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Families of Periodic Orbits: Virtual Periods and Global Continuability

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For a differential equation depending on a parameter, there have been numerous investigations of the continuation of periodic orbits as the parameter is varied. Mallet-Paret and Yorke investigated in generic situations how connected components of orbits must terminate. Here we extend the theory to the general case, dropping genericity assumptions. © 1984 Academic Press, Inc.

1. INTRODUCTION

Poincaré studied (periodic) orbits of parametrized vector fields—orbits of

$$\frac{dx}{dt} = f(x, \lambda) \tag{1.1}$$

where $f: R^n \times R \to R^n$. He observed that, by using the (Poincaré) return map for an orbit at λ_0, the implicit function theorem could be employed for continuing the fixed point of the return map to nearby values of λ, and therefore the orbit itself could be continued locally. In this paper "orbit" always refers to periodic orbit and "period" to the orbit's minimum period.

The global behavior of families of orbits, however, is somewhat different from that of fixed points, since return maps are only defined locally in an

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(x, λ)-neighborhood of an orbit. Global results have been obtained for generic families (see Section 2 for a discussion of the residual set of vector fields considered)—provided the initial orbit is not a Mobius-type orbit. (Briefly, a Mobius orbit is one whose unstable manifold is nonorientable.) For a connected component $\Gamma \subset R^n \times R$ of the set of points (x, λ) on the orbits of such an equation and for a locally continuable non-Mobius orbit γ in Γ, at least one of the following must hold (see [9]):

(a) $\Gamma - \gamma$ is connected;

or each of two components Γ_i $(i = 1, 2)$ of $\Gamma - \gamma$ satisfies one of the following:

(b1) Γ_i is unbounded in (x, λ)-space,

(b2) $\bar{\Gamma}_i$ contains a generalized center (i.e., a stationary point (x_0, λ_0) such that $D_x f(x_0 \lambda_0)$ has some purely imaginary eigenvalues), or

(b3) the periods of orbits in Γ_i are unbounded.

DEFINITION 1.2. If an orbit γ of (1.1) satisfies any of the above conditions, we say γ is *P-globally continuable.*

In the case of a Mobius orbit, it has been shown that, even if the orbit is locally continuable, a family of orbits emanating from it can terminate in such a way as not to satisfy any of the above conditions (see [4]).

The question remained as to whether orbits of a general C^1 system exhibit the same global behavior as those in the generic class. The principal aim of this paper is to demonstrate that for non-Mobius orbits of a general system local continuability does in fact imply global continuability (which we define after some preliminary concepts). We obtain this result by approximating f in (1.1) by a sequence $\{g_n\}$, where the families of orbits of each parameterized vector field g_n are of generic type.

Let $f_i: R^n \to R^n$, $i \in N$, be a sequence of C^1 functions converging to a C^1 function f, and for each i, let γ_i be a periodic orbit of the differential equation $\dot{x} = f_i(x)$. Assume there is an upper bound on the periods of the γ_i's. The limit of $\{\gamma_i\}$, if it exists, will be a periodic orbit β of $\dot{x} = f(x)$. For the purpose of this discussion, we assume β to be non-constant. Let τ_i be the period of γ_i, for each i. If $\lim_{i \to \infty} \tau_i$ exists and is finite, then this limit is some integer multiple of the period τ of the orbit β. We formalize this idea in the following definition:

Let A be the $n-1$-dimensional square matrix $D_x T(x_0)$, where T is the Poincaré map associated with β at x_0. If there exists a point $y \in R^{n+1}$ with $y, Ay, \ldots, A^{m-1}y$ distinct, but $A^m y = y$, for some $m \geqslant 1$, then we say $\bar{\tau} = m\tau$ is a *virtual period* of β, and m is the *order* of the virtual period $\bar{\tau}$.

For $m = 1$, the definition is satisfied by $y = 0$; thus the period is a virtual period of the orbit. For $m > 1$, $y \neq 0$. It has been shown in [8] that in the situation described above, where τ_i is the period of the orbit γ_i, $\bar{\tau} = \lim_{i \to \infty} \tau_i$ is a virtual period of the limit orbit β. Notice that there can be only a finite number of virtual periods for β. In fact, if the above matrix A has no eigenvalues that are roots of unity, then the period is the only virtual period.

We incorporate the idea of virtual periods into our definition of global continuability.

DEFINITION 1.3. For an orbit β of (1.1), let $B \subset R^n \times R$ be the component of the set of points on (periodic) orbits that contains β. We say β is *globally continuable* if either of the following conditions hold:

(a) $B - \beta$ is connected;

or each of two components B_i $(i = 1, 2)$ of $B - \beta$ satisfies one of the following:

(b1) B_i is unbounded in (x, λ)-space,

(b2) $\bar{B}_i$ contains a generalized center, or

(b3) the virtual periods of orbits in B_i are unbounded.

