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On the Mullineux involution for Ariki–Koike algebras

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ABSTRACT

This note is concerned with a natural generalization of the Mullineux involution for Ariki–Koike algebras. By using a result of Fayers together with previous results by the authors, we give an efficient algorithm for computing this generalized Mullineux involution. Our algorithm notably does not involve the determination of paths in affine crystals.

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1. Introduction

It is known that the set of simple modules of the symmetric group over a field of characteristic p can be labeled by the set $\mathcal{P}_p$ of p-regular partitions, that is, the partitions where no nonzero part is repeated p or more times. The tensor product of one of these simple modules with the sign representation is again simple. This gives a natural involution I on $\mathcal{P}_p$. In 1979, Mullineux [22] introduced a combinatorial algorithm yielding an involution m on $\mathcal{P}_p$ and conjectured the equality $m = I$. In 1994, Kleshchev [19] obtained the first combinatorial description of the involution I by using sequences of p-regular partitions. In particular, the corresponding algorithm k is different from the combinatorial procedure m of Mullineux. Finally, the conjecture $m = I$ (and thus the equality $m = k$) was proved by Ford and Kleshchev in [14] (see also [6] and [8] for different proofs). For the sake of simplicity, we will refer to the involution $I = m$ as the Mullineux involution in the sequel.

A natural quantum analogue of this problem has been considered by Richards [23] and by Brundan [7] in the context of the Hecke algebra of type A (see also [8] and [20]). The Mullineux involution can then be interpreted as a skew-isomorphism of $A_{e-1}^{(1)}$-crystals, that is as an isomorphism of oriented

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graphs switching the sign of each arrow. The resulting algorithm then coincides with that introduced by Kleshchev in [19].

In this note, we study an analogue of the Mullineux involution for the Ariki–Koike algebras (also called cyclotomic Hecke algebras of type $G(l,1,n)$ in the literature). Let l, n be positive integers, R be a field of arbitrary characteristic and consider $(q, Q_0, Q_1, \dots, Q_{l-1})$ an $l+1$-tuple of invertible elements in R such that $q \neq 1$. The Ariki–Koike algebra $\mathcal{H}_{R,n} := \mathcal{H}_{R,n}(q; Q_0, Q_1, \dots, Q_{l-1})$ over R is the unital associative R-algebra presented by:

- generators: $T_0, T_1, \dots, T_{n-1}$,
- relations:

$$\begin{aligned}
&T_0T_1T_0T_1 = T_1T_0T_1T_0,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (i = 1, \dots, n-2),\\
&T_iT_j = T_jT_i \quad \bigl(|j-i| > 1\bigr),\\
&(T_0 - Q_0)(T_0 - Q_1)\dots(T_0 - Q_{l-1}) = 0,\\
&(T_i - q)(T_i + 1) = 0 \quad (i = 1, \dots, n-1).
\end{aligned}$$

If $l = 1$ (respectively $l = 2$), we obtain a Hecke algebra of type A (respectively of type B). To study these algebras, by a theorem of Dipper and Mathas [9], it is sufficient to consider the case where $Q_0 = q^{s_0}$, $Q_1 = q^{s_1}$, $\dots$, $Q_{l-1} = q^{s_{l-1}}$ for $\mathbf{s} = (s_0, s_1, \dots, s_{l-1})$ an l-tuple of integers. The algebra $\mathcal{H}_{R,n}$ is then a cellular algebra in the sense of Graham and Lehrer [13]. As a consequence, we can define Specht modules which are indexed by the l-partitions of rank n. Recall that an l-partition $\boldsymbol{\lambda}$ of rank n is a sequence of l partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(l-1)})$ such that $\sum_{k=0}^{l-1} |\lambda^{(k)}| = n$. We denote by $\Pi_{l,n}$ the set of l-partitions of rank n. Hence, to each $\boldsymbol{\lambda} \in \Pi_{l,n}$ is associated a Specht module $S^{\boldsymbol{\lambda}}$. Let e be the multiplicative order of q (we put $e = \infty$ if q is not a root of unity). The Specht modules are in general not irreducible. Nevertheless, there exists a natural bilinear form, $\mathcal{H}_{R,n}$-invariant, on each of these modules and an associated radical such that the quotients $D^{\boldsymbol{\lambda}} := S^{\boldsymbol{\lambda}}/\operatorname{rad}(S^{\boldsymbol{\lambda}})$ are 0 or irreducible. Moreover, the nonzero $D^{\boldsymbol{\lambda}}$ provide a complete set of nonisomorphic simple modules. Let $\Phi_e^{\mathfrak{s}}(n)$ be the set of l-partitions such that $D^{\boldsymbol{\lambda}}$ is nonzero (where we set $\mathfrak{s} = (s_0 (\operatorname{mod} e), \dots, s_{l-1} (\operatorname{mod} e))$). It has been shown by Ariki [1] and Ariki and Mathas [4], that this set equals to the set of Kleshchev l-partitions. These l-partitions are generalizations of the e-regular partitions. They appear as the vertices of a distinguished realization of the abstract $A_{e-1}^{(1)}$-crystal $B_e(\Lambda_{\mathfrak{s}})$ associated to the irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module of highest weight $\Lambda_{\mathfrak{s}}$ (see Section 2.1).

Following [10], denote by $\widetilde{\mathcal{H}}_{R,n}$ the algebra $\mathcal{H}_{R,n}(q^{-1}; s_{l-1}, \dots, s_0)$ and write $\widetilde{T}_0, \dots, \widetilde{T}_{l-1}$ for the standard generators of $\widetilde{\mathcal{H}}_{R,n}$. If $\boldsymbol{\lambda} \in \Pi_{l,n}$, we write $\widetilde{S}^{\boldsymbol{\lambda}}$ for the corresponding Specht module on $\widetilde{\mathcal{H}}_{R,n}$ and $\widetilde{D}^{\boldsymbol{\lambda}}$ for the corresponding irreducible one. We have an isomorphism $\theta : \mathcal{H}_{R,n} \to \widetilde{\mathcal{H}}_{R,n}$ given by

$$T_0 \mapsto \widetilde{T}_0, \qquad T_i \mapsto -q\widetilde{T}_i \quad (i = 1, \dots, n-1).$$

Put $\widetilde{\mathfrak{s}} = (-s_{l-1} (\operatorname{mod} e), \dots, -s_0 (\operatorname{mod} e))$. Then, θ induces a functor F_l from the category of $\widetilde{\mathcal{H}}_{R,n}$-modules to the category of $\mathcal{H}_{R,n}$ modules. As a consequence, we obtain a bijective map

$$m_l : \Phi_e^{\mathfrak{s}} \to \Phi_e^{\widetilde{\mathfrak{s}}},$$

satisfying

$$F_l\bigl(\widetilde{D}^{m_l(\boldsymbol{\lambda})}\bigr) \simeq D^{\boldsymbol{\lambda}},$$

for all $\lambda \in \Phi_e^{\mathfrak{s}}$. This map can be viewed as a generalization of the Mullineux involution. In [10], Fayers describes this generalized Mullineux involution by using skew-isomorphisms of $A_{e-1}^{(1)}$-crystals. This description is similar to a former one obtained in [21] for the analogue of the Mullineux involution, namely the Zelevinsky involution, in affine Hecke algebras of type A. Both rest on the determination of paths in crystal graphs. The aim of this note is to give an alternative efficient description of the map m_l in the spirit of the original procedure of Mullineux, that is without using paths in crystals. We obtain an algorithm to compute the Mullineux involution for all positive integer l. This is achieved by combining the known description of the original Mullineux involution (i.e. for $l = 1$) with results of [17] on isomorphisms of $A_{e-1}^{(1)}$-crystals.

The following sections are structured as follows. Section 2 is devoted to a brief review on $A_{e-1}^{(1)}$-crystals and their labellings by Uglov and Kleshchev l-partitions. In Section 3, we mainly recall and reformulate in an appropriate language results of [16] and [17]. Section 4 contains the main result of this paper, namely the description of the map m_l on Kleshchev l-partitions.

