g: I_c \rightarrow \mathcal{C}[C, C']$.

$$\begin{array}{ccc} & \mathcal{C}[C, C'] & \\ & \nearrow g & \downarrow f_* \\ I_c & \xrightarrow{\tilde{\text{id}}_C} & \mathcal{C}[C, C] \end{array}$$

Composition gives a map

$$\mathcal{C}[C', D] \wedge_{I_c} \rightarrow \mathcal{C}[C', D] \wedge \mathcal{C}[C, C'] \rightarrow \mathcal{C}[C, D]$$

and adjoint to this map, we have a map

$$\hat{g}: \mathcal{C}[C', D] \rightarrow [I_c, \mathcal{C}[C, D]].$$

By construction, the composite map $\hat{g} \circ f^*: \mathcal{C}[C, D] \rightarrow [I_c, \mathcal{C}[C, D]]$ is the map $\tilde{\ell}_c$, which is a weak equivalence by the Unit Axiom in \mathcal{M} (since $\mathcal{C}[C, D]$ is fibrant).

We have constructed a commutative diagram

$$\begin{array}{ccc} \mathcal{C}[C, D] & \xrightarrow{\tilde{\ell}_c} & [I_c, \mathcal{C}[C, D]] \\ f^* \downarrow & \nearrow \hat{g} & \downarrow (f^*)_* \\ \mathcal{C}[C', D] & \longrightarrow & [I_c, \mathcal{C}[C', D]] \end{array}$$

where the bottom map is the composite $(f^*)_* \circ \hat{g}$. If this map were $\tilde{\ell}_c$, then the argument would be complete (cf. Proposition 6.7 below); the remainder of the argument is to show that it is homotopic to $\tilde{\ell}_c$.

Note that $(f^*)_* \circ \hat{g}$ is induced by “composition” with the map $h = f^* \circ g: I_c \rightarrow \mathcal{C}[C', C']$. If we write \tilde{f} for the map $I_c \rightarrow \mathcal{C}[C', C]$ adjoint to the map $I_c \rightarrow I \rightarrow \mathcal{C}[C', C]$ representing f , then we have

$$f_* \circ h = f_* \circ f^* \circ g = f^* \circ \text{id}_C = \tilde{f}: I_c \rightarrow \mathcal{C}[C', C].$$

Let J be a Quillen left cylinder object for I_c , i.e., factor the codiagonal map $I_c \amalg I_c \rightarrow I_c$ as a cofibration $I_c \amalg I_c \rightarrow J$ followed by an acyclic fibration $J \rightarrow I_c$. Then writing $\tilde{\text{id}}_{C'}: I_c \rightarrow \mathcal{C}[C', C']$ for the map $I_c \rightarrow I \rightarrow \mathcal{C}[C', C']$ induced by the identity on C' , we have the following solid arrow commuting diagram:

$$\begin{array}{ccc}
 I_c \amalg I_c & \xrightarrow{\tilde{\text{id}}_{C'} \amalg h} & \mathcal{C}[C', C'] \\
 \downarrow & \nearrow \phi & \downarrow \sim f_* \\
 J & \xrightarrow{\tilde{f}} & I_c \xrightarrow{f} \mathcal{C}[C', C]
 \end{array}$$

Choose a lift ϕ as indicated by the dashed arrow in the diagram. Then composition gives us a map

$$\mathcal{C}[C', D] \wedge J \rightarrow \mathcal{C}[C', D] \wedge \mathcal{C}[C', C'] \rightarrow \mathcal{C}[C', D].$$

Let $\hat{\phi}$ denote the adjoint map $\mathcal{C}[C', D] \rightarrow [J, \mathcal{C}[C', D]]$.

