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Artin t-motifs

Lenny Taelman [1]

Universiteit Leiden, Mathematisch Instituut, Postbus 9512, Leiden, Netherlands

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ABSTRACT

We show that analytically trivial t-motifs satisfy a Tannakian duality, without restrictions on the base field, save for that it be of generic characteristic. We show that the group of components of the t-motivic Galois group coincides with the absolute Galois group of the base field.

1. Introduction

Let k be a finite field with q elements and denote by $k[t]$ the polynomial ring in one variable t over k. Let K be a field containing k and $\theta \in K$ a distinguished element. Write τ for the endomorphism 'Frobenius $\otimes$ identity' of $K[t] = K \otimes_k k[t]$, in other words: $\tau(\sum a_i t^i) = \sum a_i^q t^i$. In [1] Anderson defined a t-motive to be a pair (M, σ) consisting of a $K[t]$-module M and a function $\sigma : M \to M$ subject to the conditions that

(i) M is free and finitely generated over $K[t]$;
(ii) σ is semi-linear with respect to τ ($\sigma(m_1 + m_2) = \sigma(m_1) + \sigma(m_2)$ and $\sigma(fm) = \tau(f)\sigma(m)$ for all $m, m_1, m_2 \in M$ and $f \in K[t]$);
(iii) the determinant of σ with respect to one (equivalently: any) $K[t]$-basis of M vanishes only at $t = \theta$;
(iv) there exists a finite set $S \subset M$ such that $\bigcup_n \sigma^n(S)$ spans M as a K-vector space. (M is 'finitely generated over $K[\sigma]$');

He showed in [1] that the category of t-motives contains the (opposite) category of Drinfeld modules as a full subcategory. But there are plenty of interesting t-motives that are not Drinfeld modules: for example, the category of t-motives is closed under direct sums and tensor products while the subcategory of Drinfeld modules is not closed under either operation.

E-mail address: lenny.taelman@gmail.com.

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Recently, Papanikolas has shown [8] that *analytically trivial* (or *uniformisable*) t-motifs (*see* Section 3 for the definition) satisfy a Tannakian duality, at least when K is algebraically closed. For the transcendence applications of [8] it is of course sufficient to work over an algebraically closed field, but there is also a very arithmetic flavour to t-motifs and it is therefore desirable to have a Tannakian duality over any base field K.

In this paper we show that such a duality indeed holds over general K: that a suitable category of t-motifs over K is equivalent to the category of representations of some affine group scheme Γ_K over $k(t)$ (the subscript K is present only to denote dependency on K). (Sections 2 and 3.)

We follow [8] closely, save in two instances where we simplify the construction a bit:

(i) the closure under internal hom is effected by a formal inversion of the 'Carlitz twist';
(ii) there is no condition 'finitely generated over $K[\sigma]$' in the definitions.

The first one is of course completely analogous to the inversion of the Tate twist in Algebraic Geometry. Some form of such inversion in the function field context is already present in [11].

The second one is justified in Section 5.3, where we show that any t-motif (in our terminology) becomes finitely generated over $K[\sigma]$ after a suitable Tate twist. Hence the resulting Tannakian category coincides with the one generated by those t-motifs that are finitely generated over $K[\sigma]$.

Having established a Tannakian duality for t-motifs over a general K, we show that the group of components of Γ_K is isomorphic with the absolute Galois group of K. (In particular we obtain that when K is separably closed, the fundamental group Γ_K is connected.) This is obtained by a careful analysis of those t-motifs that are trivialised by a finite separable extension of K. This class of t-motifs forms the proper analogue of the so-called Artin motifs in Algebraic Geometry; this is why we refer to them as *Artin t-motifs*. (Sections 4 and 6.)

1.1. Notation

A few words on notation are in order. Let R be a ring and $\tau : R \to R$ an endomorphism of R. An additive function σ between two left R-modules is said to be semi-linear with respect to τ if it satisfies the identity $\sigma(rm) = \tau(r)\sigma(m)$. In this note (almost) all semi-linear functions are denoted by the Greek letter σ, and the different endomorphisms according to which they are semi-linear are all denoted by τ. This should not lead to confusion.

2. Constructing the category of t-Motifs

2.1. Effective t-Motifs

Let k be a finite field of q elements and K a field containing k. Fix a homomorphism $k[t] \to K$ of k-algebras and denote the image of t by θ. We do not demand that $k[t] \to K$ be injective now. We shall frequently refer to 'the field K', this is silently understood to contain the structure homomorphism $k[t] \to K$.

Denote by τ the endomorphism of $K[t]$ determined by $\tau(x) = x^q$ for all $x \in K$ and $\tau(t) = t$. The following definition goes back to [1], although here a slightly less restrictive form is used.

Definition 2.1.1. An *effective t-motif* of rank r over K is a pair $M = (M, \sigma)$ consisting of

- a free and finitely generated $K[t]$-module M of rank r, and
- a map $\sigma : M \to M$ satisfying $\sigma(fm) = \tau(f)\sigma(m)$ for all $f \in K[t]$ and $m \in M$,

such that the determinant of σ with respect to some (and hence any) $K[t]$-basis of M is a power of $t - \theta$ up to a unit in K.

A morphism of effective t-motifs is a morphism of $K[t]$-modules making the obvious square commute. The group of morphisms is denoted $\mathrm{Hom}_\sigma(M_1, M_2)$. The resulting category of effective t-motifs over K is denoted by $t\mathcal{M}_{\mathrm{eff}}(K)$ or simply by $t\mathcal{M}_{\mathrm{eff}}$.

Example 2.1.2. For any field $k[t] \to K$, the pair

$$C \stackrel{\mathrm{def}}{=} K[t]e \quad \text{with } \sigma(fe) \stackrel{\mathrm{def}}{=} \tau(f)(t-\theta)e,$$

is an effective t-motif and we will call it the *Carlitz t-motif*. This is the function field counterpart to the Lefschetz motif.

2.1.3. Define the *tensor product* of two effective t-motifs as

$$M_1 \otimes M_2 \stackrel{\mathrm{def}}{=} M_1 \otimes_{K[t]} M_2 \quad \text{with } \sigma(m_1 \otimes m_2) \stackrel{\mathrm{def}}{=} \sigma(m_1) \otimes \sigma(m_2).$$

This is again an effective t-motif.