2. $A_{e-1}^{(1)}$-crystals and the Mullineux involution

Let $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ be the quantum group of affine type $A_{e-1}^{(1)}$. This is an associative $\mathbb{Q}(v)$-algebra with generators e_i, f_i, t_i, t_i^{-1} (for $i = 0, \dots, e-1$) and ∂ and relations given in [24, §2.1]. We denote by $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$ the subalgebra generated by e_i, f_i, t_i, t_i^{-1} (for $i = 0, \dots, e-1$). We begin this section, by reviewing some background on the crystal graph theory of the irreducible highest weight $\mathcal{U}'_v(\widehat{\mathfrak{sl}_e})$-modules. In particular, we recall the notion of Uglov and Kleshchev l-partitions and describe Fayers work on the Mullineux involution. We refer to [18] and to [2] for the general theory of crystals. [12, §7] gives a nice survey on some of the problems we will consider.

2.1. Realizations of abstract $A_{e-1}^{(1)}$-crystals

By slightly abuse of notation, we identify the elements of $\mathbb{Z}/e\mathbb{Z}$ with their corresponding labels in $\{0, \dots, e-1\}$ when there is no risk of confusion. Let v be an indeterminate and $e > 1$ be an integer. Write $\{\Lambda_0, \dots, \Lambda_{e-1}, \delta\}$ for the set of dominant weights of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$. Let $l \geqslant 1 \in \mathbb{N}$ and consider $\mathbf{s} = (s_0, \dots, s_{l-1}) \in \mathbb{Z}^l$. We set $\mathfrak{s} = (s_0 (\operatorname{mod} e), \dots, s_{l-1} (\operatorname{mod} e)) \in (\mathbb{Z}/e\mathbb{Z})^l$ and $\Lambda_{\mathfrak{s}} := \Lambda_{s_0 (\operatorname{mod} e)} + \dots + \Lambda_{s_{l-1} (\operatorname{mod} e)}$.

As a $\mathbb{C}(v)$-vector space, the Fock space $\mathfrak{F}_e$ of level l admit the set of all l-partitions as a natural basis. Namely the underlying vector space is

$$\mathfrak{F}_e = \bigoplus_{n \geqslant 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{l,n}} \mathbb{C}(v) \boldsymbol{\lambda},$$

where $\Pi_{l,n}$ is the set of l-partitions with rank n. One can put different structures of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-modules on $\mathfrak{F}_e$. To describe them, we need some combinatorics.

Let $\boldsymbol{\lambda}$ be an l-partition (identified with its Young diagram). Then, the nodes of $\boldsymbol{\lambda}$ are the triplets $\gamma = (a, b, c)$ where $c \in \{0, \dots, l-1\}$ and a, b are respectively the row and column indices of the node γ in $\lambda^{(c)}$. The content of γ is the integer $c(\gamma) = b - a + s_c$ and the residue $\operatorname{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$\operatorname{res}(\gamma) \equiv c(\gamma) (\operatorname{mod} e). \tag{1}$$

We will say that γ is an i-node of $\boldsymbol{\lambda}$ when $\operatorname{res}(\gamma) \equiv i (\operatorname{mod} e)$. Finally, We say that γ is removable when $\gamma = (a, b, c) \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \backslash \{\gamma\}$ is an l-partition. Similarly γ is addable when $\gamma = (a, b, c) \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{\gamma\}$ is an l-partition.

We now define distinct total orders on the set of addable and removable i-nodes of the multipartitions. Consider $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ two i-nodes in $\boldsymbol{\lambda}$. We define an order $\prec_{\mathbf{s}}$ by setting

$$\gamma_1 \prec_{\mathbf{s}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2) & \text{or} \\ c(\gamma_1) = c(\gamma_2) & \text{and} \quad c_1 > c_2. \end{cases}$$

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two ℓ-partitions of rank n and $n+1$ and assume there exists an i-node γ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$. We define the following numbers:

$$\begin{aligned} N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \sharp\{\text{addable } i\text{-nodes } \gamma \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma\} \\ &\quad - \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ_{\mathbf{s}} \gamma\}, \\ N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu}) &= \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma\} \\ &\quad - \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec_{\mathbf{s}} \gamma\}, \\ N_i(\boldsymbol{\lambda}) &= \sharp\{\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\} \\ &\quad - \sharp\{\text{removable } i\text{-nodes of } \boldsymbol{\lambda}\} \quad \text{for } 0 \leqslant i \leqslant e-1 \\ \text{and} \quad M_0(\boldsymbol{\lambda}) &= \sharp\{0\text{-nodes of } \boldsymbol{\lambda}\}. \end{aligned}$$

The following theorem is [24, Theorem 2.1].

Theorem 2.1 *(Jimbo et al. [15], Foda et al. [11], Uglov [24]). Let $\mathbf{s} \in \mathbb{Z}^l$. Then we can put an action of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$ on $\mathcal{F}_e$ as follows*:

$$e_i \boldsymbol{\lambda} = \sum_{\operatorname{res}([\boldsymbol{\lambda}]/[\boldsymbol{\mu}])=i} q^{-N_i^{\succ}(\boldsymbol{\mu},\boldsymbol{\lambda})} \boldsymbol{\mu}, \qquad f_i \boldsymbol{\lambda} = \sum_{\operatorname{res}([\boldsymbol{\mu}]/[\boldsymbol{\lambda}])=i} q^{N_i^{\prec}(\boldsymbol{\lambda},\boldsymbol{\mu})} \boldsymbol{\mu},$$
$$t_i \boldsymbol{\lambda} = q^{N_i(\boldsymbol{\lambda})} \boldsymbol{\lambda}, \qquad \partial \boldsymbol{\lambda} = -\big(\Delta + M_0(\boldsymbol{\lambda})\big)\boldsymbol{\lambda} \quad (0 \leqslant i \leqslant e-1)$$

where Δ is a rational number defined in [24, *Theorem* 2.1] *which does not depend on $\boldsymbol{\lambda}$ (but depends on $\mathbf{s}$). The associated $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module which is denoted by $\mathcal{F}_e^{\mathbf{s}}$ is an integrable module.*

Now, one can define another structure of $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module on the Fock space $\mathcal{F}_e$. This is achieved by considering the following order on the set of i-nodes of an l-partition:

$$\gamma_1 \prec_{\mathfrak{s}} \gamma_2 \iff \begin{cases} c_1 > c_2 & \text{or} \\ c_1 = c_2 & \text{and} \quad c(\gamma_1) < c(\gamma_2). \end{cases}$$

Using the same formulas as in Theorem 2.1 where $\prec_{\mathbf{s}}$ is replaced by $\prec_{\mathfrak{s}}$ and $\Delta = 0$, the space $\mathcal{F}_e$ is endowed with a different structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module that we will denote by $\mathcal{F}_e^{\mathfrak{s}}$. We refer to [2, §10.2] for the proof of this assertion.

Observe that the order $\prec_{\mathbf{s}}$ and thus $\mathcal{F}_e^{\mathbf{s}}$ really depends on $\mathbf{s}$ whereas $\prec_{\mathfrak{s}}$ and thus $\mathcal{F}_e^{\mathfrak{s}}$ only depends on the class $\mathfrak{s} \in (\mathbb{Z}/e\mathbb{Z})^l$.

Remark 2.2.

(1) For $l = 1$, the Fock spaces $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathfrak{s}}$ coincide.
(2) With $\mathfrak{s} = (s_0 (\operatorname{mod} e), \dots, s_{l-1} (\operatorname{mod} e))$, we have

$$\mathfrak{F}_e^{\mathfrak{s}} = \mathfrak{F}_e^{s_0} \otimes \cdots \otimes \mathfrak{F}_e^{s_{l-1}}. \tag{2}$$

The Fock space $\mathfrak{F}_e^{\mathfrak{s}}$ is a tensor product of Fock spaces of level 1.

