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Maximal Intersecting Families and Affine Regular Polygons in $PG(2, q)$

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A family of mutually intersecting k-sets is called a k-clique. A k-clique is maximal if it is not contained in any larger k-clique. Using a classification result of Wettl we give a new upper bound for $m(k)$, the minimum number of members of a maximal k-clique, proving $m(k) \leqslant k^2/2 + 5k + o(k)$ whenever $k-1$ is a prime power. The proof is based on finite geometric results which are thought to be of independent interest.

1. PRELIMINARIES ON MAXIMAL CLIQUES

Let k be a positive integer. A k-clique (or intersecting family of rank k) is a collection of pairwise nondisjoint k-sets. A k-clique is *maximal* if it cannot be extended to another k-clique by adding a new k-set (and possible new elements).

A set B is called a *blocking set* of the hypergraph $\mathscr{F}$ if $B \cap F \neq \varnothing$ for

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every $F \in \mathscr{F}$. Clearly if $\mathscr{F}$ is a k-clique then every superset of any $F \in \mathscr{F}$ is a blocking set, so we say a blocking set is *nontrivial* if it contains no member of $\mathscr{F}$. Thus a k-clique is *maximal* if and only if it contains no nontrivial blocking set of k or fewer elements. For example, the following hypergraphs are maximal k-cliques:

all the k-subsets of a given $(2k-1)$-element set, (1.1)

the system of lines of a finite projective plane of order $k-1$. (1.2)

Denote by $m(k)$ the minimum size of a maximal k-clique. Erdös and Lovász [10] have given bounds for $m(k)$, proving in particular that $m(k) \geqslant \frac{8}{3}k - 3$. This was improved by Dow, Drake, Füredi, and Larson [4] to

$$m(k) \geqslant 3k \qquad \text{for} \quad k \geqslant 4. \tag{1.3}$$

J.-C. Meyer [16] observed that $m(1)=1$, $m(2)=3$, and $m(3)=7$. That $m(4)=12$ follows from (1.3) and the fact, proved in [11], that

$m(k) \leqslant \frac{3}{4}k^2$ whenever k is even and a projective plane of order $k/2$ exists. (1.4)

For $k>4$ the value of $m(k)$ remains open.

The determination of $m(k)$ is one of a number of questions concerning minimum cardinality families which are maximal with respect to various restrictions. This type of problem was raised by Erdös and Kleitman [9], and there has been little progress in these investigations. It was conjectured by Meyer [17] and Erdös [7] that $m(k) \geqslant k^2 - k + 1$ (the bound being attained by (1.2) where $k-1$ is a prime power). This was disproved in [11] (see (1.4) above), but it is still true that known constructions depend heavily on projective and affine geometries. There are at present just two other known classes of k-cliques of size less than k^2. These give the following upper bounds:

$$m(q^n + q^{n-1}) \leqslant q^{2n} + q^{2n-1} + q^{2n-2} \tag{1.5}$$

obtained from any n-uniform projective Hjelmslev plane of order q (Babai and Füredi [11] for $n=2$, Drake and Sane [6] for all n):

$m(k) \leqslant \frac{3}{4}k^2 + \frac{3}{2}k - 1$ if $k-1$ is an odd prime power, $k \geqslant 8$ (Blokhuis [1]). (1.6)

The construction giving (1.6) was the first counterexample to the conjecture of [11] that any maximal k-clique $\mathscr{F}$ satisfies $|\mathscr{F}| \geqslant |\bigcup \mathscr{F}|$. Using an idea of Drake [5], Blokhuis [1] recently gave an ingenious inductive proof of

$$m(k) < k^5 \qquad \text{for every } k. \tag{1.7}$$

It seems reasonable to conjecture that at least $m(k)/k \to \infty$ $(k \to \infty)$. (Erdős [8, problem I.3] offers $500 for resolution of a somewhat stronger conjecture.)

In this paper we show

THEOREM 1.8. *If $k-1$ is a prime power then $m(k) \leqslant k^2/2 + 5k + o(k)$.*

(Actually we give the proof only for $q \equiv -1 \pmod 6$, but the other cases are similar.) The proof, given in Section 4, is based on a result (Theorem 2.5) which is thought to be of independent interest. In particular it is shown (Corollary 2.6) that if B is a set of $q+1$ points meeting all lines meeting a conic C of the Galois plane PG(2, q), then the points $B \backslash C$ are collinear.

2. MINIMAL COVERINGS OF THE SET OF LINES MEETING A CONIC

Let $(\mathscr{P}, \mathscr{L})$ be the hypergraph formed by the points and lines of the Galois plane PG(2, q). We write $L(x, y)$ for the line spanned by (distinct) points x, y, and $P(L, M)$ for the point of intersection of the lines L, M.