The pair $(K[t], \tau)$ is an effective t-motif which we shall denote $\mathbf{1}$. We call it the *unit t-motif*, since for every M, one has natural isomorphisms $M \otimes \mathbf{1} = M$ and $\mathbf{1} \otimes M = M$.

Remark 2.1.4. We shall follow a convention sometimes used in representation theory and write nM for the direct sum of n copies of M and M^n for the n-fold tensor power of M.

If M_1 and M_2 are effective t-motifs then $\mathrm{Hom}(M_1, M_2)$ is naturally a $k[t]$-module. We have the following [1, Theorem 2]:

Proposition 2.1.5. $\mathrm{Hom}(M_1, M_2)$ *is free and finitely generated over* $k[t]$.

2.2. Duality

2.2.1. Let M_1 and M_2 be effective t-motifs over K. Inspired by the theory of linear representations of groups we could try to assign to M_1 and M_2 an effective t-motif of internal homomorphisms as

$$\mathcal{H}om(M_1, M_2) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{K[t]}(M_1, M_2) \quad \text{with } \sigma(f) \stackrel{\mathrm{def}}{=} \sigma \circ f \circ \sigma^{-1},$$

where $\sigma \circ f \circ \sigma^{-1}$ is to be read as $\sigma_2 \circ f \circ \sigma_1^{-1}$. This does, however, not make sense, since σ_1 need not be invertible. First of all, K need not be perfect, and secondly—more seriously—the determinant of σ_1 is $(t-\theta)^d$ up to a constant, and hence not invertible if $d > 0$.

2.2.2. This can be partially resolved. Write K^a for some algebraic closure of K. Note that after extension of scalars from $K[t]$ to $K^a(t)$ the induced action of σ on $M_1 \otimes_{K[t]} K^a(t)$ *is* invertible.

Proposition 2.2.3. *For n sufficiently large, the subgroup*

$$\mathrm{Hom}_{K[t]}\big(M_1, M_2 \otimes C^n\big) \subset \mathrm{Hom}_{K^a(t)}\big(M_1 \otimes K^a(t), M_2 \otimes C^n \otimes K^a(t)\big)$$

is stable under $f \mapsto \sigma \circ f \circ \sigma^{-1}$.

Proof. Choose bases and express σ on M_1 and M_2 by matrices S_1 and S_2 respectively. Then $M_2 \otimes C^n$ has a basis on which σ is expressed by the matrix $(t-\theta)^n S_2$. The map $f \mapsto \sigma \circ f \circ \sigma^{-1}$ translates to a map

$$\mathrm{M}\big(r_2 \times r_1, K^a(t)\big) \to \mathrm{M}\big(r_2 \times r_1, K^a(t)\big) : F \mapsto F'$$

with

$$F' = (t-\theta)^n S_2 \tau\big(F\tau^{-1}\big(S_1^{-1}\big)\big) = (t-\theta)^n S_2 \tau(F) S_1^{-1}.$$

The Proposition claims that $\mathrm{M}(r_2 \times r_1, K[t])$ is mapped into itself. But since the determinant of S_1 is a power of $(t-\theta)$, the matrix $(t-\theta)^n S_1^{-1}$ has entries in $K[t]$ when n is sufficiently large. This immediately implies that $\mathrm{M}(r_2 \times r_1, K[t])$ is mapped into itself. □

2.2.4. It follows from the proof that $\sigma(f) \stackrel{\text{def}}{=} \sigma \circ f \circ \sigma^{-1}$ induces the structure of an effective t-motif on $\mathrm{Hom}_{K[t]}(M_1, M_2 \otimes C^n)$ for large n. We shall denote it by $\mathcal{H}om(M_1, M_2 \otimes C^n)$. These internal homs are stable for growing n in the sense that there are natural isomorphisms

$$\mathcal{H}om\big(M_1, M_2 \otimes C^n\big) \otimes C \to \mathcal{H}om\big(M_1, M_2 \otimes C^{n+1}\big) \tag{1}$$

relating them.

2.3. Carlitz twist and t-motifs

The previous section hints that the obstruction to having internal homs will be lifted as soon as the Carlitz t-motif is made invertible. (Very reminiscent of the inversion of the Lefschetz motif in the construction of the category of pure motifs: If X is a smooth and projective variety of dimension d, then ℓ-adic Poincaré duality defines a perfect pairing

$$H^i_{\text{ét}}(X_{K^s}, \mathbf{Q}_\ell) \times H^{2d-i}_{\text{ét}}(X_{K^s}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell(-d)$$

which suggests that the motif $h^i(X, \mathbf{Q})$ is dual to $h^{2d-i}(X, \mathbf{Q})$ shifted by the dth power of the Lefschetz motif, see [3, §4.1].) This can be done quite easily, because of the following lemma, whose verification is straightforward.

Lemma 2.3.1. *If M_1 and M_2 are effective t-motifs, then the natural map*

$$\mathrm{Hom}_\sigma(M_1, M_2) \to \mathrm{Hom}_\sigma\big(M_1 \otimes C^n, M_2 \otimes C^n\big)$$

that takes f to $f \otimes \mathrm{id}$ is an isomorphism.

We are now ready to make the following definition.

Definition 2.3.2. A *t-motif* is a pair (M, i) consisting of an effective t-motif M and an integer $i \in \mathbf{Z}$. *Morphisms* between t-motifs are defined by

$$\mathrm{Hom}_\sigma\big((M_1, i_1), (M_2, i_2)\big) \stackrel{\text{def}}{=} \mathrm{Hom}_\sigma\big(M_1 \otimes C^{n+i_1}, M_2 \otimes C^{n+i_2}\big),$$

for n sufficiently large. The resulting category is denoted by $t\mathcal{M}(K)$ or simply by $t\mathcal{M}$.

It suffices to take $n \geqslant \max(-i_1, -i_2)$ in the definition. The module of morphisms is independent of n by the preceding lemma.

2.3.3. The functor $M \mapsto (M, 0)$ is fully faithful and we will identify $t\mathcal{M}_{\text{eff}}$ with its image in $t\mathcal{M}$.

The natural isomorphism between $M \otimes C^{n+1}$ and $M \otimes C^n \otimes C$ defines a distinguished isomorphism of t-motifs

$$(M, i+1) = (M \otimes C, i). \tag{2}$$

In particular, for $i > 0$ we can identify C^i with $(\mathbf{1}, i)$. But note that $(\mathbf{1}, i)$ is an object in $t\mathcal{M}$ even when i is negative.