The two acyclic cofibrations $I_c \rightarrow J$ induces two acyclic fibrations $[J, \mathcal{C}[C', D]] \rightarrow [I_c, \mathcal{C}[C', D]]$. By composition with $\hat{\phi}$, we obtain two maps $\mathcal{C}[C', D] \rightarrow [I_c, \mathcal{C}[C', D]]$, which by construction are $\tilde{\ell}_c$ and $(f^*)_* \circ \hat{g}$. It follows that $[J, \mathcal{C}[C', D]]$ is a Quillen path object for $[I_c, \mathcal{C}[C', D]]$ and that $\hat{\phi}$ is a Quillen right homotopy between $(f^*)_* \circ \hat{g}$ and $\tilde{\ell}_c$.

In particular, since $\tilde{\ell}_c$ is a weak equivalence, $\hat{\phi}$ is a weak equivalence, and therefore $(f^*)_* \circ \hat{g}$ is a weak equivalence. This shows that of the three composable maps

$$\begin{aligned}
 f^*: \mathcal{C}[C, D] &\rightarrow \mathcal{C}[C', D] \\
 \hat{g}: \mathcal{C}[C', D] &\rightarrow [I_c, \mathcal{C}[C, D]] \\
 (f^*)_*: [I_c, \mathcal{C}[C, D]] &\rightarrow [I_c, \mathcal{C}[C', D]]
 \end{aligned}$$

both $\hat{g} \circ f^*$ and $(f^*)_* \circ \hat{g}$ are weak equivalences. The following “two out of six” principle of Dwyer, Hirschhorn, Kan, and Smith [2, 8.2.(ii)] implies that f^* is a weak equivalence and completes the proof of the theorem.

Proposition 6.7 (Two out of Six Principle [2, Section 9]). *Let \mathcal{C} be a closed model category, and let*

$$a: W \rightarrow X, \quad b: X \rightarrow Y, \quad c: Y \rightarrow Z$$

be maps in \mathcal{C} . If $b \circ a$ and $c \circ b$ are weak equivalences, then so are a , b , and c .

Proof. For any object V in \mathcal{C} , the map $(b \circ a)_*: \text{Ho}\mathcal{C}(V, W) \rightarrow \text{Ho}\mathcal{C}(V, Y)$ is a bijection, and so $b_*: \text{Ho}\mathcal{C}(V, X) \rightarrow \text{Ho}\mathcal{C}(V, Y)$ is a surjection. The map $(c \circ b)_*: \text{Ho}\mathcal{C}(V, X) \rightarrow \text{Ho}\mathcal{C}(V, Z)$ is a bijection, and so $b_*: \text{Ho}\mathcal{C}(V, X) \rightarrow \text{Ho}\mathcal{C}(V, Y)$ is an injection. Thus, b_* is a bijection for every V in \mathcal{C} , and so by the Yoneda lemma, b is an isomorphism in $\text{Ho}\mathcal{C}$. It follows that b is a weak equivalence, and by the two out of three axiom, that a and c are weak equivalences. \square

7. Proofs of the main results

In this section, we apply the theory of enriched derived functors developed in Sections 3–5 to prove the theorems stated in the introduction. Throughout, we assume that \mathcal{M} is a monoidal model category. We use A generally to denote an arbitrary monoid in \mathcal{M} and B to denote a monoid whose underlying object in \mathcal{M} is semifibrant. Also, we assume that all of the categories of modules being discussed are closed model categories with fibrations and weak equivalences created in \mathcal{M} .

In order to complete the proofs of the results stated in the introduction, we must show that our results on enriched parametrized adjunctions can be applied to the fundamental parametrized adjunctions arising in the study of bimodules. The following proposition provides a general statement.

Proposition 7.1. *Let $A, B,$ and C be monoids in \mathcal{M} . If the underlying object of B is semifibrant in \mathcal{M} , then:*

(a) *For cofibrations $f: M \rightarrow M'$ in ${}_A\mathcal{M}_B$ and $g: N \rightarrow N'$ in ${}_B\mathcal{M}_C$, the map*

$$(M \wedge_B N') \cup_{(M \wedge_B N)} (M' \wedge_B N) \rightarrow M' \wedge_B N'$$

is a cofibration in ${}_A\mathcal{M}_C$ and is a weak equivalence if either f or g is.