Let $C \subset \mathscr{P}$ be a proper conic, and define

$$\mathscr{L}(C) = \{L \in \mathscr{L} : L \cap C \neq \varnothing\},$$

the set of secants and tangents of C. Thus

$$|\mathscr{L}(C)| = \binom{q+1}{2} + q + 1. \tag{2.1}$$

The following result was proved by Bruen and Thas [3] for q even and by Korchmaros and Segre [15] for all q:

> If B is a subset of $\mathscr{P} \backslash C$ of size at most $q+1$ meeting all lines of $\mathscr{L}(C)$, then B is an exterior line of C. (2.2)

(The hypotheses immediately give $|B| = q+1$ and $|B \cap L| = 1$ for each $L \in \mathscr{L}(C)$.) In this section we generalize (2.2), describing all the minimal blocking sets of $\mathscr{L}(C)$. For another nice generalization of (2.2) see [2].

We are interested in blocking sets of size $q+1$ (the obvious lower bound) of the hypergraph $\mathscr{L}(C)$. Let us denote by $\mathscr{B}(C)$ the collection of such blocking sets which are not equal to C and are not lines of $\mathcal{L}$. For example, it is an easy exercise to verify

PROPOSITION 2.3. *If* $B \in \mathcal{B}(C)$ *satisfies* $|B \backslash C| \leqslant 3$, *then one of the following holds*:

(a) $B = C \backslash \{x\} \cup \{y\}$ *with* $x \in C$ *and* $y \in L_x \backslash \{x\}$, *where* L_x *denotes the tangent at* x;

(b) $B = C \backslash \{u, v\} \cup \{w, x\}$ *with* $u, v \in C$, $w \in L(u, v) \backslash \{u, v\}$ *and* $x = P(L_u, L_v)$, *where* L_u *and* L_v *are the tangents at* u *and* v.

(c) $B = C \backslash \{u_1, u_2, u_3\} \cup \{x_1, x_2, x_3\}$, *where* $u_i \in C$ *and* x_i *is the intersection of the tangent at* u_i *and the secant spanned by* u_{i+1}, u_{i+2} (*subscripts* mod 3).

Before giving a complete description of $\mathcal{B}(C)$ we recall a few geometric facts. Let $L \in \mathcal{L}$ and let $c_1, ..., c_m$ be the points of $C \backslash L$, $l_1, ..., l_m$ the points of $L \backslash C$ $(m \in \{q-1, q, q+1\})$ indexed so that l_i lies on $L(c_1, c_i)$. (As usual $L(c, c)$ is the tangent at $c \in C$.) If we multiply subscripts by the rule

$$xy = z \qquad \text{if } l_z \text{ lies on } L(c_x, c_y),$$

then the set $\{1, ..., m\}$ becomes an Abelian group (a consequence of Pascal's theorem), which we denote $G(L, c_1)$. It was shown by Korchmáros [13, 14] that $G(L, c_1)$ is cyclic if $m \in \{q-1, q+1\}$ and elementary Abelian if $m = q$.

EXAMPLE 2.4. Let N be a coset of a subgroup of $G(L, c_1)$, and let B be obtained by deleting from C the points of $C \backslash L$ corresponding to N and adding the points of $L \backslash C$ corresponding to $N \cdot N$. Then $|B| = q+1$ (since $|N \cdot N| = |N|$) and so $B \in \mathcal{B}(C)$.

Notice that Example 2.4 includes the examples in Proposition 2.3. It is also easy to see that replacing c_1 by c_i gives the same examples since the system of cosets does not change. Our main geometric result is

THEOREM 2.5. *The only sets in* $\mathcal{B}(C)$ *are those given by Example* 2.4.

The proof of Theorem 2.5 depends mainly on establishing

LEMMA 2.6. *If* $B \in \mathcal{B}(C)$ *then the points of* $B \backslash C$ *are collinear.*

We remark that Theorem 2.5 is closely related to results of Wettl [19] and Szönyi and Wettl [18]. They described the $(q+1)$-element sets Q with the property that for some line L, $Q \backslash L$ is an arc (i.e., contains no three collinear points) and every line containing 2 points from $Q \backslash L$ avoids $L \backslash Q$.

3. Proof of Theorem 2.5

We begin with a lemma of Korchmáros and Segre [15].

LEMMA 3.1. *Suppose a, b, c are noncollinear points of* PG$(2, q)$ *and D is a set of $q+1$ points disjoint from $\{a, b, c\}$ but meeting every line which meets $\{a, b, c\}$. Then the points $D \cap L(a, b)$, $D \cap L(a, c)$, and $D \cap L(b, c)$ are collinear.*

Let a, b, c be collinear points disjoint from C and suppose that p_a, p_b, p_c are points of the conic C such that each of $\{a, p_b, p_c\}$, $\{p_a, b, p_c\}$, $\{p_a, p_b, c\}$ is a collinear triple. Then the triangle $\{p_a, p_b, p_c\}$ is said to be *associated* with the triple $\{a, b, c\}$.