Section 2 contains a description of the generic properties of orbits for the residual set of vector fields used in the limit arguments and a summary of the generic theory. The main theorem—global continuability for locally continuable non-Mobius orbits of general C^1 systems—is proved in Section 3. In Section 4, we consider a smaller class of C^1 vector fields, for which the generic notion of global continuability (P-global continuability) holds.

In introducing the idea of virtual period and using it to obtain global results, we have restricted ourselves to results that use no index or degree theory. Also, the results we use from other papers do not depend on index theory of any kind. (the development of an orbit index paralleling this theory appears in [5]. See also [8] and [9].) For more difficult problems, such as the global continuation of orbits emanating from a generalized center, the orbit index appears to be necessary, and such problems are not discussed here.

The results in this paper and in [9] should be contrasted with those of [2], [7], and others, in that those papers studied the boundedness of components of the set $Q = \{(t, x, \lambda)$: (x, λ) is on an orbit of (1.1), and $t \geqslant 0$ is *any* integer multiple of the orbit's period$\}$. Since t was not necessarily even a virtual period, these results were incomplete.

While we have emphasized differential equations in $R^n \times R$, the theory extends to $M \times R$, where M is any manifold. It also extends to cases where f is defined on an open subset U of $R^n \times R$, in which case condition (b1) of the definition of global continuability should be interpreted as saying "B_i does not lie in a compact subset of U."

2. A Description of Global Results for Generic Families of Periodic Orbits

The continuation of periodic orbits of a parametrized system of ordinary differential equations can be studied locally through techniques of fixed point theory and globally in terms of the topological structure of maximal connected families of such orbits. Given a differential equation depending on a parameter

$$\frac{dx}{dt} = f(x, \lambda) \tag{2.1}$$

where $f: R^n \times R \to R^n$ is C^1, the main tool for analyzing a periodic solution γ of (2.1) is the Poincaré (or first-return) map. Let (x_0, λ_0) be a point on γ, and let D be an n-dimensional disk perpendicular to $(f(x_0, \lambda_0), 0)$ at (x_0, λ_0). The C^1 Poincaré map T is defined for (x_1, λ_1) in D sufficiently close to (x_0, λ_0) as follows: let $T(x_1, \lambda_1)$ be the x-coordinate of the point where the trajectory through (x_1, λ_1) next hits D. (The λ-coordinate is λ_1.) We say μ is a *multiplier* of γ if it is an eigenvalue of the $(n-1) \times (n-1)$ matrix of partial derivatives $D_x T(x_0, \lambda_0)$. Notice that each point on a periodic orbit is a fixed point of the Poincaré map associated with the orbit at that point. The Poincaré index of the orbit is the Brouwer fixed point index of T. Orbits which have an odd number of multipliers (counted with multiplicities) in $(-\infty, -1)$ and which, in addition, have no multipliers equal to -1 are called *Mobius orbits*.

For the remainder of this section we assume that the function f in (2.1) is C^3. We consider a set of properties of periodic orbits of (2.1) which are "generic" in the following sense: the set of C^3 vector fields all of whose periodic orbits possess these properties is residual in the C^3-topology (i.e., the set has a subset which is the countable intersection of open dense sets).

If an orbit γ containing (x_0, λ_0) does not have $+1$ as a multiplier, then $D_x(T - id_x)$ will be non-singular. By the implicit function theorem, there exists a neighborhood W of (x_0, λ_0) in R^{n+1} such that the component of zeroes of $T - id_x$ containing (x_0, λ_0) extends through W. Since these zeroes lie on periodic orbits, the component of orbits containing γ will similarly extend through W, and we say that γ is *locally continuable*. If an eigenvalue of $D_x T(x_0, \lambda_0)$—or of the derivative of some iterate of T—crosses the value $+1$, there will be a bifurcation of orbits from γ. Notice that if γ has a multiplier which is the jth root of unity, then $D_x T^j(x_0, \lambda_0)$ will have an eigenvalue equal to 1. When $j \neq 1$, a family of longer period orbits can bifurcate from γ. In the Appendix to [5], a set of generic bifurcations is described. We summarize these ideas in the following list of types of orbits that may be contained in our "generic" families:

Type 0 orbit: an orbit with no multipliers which are roots of unity. In particular, since these orbits have no multipliers equal to 1, they are locally continuable.