2.3.4. The operations $\oplus$ and $\otimes$ and $\mathcal{H}om$ extend from the category of effective t-motifs—or parts thereof—to the full category of t-motifs:

$$(M_1, i_1) \oplus (M_2, i_2) \stackrel{\text{def}}{=} \left(M_1 \otimes C^{n+i_1} \oplus M_2 \otimes C^{n+i_2}, -n\right)$$
$$(M_1, i_1) \otimes (M_2, i_2) \stackrel{\text{def}}{=} (M_1 \otimes M_2, i_1 + i_2)$$
$$\mathcal{H}om\big((M_1, i_1), (M_2, i_2)\big) \stackrel{\text{def}}{=} \left(\mathcal{H}om\big(M_1, M_2 \otimes C^{i_1-i_2+n}\big), -n\right)$$

The occurrences of n in these definitions should be read 'with n sufficiently large'. Using the isomorphisms (1) and (2), one verifies that these are independent of n and coincide with the operations on effective t-motifs, whenever defined.

From now on we will often drop the integer i from the notation and write M for a t-motif, effective or not.

2.3.5. As usual, we define the dual of a t-motif M to be $M^\vee \stackrel{\text{def}}{=} \mathcal{H}om(M, \mathbf{1})$. The operations of direct sum, tensor product, duality and internal hom satisfy the expected relations—those familiar from the theory of linear representations of groups. In particular, there is an *adjunction formula*

$$\mathrm{Hom}(M_1 \otimes M_2, M_3) = \mathrm{Hom}\big(M_1, \mathcal{H}om(M_2, M_3)\big). \tag{3}$$

Also, taking duals is *reflexive*: the natural morphism

$$M \to \left(M^\vee\right)^\vee \tag{4}$$

is an isomorphism. And finally, $\mathcal{H}om$ is *distributive* over $\otimes$ in the sense that the natural morphism

$$\mathcal{H}om(M_1, M_3) \otimes \mathcal{H}om(M_2, M_4) \to \mathcal{H}om(M_1 \otimes M_2, M_3 \otimes M_4), \tag{5}$$

is an isomorphism. These identities are easily verified.

Remark 2.3.6. The category $t\mathcal{M}$ has kernels and cokernels. This can be seen as follows. All morphisms in $t\mathcal{M}$ become morphisms of effective t-motifs after an appropriate shift with a tensor power of the Carlitz motif. It is thus sufficient to show that $t\mathcal{M}_{\mathrm{eff}}$ has kernels and cokernels.

Let $M_1 \to M_2$ be a morphism of effective t-motifs. Its group-theoretic kernel is automatically a t-motif and a kernel in the category $t\mathcal{M}_{\mathrm{eff}}$. The cokernel of f in the pre-abelian category of *free* $K[t]$-modules—the ordinary cokernel modulo torsion—inherits an action of σ and one verifies that this defines an effective t-motif and a cokernel of f in $t\mathcal{M}_{\mathrm{eff}}$. Hence $t\mathcal{M}$ is pre-abelian.

Of course $t\mathcal{M}_{\mathrm{eff}}$ is not abelian, since for example the multiplication-by-t map $\mathbf{1} \to \mathbf{1}$ has trivial kernel and cokernel, but is not an isomorphism.

The existence of kernels and cokernels, and the existence of an internal hom bifunctor satisfying (3), (4) and (5) is summarised in

Theorem 2.3.7. *$t\mathcal{M}_K$ is a rigid $k[t]$-linear pre-abelian tensor category.*

3. The t-motivic Galois group Γ

Passing to the category $t\mathcal{M}_K^\circ$ of objects 'up to isogeny' (see Section 3.1 below for the definition) we obtain a rigid $k(t)$-linear abelian tensor category.

On a suitable subcategory of $t\mathcal{M}_K^\circ$ (closed under $\otimes$, $\oplus$, $\ldots$) we define a neutral fibre functor: a fully faithful exact functor to the category of finite-dimensional $k(t)$-vector spaces. The main theorem on Tannakian duality then asserts that this subcategory is equivalent with the category of $k(t)$-linear representations of some affine group scheme $\Gamma = \Gamma_K$ over $k(t)$. (See Section 3.2.)

In constructing this subcategory and fibre functor, we follow closely Papanikolas [8].

3.1. Isogenies

Definition 3.1.1. An *isogeny* between two effective t-motifs M_1 and M_2 is by definition a morphism $f \in \mathrm{Hom}_\sigma(M_1, M_2)$ such that there exist a $g \in \mathrm{Hom}_\sigma(M_2, M_1)$ and a nonzero h in $k[t]$ with $fg = h\,\mathrm{id} = gf$.

The category whose objects are effective t-motifs over K and whose hom-sets are the modules $\mathrm{Hom}_\sigma(-,-) \otimes_{k[t]} k(t)$ is denoted by $t\mathcal{M}_{\mathrm{eff}}^\circ(K)$. Sometimes we will refer to its objects as *effective t-motifs up to isogeny*.

Denote by $M(t)$ the $K(t)$-module $M \otimes_{K[t]} K(t)$. The action of σ on M extends naturally and makes $M(t)$ into a $K(t)[\sigma]$-module.

Proposition 3.1.2. *The natural map*

$$\mathrm{Hom}_\sigma(M_1, M_2) \otimes_{k[t]} k(t) \to \mathrm{Hom}_{K(t)[\sigma]}\big(M_1(t), M_2(t)\big)$$

is an isomorphism.

Hence the functor $M \mapsto M(t)$ is fully faithful on $t\mathcal{M}_{\mathrm{eff}}^\circ$. We shall identify $t\mathcal{M}_{\mathrm{eff}}^\circ$ with its image in the category of $K(t)[\sigma]$-modules. If we take M_1 and M_2 in the proposition to be the unit t-motif $\mathbf{1}$, we obtain that the field of invariants $K(t)^\sigma$ equals $k(t)$.

Proof of Proposition 3.1.2. (See also [8].) Note that the map is $k(t)$-linear. Injectivity is clear.