The $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-submodule of $\mathfrak{F}_e^{\mathbf{s}}$ generated by the empty l-partition is denoted by $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})$. By [24, Remark 2.2], this is an irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with weight $\Lambda_{\mathbf{s}} - \Delta\delta$. Similarly, the $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-submodule of $\mathfrak{F}_e^{\mathfrak{s}}$ generated by the empty l-partition is denoted by $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$. By [2, Theorem 10.10], this is an irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with weight $\Lambda_{\mathfrak{s}}$.

We rather need in the sequel the restrictions $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})'$ and $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})'$ of $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})$ and $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ to the subalgebra $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$. They are both irreducible as $\mathcal{U}_v'(\widehat{\mathfrak{sl}_e})$-modules of highest weight $\Lambda_{\mathfrak{s}}$. In particular, all the modules $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})'$, $\mathbf{s} \in \mathfrak{s}$ are isomorphic to $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})'$. However, as explained above, the corresponding actions of the Chevalley operators on these modules do not coincide in general.

The modules $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathfrak{s}}$ are integrable. Hence, by the general theory of Kashiwara crystal bases, $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathfrak{s}}$ have crystal graphs that we denote $B_e^{\mathbf{s}}$ and $B_e^{\mathfrak{s}}$, respectively. Since the module structures $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathfrak{s}}$ depend on the total orders $\prec_{\mathbf{s}}$ and $\prec_{\mathfrak{s}}$, the graph structures on $B_e^{\mathbf{s}}$ and $B_e^{\mathfrak{s}}$ do not coincide in general although they both admit the set of l-partitions as set of vertices.

To describe the graph structure of $B_e^{\mathbf{s}}$ (for a fixed $\mathbf{s} \in \mathfrak{s}$) we proceed as follows. Starting from any l-partition $\boldsymbol{\lambda}$, we can consider its set of addable and removable i-nodes. Let w_i be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\prec_{\mathbf{s}}$ next by encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\widetilde{w}_i = A^p R^q$ for the word derived from w_i by deleting as many of the factors RA as possible. If $p > 0$, let γ be the rightmost addable i-node in $\widetilde{w}_i$. When $\widetilde{w}_i \neq \emptyset$, the node γ is called the good i-node. Now, the crystal $B_e^{\mathbf{s}}$ is the graph with

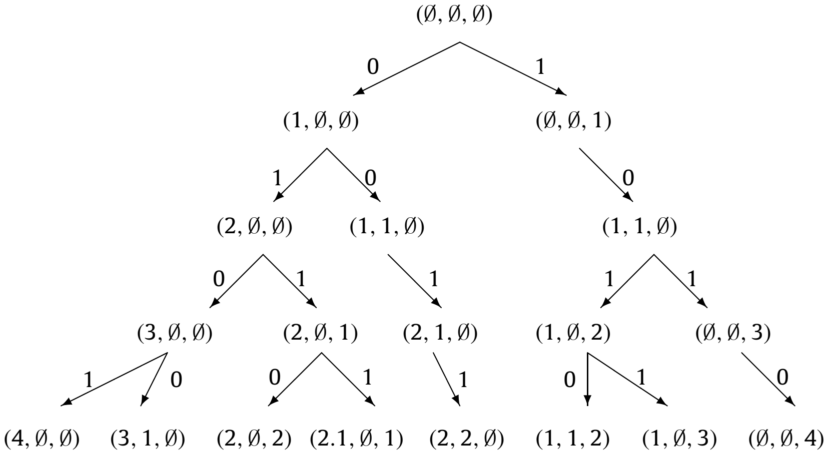
- vertices: the l-partitions,
- edges: $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is obtained by adding to $\boldsymbol{\lambda}$ a good i-node.

The graph structure on and $B_e^{\mathfrak{s}}$ is obtained similarly by using the order $\prec_{\mathfrak{s}}$ instead of $\prec_{\mathbf{s}}$.

2.2. Uglov and Kleshchev multipartitions

By definition $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})$ and $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ are the irreducible components with highest weight vector $\emptyset$ in $\mathfrak{F}_e^{\mathbf{s}}$ and $\mathfrak{F}_e^{\mathfrak{s}}$, their crystal graphs $B_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})$ and $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ (which are also the crystal graphs of $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})'$ and $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})'$) can be realized respectively as the connected components of highest weight vertex $\emptyset$ in $B_e^{\mathbf{s}}$ and $B_e^{\mathfrak{s}}$.

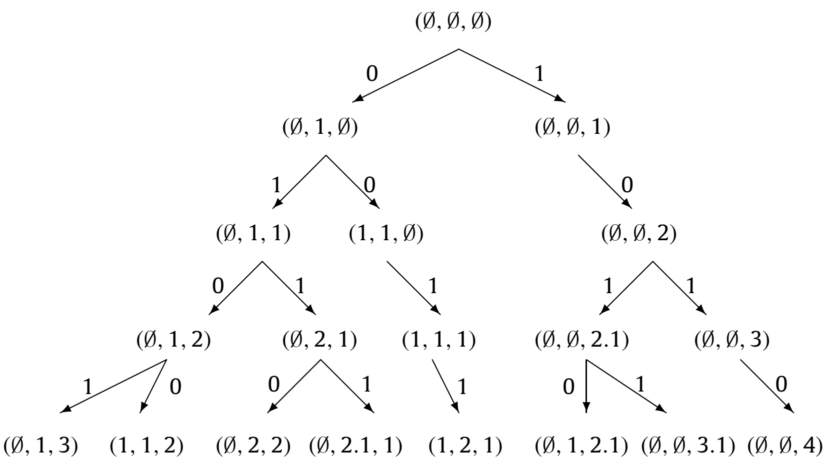
Example 2.3. The graph below is the subgraph of $B_4^{(0,0,1)}(2\Lambda_0 + \Lambda_1)$ containing the 3-partitions of rank less or equal to 4.



We denote by $\Phi_e^{\mathbf{s}}$ the set of vertices in $B_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})$. The elements of $\Phi_e^{\mathbf{s}}$ are called the Uglov l-partitions. We also write $\Phi_e^{\mathbf{s}}(n)$ for the subset of $\Phi_e^{\mathbf{s}}$ containing the Uglov l-partitions with rank n. In general the set $\Phi_e^{\mathbf{s}}$ is very difficult to describe without computing the underlying crystal. Nevertheless, in the case where $0 \leqslant s_0 \leqslant \cdots \leqslant s_{l-1} \leqslant e-1$, Foda et al. have given a simple nonrecursive description of the Uglov l-partitions (see [11, Theorem 2.10]).

We denote by $\Phi_e^{\mathfrak{s}}$ the set of vertices in $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$. The elements of $\Phi_e^{\mathfrak{s}}$ are called the Kleshchev l-partitions. We define $\Phi_e^{\mathfrak{s}}(n)$ as we have done in the Uglov case. Note that there exists only a recursive description of the Kleshchev l-partitions $\Phi_e^{\mathfrak{s}}$, except when the number of fundamental weights appearing in the decomposition of $\Lambda_{\mathfrak{s}}$ is less or equal to 2, see [3] and [5]. In the other cases, they can only be obtained by computing the underlying crystal or using the procedure described in [17, §5.4].

Example 2.4. The graph below is the subgraph of $B_4^{\mathfrak{s}}(2\Lambda_0 + \Lambda_1)$ with $\mathfrak{s} = (0(\mathrm{mod}\,4), 0(\mathrm{mod}\,4), 1(\mathrm{mod}\,4))$ containing the 3-partitions of rank less than 4.