Type I orbit: an orbit γ_1, with a single multiplier $\mu = +1$ (algebraically simple), and no other multipliers which are roots of unity, in a system satisfying the following additional conditions:

(1) the $n \times (n-1)$ matrix $D_{(x,\lambda)}(T - id_x)$ has full rank (thus, by the implicit function theorem applied to $T - id_x$, there is a path of orbits through γ_1); and

(2) the rate of change of the multiplier along the path of orbits through γ_1 is non-zero.

In this case two families of orbits (of approximately the same period) approach each other as λ increases (respectively, decreases), coalesce when $\lambda = \lambda_0$ at the bifurcation orbit γ_1, and disappear for $\lambda > \lambda_0$ (resp., $\lambda < \lambda_0$).

Type II orbit: an orbit γ_2 with a single multiplier $\mu = -1$ (algebraically simple), and no other multipliers which are roots of unity, in a system satisfying the following additional conditions:

(1) the derivative of the multiplier with respect to the parameter is non-zero, in which case (following from a Liapunov–Schmidt argument) a unique path of orbits with periods approximately twice those of orbits on the original family bifurcates from γ_2; and

(2) $+1$ is not a multiplier for orbits on the double period branch within some neighborhood of γ_2.

Since $+1$ is not a multiplier of γ_2, the original family continues through the bifurcation orbit with approximately the same periods. In order for a periodic orbit to have a negative multiplier, nearby solutions must twist around the orbit (as, for example, in the case of a flow on a Mobius band about the center circle). Thus period-doubling bifurcations can occur in R^n only for $n \geqslant 3$. Under the above hypotheses, period-doubling bifurcations are isolated in the following sense: for any bounded subset Ω of $R^n \times R$ and any number q, there are at most a finite number of type II orbits lying wholly in Ω that have period less than q.

Notice that it is possible for type 0 or I orbits to be Mobius, but not possible for type II orbits (since -1 is a multiplier). Also note that for type 0 and I orbits the only virtual period is the period itself, (a virtual period of order 1), while each type II orbit has virtual periods of orders 1 and 2.

We let K be the set of those C^3 functions $f: R^n \times R \to R^n$ such that every periodic orbit of $dx/dt = f(x, \lambda)$ is of type 0, I, or II. We refer the reader to [5] for a justification that K is residual in the C^3-topology. (This justification is based on the local fixed-point results of Brunovský [6] coupled with the global methods of Peixoto [10]. For an alternate treatment, see Sotomayor [11].) Throughout the remainder of the paper, we will continue to apply the term "generic" not only to the properties of vector fields in K, but also more loosely to the vector fields themselves and to families (i.e., components) of orbits of these vector fields.

In the case of fixed points or zeroes of parametrized maps, local continuability in fact implies global continuability. (See [1] for the appropriate definition of global continuability in this case and the theorem.) Unfortunately, the analogus result for periodic orbits is impossible to obtain, even in the generic case, $f \in K$. An example of a Mobius orbit γ with non-zero Poincaré index which fails to satisfy any of the conditions for P-global continuability in Definition 1.2 is presented in [4]. It has been shown in [3] that this phenomenon is only realizable when the dimension of the x-space is strictly greater than 3.

If we examine the continuability from the three generic types of orbits, we see that in each case there is a 1-dimensional path of orbits (branched in the case of type II) passing through the given orbit. The two branches emanating from a type I orbit must be either both Mobius or both non-Mobius, depending on whether the orbit itself is Mobius. Two of the three branches emanating from a type II orbit must be non-Mobius, and the third must be Mobius. (We refer the reader to [9] for details.) Thus a non-Mobius orbit always lies on an (unbranched) path of non-Mobius orbits. We define an equivalence relation E on a subset of $R^n \times R$: $((x_1, \lambda), (x_2, \lambda)) \in E$ if and only if x_1 and x_2 lie on the same orbit of (2.1). If we add the hypothesis that for a given non-Mobius orbit γ, the component Π of non-Mobius orbits containing γ has uniformly bounded periods, then we are able to conclude that $\Pi^* = \Pi/E$ is in fact a 1-manifold. (It is a metric space, using, for example, the Hausdorff metric on equivalence classes. Since Π does not contain orbits of arbitrarily high period, we are assured that there exists a neighborhood W^* of the point $\gamma^* = \gamma/E$ such that $\Pi^* \cap W^*$ is homeomorphic to $(0, 1)$.)

We end this section with a global theorem for generic families (a special case of Theorem 4.2 in [9]) and include a proof which is rather different from and somewhat simpler than the original.