To show surjectivity, choose $K[t]$-bases for M_1 and M_2 and express the action of σ on them through matrices S_1 and S_2. Expressed on the induced bases for $M_1(t)$ and $M_2(t)$, a $K(t)[\sigma]$-homomorphism from $M_1(t)$ to $M_2(t)$ is a matrix F over $K(t)$ that satisfies

$$S_2^{-1} F S_1 = \tau(F). \tag{6}$$

Let h be the minimal common denominator of the entries of F, that is, the minimal monic polynomial in $K[t]$ with the property that hF has entries in $K[t]$. The minimal common denominator of the entries of the right-hand side $\tau(F)$ is $\tau(h)$ and the minimal common denominator of the left-hand side is $(t-\theta)^r h$ for some r. Equating them yields $r = 0$ and $\tau(h) = h$, hence the proposition. □

This proposition has an important consequence:

Corollary 3.1.3. *$t\mathcal{M}_{\mathrm{eff}}^\circ$ is an abelian $k(t)$-linear tensor category. $t\mathcal{M}^\circ$ is a rigid abelian $k(t)$-linear tensor category.*

Recall that $t\mathcal{M}_{\mathrm{eff}}$ is not abelian (Remark 2.3.6).

Proof of Corollary 3.1.3. The kernels and cokernels in $t\mathcal{M}_{\mathrm{eff}}^\circ$ are just the ordinary group-theoretic kernels and cokernels in the category of left $K(t)[\sigma]$-modules, and it is clear that a morphism whose kernel and cokernel vanish is an isomorphism, and that $t\mathcal{M}_{\mathrm{eff}}^\circ$ is abelian.

That $t\mathcal{M}^\circ$ is abelian is implied by the abelianness of $t\mathcal{M}^\circ_{\mathrm{eff}}$ and that it is rigid is implied by the rigidity of $t\mathcal{M}$, the required properties of $\mathcal{H}om$ are preserved under extension of scalars from $k[t]$ to $k(t)$. □

3.2. *A fibre functor*

In this section we construct a neutral fibre functor on a sub-category of $t\mathcal{M}^\circ(K)$, where $k[t] \to K$ is assumed to be injective. This construction occurs already in [1] and is interpreted as a fibre functor in [8]. I do not know if there exists a neutral fibre functor on *all* of $t\mathcal{M}^\circ$.

3.2.1. Let $K^\dagger$ be a field containing K that is algebraically closed and complete with respect to a valuation $\|\cdot\|$. Denote by $K^\dagger\{t\} \subset K^\dagger[[t]]$ the subring of restricted power series, that is, those power series whose coefficients converge to 0. In particular, these series have a radius of convergence greater than or equal to 1. Note that $K^\dagger\{t\}$ is closed under τ—raising all coefficients to the qth power. A τ-invariant power series has coefficients in the finite field k and hence is restricted if and only if it is a polynomial in t. That is, $K^\dagger\{t\}^\tau = k[t]$. Denote by $K^\dagger(\{t\})$ the field of fractions of $K^\dagger\{t\}$. In the next paragraph we shall show that $K^\dagger(\{t\})^\tau = k(t)$.

3.2.2. Define the functors $H_{\mathrm{an}}(-,k[t])$ and $H_{\mathrm{an}}(-,k(t))$ on the category $t\mathcal{M}_{\mathrm{eff}}$ of effective t-motifs as

$$H_{\mathrm{an}}\big(M,k[t]\big) \stackrel{\mathrm{def}}{=} \big(M \otimes_{K[t]} K^\dagger\{t\}\big)^\sigma,$$

$$H_{\mathrm{an}}\big(M,k(t)\big) \stackrel{\mathrm{def}}{=} \big(M \otimes_{K[t]} K^\dagger\big(\{t\}\big)\big)^\sigma.$$

The functors $H_{\mathrm{an}}(-,k[t])$ and $H_{\mathrm{an}}(-,k(t))$ are related.

Proposition 3.2.3. $H_{\mathrm{an}}\big(M,k(t)\big) = H_{\mathrm{an}}\big(M,k[t]\big) \otimes k(t)$.

Taking M to be **1** yields:

Corollary 3.2.4. $K^\dagger\big(\{t\}\big)^\tau = k(t)$.

Proof of Proposition 3.2.3. The ring $K^\dagger\{t\}$ is a principal ideal domain and every nonzero ideal is of the form

$$(t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)K^\dagger\{t\} \subset K^\dagger\{t\}$$

with $\|\alpha_i\| \leqslant 1$ for all i [13].

Take a $g \in H_{\mathrm{an}}(M,k(t))$ and write it as $h^{-1}m$ with $m \in M \otimes K^\dagger\{t\}$ and h a finite product $h = \prod(t-\alpha_i)$. Assume that the degree of h is minimal. The invariance of $g = h^{-1}m$ gives

$$\tau(h)m = h\sigma(m) \in M \otimes K^\dagger\{t\},$$

hence the minimality of h implies $\tau(h) = h$. □

Proposition 3.2.5. *If* $\|\theta\| > 1$, *then* $H_{\mathrm{an}}(C,k[t]) \approx k[t]$ *and* $H_{\mathrm{an}}(C,k(t)) \approx k(t)$.

Proof. (Cf. Lemma 2.5.4 of [2].) Consider the product expansion

$$\Omega \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt[q-1]{-\theta}} \prod_{i\geqslant 0}\left(1 - \frac{t}{\theta^{q^i}}\right),$$

where $\sqrt[q-1]{-\theta}$ is any root in $K^\dagger$. (Any two such roots differ by a scalar in $k^\times$.) The infinite product converges for all values of t and all zeroes have absolute value greater than or equal to $\|\theta\| > 1$, thus $\Omega \in K\{t\}^\times$. By construction $\Omega = (t-\theta)\tau(\Omega)$ and therefore $H_{\mathrm{an}}(C, k[t])) = k[t]\Omega e$. Similarly $H_{\mathrm{an}}(C, k(t)) = k(t)\Omega e$. □

Henceforth, when considering the functors H_{an}, we shall always assume that $\|\theta\| > 1$.

3.2.6. So far, we have only considered *effective* t-motifs. Shifting back and forth with powers of the Carlitz motif, we can extend the functors H_{an} to functors defined on all t-motifs as follows

$$H_{\mathrm{an}}\big((M,i), k[t]\big) \stackrel{\mathrm{def}}{=} H_{\mathrm{an}}\big(M, k[t]\big) \otimes_{k[t]} H_{\mathrm{an}}\big(C, k[t]\big)^{\otimes i},$$

$$H_{\mathrm{an}}\big((M,i), k(t)\big) \stackrel{\mathrm{def}}{=} H_{\mathrm{an}}\big(M, k(t)\big) \otimes_{k(t)} H_{\mathrm{an}}\big(C, k(t)\big)^{\otimes i}.$$

The resulting functor is well-defined by the canonical isomorphisms

$$H_{\mathrm{an}}\big(M \otimes C, k[t]\big) = H_{\mathrm{an}}\big(M, k[t]\big) \otimes_{k[t]} H_{\mathrm{an}}\big(C, k[t]\big).$$

3.2.7. These functors are not faithful. In fact, we shall shortly see that there exist non-trivial M with $H_{\mathrm{an}}(M, k[t]) = 0$.