LEMMA 3.2 (folklore; see [15]). *The number of triangles associated with a collinear triple spanning a line l is at most*

1. *if q is even and l is not a tangent or*
   q is odd and l is a tangent
2. *otherwise.*

Let $B \in \mathscr{B}(C)$ and denote $C \backslash B$ by K, $B \backslash C$ by W, and $|K| = |W|$ by k. We first prove Lemma 2.6, after which Theorem 2.5 follows easily. Of course we may suppose $k \geqslant 3$. Consider the following 2-design $\mathcal{A}$ on W:

$$\mathcal{A} = \{L \cap W : L \cap W \geqslant 2, L \in \mathscr{L}\}.$$

Our aim is to prove that $\mathcal{A}$ is a trivial design, i.e., $\mathcal{A} = \{W\}$. Clearly,

$$\sum_{A \in \mathcal{A}} \binom{|A|}{2} = \binom{k}{2}. \tag{3.3}$$

CLAIM 3.4. $\sum_{A \in \mathcal{A}} \binom{|A|}{3} \geqslant \frac{1}{2}\binom{k}{3}$.

Proof. Indeed by Lemma 3.1 every triangle $\{a, b, c\} \subseteq K$ is associated with a collinear triple of W. So by Lemma 3.2 the number of collinear triples in W is at least $\frac{1}{2}\binom{k}{3}$. On the other hand, the number of collinear triples in W is exactly $\sum \binom{|A|}{3}$. ∎

This immediately gives Lemma 2.6 if $k = 3$ or 4, so from now on we suppose that $k \geqslant 5$. Suppose for Claims 3.5 and 3.7 that q is odd.

CLAIM 3.5. *For all $A \in \mathcal{A}$ we have $|A| \geqslant 3$.*

Our main tool in the proof of Claim 3.5 is the fact:

If $L \in \mathscr{L}$, $L \cap C = \{a, b\}$, $L \cap K \neq 0$, and $L \cap W \neq 0$, then $\{a, b\} \subseteq K$. (3.6)

For if $a \in K$, e.g., then each line through a carries just one point of B, and since $B \cap L$ contains a point of W it cannot also contain b.

Proof of 3.5. Let $p, q \in W$. Since $k \geqslant 5$ there is some $a \in K$ such that both $L = L(p, a)$ and $L' = L(q, a)$ are secants of C. Set $L \cap C = \{a, b\}$, $L' \cap C = \{a, c\}$. By (3.6), $b, c \in K$ and applying Lemma 3.1 (with $D = B$) we find that $L(b, c) \cap W$ is a third point of the block A of $\mathscr{A}$ containing p, q. ∎

CLAIM 3.7. *There is an* $A_0 \in \mathscr{A}$ *with* $|A_0| \geqslant \frac{1}{2}k + 1$.

Proof. Indeed with $d = \max\{|A| : A \in \mathscr{A}\}$ we have

$$\frac{1}{2}\binom{k}{3} \leqslant \sum_{A \in \mathscr{A}} \binom{|A|}{3} \leqslant \frac{d-2}{3} \sum_{A \in \mathscr{A}} \binom{|A|}{2} = \frac{d-2}{3}\binom{k}{2}$$

(the last equality by (3.3)), and the claim follows. ∎

Claims 3.5 and 3.7 quickly imply 2.6. For if A_0 is as in Claim 3.7 and $p \in W \backslash A_0$, then the lines joining p to A_0 contain at least

$$1 + 2|A_0| > k$$

points of W, a contradiction. ∎

For q even a similar argument shows that the points of W are collinear with the possible exception of the core (say r) of C. But in this exceptional case Lemma 3.2 demands at least $\binom{k}{3}$ triples from the $k-1$ points of $W \backslash \{t\}$, which is impossible. So again W is collinear.

Proof of Theorem 2.5. We may suppose $k \geqslant 2$. Denote the line containing W by L. We work in the group $G = G(L, c_1)$, where c_1 is some point of $C \backslash L$. Denote by $\hat{K}$ and $\hat{W}$ the subsets of G corresponding to K and W. By the definitions of B, K, W we have $\hat{K} \cdot \hat{K} \subseteq \hat{W}$, whence $|\hat{K} \cdot \hat{K}| = |\hat{K}|$ implying that $\hat{K}$ is a coset of some subgroup of G. ∎