(b) *For a cofibration $f: M \rightarrow M'$ in ${}_A\mathcal{M}_B$ and a fibration $p: P' \rightarrow P$ in ${}_A\mathcal{M}_C$, the map*

$${}^A[M', P'] \rightarrow {}^A[M', P] \times_{{}^A[M, P]} {}^A[M, P']$$

is a fibration in ${}_B\mathcal{M}_C$ and is a weak equivalence if either f or p is.

(c) *For a cofibration $g: N \rightarrow N'$ in ${}_B\mathcal{M}_C$ and a fibration $p: P' \rightarrow P$ in ${}_A\mathcal{M}_C$, the map*

$$[N', P']^C \rightarrow [N', P]^C \times_{[N, P]^C} [N, P']^C$$

is a fibration in ${}_A\mathcal{M}_B$ and is a weak equivalence if either g or p is.

Proof. By the usual Quillen adjunction argument, part (a) is equivalent to both part (b) and part (c). Part (b) follows from Proposition 6.6 and the Enrichment Axiom for ${}_A\mathcal{M}$ (Proposition 3.9). \square

As previously indicated, Theorem 1.1 is a special case of Theorem 3.10, and Proposition 1.3 follows from the results proved in the previous section. We now go through the proofs of the remaining theorems from the introduction:

Proof of Theorem 1.4. Applying Theorem 4.9, part (a) is clear from the hypothesis that the fibrations and weak equivalences are created in \mathcal{M} and part (b) is a formal consequence of part (a) since enriched right adjoints preserve cotensors. Parts (c) and (d) follow similarly from Theorem 4.9 and the definition of semifibrant. \square

Proof of Theorem 1.7. Parts (a) and (b) of this theorem follow from Theorem 5.8 and Proposition 7.1. The claim in part (c) of the theorem that $M \wedge_B (-)$ and ${}^A[M, -]$ form a Quillen adjoint pair is just a reformulation of the hypothesis that M is semifibrant in ${}_A\mathcal{M}$. Theorem 4.9 provides enriched derived adjoint functors for this Quillen pair. The equivalence of $\text{Ext}_A(M_B, -_C)$ with the right derived functor is a consequence of Theorem 6.2 and the equivalence of $\text{Tor}_B({}_A M, -_C)$ with the left derived functor follows by the uniqueness of left adjoints. \square

Proof of Theorem 1.10. Note that for each of the natural transformations whose existence is asserted by this theorem, there is an obvious corresponding natural transformation before passage to the homotopy categories. Applying Theorem 5.4 and the universal property of the enriched total left and right derived bifunctors to these known natural transformations yields the desired natural transformations between the derived functors. \square

Proof of Theorem 1.11. By hypothesis the forgetful functor preserves fibrant objects and by Proposition 6.6 it preserves cofibrant objects when the monoid (whose action is being forgotten) is semifibrant in \mathcal{M} . The theorem follows by applying Proposition 5.6:

(a) The right adjoint version, with $\Phi = {}^A[-, -]$, Ψ the forgetful functor ${}_A\mathcal{M}_B \rightarrow {}_A\mathcal{M}$, and Ξ the forgetful functor ${}_A\mathcal{M}_C \rightarrow {}_A\mathcal{M}$ in part (a).

(b) With $\Phi = \wedge_B$, Ψ the forgetful functor ${}_A\mathcal{M}_B \rightarrow \mathcal{M}_B$, and Ξ the identity functor in part (b).