Definition 3.2.8. A t-motif (M,i) over K is said to be *analytically trivial* if one of the following equivalent conditions holds:

- $M \otimes_{K[t]} K^\dagger\{t\}$ has a σ-invariant $K^\dagger\{t\}$-basis,
- $M \otimes_{K[t]} K^\dagger(\{t\})$ has a σ-invariant $K^\dagger(\{t\})$-basis,
- $\mathrm{rk}_{k[t]} H_{\mathrm{an}}(M, k[t]) = \mathrm{rk}\, M$,
- $\dim_{k(t)} H_{\mathrm{an}}(M, k(t)) = \mathrm{rk}\, M$.

Denote by $t\mathcal{M}_{\mathrm{a.t.}} \subset t\mathcal{M}$ and $t\mathcal{M}^\circ_{\mathrm{a.t.}} \subset t\mathcal{M}^\circ$ the full subcategories consisting of the analytically trivial objects.

The analytic triviality of a t-motif M depends on the embedding of K into $K^\dagger$—see 3.2.10 for an example. When we are dealing with the category $t\mathcal{M}_{\mathrm{a.t.}}$ we will assume that such an embedding has been fixed.

Proof of equivalence. By paragraph 3.2.2 the first condition is equivalent with the second, and the third with the fourth. Clearly the first implies the third. To conclude the converse (that the third implies the first), it suffices to show that for all effective t-motifs M the natural map

$$\big(M \otimes K^\dagger\{t\}\big)^\sigma \otimes_{k[t]} K^\dagger\{t\} \to M \otimes K^\dagger\{t\}$$

is injective. This can be done exactly as in [1, Theorem 2], using $K^\dagger\{t\}^\tau = k[t]$. □

Some immediate consequences of the definition are:

Theorem 3.2.9. *The class of analytically trivial t-motifs is closed under tensor product and duality. Moreover*

- $H_{\mathrm{an}}(-, k[t])$ *is a faithful $k[t]$-linear $\otimes$-functor on $t\mathcal{M}_{\mathrm{a.t.}}$,*
- $H_{\mathrm{an}}(-, k(t))$ *is an exact, faithful, $k(t)$-linear $\otimes$-functor on $t\mathcal{M}^\circ_{\mathrm{a.t.}}$.*

In particular $t\mathcal{M}^\circ_{\mathrm{a.t.}}(K)$ is neutral Tannakian with fibre functor $H_{\mathrm{an}}(-, k(t))$.

It follows from the main theorem on neutral Tannakian categories [4, Theorem 2.11] that $t\mathcal{M}^{\circ}_{\text{a.t.}}(K)$ is equivalent with the category of $k(t)$-linear representations of some affine group scheme Γ_K over $k(t)$.

But note that Γ_K depends on the chosen valuation on K. We shall usually tacitly assume that a valuation (with $\|\theta\| > 1$) has been fixed.

We will end this subsection with an example of a t-motif that is not analytically trivial.

Example 3.2.10. Not all t-motifs are analytically trivial. Consider for example, the rank 2 effective t-motif

$$M_\xi = K[t]e_1 + K[t]e_2 \quad \text{with} \begin{cases} \sigma(e_1) = \xi t e_1 + e_2, \\ \sigma(e_2) = e_1, \end{cases}$$

depending on a parameter $\xi \in K$.

Claim. *M_ξ is analytically trivial if and only if $\|\xi\| < 1$.*

In particular there exists no valuation on K for which M_ξ is analytically trivial when ξ is algebraic over k.

Proof of Claim. Assume that $K^\dagger\{t\}e_1 + K^\dagger\{t\}e_2$ has an invariant vector $ae_1 + be_2$, with a, b in $K^\dagger\{t\}$. Expressing the invariance under σ gives

$$\begin{cases} a = \tau(a)\xi t + \tau^2(a), \\ b = \tau(a). \end{cases}$$

Expand $a = a_0 + a_1 t + \cdots$ with $a_i \in K^\dagger$. Then it follows that $a_0^{q^2} = a_0$, that is, a_0 lies in the quadratic extension l/k inside $K^\dagger$, and in particular $\|a_0\| = 1$ (assuming $a_0 \neq 0$). The higher a_i satisfy the recurrence equation

$$a_n - a_n^{q^2} = \xi a_{n-1}^q. \tag{7}$$

If $\|\xi\| \geqslant 1$ then $\|a_n\| \geqslant 1$ for all n and the series $a_0 + a_1 t + \cdots$ is therefore not a restricted series, confirming one direction of the proposition. If on the other hand $\|\xi\| < 1$ then define an a recursively by taking at every step the unique solution a_n of (7) that has $\|a_n\| < 1$. This produces a restricted power series for every $a_0 \in l^\times$ and it suffices to take two a_0's independent over k to obtain two independent invariant vectors for $M_\xi \otimes_{K[t]} K^\dagger\{t\}$. □

4. Artin *t*-motifs

4.1. Definition

A corollary to Lang's Theorem [6] asserting the surjectivity of the map

$$\mathrm{GL}(n, K^s) \to \mathrm{GL}(n, K^s) : A \mapsto \tau(A)A^{-1}$$

is [9, Prop. 4.1]:

Theorem 4.1.1. *The category of pairs (V, σ), consisting of a finite-dimensional K-vector space V and an additive map $\sigma : V \to V$ satisfying $\sigma(xv) = x^q\sigma(v)$ and such that $K\sigma(V) = V$ is neutral Tannakian k-linear with fibre functor*

$$V \mapsto \left(V \otimes_K K^s\right)^\sigma$$

and fundamental group $G_K \stackrel{\text{def}}{=} \mathrm{Gal}(K^s/K)$.

Remarks 4.1.2. The invariants $(-)^\sigma$ are taken for the diagonal action $V \otimes_K K^s$ induced by the given action of σ on V and the qth power map on K^s. This is the unique extension of $\sigma : V \to V$ to a semi-linear map $V \otimes K^s \to V \otimes K^s$.