Let $\mathbf{s}' := (s'_0, \ldots, s'_{l-1}) \in \mathbb{Z}^l$ be such that $\mathbf{s}' \in \mathfrak{s}$. Then the two modules $V_e^{\mathbf{s}}(\Lambda_{\mathfrak{s}})'$ and $V_e^{\mathbf{s}'}(\Lambda_{\mathfrak{s}})$ are both isomorphic to $V_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})'$. Hence the corresponding three crystal graphs are isomorphic. These crystal isomorphisms define the following bijections

$$\Psi_{e,n}^{\mathbf{s}\to\mathbf{s}'} : \Phi_e^{\mathbf{s}}(n) \to \Phi_e^{\mathbf{s}'}(n),$$
$$\Psi_{e,n}^{\mathbf{s}\to\mathfrak{s}} : \Phi_e^{\mathbf{s}}(n) \to \Phi_e^{\mathfrak{s}}(n).$$

As it can be easily checked on the previous examples, the sets $\Phi_e^{\mathbf{s}}(n)$ and $\Phi_e^{\mathfrak{s}}(n)$ do not coincide in general. However, if we fix n, it is possible to realize the set $\Phi_e^{\mathfrak{s}}(n)$ of Kleshchev l-partitions with rank n as a special set $\Phi_e^{\mathbf{s}_\infty}(n)$, $\mathbf{s}_\infty \in \mathfrak{s}$ of Uglov l-partitions. We refer the reader to Lemma 5.4.1 in [17] for the proof of the following proposition.

Proposition 2.5. *Let $n \in \mathbb{N}$ and consider $\mathbf{s}_\infty := (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$ such that*

$$s_{i+1} - s_i > n - 1 \tag{3}$$

for all $i = 0, \ldots, l-2$. Then $\Psi_{e,n}^{\mathbf{s}_\infty\to\mathfrak{s}}$ is the identity. Hence, we have:

$$\Phi_e^{\mathfrak{s}}(n) = \Phi_e^{\mathbf{s}_\infty}(n).$$

In particular, $\Phi_e^{\mathbf{s}_\infty}(n)$ does not depend on the multicharge $\mathbf{s}_\infty$ provided (3) *is satisfied.*

This shows that the description of the bijections $\Psi_{e,n}^{\mathbf{s}\to\mathbf{s}'}$ for all $\mathbf{s},\mathbf{s}'\in\mathfrak{s}$ will lead that of $\Psi_{e,n}^{\mathbf{s}\to\mathfrak{s}}$ as a special case. A multicharge verifying (3) is called asymptotic (this is called n-dominant in [25]). This justifies the notation $\Phi_e^{\mathbf{s}_\infty}(n)$ we have adopted.

3. Isomorphisms of $A_{e-1}^{(1)}$-crystals

We now review some results obtained in [16] and [17] making explicit the bijections $\Psi_{e,n}^{\mathbf{s}\to\mathbf{s}'}$. In [17], we have chosen to explain our results in the language of column shaped tableaux to make apparent the link with the combinatorics of $\mathfrak{sl}_n$. Here, we rather adopt the equivalent language of symbols also used in [16] which permits to give a very simple description of the elementary steps involved in our algorithms.

3.1. Elementary transformations in $\widehat{S}_l$

We write as usual $\widehat{S}_l$ for the extended affine symmetric group in type A_{l-1}. This group is generated by the elements $\sigma_1,\dots,\sigma_{l-1}$ and $y_0,\dots,y_{l-1}$ together with the relations

$$\sigma_c\sigma_{c+1}\sigma_c=\sigma_{c+1}\sigma_c\sigma_{c+1},\qquad \sigma_c\sigma_d=\sigma_d\sigma_c\quad\text{for } |c-d|>1,\ \sigma_c^2=1,$$

with all indices in $\{1,\dots,l-1\}$ and

$$y_cy_d=y_dy_c,\qquad \sigma_cy_d=y_d\sigma_c\quad\text{for } d\neq c,c+1,\qquad \sigma_cy_c\sigma_c=y_{c+1}.$$

For any $c\in\{0,\dots,l-1\}$, we set $z_c=y_1\cdots y_c$. Write also $\xi=\sigma_{l-1}\cdots\sigma_1$ and $\tau=y_l\xi$. Since $y_c=z_{c-1}^{-1}z_c$, $\widehat{S}_l$ is generated by the transpositions σ_c with $c\in\{1,\dots,l-1\}$ and the elements z_c with $c\in\{1,\dots,l\}$. Observe that for any $c\in\{1,\dots,l-1\}$, we have

$$z_c=\xi^{l-c}\tau^c. \tag{4}$$

This implies that $\widehat{S}_l$ is generated by the transpositions σ_c with $c\in\{1,\dots,l-1\}$ and τ. Consider e a fixed positive integer. We obtain a faithful action of $\widehat{S}_l$ on $\mathbb{Z}^l$ by setting for any $\mathbf{s}=(s_0,\dots,s_{l-1})\in\mathbb{Z}^l$

$$\sigma_c(\mathbf{s})=(s_0,\dots,s_c,s_{c-1},\dots,s_{l-1})\quad\text{and}\quad y_c(\mathbf{s})=(s_0,\dots,s_{c-1},s_c+e,\dots,s_{l-1}).$$

Then $\tau(\mathbf{s})=(s_1,s_2,\dots,s_{l-1},s_0+e)$.

Remark 3.1. From the preceding considerations, $\{s_1,\dots,s_{l-1},\tau\}$ is a minimal set of generators for $\widehat{S}_l$. Hence, the bijections $\Psi_{e,n}^{\mathbf{s}\to\mathbf{s}'}:\Phi_e^{\mathbf{s}}(n)\to\Phi_e^{\mathbf{s}'}(n)$ with $\mathbf{s}\in\mathfrak{s}$ and $\Psi_{e,n}^{\mathbf{s}\to\mathfrak{s}}:\Phi_e^{\mathbf{s}}(n)\to\Phi_e^{\mathfrak{s}}(n)$ can be obtained by composing bijections corresponding to the cases $\mathbf{s}'=\tau(\mathbf{s})$ and $\mathbf{s}'=\sigma_c(\mathbf{s})$ with $c=1,\dots,l-1$.

3.2. Elementary crystals isomorphisms

The following proposition is stated in [17, Proposition 5.2.1]. We give the proof for the convenience of the reader.

Proposition 3.2. *Let $\mathbf{s}\in\mathbb{Z}^l$ and $e\in\mathbb{N}$. Let $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(l-1)})\in\Phi_e^{\mathbf{s}}(n)$. Then*

$$\Psi_{e,n}^{\mathbf{s}\to\tau(\mathbf{s})}=\bigl(\lambda^{(1)},\dots,\lambda^{(l-1)},\lambda^{(0)}\bigr).$$

Proof. Set $\mathbf{s}=(s_0,\dots,s_{\ell-1})$. Then $\tau(\mathbf{s})=(s_1,\dots,s_{\ell-1},s_0+e)$. Consider $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(\ell-1)})$ a multipartition and set $\boldsymbol{\lambda}^{\diamondsuit}=(\lambda^{(1)},\dots,\lambda^{(\ell-1)},\lambda^{(0)})$. Consider $i\in\{0,1,\dots,e-1\}$ and $\gamma_1=(a_1,b_1,c_1)$, $\gamma_2=(a_2,b_2,c_2)$ two i-nodes of $\boldsymbol{\lambda}$. Then $\gamma_1^{\diamondsuit}=(a_1,b_1,c_1-1(\operatorname{mod} e))$ and $\gamma_2^{\diamondsuit}=(a_2,b_2,c_2-1(\operatorname{mod} e))$ are two i-nodes of $\boldsymbol{\lambda}^{\diamondsuit}$. We then easily check that $\gamma_2\prec_{\mathbf{s}}\gamma_1$ if and only if $\gamma_2^{\diamondsuit}\prec_{\tau(\mathbf{s})}\gamma_1'$. This implies that $\Psi_{e,n}^{\mathbf{s}\to\tau(\mathbf{s})}(\boldsymbol{\lambda})=\boldsymbol{\lambda}^{\diamondsuit}$. $\square$

The description of the bijections $\Psi_{e,n}^{\mathbf{s}\to\sigma_c(\mathbf{s})}$, $c=1,\dots,l-1$ are more complicated. It essentially rests on the following basic procedure on pairs (U,D) of strictly increasing sequences of integers. Let $U=[x_1,\dots,x_r]$ and $D=[y_1,\dots,y_s]$ two strictly increasing sequences of integers. We compute from (U,D) a new pair (U',D') with $U'=[x_1',\dots,x_r']$ and $D'=[y_1',\dots,y_s']$ of such sequences by applying the following algorithm:

Algorithm.