(c) With $\Phi = \wedge_B$, Ψ , the identity functor, and Ξ the forgetful functor ${}_B\mathcal{M}_C \rightarrow {}_B\mathcal{M}$ in part (c). \square

Proof of Theorem 1.12. The statement about Tor is a straightforward application of Proposition 5.6, which can be applied inductively to any association. The statement about Ext is adjoint. \square

8. Accommodating non-semicofibrant monoids

In the theorems of the introduction we needed to impose the hypothesis that certain monoids have semifibrant underlying objects in \mathcal{M} . While the results there appear to be the best possible for an arbitrary monoidal model category, the monoidal model categories used in practice tend to satisfy even stronger properties which allow the semifibrancy hypothesis to be partially dropped. Specifically, in this section we consider monoidal model categories \mathcal{M} where all categories of modules are closed model categories with fibrations and weak equivalences created in \mathcal{M} , and satisfy in addition the following properties:

- (i) For any monoid A , there exists a monoid A' with underlying object in \mathcal{M} semifibrant and a map of monoids $A' \rightarrow A$ that is a weak equivalence.
- (ii) For any monoid A and any cofibrant left A -module M , the functor $(-) \wedge_A M$ preserves weak equivalences between all right A -modules.

For the statements in this section, the monoidal model category \mathcal{M} is always assumed to satisfy properties (i) and (ii) above.

The first property holds in particular when the conclusions of [12, 4.1] hold: The category of monoids in \mathcal{M} is then itself a closed model category and the cofibrant objects have their underlying object in \mathcal{M} semifibrant. Although the we do not know of a general principle that would imply the second property, it holds in all presently known monoidal model categories of spectra [3,4,9] and equivariant spectra on complete universes [7,8,10] as well as the most common monoidal model categories coming from algebra. The purpose of this section is to indicate specifically which of the semifibrancy hypotheses of the theorems of the introduction can be eliminated under the assumptions above.

Theorem 1.1 requires no semifibrancy hypothesis. Property (ii) above, applied with the monoid I , shows that the comparison map between tensors in $\text{Ho}\mathcal{M}$ and tensors in $\text{Ho}_A \mathcal{M}$ is a natural isomorphism.

Theorem 8.1. *Let A be a monoid in \mathcal{M} . Then $\text{Ho}_A \mathcal{M}$ is enriched over $\text{Ho}\mathcal{M}$ by the right derived functor ${}^A\llbracket -, - \rrbracket$ of ${}^A[-, -]$, tensored by the left derived functor of \wedge and cotensored by the right derived functor of $[-, -]$. Moreover, the derived forgetful functor $\text{Ho}_A \mathcal{M} \rightarrow \text{Ho}\mathcal{M}$ preserves tensors and cotensors.*

Property (ii) above implies that \wedge_A satisfies the hypotheses of Theorem 5.4, and we can therefore define Tor_A to be its enriched total left derived bifunctor. In general, \wedge_A does not satisfy the hypotheses of Theorem 5.8, and we need to work a bit harder to find a right adjoint. Applying property (i) above to find a weak equivalence $A' \rightarrow A$ with A' semifibrant in \mathcal{M} , property (ii) implies both that the extension of scalars and forgetful functor adjunction between ${}_A \mathcal{M}$ and ${}_{A'} \mathcal{M}$ is a Quillen equivalence and also that the natural transformation $\wedge_{A'} \rightarrow \wedge_A$ induces an enriched natural isomorphism of left derived functors $\text{Tor}_{A'} \rightarrow \text{Tor}_A$. This implies that Tor_A fits into an enriched parametrized adjunction. The right adjoint is a refinement of Ext_I and so has some justification to be denoted as $\text{Ext}(-_A, -)$, but in general will not be the right derived functor of

$$[-, -]: \mathcal{M}_A \times \mathcal{M} \rightarrow {}_A \mathcal{M}.$$

Since comparison map $\text{Ext}(-_{A'}, -) \xrightarrow{\cong} \llbracket -, - \rrbracket$ is adjoint to the map $\text{Tor}(-, -) \rightarrow \text{Tor}_{A'}(-, -)$ induced by $\wedge \rightarrow \wedge_A$, the map $\text{Ext}(-_A, -) \xrightarrow{\cong} \llbracket -, - \rrbracket$ has an analogous description. We summarize this in the following theorems.