If K is not perfect $K\sigma(V)$ need not coincide with $\sigma(V)$, as can be seen already when $(V, \sigma) = (K, x \mapsto x^q)$.

We abusively write G_K for both the pro-finite group and the corresponding constant affine group scheme over k (obtained as the limit of the system of finite constant group schemes corresponding to the finite quotients of the pro-finite group). Their categories of representations on finite-dimensional k-vector spaces coincide.

A pair (V, σ) induces an effective t-motif $M(V) \stackrel{\text{def}}{=} V \otimes_K K[t]$ where the action of σ is induced from the action on V.

We would like to interpret the collection of t-motifs $M(V)$ as a Tannakian subcategory of $t\mathcal{M}^\circ$, but there are of course many more morphisms $M(V_1) \to M(V_2)$ than morphisms $V_1 \to V_2$ and the kernel and cokernel of a morphism from $M(V_1)$ to $M(V_2)$ are typically not of the form $M(V)$.

Proposition 4.1.3. *Let M be an effective t-motif over K. The following are equivalent*:

- *M is isomorphic to a sub-quotient of $M(V)$ for some V,*
- *$M \otimes_K K^s \approx n\mathbf{1}$ for some n.*

Definition 4.1.4. An *Artin t-motif* is an effective t-motif M satisfying the above equivalent conditions.

Proof of Proposition 4.1.3. If M is a sub-quotient of $M(V)$, then M_{K^s} is a sub-quotient of $M(V_{K^s}) \approx m\mathbf{1}$ and therefore $M_{K^s} \approx n\mathbf{1}$.

Conversely, assume that M_{K^s} has a basis of σ-invariant vectors. There exists some finite extension K'/K inside K^s such that this basis is already defined over K'. The natural map $K[t] \to K'[t]$ defines the structure of a $K[t]$-module on M'. Denote it by $R_{K'/K}M'$ in order to distinguish it from the $K'[t]$-module M'. It is clear that $R_{K'/K}M'$ is naturally an effective t-motif over K of rank $\mathrm{rk}(M)[K' : K]$. (Call it the *Weil restriction* of M' from K' to K.) But, M is a submodule of $R_{K'/K}M' \otimes_K K^s$ and the latter is isomorphic to $M(R_{K'/K}W)$ with W the sum of a number of copies of K' with the diagonal action of σ, whence the proposition. □

The full subcategory $t\mathcal{M}^\circ_{\mathrm{Artin}}(K)$ of $t\mathcal{M}^\circ(K)$ consisting of the Artin t-motifs is rigid abelian $k(t)$-linear and has a fibre functor

$$M \rightsquigarrow \left(M \otimes_{K[t]} K^s[t]\right)^\sigma \otimes_{k[t]} k(t) \tag{8}$$

and with this fibre functor we have

Proposition 4.1.5. *$t\mathcal{M}^\circ_{\mathrm{Artin}}(K)$ is neutral Tannakian $k(t)$-linear with fundamental group G_K.*

Note that it is not needed to use analytic methods to obtain a fibre functor on Artin t-motifs and in particular it is not needed to demand that $k[t] \to K$ be injective.

Proof of Proposition 4.1.5. The functor $M(V) \rightsquigarrow H(V) \otimes_k k(t)$ induces a fully faithful embedding of $t\mathcal{M}^\circ_{\mathrm{Artin}}(K)$ into the category of $k(t)$-linear representations of G_K. It will be essentially surjective as soon as every continuous $k(t)$-linear representation of G_K is a sub-quotient of $H \otimes_k k(t)$ for some

k-linear representation H. This is indeed so, since every (algebraic, or continuous) representation of G_K factors though a finite group G and every representation of G is a sub-quotient of the direct sum of a number of copies of the regular representation $k(t)[G]$, which is nothing but the regular representation $k[G]$ over k, tensored with $k(t)$. □

4.1.6. Artin t-motifs are the t-counterparts of the algebro-geometric Artin motifs. Let $\mathbf{Z} \to K$ be any field. Consider the category of smooth and projective varieties X over K that are of dimension zero. These are the spectra of the finite étale K-algebras and by Grothendieck's formulation of Galois theory the category of such X is equivalent to the category of finite G_K-sets. The motifs that are sub-quotients of the $h(X, \mathbf{Q})$ for zero-dimensional X are called Artin motifs. They form a category which is equivalent to the category of $\mathbf{Q}$-linear representations of G_K, see [3, §4.1].

4.2. Relation between Γ_K and Γ_{K^s}

Suppose now that $k[t] \to K$ is actually injective. Choose $K^\dagger \supset K$ to be algebraically closed, complete and with $\|\theta\| > 1$. Let K^s be the separable closure of K inside $K^\dagger$. For an Artin t-motif M we have that

$$\big(M \otimes_{K[t]} K^s[t]\big)^\sigma \otimes_{k[t]} k(t) = \big(M \otimes_{K[t]} K^\dagger\big(\{t\}\big)\big)^\sigma.$$

That is to say, $t\mathcal{M}(K)^\circ_{\mathrm{Artin}}$ is a full sub-category of $t\mathcal{M}(K)^\circ_{\mathrm{a.t.}}$ and the analytic fibre functor on the latter extends the algebraic fibre functor on the former.

Theorem 4.2.1. *There is a short exact sequence*

$$0 \to \Gamma_{K^s} \to \Gamma_K \to G_K \to 0$$

of affine group schemes over $k(t)$.

Proof. The full subcategory $t\mathcal{M}^\circ_{\mathrm{Artin}}(K)$ of $t\mathcal{M}^\circ_{\mathrm{a.t.}}(K)$ is Tannakian with fundamental group G_K (4.1.5) and is closed under sub-quotients in $t\mathcal{M}^\circ_{\mathrm{a.t.}}$ by definition. This implies the existence of a faithfully flat, and hence surjective, morphism $\Gamma_K \to G_K$ of affine group schemes (see for example [4, Prop. 2.21(a)]).

If M is an effective t-motif over K^s, then it has a model M' over a finite extension K' of K. The t-motif M is a submotif of $R_{K'/K} M' \otimes_K K^s$. Thus every t-motif over K^s is a submotif of a t-motif that is already defined over K. It follows that the fully faithful functor $M \rightsquigarrow M_{K^s}$ from $t\mathcal{M}^\circ_{\mathrm{a.t.}}(K)$ to $t\mathcal{M}^\circ_{\mathrm{a.t.}}(K^s)$ defines a closed immersion $\Gamma_{K^s} \to \Gamma_K$ (see [4, Prop. 2.21(b)]).