4. A MAXIMAL k-CLIQUE FROM $\mathscr{B}(C)$

For q a prime power, $q + 1 \equiv 0 \pmod 6$, let C be a proper conic of $PG(2, q)$, L an exterior line, and c_0 an arbitrary point of C. Let H be the 6-element subgroup of the (cyclic) group $G(L, c_0)$ and denote by K and W

the subsets of C and L corresponding to H. Then $B=(C\backslash K)\cup W$ is in $\mathscr{B}(C)$. Set

$$\mathscr{F}=\mathscr{L}(C)\cup\{L'\in\mathscr{L}: L'\cap W\neq\phi\}\cup\{B'\in\mathscr{B}(C): W\subseteq B'\}.$$

THEOREM 4.1. *$\mathscr{F}$ is a maximal $(q+1)$-clique with*

$$|\mathscr{F}|<\tfrac{1}{2}(q+1)^2+4(q+1).$$

Proof. Clearly $\mathscr{F}$ is intersecting. It is also easy to see that no line not in $\mathscr{F}$ is a blocking set of $\mathscr{F}$, so the maximality of $\mathscr{F}$ follows from

PROPOSITION 4.2. *If B' is a nontrivial $(q+1)$-point blocking set of*

$$\mathscr{F}_0:=\mathscr{L}(C)\cup\{L\in\mathscr{L}: L\cap W\neq\varnothing\}\cup\{B\},$$

then $W\subseteq B'\in\mathscr{B}(C)$.

Proof. That $B'\in\mathscr{B}(C)$ follows from $\mathscr{L}(C)\subseteq\mathscr{F}_0$. Set $B'=(C\backslash K')\cup W'$ with $|K'|=|W'|=k$ and suppose by way of contradiction that there exists $x\in W\backslash W'$.

Set $L'=L(W')$. We must have $L'\neq L$, since otherwise any exterior line through x avoids B'. Thus

$$|W\cap L'|\leqslant 1. \tag{4.3}$$

We will show

$$|K\backslash K'|\leqslant 2. \tag{4.4}$$

For suppose $\{a,b,c\}\subseteq K\cap B'$. Since $L'\neq L$, one of the lines $L(a,b)$, $L(a,c)$, $L(b,c)$ (say $L(a,b)$) does not contain $p(L,L')$. Let $L(a,b)\cap W=\{y\}$. Then $y\in W\backslash B'$ implies that every line through y contains exactly one point of B', a contradiction since $L(a,b)$ contains two such points. This proves (4.4).

By (4.3) some tangent to C at a point of K meets W in a point $w\notin L'$. Then w is on 4 lines which meet C only in K, at least 2 of which avoid $K\backslash K'$ (by (4.4)). These two lines, together with the $(q-1)/2$ exterior lines through w, must be met by W', whence $|W'|>(q+1)/2$. It follows that $W'=L'=B'$, contradicting the assumption that B' is nontrivial. This proves Proposition 4.2. ∎

It remains to prove the upper bound on $|\mathscr{F}|$. As noted in (2.1) we have

$$|\mathscr{L}(C)|=\binom{q+2}{2}. \tag{4.5}$$

Since at least 3 of the 6 points of W are exterior points (all 6 if $4 \mid q+1$),

$$|\{L \in \mathscr{L} \backslash \mathscr{L}(C) : L \cap W \neq \phi\}| \leqslant 3q-5. \tag{4.6}$$

To bound the size of $\{B' \in \mathscr{B}(C) : W \subset B'\}$ note that for any such B', $B' \backslash C = W'$ corresponds to a (proper) subgroup of $G(L, c_0)$ containing H. The number of such subgroups is the number of proper divisors of $(q+1)/6$ and so is less than $2\sqrt{(q+1)/6} - 1$. The assertion

$$|\{B' \in \mathscr{B}(C) : W \subset B'\}| < 4\sqrt{(q+1)/6} - 2 \tag{4.7}$$

thus follows from

PROPOSITION 4.8. *For any $B' \in \mathscr{B}(C)$ there is at most one $B'' \in \mathscr{B}(C) \backslash \{B'\}$ for which $B'' \backslash C = B' \backslash C$.*

Proof. Let $B' \backslash C = W'$. It is clear that the complements of any two B'' satisfying $B'' \backslash C = W'$ are disjoint inside C, since for such a B'' and any $c \in C \backslash B''$ we have

$$C \backslash B'' = \left(\bigcup \{L(c, d) : d \in W'\} \right) \cap C.$$

But since all tangents to C at points outside such a B'' meet W', at most $2|W'|$ points of C lie in $C \backslash B''$ for some B'', so the proposition is proved. ▮

Finally, the desired bound on $|\mathscr{F}|$ follows from (4.5)–(4.7) and the proof of Theorem 4.1 is complete. ▮

Similar constructions can be given for the remaining values of q. (Briefly: for $q \equiv 1 \pmod 6$ we may take L a secant and $|H| = 6$; for $q = 2^k$ we may take L a tangent and $|H| = 4$; the linear term in the bound of Theorem 1.8 derives from the cases $q = 3^k$, where for some values of k we must take $|H|$ as large as 9.)

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