- Assume $r\geqslant s$. We associate to y_1 the integer $x_{i_1}\in U$ such that

$$x_{i_1}=\begin{cases}\max\{x\in U\mid y_1\geqslant x\} & \text{if } y_1\geqslant x_1,\\ x_r & \text{otherwise.}\end{cases}\tag{5}$$

  We repeat the same procedure to the ordered pair $(U\setminus\{x_{i_1}\},D\setminus\{y_1\})$. By induction this yields a subset $\{x_{i_1},\dots,x_{i_s}\}\subset U$. Then we define D' as the increasing reordering $\{x_{i_1},\dots,x_{i_s}\}$ and U' as the increasing reordering of $U\setminus\{x_{i_1},\dots,x_{i_s}\}\sqcup D$.
- Assume $r<s$. We associate to x_1 the integer $y_{i_1}\in D$ such that

$$y_{i_1}=\begin{cases}\min\{y\in D\mid x_1\leqslant y\} & \text{if } x_1\leqslant y_s,\\ y_1 & \text{otherwise.}\end{cases}\tag{6}$$

  We repeat the same procedure to the ordered sequences $U\setminus\{x_1\}$ and $D\setminus\{y_{i_1}\}$ and obtain a subset $\{y_{i_1},\dots,y_{i_r}\}\subset D$. Then we define U' as the increasing reordering $\{y_{i_1},\dots,y_{i_r}\}$ and D' as the increasing reordering of $D\setminus\{y_{i_1},\dots,y_{i_r}\}\sqcup U$.

Example 3.3. Take $U=[0,1,3]$ and $D=[0,1,2,4,7]$. The subset of U determined by the previous algorithm is $\{0,1,4\}$. This gives $U'=[0,1,4]$ and $D'=[0,1,2,3,7]$.

Now consider $c\in\{1,\dots,e-1\}$ and $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(l-1)})\in\Phi_e^{\mathbf{s}}(n)$. In order to compute $\Psi_{e,n}^{\mathbf{s}\to\sigma_c(\mathbf{s})}(\boldsymbol{\lambda})$, we need the symbol S_c of the bipartition $(\lambda^{(c-1)},\lambda^{(c)})$. Let d_{c-1} and d_c be the number of nonzero parts in $\lambda^{(c-1)}$ and $\lambda^{(c)}$ and put $m=\max(d_{c-1}-s_{c-1},d_c-s_c)+1$. Set

$$\begin{cases}\beta_i^{(c-1)}=\lambda_i^{(c-1)}-i+s_{c-1}+m & \text{for any } i=1,\dots,m+s_{c-1},\\ \beta_i^{(c)}=\lambda_i^{(c)}-i+s_c+m & \text{for any } i=1,\dots,m+s_c\end{cases}$$

where, by convention the partitions $\lambda^{(c-1)}$ and $\lambda^{(c)}$ are taken with an infinite number of zero parts. The symbol S_c is the two-row tableau

$$S_c=\begin{pmatrix}\beta_{m+s_c}^{(c)} & \beta_{m+s_c-1}^{(c)} & \cdots & \cdots & \cdots & \beta_1^{(c)}\\ \beta_{m+s_{c-1}}^{(c-1)} & \beta_{m+s_{c-1}-1}^{(c-1)} & \cdots & \cdots & \beta_1^{(c-1)} & \end{pmatrix}=\begin{pmatrix}U\\ D\end{pmatrix}.$$

Observe that $\beta_i^{(k)},k\in\{c-1,c\}$ is nothing but the content of the node $\gamma=(i,\lambda_i^{(k)},k)\in\boldsymbol{\lambda}$ translated by m. This translation normalizes the symbol S_c so that $\beta_{m+s_c}^{(c)}=\beta_{m+s_{c-1}}^{(c-1)}=0$. In fact, for a fixed pair (s_{c-1},s_c) and m as above, the map $\Sigma_{(s_{c-1},s_c)}$ which associates to each bipartition its symbol is bijective. It is easy, from a symbol S_c, to recover the unique bipartition $(\lambda^{(c-1)},\lambda^{(c)})$ such that S_c is the symbol of $(\lambda^{(c-1)},\lambda^{(c)})$. We are now ready to describe the bijection $\Psi_{e,n}^{\mathbf{s}\to\sigma_c(\mathbf{s})}:\Phi_e^{\mathbf{s}}(n)\to\Phi_e^{\sigma_c(\mathbf{s})}(n)$.

Let $\mathbf{s}\in\mathbb{Z}^l$ and $e\in\mathbb{N}$. Consider $\boldsymbol{\lambda}=(\lambda^{(0)},\dots,\lambda^{(l-1)})\in\Phi_e^{\mathbf{s}}(n)$ and $c\in\{1,\dots,l-1\}$. Write $S_c=\binom{U}{D}$ for the symbol corresponding to the bipartition $(\lambda^{(c-1)},\lambda^{(c)})$. Denote by $\widetilde{S}_c=\binom{\widetilde{U}}{\widetilde{D}}$ the symbol obtained by applying to S_c the previous algorithm. Then we compute the bipartition $(\widetilde{\lambda}^{(c-1)},\widetilde{\lambda}^{(c)})=\Sigma^{-1}_{(s_{c-1},s_c)}(\widetilde{S}_c)$. Finally we consider the l-partition

$$\widetilde{\boldsymbol{\lambda}}=\left(\lambda^{(0)},\dots,\widetilde{\lambda}^{(c)},\widetilde{\lambda}^{(c-1)},\dots,\lambda^{(l-1)}\right)$$

obtained by replacing in $\boldsymbol{\lambda}$, $\lambda^{(c-1)}$ by $\widetilde{\lambda}^{(c)}$ and $\lambda^{(c)}$ by $\widetilde{\lambda}^{(c-1)}$.

Proposition 3.4. *With the above notation, we have*

$$\Psi_{e,n}^{\mathbf{s}\to\sigma_c(\mathbf{s})}(\boldsymbol{\lambda})=\widetilde{\boldsymbol{\lambda}}.$$

Proof. The above algorithm is the translation in terms of symbols of Proposition 5.2.2 in [17]. □

Example 3.5. Let $e=3$ and let $\mathbf{s}=(0,2,0)$. Then one can check that the 3-partition $(4.1,3.1,1)$ belongs to $\Phi_e^{\mathbf{s}}(10)$. We want to determine the 3-partition $\Psi_{e,n}^{\mathbf{s}\to\sigma_2(\mathbf{s})}(4.1,3.1,1)$. We first write the symbol S_2 of $(3.1,1)$ with respect to $(2,0)$ and $m=3$:

$$S_2=\begin{pmatrix}0&1&3&&\\0&1&2&4&7\end{pmatrix}.$$

By using the previous example, we obtain

$$\widetilde{S}_2=\begin{pmatrix}0&1&4&&\\0&1&2&3&7\end{pmatrix}.$$

This thus gives $\widetilde{\lambda}^{(2)}=(2)$ and $\widetilde{\lambda}^{(1)}=(3)$. Hence $\Psi_{e,n}^{\mathbf{s}\to\sigma_2(\mathbf{s})}(4.1,3.1.1)=(4.1,2,3)$.

4. Mullineux involution for Kleshchev *l*-partitions

4.1. Mullineux involution as a skew crystal isomorphism

We here keep the setting of the introduction and review a result of Fayers [10] which shows how the Mullineux involution on Kleshchev l-partitions can be computed by using paths in crystal graphs. The generalized Mullineux involution can be regarded as a bijection

$$m_l:\Phi_e^{\mathfrak{s}}\to\Phi_e^{\widetilde{\mathfrak{s}}}$$

where $\mathfrak{s}=(s_0(\text{mod }e),\dots,s_{l-1}(\text{mod }e))$ and $\widetilde{\mathfrak{s}}=(-s_{l-1}(\text{mod }e),\dots,-s_0(\text{mod }e))$.