THEOREM 2.2 (Mallet-Paret and Yorke). *Let γ be a periodic solution of* (2.1), *where $f \in K$. If γ is not a Mobius orbit, then γ is $P-$globally continuable.*

Proof. Let Γ be the component of orbits of (2.1) containing γ. Assume that $\Gamma - \gamma$ is not connected and that Γ^+, a component of $\Gamma - \gamma$, is bounded. Further assume that the periods of orbits in Γ^+ are bounded. Let Π be the subset of $\Gamma^+ \cup \gamma$ of non-Mobius orbits. By the previous discussion, Π^* is a 1-manifold, so there exists a homeomorphism h: $[0, 1) \to \Pi^*$. For an increasing sequence $\{t_i\}_{i \in N}$ of points in $[0, 1)$ such that $\lim_{i \to \infty} t_i = 1$, let

$\pi_i^* = h(t_i)$, for each i. It is easily seen that the corresponding sequence $\{\pi_i\}$ of orbits in Π converges to an invariant set ρ. Let τ_i be the period of π_i, for each i. Since $\{\tau_i\}$ is bounded, ρ must be a stationary point or a (periodic) orbit. We conclude that ρ is a stationary point; otherwise, $\rho \in \Pi$ but $\rho^* \notin h([0, 1))$. In other words, since Π^* is a manifold, it cannot double back on itself. The limit point ρ^* cannot be a point of the manifold. Thus $\overline{\Gamma^+}$ contains a generalized center, and γ is P-globally continuable. ∎

3. A Global Result for Periodic Orbits in the General Case

Having defined a generic class of C^3 vector fields and having described a global result for certain periodic orbits of such a system, we proceed in this section to prove an analogous theorem (Theorem 3.1) for general C^1 systems. Let T be the Poincaré map of β at (x_0, λ_0), and let $A = D_x T(x_0, \lambda_0)$. Suppose that β has a virtual period of order $m > 1$. Then $A^m y = y$, for some $y \neq 0$. Thus 1 is an eigenvalue of A^m. In this case, it is easily seen that the rth root of unity is an eigenvalue of A, for some divisor r of m; hence $A^r - I$ is singular. We use this fact about virtual periods in the proof of our main result:

Theorem 3.1. *Let β be a periodic orbit of* (2.1), *and let T be the Poincaré map of β at (x_0, λ_0). Assume that $(D_x T(x_0, \lambda_0))^j - I$ is non-singular for $j \geqslant 1$ (i.e., β has no multipliers that are roots of unity), and assume that β is not a Mobius orbit. Then β is globally continuable.*

The proof of this theorem depends on the following lemma:

Lemma 3.2. *Let $\{g_i : R^n \times R \to R^n\}_{i \in N}$ be a sequence of functions in K converging in the C^1-topology to a C^1 function $f : R^n \times R \to R^n$, and let γ_i be a periodic orbit of $\dot{x} = g_i(x, \lambda)$, for each i, with $\{\gamma_i\}$ converging to a (non-stationary) orbit β. If $\{\gamma_i\}_{i \in N}$ is a sequence of P-globally continuable orbits, if the periods of the γ_i's are bounded, and if the Poincaré map of β satisfies the hypothesis of Theorem* 3.1, *then β is globally continuable.*

Proof of Lemma 3.2. Assume that β is not globally continuable. Denote by B^+ and B^- the two components of $B - \beta$, and suppose that B^+ satisfies none of the conditions (b1)–(b3) of Definition 1.3. In particular, $f(x, \lambda) \neq 0$, for $(x, \lambda) \in B^+$. By the hypothesis on T, β has no virtual periods of order greater than one, and there is a neighborhood W of β in $R^n \times \{\lambda_0\}$ in which β is the only orbit—except, perhaps, for orbits of very long period.

Let

$$p_0 = \min_{\beta \in B^+} \{\sigma : \sigma \text{ is the period of } \beta\}$$

and

$$p_1 = \max_{\beta \in B^+} \{\tau : \tau \text{ is a virtual period of } \beta\}.$$

Note that $p_0 > 0$. If $\phi(t, x, \lambda)$ is the solution of (2.1) through (x, λ), we define

$$F(x, \lambda) = \min_{(1/4)p_0 \leqslant t \leqslant 3p_1} |\phi(t, x, \lambda) - x|.$$

The set of zeroes of F are points on periodic orbits of (2.1). Loosely speaking, F measures how close ϕ is to being periodic, for periods between $\frac{1}{4}p_0$ and $3p_1$. It is clear that F is continuous in x and λ.