The sequence is exact in the middle if and only if the representations of Γ_K on which Γ_{K^s} acts trivially are precisely those coming from a representation of G_K. In other words, the exactness is equivalent with the statement that a t-motif M over K satisfies $M_{K^s} \approx n\mathbf{1}$ for some n if and only if it is an Artin t-motif. This was one of the equivalent definitions of the notion of an Artin t-motif (see 4.1.3). □

5. Weights

5.1. Dieudonné t-modules

As usual k is a finite field of q elements and K a field containing k. Denote by τ the continuous endomorphism of the field of Laurent series $K((t^{-1}))$ that fixes t^{-1} and that restricts to the qth power map on K.

Definition 5.1.1. A *Dieudonné t-module* over K is a pair (V, σ) of

- a finite-dimensional $K((t^{-1}))$-vector space V, and
- an additive map $\sigma : V \to V$ satisfying $\sigma(fv) = \tau(f)\sigma(v)$ for all $f \in K((t^{-1}))$ and all $v \in V$,

such that $K\sigma(V) = V$.

A morphism of Dieudonné t-modules is of course a $K((t^{-1}))$-linear map commuting with σ.

Dieudonné t-modules are easily classified, at least over a separably closed field. The main 'building blocks' are the following modules:

Definition 5.1.2. Let $\lambda = s/r$ be a rational number with $(r, s) = 1$ and $r > 0$. The Dieudonné t-module V_λ is defined to be the pair (V_λ, σ) with

- $V_\lambda \stackrel{\text{def}}{=} K((t^{-1}))e_1 \oplus \cdots \oplus K((t^{-1}))e_r$,
- $\sigma(e_i) \stackrel{\text{def}}{=} e_{i+1}$ $(i < r)$ and $\sigma(e_r) \stackrel{\text{def}}{=} t^s e_1$.

The classification states:

Proposition 5.1.3. *If V is a Dieudonné t-module over a separably closed field K then there exist rational numbers $\lambda_1, \ldots, \lambda_n$ such that*

- *$V \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$, and*
- *the t^{-1}-adic valuations of the roots of the characteristic polynomial of σ expressed on* any *$K((t^{-1}))$-basis are $\{-\lambda_i\}_i$, each counted with multiplicity* $\dim V_{\lambda_i}$.

If $\lambda \neq \mu$, then $\mathrm{Hom}(V_\lambda, V_\mu) = 0$. *For all λ, the ring* $\mathrm{End}(V_\lambda)$ *is a division algebra over $k((t^{-1}))$. Its Brauer class is* $\lambda + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z} = \mathrm{Br}(k((t^{-1})))$.

Note that this classification is formally identical to the classification of the classical (p-adic) Dieudonné modules [5].

Proof. This is shown in [7, Appendix B]. Although the statements therein are made only for a particular field K, nowhere do the proofs make use of anything stronger then the separably closedness of K. □

The following characterisation of V_λ is useful.

Proposition 5.1.4. *Let V be a Dieudonné t-module over a separably closed field K and λ a rational number. The following are equivalent*:

- *$V \approx V_\lambda \oplus V_\lambda \oplus \cdots \oplus V_\lambda$;*
- *there exists a lattice $\Lambda \subset V$ such that $\sigma^r(\Lambda) = t^s\Lambda$ where r and s are coprime integers with $\lambda = s/r$.*

Proof. *One ⇒ Two.* If $V = V_\lambda$ and (e_i) the basis that occurs in its definition (5.1.2) then the lattice generated by the same basis (e_i) has the required property. For $V = V_\lambda \oplus \cdots \oplus V_\lambda$ it thus suffices to take the lattice $\Lambda \oplus \cdots \oplus \Lambda$.

Two ⇒ One. The operator $t^{-s}\sigma^r$ transforms a $K[[t^{-1}]]$-basis of Λ into a new $K[[t^{-1}]]$-basis of Λ and therefore has eigenvalues of valuation 0. □

5.2. Weights

5.2.1. Let K be separably closed. Let M be an effective t-motif over K. Then

$$M\big(\big(t^{-1}\big)\big) \stackrel{\text{def}}{=} M \otimes_{K[t]} K\big(\big(t^{-1}\big)\big) = M(t) \otimes_{K(t)} K\big(\big(t^{-1}\big)\big)$$

is a Dieudonné t-module. The displayed equality shows that it only depends on the isogeny class of M. By the classification of Dieudonné t-modules (5.1.2) there exist rational numbers $\lambda_1, \dots, \lambda_n$ such that

$$M\big(\big(t^{-1}\big)\big) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}.$$

We call these rational numbers the *weights* of M. If K is not separably closed then we define the weights of an effective t-motif M to be the weights of M_{K^s}. This clearly does not depend on the choice of a separable closure.

We say that M is *pure of weight* λ if the only weight occurring is λ. By Proposition 5.1.4, this coincides with the definition as given in [1].

We now collect a number of facts related to the notions of weights and purity. They are either immediate consequences of the definitions or well-known facts established in the literature.

Proposition 5.2.2. *We have the following*:

- *If M is pure of weight λ, then every sub-quotient of M is pure of weight λ*;
- *If M has a filtration in which all successive quotients are pure of weight λ, then M is pure of weight λ*;
- *If the sets of weights of M_1 and M_2 are disjoint, then* $\mathrm{Hom}(M_1, M_2) = 0$;
- *C is pure of weight* 1;
- *The weights of $M_1 \otimes M_2$ are the sums of weights of M_1 with those of M_2*;
- *The weight of a pure effective t-motif M is non-negative.*

Proofs. *One.* If M' is a sub-quotient of M, then $M'((t^{-1}))$ is a sub-quotient of $M((t^{-1}))$ and the claimed statement follows at once from the classification 5.1.2.

Two. A normal series of M induces a normal series of $M((t^{-1}))$ and again the contention follows from 5.1.2.

Three. $\mathrm{Hom}(M_1, M_2)$ is a submodule of $\mathrm{Hom}(M_1((t^{-1})), M_2((t^{-1})))$, which is zero by 5.1.2.

Four. The valuation of $t - \theta$ at t^{-1} is -1.