Theorem 8.2. *Let A be a monoid in \mathcal{M} .*

- (a) *The total left derived bifunctor Tor_A of \wedge_A exists, is enriched over $\text{Ho}\mathcal{M}$, and is an enriched parametrized left adjoint in each variable.*
- (b) *The right adjoints $\text{Ext}(-_A, -)$ and $\text{Ext}({}_A -, -)$ are naturally isomorphic to $\llbracket -, - \rrbracket$ in $\text{Ho}\mathcal{M}$ by the adjoint to the comparison map $\text{Tor}(-, -) \rightarrow \text{Tor}_A(-, -)$.*
- (c) *For each fixed right module M and each fixed left module N , $\text{Tor}_A(M, -)$ and $\text{Tor}_A(-, N)$ are the left derived functors of $M \wedge_A (-)$ and $(-) \wedge_A N$.*

Theorem 8.3. *Let $A' \rightarrow A$ be a map of monoids and a weak equivalence in \mathcal{M} . Then the forgetful functor ${}_A \mathcal{M} \rightarrow {}_{A'} \mathcal{M}$ is the right adjoint of a Quillen equivalence. The derived equivalence of homotopy categories preserves tensors and cotensors, and the universal enriched natural transformation $\text{Tor}_{A'} \rightarrow \text{Tor}_A$ is a natural isomorphism.*

The bimodule version of Tor is complicated by the fact that a pair of weak equivalences of monoids $A' \rightarrow A$ and $B' \rightarrow B$ does not necessarily induce a weak equivalence $A' \wedge B' \rightarrow A \wedge B$, and so does not necessarily induce a Quillen equivalence between categories of bimodules. However, it follows from property (ii) above, that the map is a weak equivalence when one of A', B' and one of A, B are semifibrant in \mathcal{M} .

Proposition 8.4. *Let $A' \rightarrow A$ and $B' \rightarrow B$ be maps of monoids in \mathcal{M} . Then the forgetful functor ${}_{A'}\mathcal{M}_B \rightarrow {}_{A'}\mathcal{M}_{B'}$ is the right adjoint of a Quillen adjunction. If both maps are weak equivalences and one of A', B' and one of A, B are semifibrant in \mathcal{M} , then the Quillen adjunction is a Quillen equivalence.*

When C is a monoid whose underlying object is semifibrant in \mathcal{M} , then cofibrant (B, C) -bimodules are cofibrant as left B -modules. Applying property (ii) again, we obtain the following refinement of the theorems from the introduction.

Theorem 8.5. *Let A, B , and C be monoids in \mathcal{M} and assume that C is semifibrant in \mathcal{M} . Then the left derived functor $\text{Tor}_B({}_A-, -C)$ exists, is an enriched parametrized left adjoint in the second variable (parametrized by the first variable), and the enriched natural map*

$$\text{Tor}_B(-, -) \rightarrow \text{Tor}_B({}_A-, -C)$$

is an isomorphism. Moreover, if either A or B is semifibrant in \mathcal{M} , then $\text{Tor}_B({}_A-, -C)$ is also an enriched parametrized left adjoint in the first variable.

Since by 8.2(c), we have that $\text{Tor}_A({}_A A, -)$ and therefore $\text{Ext}_A(A_A, -)$ are naturally isomorphic to the identity functor, the previous theorem can be applied as in the introduction to the case of $C = I$ to study the extension of scalars and coextension of scalars functors.

Corollary 8.6. *Let $A \rightarrow B$ be a map of monoids in \mathcal{M} . Then the derived forgetful functor $\text{Ho}_B \mathcal{M} \rightarrow \text{Ho}_A \mathcal{M}$ has both a left and a right adjoint. The left adjoint is naturally isomorphic in $\text{Ho} \mathcal{M}$ to $\text{Tor}_A(B, -)$ and the right adjoint is naturally isomorphic in $\text{Ho} \mathcal{M}$ to ${}^A\llbracket B, - \rrbracket$.*