Let $N_\varepsilon = \{(x, \lambda) \in R^n \times R : |F(x, \lambda)| \leqslant \varepsilon\}$ for $\varepsilon > 0$, and let N_ε^0 be the component of N_ε containing B. Choose ε small enough so that

(1) the component M_ε of N_ε^0 of $N_\varepsilon^0 \cap (R^n \times \{\lambda_0\})$ containing β is a subset of W;

(2) $N_\varepsilon^0 \backslash M_\varepsilon$ has two components, which we denote by N_ε^+ and N_ε^-, ($B^+ \subset N_\varepsilon^+$ and $B^- \subset N_\varepsilon^-$);

(3) there are no zeroes of f in N_ε^+;

(4) N_ε^+ is bounded;

(5) there exists $\rho_1 > 0$ such that the system $\dot{x} = g(x, \lambda)$ will have only one orbit with period $\leqslant 3p_1$ contained in M_ε, when $\|f - g\| < \rho_1$; and

(6) there exists $\rho_2 > 0$ such that the system $\dot{x} = g(x, \lambda)$ will have no orbits in $N_\in^+$ with periods in the interval $J = [(11/10)p_1, (11/5)p_1]$ or with periods smaller than $(9/10)p_0$, when $\|f - g\| < \rho_2$.

To show the existence of an $\varepsilon > 0$ satisfying condition (5), we argue as follows:

Assume that the period of β is 1. By the implicit function theorem, the C^1 map $T_g - id_x$ will have only one zero in a neighborhood of (x_0, λ_0) in a cross section transverse to β, for T_g sufficiently close to T in the C^1-topology. Thus there is an $\varepsilon_1 > 0$ such that $\dot{x} = g(x, \lambda)$ will have only one orbit of period near 1 in M_ε. We claim that for each integer j, $1 < j \leqslant 3p_1$, there exists $\varepsilon_j > 0$ such that M_{ε_j} contains no orbits of $\dot{x} = g(x, \lambda)$ of period near j. Suppose otherwise. Then there exists a sequence $\{g_i\}_{i \in N}$ of functions and a corresponding sequence $\{\gamma_i\}_{i \in N}$ of orbits with periods $\{\tau_i\}_{i \in N}$ (where γ_i is an orbit of $\dot{x} = g_i(x, \lambda)$ for each i), such that $\lim_{i \to \infty} g_i = f$, $\lim_{i \to \infty} \gamma_i = \beta$, and $\lim_{i \to \infty} \tau_i = j$. But then j is a virtual period of β, a contradiction. Let $M_\varepsilon = \bigcap_{1 \leqslant j \leqslant 3p_1} M_{\varepsilon_j}$. ($M_\varepsilon$ is in fact equal to some M_{ε_j}, since we are intersecting a finite nested family.) Then $\dot{x} = g(x, \lambda)$ will have only one orbit with period less than $3p_1$ in M_ε. (This orbit must be of type 0.)

To show that condition (6) may be satisfied, we argue as follows:

Suppose that for every $\varepsilon > 0$, there exists a C^1 function $g: R^n \times R \to R^n$ such that $\|f-g\| < \rho_2$ and $\dot{x} = g(x, \lambda)$ has an orbit entirely in N_ε^+ with period in J. Then for a decreasing sequence $\{\varepsilon_i\}_{i \in N}$, we obtain a sequence of functions $\{g_i\}$ and a corresponding sequence of orbits $\{\pi_i\}$ such that π_i is an orbit of $\dot{x} = g_i(x, \lambda)$, and the period of π_i is in J. Choose a point x_i on π_i, for each i. The infinite set $\{x_i\}_{i \in N}$ is bounded and thus will have an accumulation point x, which is easily seen to lie on an orbit σ in B^+. But then σ will have a virtual period τ in J, contradicting the definition of p_1. A similar argument shows that ε can be chosen so that no orbit of $\dot{x} = g(x, \lambda)$ in N_ε^+ will have period less than $(9/10)p_0$.

Following the choice of ε, we let g_ε be a function in K sufficiently close to f in the C^1-topology so that

(1) $\|f - g_\varepsilon\| < \min\{\rho_1, \rho_2, \mu\}$, where $\mu = \min_{(x,\lambda) \in \overline{N_\varepsilon^+}} f(x, \lambda)$; and

(2) $\|F - G_\varepsilon\| < \varepsilon/2$, where G_ε is defined analogously to F for solutions of $\dot{x} = g_\varepsilon(x, \lambda)$ (for $\frac{1}{4}p_0 \leqslant t \leqslant 3p_1$).

To show that g_ε can be chosen to satisfy condition (2), we argue as follows: Let $\psi(t, x, \lambda)$ be the solution of

$$\dot{x} = g_\varepsilon(x, \lambda) \tag{3.3}$$

through (x, λ). Since N_ε^+ is bounded, g_ε can be chosen so that $|\phi(t, x, \lambda) - \psi(t, x, \lambda)| < \varepsilon/2$, for $(x, \lambda) \in N_\varepsilon^+$, $\frac{1}{4}p_0 \leqslant t \leqslant 3p_1$. Then it is easily seen that $|\min_{(1/4)p_0 \leqslant t \leqslant 3p_1} |\phi(t, x, \lambda) - x| - \min_{(1/4)p_0 \leqslant t \leqslant 3p_1} |\psi(t, x, \lambda) - x|| < \varepsilon/2$.