Theorem 4.1 *(Fayers [10]). Let $\boldsymbol{\lambda}\in\Phi_e^{\mathfrak{s}}$ and consider in $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ a path*

$$\boldsymbol{\emptyset}\xrightarrow{i_1}\cdot\xrightarrow{i_2}\cdot\xrightarrow{i_3}\cdots\xrightarrow{i_n}\boldsymbol{\lambda}$$

from the empty l-partition to $\boldsymbol{\lambda}$. There exists a corresponding path

$$\emptyset \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \cdots \xrightarrow{-i_n} \boldsymbol{\mu}$$

in $B_e^{\widetilde{\mathfrak{s}}}(\Lambda_{\mathfrak{s}})$ from the empty l-partition to an l-partition $\boldsymbol{\mu} \in \Phi_e^{\widetilde{\mathfrak{s}}}$. We have then

$$m_l(\boldsymbol{\lambda}) = \boldsymbol{\mu}.$$

In the level 1 case (i.e. for $l = 1$), the map m_1 can be described independently of the notion of crystal graph by using an algorithm originally due to Mullineux. We refer for example to [14] for a complete exposition of this procedure. For all $s \in \mathbb{Z}$, it gives the map

$$m_1 : \Phi_e^{\mathfrak{s}} \to \Phi_e^{-\mathfrak{s}}$$

where $\mathfrak{s} = s(\operatorname{mod} e)$ Our aim is now to show how the generalized Mullineux involution (i.e. in level $l > 1$) can be computed from the map m_1 and the results of Section 3. For any $\mathfrak{s} = (s_0(\operatorname{mod} e), \ldots, s_{l-1}(\operatorname{mod} e)) \in (\mathbb{Z}/e\mathbb{Z})^l$, set

$$\begin{cases} -\mathfrak{s} = \big(-s_0(\operatorname{mod} e), \ldots, -s_{l-1}(\operatorname{mod} e)\big), \\ \widetilde{\mathfrak{s}} = \big(-s_{l-1}(\operatorname{mod} e), \ldots, -s_0(\operatorname{mod} e)\big). \end{cases}$$

Our strategy is as follows. Let $\boldsymbol{\lambda}$ be in $\Phi_e^{\mathfrak{s}}$ and consider a path from the empty l-partition to $\boldsymbol{\lambda}$ in $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ with arrows successively labeled by $i_1, \ldots, i_n$.

(1) We first describe the l-partition $\boldsymbol{\nu}$ in $\Phi_e^{-\mathfrak{s}}$ defined by considering the path

$$\emptyset \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \cdots \xrightarrow{-i_n} \boldsymbol{\nu}$$

in $B_e^{-\mathfrak{s}}(\Lambda_{-\mathfrak{s}})$.

(2) Next, we describe the bijection $\Phi_e^{-\mathfrak{s}} \to \Phi_e^{\widetilde{\mathfrak{s}}}$ defined by the crystal isomorphism between $B_e^{-\mathfrak{s}}(\Lambda_{-\mathfrak{s}})$ and $B_e^{\widetilde{\mathfrak{s}}}(\Lambda_{-\mathfrak{s}})$. This permits to compute $\boldsymbol{\mu}$ from $\boldsymbol{\nu}$.

4.2. Computing $\boldsymbol{\nu}$ from $\boldsymbol{\lambda}$

Proposition 4.2. *Let $\boldsymbol{\lambda} \in \Phi_e^{\mathfrak{s}}$ and consider in $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ a path*

$$\emptyset \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdot \xrightarrow{i_3} \cdots \xrightarrow{i_n} \boldsymbol{\lambda}$$

from the empty l-partition to $\boldsymbol{\lambda}$. Then there exists a corresponding path

$$\emptyset \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \cdots \xrightarrow{-i_n} \boldsymbol{\nu}$$

in $B_e^{-\mathfrak{s}}(\Lambda_{-\mathfrak{s}})$ from the empty l-partition to an l-partition $\boldsymbol{\nu} \in \Phi_e^{-\mathfrak{s}}$. We have then

$$\boldsymbol{\nu} = \big(m_1(\lambda^0), \ldots, m_1(\lambda^{l-1})\big).$$

Proof. We know by (2) that the Fock spaces $\mathfrak{F}_e^{\mathfrak{s}}$ and $\mathfrak{F}_e^{-\mathfrak{s}}$ can be regarded as tensor products of level 1 Fock spaces. Therefore, each l-partition $\mathbf{b} = (b^{(0)}, \ldots, b^{(l-1)})$ in the crystals $B_e^{\mathfrak{s}}$ and $B_e^{-\mathfrak{s}}$ of the Fock spaces $\mathfrak{F}_e^{\mathfrak{s}}$ and $\mathfrak{F}_e^{-\mathfrak{s}}$ can be written $\mathbf{b}= b^{(0)} \otimes \cdots \otimes b^{(l-1)}$. Recall that in Kashiwara crystal basis theory, the action of each crystal operator $\widetilde{e}_i, \widetilde{f}_i$, $i \in \{0, \ldots, l-1\}$ on $\mathbf{b} = b_0 \otimes \cdots \otimes b_{l-1}$ is determined by the integers $\varepsilon_i(b_k) = \max\{r \in \mathbb{N} \mid \widetilde{e}_i^r(b_k) \neq 0\}$ and $\varphi_i(b_k) = \max\{r \mid \widetilde{f}_i^r(b_k) \neq 0\}$ when k runs over $\{0, \ldots, l-1\}$.

We now proceed by induction on n. For $n=0$, the proposition is clear. Assume that the result is proved for $n-1$ and set $\boldsymbol{\lambda}=\widetilde{f}_{i_n}\cdots\widetilde{f}_{i_1}(\boldsymbol{\emptyset})$ in $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$. Consider $\boldsymbol{\lambda}_\flat=\widetilde{f}_{i_{n-1}}\cdots\widetilde{f}_{i_1}(\boldsymbol{\emptyset})\in B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ and $\boldsymbol{\nu}_\flat=\widetilde{f}_{-i_{n-1}}\cdots\widetilde{f}_{-i_1}(\boldsymbol{\emptyset})\in B_e^{-\mathfrak{s}}(\Lambda_{-\mathfrak{s}})$. By induction, we have then $\boldsymbol{\nu}_\flat=(m_1(\lambda_\flat^{(0)}),\dots,m_1(\lambda_\flat^{(l-1)}))$. The Mullineux involution m_1 switches the signs of the arrows. Thus $\varepsilon_i(m_1(\lambda_\flat^{(k)}))=\varepsilon_{-i}(\lambda_\flat^{(k)})$ and $\varphi_i(m_1(\lambda_\flat^{(k)}))=\varphi_{-i}(\lambda_\flat^{(k)})$ for any $k=0,\dots,l-1$. Since $\widetilde{f}_{i_n}(\boldsymbol{\lambda}_\flat)=\boldsymbol{\lambda}\neq 0$, we have $\widetilde{f}_{i_n}(\boldsymbol{\nu}_\flat)=\boldsymbol{\nu}\neq 0$ by the above arguments. Moreover $\widetilde{f}_{i_n}$ and $\widetilde{f}_{-i_n}$ act on the same component of $\boldsymbol{\lambda}_\flat$ and $\boldsymbol{\nu}_\flat$ considered as tensor products. Namely, there exists an integer $a\in\{0,\dots,l-1\}$ such that

$$\begin{cases}\boldsymbol{\lambda}=\big(\lambda_\flat^{(0)},\dots,\widetilde{f}_{i_n}\big(\lambda_\flat^{(a)}\big),\dots,\lambda_\flat^{(l-1)}\big),\\ \boldsymbol{\nu}=\big(m_1\big(\lambda_\flat^{(0)}\big),\dots,\widetilde{f}_{-i_n}\big(m_1\big(\lambda_\flat^{(a)}\big)\big),\dots,m_1\big(\lambda_\flat^{(l-1)}\big)\big).\end{cases}$$