The map $I \rightarrow A$ gives the free and cofree functors on homotopy categories. Using the map of monoids $A \cong A \wedge I \rightarrow A \wedge A$, the universal property of the left derived functors and the universal property of right adjoints induce comparison maps between the free functor and the functors

$$\mathbb{F}(-) = \text{Tor}({}_{A \wedge A^{\text{op}}} A, -) \quad \text{and} \quad \mathbb{F}^{\sharp}(-) = \text{Ext}(A_{A \wedge A^{\text{op}}}, -),$$

to $\text{Ho}_A \mathcal{M}_A$. Since the comparison maps with $\text{Tor}(A, -)$ and $\llbracket A, - \rrbracket$ are isomorphisms and the derived forgetful functor reflects isomorphisms, we obtain the first part of the following theorem. The isomorphisms in the second part follow because the derived forgetful functor $\text{Ho}_A \mathcal{M} \rightarrow \text{Ho} \mathcal{M}$ takes the comparison maps to the corresponding ones for the free and cofree functors under the natural isomorphism from the first part.

Theorem 8.7. *The free and cofree functors $\text{Ho} \mathcal{M} \rightarrow \text{Ho}_A \mathcal{M}$ are enriched naturally isomorphic to the composite of the functors $\mathbb{F}, \mathbb{F}^{\sharp}: \text{Ho} \mathcal{M} \rightarrow \text{Ho}_A \mathcal{M}_A$ and the derived forgetful functor $\text{Ho}_A \mathcal{M}_A \rightarrow \text{Ho}_A \mathcal{M}$. Moreover, the canonical comparison maps*

$$M \widehat{\otimes} X \rightarrow \text{Tor}_A({}_A \mathbb{F}X, M) \quad \text{and} \quad \llbracket X, M \rrbracket \rightarrow \text{Ext}_A(\mathbb{F}X_A, M)$$

in $\text{Ho}_A \mathcal{M}$, and

$$\text{Ext}_A(M, \mathbb{F}^{\sharp} X_A) \rightarrow \text{Ext}({}_A M, X)$$

in $\text{Ho} \mathcal{M}_A$ are isomorphisms.

Acknowledgements

The second author was supported in part by NSF grant DMS-0504069. The second author would like to thank Brooke Shipley and Andrew Blumberg for helpful comments.

References

- [1] W.G. Dwyer, J. Spaliński, Homotopy theories and model categories, in: *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995, pp. 73–126.
- [2] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, J.H. Smith, Homotopy Limit Functors on Model Categories and Homotopical Categories, in: *Mathematical Surveys and Monographs*, vol. 113, American Mathematical Society, Providence, RI, 2004.
- [3] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, Rings, Modules, and Algebras in Stable Homotopy Theory, in: *Mathematical Surveys and Monographs*, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.
- [4] M. Hovey, B. Shipley, J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* 13 (1) (2000) 149–208.
- [5] M. Hovey, *Model Categories*, in: *Mathematical Surveys and Monographs*, vol. 63, American Mathematical Society, Providence, RI, 1999.
- [6] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, Cambridge University Press, Cambridge, 1982.
- [7] L.G. Lewis Jr., M.A. Mandell, Equivariant universal coefficient and Künneth spectral sequences, *Proc. London Math. Soc.* (3) 92 (2) (2006) 505–544.
- [8] M.A. Mandell, J.P. May, Equivariant orthogonal spectra and S -modules, *Mem. Amer. Math. Soc.* 159 (755) (2002) x+108.
- [9] M.A. Mandell, J.P. May, S. Schwede, B. Shipley, Model categories of diagram spectra, *Proc. London Math. Soc.* (3) 82 (2) (2001) 441–512.
- [10] M.A. Mandell, Equivariant symmetric spectra, in: *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, in: *Contemp. Math.*, vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 399–452.
- [11] D.G. Quillen, *Homotopical Algebra*, in: *Lecture Notes in Mathematics*, vol. 43, Springer, Berlin, 1967.
- [12] S. Schwede, B.E. Shipley, Algebras and modules in monoidal model categories, *Proc. London Math. Soc.* (3) 80 (2) (2000) 491–511.