Notice that condition (2) implies that no orbits of (3.3) with periods in $[\frac{1}{4}p_0, 3p_1]$ will intersect ∂N_ε (the boundary of N_ε); condition (1) implies that N_ε^+ will contain no stationary solutions of (3.3).

Suppose that γ_ε is the unique orbit of (3.3) with period less than $3p_1$ contained in M_ε. Let Γ_ε be the component of periodic solutions of (3.3) containing γ_ε. In a small neighborhood U of γ_ε in R^{n+1}, $\Gamma_\varepsilon - \gamma_\varepsilon$ has two components. Let Γ_ε^+ be the component of $\Gamma_\varepsilon - \gamma_\varepsilon$ which intersects $U \cap N_\varepsilon^+$ (i.e., Γ_ε^+ is the component which extends into N_ε^+). We want to show that Γ_ε^+ is in fact contained in N_ε^+. Now Γ_ε^+ can only escape from N_ε^+ in one of two ways: (1) through $Q_\varepsilon = \partial N_\varepsilon \cap N_\varepsilon^+$ or (2) through M_ε. We discuss each case separately.

Suppose $x \in \Gamma_\varepsilon^+ \cap Q_\varepsilon$. Then x must be on an orbit π with period τ, where $\tau \in (-\infty, \frac{1}{4}p_0) \cup (3p_1, \infty)$. We assume that γ_ε is sufficiently near β that the period of γ_ε is less that p_1. Suppose $\tau > 3p_1$. Then Γ_ε^+ contains orbits with periods less that p_1 and orbits with periods greater than $3p_1$. Since periods on a generic family increase continuously or with gaps a factor of 2, there must be an orbit σ on Γ_ε^+ with period in $[(11/10)p_1, (11/5)p_1]$, and no orbits on the "path" from γ_ε to σ with periods greater than $(11/5)p_1$. But then it is easily seen that σ is in N_ε^+, contradicting condition (6) on the choice of ε. A similar argument shows $\tau \geqslant \frac{1}{4}p_0$. Thus Γ_ε^+ and Q_ε are disjoint.

By condition (5) on the choice of ε, γ_ε is the only orbit of Γ_ε with period less than $3p_1$ in M_ε. By condition (6) on the choice of ε, Γ_ε contains no orbits with periods greater than $(11/10)p_1$. Thus Γ_ε^+ and M_ε are disjoint.

By the hypothesis of the lemma, we are assuming that γ_ε is P-globally continuable. Hence either $\Gamma_\varepsilon - \gamma_\varepsilon$ is connected, or Γ_ε^+ is unbounded, contains a generalized center, or has unbounded periods. Since Γ_ε^+ is contained in the bounded set N_ε^+, there are no stationary solutions of (3.3) in N_ε^+, and $(11/10)p_1$ is an upper bound on periods in Γ_ε^+, we see that $\Gamma_\varepsilon - \gamma_\varepsilon$ must be connected. However, since γ_ε is not a type I orbit, the only way in which $\Gamma_\varepsilon - \gamma_\varepsilon$ can be connected is for Γ_ε^+ to leave N_ε^+, a contradiction. We conclude that β is globally continuable. ∎

Proof of Theorem 3.1. For a sequence $\{g_i\}_{i\in N}$ of functions in K such that $\lim_{i\to\infty} g_i = f$ (as in Lemma 3.2), let $\{\gamma_i\}$ be the corresponding sequence of periodic solutions converging to β. If β is not a Mobius orbit, then for i sufficiently large, γ_i will not be a Mobius orbit and hence will be P-globally continuable. By Lemma 3.2, β is globally continuable. ∎

Although the structure of families of orbits in the general case does not lend itself to analysis as easily as that of generic families, in certain cases the approximation techniques of the previous theorem will yield more information about the set of virtual periods of such a family. Namely, if the virtual periods are shown to be unbounded by approximating the family with generic ones on which the periods are unbounded, then the set of virtual periods will be in some sense as "complete" as the set of periods of the nearby generic families. In particular, Proposition 3.5 shows that, given any interval I (beginning with a sufficiently large number), there will be a compact, connected subset of the component whose virtual periods lie in I and take on all values in I, except perhaps for gaps of a factor of two. For an application of the following Proposition and Corollary to continuation in dimensions 3 and 4, see Section 4 of [5], where it is shown that global continuation implies P-global continuation in these cases.