We have $m_1(\widetilde{f}_{i_n}(\lambda_\flat^{(a)}))=\widetilde{f}_{-i_n}(m_1(\lambda_\flat^{(a)}))$ since m_1 switches the sign of each arrow. This gives $\boldsymbol{\nu}=(m_1(\lambda^0),\dots,m_1(\lambda^{l-1}))$ which establishes the proposition by induction. $\square$

4.3. Computing μ from ν

Now, assume we have computed the Kleshchev l-partition $\boldsymbol{\nu}\in\Phi_e^{-\mathfrak{s}}$ from the Kleshchev l-partition $\boldsymbol{\lambda}\in\Phi_e^{\mathfrak{s}}$ as in the previous section. To compute $\boldsymbol{\mu}\in\Phi_e^{\widetilde{\mathfrak{s}}}$ from the required crystal isomorphism, we need to realize $\boldsymbol{\nu}$ as an Uglov l-partition. To do this, we consider an asymptotic multicharge $-\mathbf{s}\in-\mathfrak{s}$. The multicharge $\mathfrak{s}$ can be written $\mathfrak{s}=(s_0(\operatorname{mod} e),\dots,s_{l-1}(\operatorname{mod} e))$ with s_c in $\{0,\dots,e-1\}$ for any $c=0,\dots,l-1$. Now for any $c=1,\dots,l-1$, write p_c for the minimal nonnegative integer such that

$$s_{c-1}-s_c+p_ce>n-1.$$

Put then:

$$-\mathbf{s}:=\left(-s_0,-s_1+p_1e,-s_2+p_1e+p_2e,\dots,-s_{l-1}+\sum_{i=1}^{l-1}p_ie\right).\tag{7}$$

Since $-\mathbf{s}$ is asymptotic, we have $\boldsymbol{\nu}\in\Phi_e^{-\mathbf{s}}$ by Proposition 2.5 and

$$\Psi_{e,n}^{-\mathbf{s}\to-\mathfrak{s}}(\boldsymbol{\nu})=\boldsymbol{\nu}.$$

Now, with the notation of Section 2.1 for the generators of the affine Weyl group of type A, we set

$$w_{(c,d)}=\sigma_c\sigma_{c-1}\dots\sigma_d\quad\text{for any } c\geqslant d>0.$$

Given $p\in\{1,\dots,l-1\}$, we also introduce

$$\gamma_p=w_{(l-p,1)}\dots w_{(l-2,p-1)}w_{(l-1,p)}.$$

Note that for $\mathbf{v}:=(v_0,\dots,v_{l-1})\in\mathbb{Z}^l$ and $p=1,\dots,l-1$, we have:

$$\gamma_p.\mathbf{v}=(v_p,\dots,v_{l-1},v_0,\dots,v_{p-1}).$$

For each $p=1,\dots,l-1$, set

$$\alpha_p=\tau^{l-p}\gamma_p.$$

We have thus, for $\mathbf{v} = (v_0, \dots, v_{l-1}) \in \mathbb{Z}^l$

$$\alpha_p.\mathbf{v} = (v_0, \dots, v_{p-1}, v_p + e, \dots, v_{l-1} + e).$$

Finally, let $w_0 = w_{(1,1)} w_{(2,1)} \dots w_{(l-1,1)}$ be the longest element of the symmetric group S_l and put

$$\eta := \alpha_1^{2(p_{l-1}+1)} \dots \alpha_{l-1}^{2(p_1+1)} w_0. \tag{8}$$

Proposition 4.3. *Consider the multicharge $-\mathbf{s}$ of (7) and set*

$$\widetilde{\mathbf{s}} = (\widetilde{s_0}, \dots, \widetilde{s_{l-1}}) := \eta.(-\mathbf{s}). \tag{9}$$

Then for any $i = 0, \dots, l-1$, we have

$$\widetilde{s_i} \equiv -s_{l-i-1} (\operatorname{mod} e).$$

Therefore $\widetilde{\mathbf{s}}$ belongs to $\widetilde{\mathfrak{s}}$. Moreover, the multicharge $\widetilde{\mathbf{s}}$ is asymptotic.

Proof. By definition of $\widetilde{\mathbf{s}} = (\widetilde{s_0}, \dots, \widetilde{s_{l-1}})$, we have for all $d = 0, \dots, l-2$:

$$\widetilde{s_{d+1}} - \widetilde{s_d} = -s_{l-d-2} + s_{l-d-1} + p_{l-d-1}e + 2e. \tag{10}$$

Now, since the integers s_c, $c = 0, \dots, \ell - 1$ belong to $\{0, \dots, e-1\}$, we can write

$$-s_{l-d-2} + s_{l-d-1} + 2e \geqslant s_{l-d-2} - s_{l-d-1}.$$

Hence, we derive from (10) and the definition of the integers p_d that

$$\widetilde{s_{d+1}} - \widetilde{s_d} \geqslant s_{l-d-2} - s_{l-d-1} + p_{l-d-1}e > n-1.$$

Thus, the multicharge $\eta.(\mathbf{s})$ is asymptotic which proves our proposition. □

Recall that we have a Kleshchev l-partition $\boldsymbol{\nu} \in \Phi_e^{-\mathfrak{s}}$. Using Section 3, we can compute

$$\widehat{\boldsymbol{\nu}} := \Psi_{e,n}^{-\mathbf{s} \to \widetilde{\mathbf{s}}}(\boldsymbol{\nu})$$

where $\widetilde{\mathbf{s}}$ is defined by (9). Since $\widetilde{\mathbf{s}}$ is asymptotic and belongs to $\widetilde{\mathfrak{s}}$, we derive from Proposition 2.5

$$\Psi_{e,n}^{\widetilde{\mathbf{s}} \to \widetilde{\mathfrak{s}}}(\widehat{\boldsymbol{\nu}}) = \widehat{\boldsymbol{\nu}}.$$

Let us summarize the transformations we have computed from $\boldsymbol{\lambda} \in \Phi_e^{\mathfrak{s}}$.

$$\boldsymbol{\lambda} \in \Phi_e^{\mathfrak{s}} \xrightarrow{\Psi_{e,n}^{\mathfrak{s} \to -\mathfrak{s}}} \boldsymbol{\nu} \in \Phi_e^{-\mathfrak{s}} \xrightarrow{\Psi_{e,n}^{-\mathfrak{s} \to -\mathbf{s}}} \boldsymbol{\nu} \in \Phi_e^{-\mathbf{s}} \xrightarrow{\Psi_{e,n}^{-\mathbf{s} \to \widetilde{\mathbf{s}}}} \widehat{\boldsymbol{\nu}} \in \Phi_e^{\widetilde{\mathbf{s}}} \xrightarrow{\Psi_{e,n}^{-\widetilde{\mathbf{s}} \to \widetilde{\mathfrak{s}}}} \widehat{\boldsymbol{\nu}} \in \Phi_e^{\widetilde{\mathfrak{s}}} \tag{11}$$

Recall now that we have a path

$$\boldsymbol{\emptyset} \xrightarrow{i_1} \cdot \xrightarrow{i_2} \cdot \xrightarrow{i_3} \dots \xrightarrow{i_n} \boldsymbol{\lambda}$$

in $B_e^{\mathfrak{s}}$. By (11), we have the path

$$\emptyset \xrightarrow{-i_1} \cdot \xrightarrow{-i_2} \cdot \xrightarrow{-i_3} \cdots \xrightarrow{-i_n} \widehat{\boldsymbol{\nu}}$$

in $B_e^{\widetilde{\mathfrak{s}}}$. This gives

$$m_l(\boldsymbol{\lambda}) = \boldsymbol{\mu} = \widehat{\boldsymbol{\nu}}.$$

Hence, the procedure (17) gives a purely combinatorial algorithm for computing the Mullineux involution which does not use the crystal (and thus the notion of good nodes) of irreducible highest weight affine modules.