Five. Immediate since the zeroes of the characteristic polynomials are multiplied.

Six. Clear for rank one M, for a general M take the top exterior power. □

5.2.3. If M is an effective t-motif and

$$M\big(\big(t^{-1}\big)\big) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n},$$

then by the proposition

$$(M \otimes C)\big(\big(t^{-1}\big)\big) \approx V_{\lambda_1+1} \oplus \cdots \oplus V_{\lambda_n+1}.$$

It is thus natural to define the *weights* of a t-motif (M, i) to be the set of $\lambda + i$ where λ runs through the weights of M. To be consistent, a t-motif (M, i) is then said to be *pure of weight* λ if and only if M is pure of weight $\lambda - i$.

5.3. Finite generation over $K[\sigma]$

Let K be algebraically closed.

Theorem 5.3.1. *If M is an effective t-motif and all weights of M are positive, then M is finitely generated as a $K[\sigma]$-module.*

Corollary 5.3.2. *If M is an effective t-motif then for all n sufficiently large $M \otimes C^n$ is finitely generated over $K[\sigma]$.*

Remark 5.3.3. It follows in particular that the analytically trivial effective t-motifs that are finitely generated over $K[\sigma]$ generate the Tannakian category $t\mathcal{M}^{\circ}_{\text{a.t.}}$, and that the t-motivic Galois groups constructed here coincide with those of [8].

Proof of Theorem 5.3.1. (Cf. [1, Prop. 1.9.2].) There is an isomorphism

$$M\big((t^{-1})\big) \approx V_{u_1/v} \oplus \cdots \oplus V_{u_k/v}.$$

Let Λ be the $K[[t^{-1}]]$-lattice in $M((t^{-1}))$ that corresponds with the standard lattice in the right-hand side (see 5.1.4). Let u be the minimum of the u_i. By the hypothesis $u > 0$. Then

$$t^u \Lambda \subset \sigma^v \Lambda.$$

Define an increasing filtration

$$M_0 \subset M_1 \subset M_2 \subset \cdots$$

by

$$M_n \stackrel{\text{def}}{=} M \cap t^{nu} \Lambda,$$

the intersection taken inside $M((t^{-1}))$. Clearly $M = \bigcup_n M_n$.

Claim: $M_{n+1} \subset M_n + \sigma^v M_n$ for all sufficiently big n.

Since the M_n are of finite dimension over K, the claim implies at once that M is finitely generated over $K[\sigma]$.

To prove the claim, note that for all n sufficiently large

$$M + t^n \Lambda = M\big((t^{-1})\big). \tag{9}$$

For such n we have

$$M_{n+1} = M \cap t^{nu} t^u \Lambda \subset M \cap t^{nu} \sigma^v \Lambda \subset \big(M \cap t^{nu}\big) + \big(\sigma^v M \cap \sigma^v t^{nu} \Lambda\big) = M_n + \sigma^v M_n$$

where the second inclusion needs (9). □

6. Connected components of Γ

The morphism $k[t] \to K$ is assumed to be injective.

6.1. The Tate conjecture

Let λ be a monic irreducible element of $k[t]$. Denote by $k(t)_\lambda$ the λ-adic completion of $k(t)$. To an effective t-motif $M = (M, \sigma)$ over K we associate a λ-adic Galois representation as follows:

$$M \mapsto H_\lambda(M) := \varprojlim_n \left(M_{K^s}/\lambda^n M_{K^s}\right)^\sigma \otimes_{k[t]} k(t).$$

Here $(-)^\sigma$ denotes invariants for the induced action of σ on $M_{K_s}/\lambda^n M_{K^s}$ (note that σ and λ^n commute). It follows from Lang's Theorem [6] that $H_\lambda(M)$ is a vector space over $k(t)_\lambda$ whose dimension equals the rank of M. By transport of structure G_K acts continuously on $H_\lambda(M)$.

The construction extends to give a $k(t)$-linear $\otimes$-functor H_λ from $t\mathcal{M}^\circ$ to the category of finite-dimensional continuous λ-adic G_K-representations.

Let us now restrict attention to analytically trivial t-motifs. Fix a λ as above. On the category $t\mathcal{M}^\circ_{\text{a.t.}}$ we now have two fibre functors to $k(t)_\lambda$-vector spaces: $H_{\text{an}}(-, k(t)_\lambda)$ (defined as $H_{\text{an}}(-, k(t)) \otimes k(t)_\lambda$) and $H_\lambda(-)$. By the formalism of Tannakian categories there exists an isomorphism α_λ of fibre functors

$$\alpha_\lambda(-) : H_{\text{an}}\left(-, k(t)_\lambda\right) \to H_\lambda(-).$$

Also it follows that if M is a t-motif with Tannakian fundamental group G, then the image of the λ-adic Galois representation is contained in $G(k(t)_\lambda)$, via the identification $\alpha_\lambda(M)$.

We have the following fundamental result ([11,12]; see also [10, §19]):

Theorem 6.1.1 *("Tate conjecture"). Assume K is finitely generated and let λ be a monic irreducible element of $k[t]$, coprime with the kernel of $k[t] \to K$. Then for all $M_1, M_2 \in t\mathcal{M}^\circ(K)$ the natural homomorphism*

$$\mathrm{Hom}(M_1, M_2) \otimes_{k(t)} k(t)_\lambda \to \mathrm{Hom}_{k(t)_\lambda[G_K]}\left(H_\lambda(M_1), H_\lambda(M_2)\right)$$

is an isomorphism.

6.2. Connected components of Γ

Theorem 6.2.1. *Γ_{K^s} has no finite quotients. In particular it is connected.*

Proof. Note that $\Gamma \to \pi_0(\Gamma)$ is a pro-finite étale quotient, hence the second statement indeed follows from the first.

Let G be a finite quotient of Γ_{K^s}. To a faithful representation (say, the regular representation) of G corresponds an analytically trivial t-motif M over K^s. It suffices to show that M is constant, for then it is trivial (over K^s) and consequently G is trivial.

Now, M is defined over some finitely generated $L \subset K^s$, hence the map $\Gamma_{K^s} \to G$ factors as

$$\Gamma_{K^s} \to \Gamma_L \to G.$$

Since $G(k_\lambda)$ is finite, the λ-adic representation of G_L associated with M_L becomes trivial over some finite extension L'/L inside K^s. Thus by the Tate conjecture, $M_{L'}$ is trivial, hence M was constant. □

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