In the following, let $f \in C^1(R^n \times R; R^n)$; let $\{g_i\}_{i\in N}$ be a sequence of functions in K such that $\lim_{i\to\infty} g_i = f$; and, for each i, let γ_i be an orbit of

$$\dot{x} = g_i(x, \lambda) \tag{3.4}$$

at $\lambda = \lambda_0$. We denote by Γ_i the component of orbits of (3.4) containing γ_i, for each i. Assuming that the set of periods of $\{\gamma_i\}_{i\in N}$ is bounded, then $\lim_{i\to\infty} \gamma_i$ is an orbit or stationary point of (2.1). In Proposition 3.5, we let $\beta = \lim_{i\to\infty} \gamma_i$ be an orbit of (2.1) at $\lambda = \lambda_0$ and let B be the component of orbits of (2.1) containing β. We further assume $\bar{B}$ has no generalized centers.

PROPOSITION 3.5. *Suppose that for each i, one component of* $\Gamma_i - \gamma_i$, *say,* Γ_i^+, *has the property that the set of periods of its orbits is unbounded. In* *addition, we assume that* $\bigcup_i \Gamma_i^+$ *is bounded. Then if* α_1 *and* $\alpha_2 > \alpha_1$ *are sufficiently large, there will be a compact, connected set S such that* $S \subset B$ *and each orbit in S has virtual period in* $[\alpha_1, 2\alpha_2]$. *Furthermore, for each* $\alpha \in [\alpha_1, \alpha_2]$, *there is an orbit in S with virtual period in* $[\alpha, 2\alpha)$.

Proof. First, notice that $\lim_{i\to\infty} \Gamma_i$ is a subset of B ($\lim_{i\to\infty} \Gamma_i$ is the set of all limit points of sequences $\{x_i\}_{i\in N}$, where $x_i \in \Gamma_i$). Let $\alpha_1 > \max\{\tau_i : \tau_i$ is the period of $\gamma_i\}$. Throughout the remainder of the proof, we use the following fact about generic families: for any two orbits γ_1, γ_2 on Γ_i, there is a path of orbits beginning with γ_1 and ending with γ_2, along which the periods vary continuously or jump by a factor of 2 (or $\frac{1}{2}$).

For each i, let A_i be the subset of Γ_i^+ consisting of all orbits with period in $[\alpha_1, 2\alpha_2]$. Since each Γ_i^+ has orbits of period less than α_1 and orbits of period greater than $2\alpha_2, A_i$ is non-empty, for each i. Let $\{A_{ij}\}_{1\leqslant j\leqslant m_i}$ be the set of components of A_i. (The number of components is finite, since Γ_i^+ is bounded.) We argue that at least one component of each A_i has the following property: for each $\alpha \in [\alpha_1, \alpha_2]$ there exists an orbit in the set with period in $[\alpha, 2\alpha)$. Otherwise, for each j $(1 \leqslant j \leqslant m_i)$ there exists a number α_j such that $\alpha_j \in [\alpha_1, \alpha_2]$ and A_{ij} has no orbits with period in $[\alpha_j, 2\alpha_j)$. Since the period of γ_i is less than α_1, and $\alpha_j \leqslant \alpha_2$ for all j, all orbits in each A_{ij} must have periods less than α_2. But Γ_i^+ has orbits with period greater than $2\alpha_2$. Let ρ_i be one such orbit. Then there exists a path of orbits starting with γ_i and ending with ρ_i. Such a path contains an orbit with period in $[\alpha_2, 2\alpha_2)$, a contradiction.

Let S_i be a component of A_i which satisfies the above property, and let $S = \lim_{i\to\infty} S_i$. Since each S_i is connected and $\{S_i\}_{i\in N}$ is uniformly bounded, we conclude by a standard point-set argument that S is connected. Also note that S is closed and that every point in S lies on an orbit with virtual period in $[\alpha_1, 2\alpha_2]$. Finally, for every $\alpha \in [\alpha_1, \alpha_2]$, each S_i contains an orbit η_i with period in $[\alpha, 2\alpha)$. Thus $\eta = \lim_{i\to\infty} \eta_i$ is an orbit in S with virtual period in $[\alpha, 2\alpha)$. ∎

COROLLARY. *Let* β *be an orbit of* (2.1) *at* $\lambda = \lambda_0$, *satisfying the hypotheses of Theorem* 3.1. *Suppose that a component* B^+ *of* $B - \beta$ *satisfies only condition* (b3) *of Definition* 1.3. *Then if* α_1 *and* $\alpha_2 > \alpha_1$ *are sufficiently large, there will be a compact, connected set S such that* $S \subset B^+$ *and each orbit in S has virtual period in* $[\alpha_1, 2\alpha_2]$. *Furthermore, for each* $\alpha \in [\alpha_1, \alpha_2]$, *there is an orbit in S with virtual period in* $[\alpha, 2\alpha)$.