4.4. The case $e = \infty$

We now briefly focus on the case $e = \infty$. Our algorithm considerably simplifies because in this case, the Kleshchev and FLOTW l-partitions coincide. Moreover, the Mullineux involution m_1 reduces to the ordinary conjugation of partitions. Consider $\boldsymbol{\lambda} \in \Phi_\infty^{\mathbf{s}}$. The computation of $m_l(\boldsymbol{\lambda})$ can be described as follows:

- First, by Proposition 4.2, the skew-isomorphism of crystals from $B_e^{\mathfrak{s}}(\Lambda_{\mathfrak{s}})$ to $B_e^{-\mathfrak{s}}(\Lambda_{-\mathfrak{s}})$ which switches the sign of the arrows sends $\boldsymbol{\lambda}$ to

$$\boldsymbol{\mu} = \left(\left(\lambda^0\right)', \dots, \left(\lambda^{l-1}\right)'\right)$$

  where m_1 just corresponds to the usual conjugation of partitions.
- Next, w_0 being the longest element of the symmetric group, we have

$$w_0.(-s_0, \dots, -s_{l-1}) = (-s_{l-1}, \dots, -s_0).$$

  Thus we derive

$$m_l(\boldsymbol{\lambda}) = \boldsymbol{\nu} := \Psi_n^{\mathbf{s} \to w_0.\mathbf{s}}(\boldsymbol{\mu}).$$

4.5. Concluding remarks

The algorithm of Section 3.2 also permits to compute the bijections $\Psi_{e,n}^{\mathfrak{s} \to \sigma(\mathfrak{s})}$ between $\Phi_e^{\mathfrak{s}}$ and $\Phi_e^{\sigma(\mathfrak{s})}$ for any $\sigma \in S_n$. By decomposing σ in terms of the generators s_c, $c = 1, \dots, l-1$, it suffices to compute the isomorphisms $\Psi_{e,n}^{\mathfrak{s} \to s_c(\mathfrak{s})}$. Since $B_e^{\mathfrak{s}}$ can be regarded as a tensor product of crystals, we can assume $l = 2$ and describe the bijection

$$\Psi_{e,n}^{(\mathfrak{s}_0,\mathfrak{s}_1) \to (\mathfrak{s}_1,\mathfrak{s}_0)} : \Phi_e^{(\mathfrak{s}_0,\mathfrak{s}_1)} \to \Phi_e^{(\mathfrak{s}_1,\mathfrak{s}_0)}.$$

Consider $\boldsymbol{\lambda} \in \Phi_e^{(\mathfrak{s}_0,\mathfrak{s}_1)}$ and choose (a_0, a_1) an asymptotic charge with $a_0 \in \mathfrak{s}_0$ and $a_1 \in \mathfrak{s}_1$. We have $\boldsymbol{\lambda} \in \Phi_e^{(a_0,a_1)}$. Then one computes $\Psi_{e,n}^{(a_0,a_1) \to (a_1,a_0)}(\boldsymbol{\lambda}) = \boldsymbol{\nu}$ as in Proposition 3.4. Now the problem is that (a_1, a_0) is not asymptotic in general. To make it asymptotic, we need to compose translations by e in $\mathbb{Z}^2$ acting on the right coordinate. Observe that for any $(v_0, v_1) \in \mathbb{Z}^2$, we have $\tau s_1(v_0, v_1) = (v_0, v_1 + e)$. So if we set $\kappa = \tau s_1$, we have to compute the bipartitions

$$\boldsymbol{\mu}^{(k)} = \Psi_{e,n}^{(a_1,a_0) \to \tau^k (a_1,a_0)}(\boldsymbol{\nu}) \tag{12}$$

with $k \in \{0, \dots, m\}$ where $m \in \mathbb{N}$ is minimal such that $a_1 - a_0 + me > n - 1$. Nevertheless, in practice one can restrict our computations and take m minimal such $\boldsymbol{\mu}^{(m)} = \boldsymbol{\mu}^{(m+1)}$. This is because the Kleshchev bipartition is fixed by the bijection $\Psi_{e,n}^{(\mathfrak{s}_1,\mathfrak{s}_0) \to \tau(\mathfrak{s}_1,\mathfrak{s}_0)}$. Finally, one has

$$\Psi_{e,n}^{(\mathfrak{s}_0,\mathfrak{s}_1)\to(\mathfrak{s}_1,\mathfrak{s}_0)}(\boldsymbol{\lambda})=\boldsymbol{\mu}^{(m)}.$$

Such a stabilization phenomenon also happens during the computation of $\boldsymbol{\mu}$ from $\boldsymbol{\nu}$ (see Section 4.3). This means that, in practice, we have

$$\Psi_{e,n}^{-\mathbf{s}\to\eta(-\mathbf{s})}(\boldsymbol{\nu})=\Psi_{e,n}^{-\mathbf{s}\to\eta^\flat(-\mathbf{s})}(\boldsymbol{\nu})$$

where $\eta:=\alpha_1^{2(p_{l-1}+1)}\cdots\alpha_{l-1}^{2(p_1+1)}w_0$ is defined as in (8) and where $\eta^\flat:=\alpha_1^{2(p_{l-1}^\flat+1)}\cdots\alpha_{l-1}^{2(p_1^\flat+1)}w_0$ with $p_c^\flat\leqslant p_c$ for $c\in\{0,\dots,l-1\}$.

The previous description of the bijections $\Psi_{e,n}^{\mathfrak{s}\to\sigma(\mathfrak{s})}$, $\sigma\in S_l$ yields an alternative for computing $\Psi_{e,n}^{-\mathfrak{s}\to\widetilde{\mathfrak{s}}}(\boldsymbol{\nu})=\boldsymbol{\mu}$ in Section 4.3. Nevertheless, one can verify that the number of elementary transformations it requires (i.e. the number of bijections $\Psi_{e,n}^{\mathbf{s}\to s_c(\mathbf{s})}$, $c=1,\dots,l-1$ or $\Psi_{e,n}^{\mathbf{s}\to\tau(\mathbf{s})}$ to compute) is in general greater than the number of transformations used in $\Psi_{e,n}^{-\mathbf{s}\to\eta(-\mathbf{s})}$.

Example 4.4. We conclude this paper with a computation of the Mullineux involution on a 3-partition. Take $e=4$ and $\mathfrak{s}=(0(\mathrm{mod}\,4),1(\mathrm{mod}\,4),3(\mathrm{mod}\,4))$. Then, one can check that the 3-partition $\boldsymbol{\lambda}:=(4,3,1.1.1)$ is a Kleshchev 3-partition of rank $n=10$ associated to $\mathfrak{s}$. Now we have

$$m_1(4)=(2.1.1),\qquad m_1(3)=(1.1.1),\qquad m_1(1.1.1)=(3).$$

Hence, to $\boldsymbol{\lambda}$, one can attach the Kleshchev 3-partition $\boldsymbol{\nu}=(2.1.1,1.1.1,3)$ which is Kleshchev for $-\mathfrak{s}=(0(\mathrm{mod}\,4),-1(\mathrm{mod}\,4),-3(\mathrm{mod}\,4))$. Since $p_1=p_2=3$, this 3-partition is an Uglov 3-partition associated with the multicharge

$$-\mathbf{s}=(0,-1+3\times 4,-3+3\times 4+3\times 4)=(0,11,21).$$

Following Section 4.3, we put:

$$\eta=\alpha_1^8\alpha_2^8w_0.$$

We compute

$$\Psi_{e,n}^{-\mathbf{s}\to\eta.(-\mathbf{s})}(\boldsymbol{\nu})=(\emptyset,1.1.1,4.1.1.1).$$

Hence, we obtain

$$m_3(\boldsymbol{\lambda})=(\emptyset,1.1.1,4.1.1.1).$$

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