Proof. By the hypothesis, B^+ is bounded. As in the proof of Lemma 3.2, B^+ is contained in a bounded neighborhood N_ε^+. We choose a sequence of functions $\{g_i\}_{i\in N}$ such that $g_i \in K$, for each i, and $\lim_{i\to\infty} g_i = f$. Let γ_i be an orbit of (3.4) at $\lambda = \lambda_0$ such that $\lim_{i\to\infty} \gamma_i = \beta$; and let Γ_i be the component of orbits of (3.4) containing γ_i. Then for g_i sufficiently near f (as in the proof of Lemma 3.2), Γ_i^+ is contained in N_ε^+, and $\overline{\Gamma_i^+}$ has no generalized centers; hence, the periods of orbits on Γ_i^+ are unbounded. Choose α_1 as in the proof of Proposition 3.5. Since β has no virtual periods (except the actual period), the set S guaranteed by Proposition 3.5 is contained in B^+. ∎

Remarks. Notice that if the periods of orbits on a generic family are bounded by m, then the virtual periods are bounded by $2m$. Thus Theorem 3.1 is a generalization of Theorem 2.2 (P-global continuability): for $f \in K$ the condition that the virtual periods are unbounded implies that the actual periods are unbounded.

We reiterate that the Corollary to Proposition 3.5 only applies to cases in which no other conditions of Definition 1.3 are satisfied for a component B^+ of orbits. Otherwise, virtual periods may become unbounded on a branch of B^+ which is not close to a generic family. As f is perturbed to a function in K, that particular branch may even disappear, in which case another branch will be globally continuable— but will not necessarily satisfy (b3).

4. Generic Families Revisited

In practice, differential equations often arise that have special symmetries. Some of the bifurcations that are likely to occur are not permissible in K. The question naturally arises as to how the hypotheses of Theorem 3.1 might be strengthened in order to obtain P-global continuability for periodic orbits of a class of C^1 vector fields larger than K. In this section we define such a class and prove another P-global continuability result.

We let Ω be the set of C^1 functions $f: R^n \times R \to R^n$ such that the number of periodic solutions of (2.1) with virtual periods of order greater than k is countable, for k sufficiently large. Clearly $K \subsetneq \Omega$. For comparison, note that for $f \in K$ there are at most countably many orbits with virtual periods of order strictly greater than one (i.e., only type II orbits); there are no orbits with virtual periods of order greater than 2.

PROPOSITION 4.1. *Let β be a periodic orbit of* (2.1), *where $f \in \Omega$, and let T be the Poincare map of β at (x_0, λ_0). If $(D_x T(x_0, \lambda_0))^j - I$ is non-singular for all $j \geqslant 1$, and if β is not a Mobius orbit, then β is P-globally continuable.*

Proof. Assume β is not P-globally continuable. As in the proof of Lemma 3.2, let B be the component of periodic solutions of (2.1) containing β, B^+ be a bounded component of $B - \beta$, and N_ε^+ be a bounded neighborhood of B^+. By Theorem 3.1, we know that the virtual periods of orbits in B^+ are unbounded. Since the actual periods in B^+ are bounded, the orders of the virtual periods must go to infinity. We will show that the set of orbits with orders greater than k is uncountable, $\forall\, k \in N$.

Fix k. As before, we have connected families Γ_i^+ of periodic solutions of (3.4), where each $g_i \in K$ and $\lim_{i\to\infty} g_i = f$. For i sufficiently large, $\Gamma_i^+ \in N_\varepsilon^+$. Arguing as in the proof of Lemma 3.2, we see that, by the conditions imposed on g_i, the actual periods of each Γ_i^+ in N_ε^+ must go to infinity. On such a generic family, the periods change continuously or with gaps a factor of 2. Thus, on each Γ_i^+, a continuum P_i of orbits must have periods greater than k. Let P be the set of limit points of sequences $\{x_i\}_{i\in N}$, where $x_i \in \pi_i$, for orbits π_i on P_i. Clearly, $P \subset B$. Each P_i is connected and $\{P_i\}_{i\in N}$ is uniformly bounded, implying that P is connected. Since P is easily seen to be non-empty, P must be a single orbit or a continuum or orbits. If P is a single orbit π, then π has virtual periods of arbitrarily high order, contradicting the fact that an orbit can have only a finite number of virtual periods. Thus P is a continuum of orbits each of which has a virtual period of order greater than k. Such a continuum exists for each k, contradicting the hypothesis on f. ∎

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