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Qualitative aspects for the cubic nonlinear Schrödinger equations with localized damping: Exponential and polynomial stabilization

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Results of exponential/polynomial decay rates of the energy in L^2-level, related to the cubic nonlinear Schrödinger equation with localized damping posed on the whole real line, will be established in this paper. We use Kato's theory and a priori estimates to obtain the result of global well-posedness and we determine exponential/polynomial stabilization combining the ideas of unique continuation due to Zhang, semigroup property and Komornik's approach.

1. Introduction

This manuscript presents new results regarding the study of uniform decay rates of the energy related to the damped cubic nonlinear Schrödinger equations (DCNLS henceforth),

$$\begin{cases} iu_t + u_{xx} + \varepsilon|u|^2u + ia(x)u = 0, & \text{in } \mathbb{R}\times(0,+\infty),\\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

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and

$$\begin{cases} iu_t + u_{xx} + \varepsilon|u|^2u + ib(x)|u|^2u = 0, & \text{in } \mathbb{R}\times(0,+\infty),\\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.2}$$

where $\varepsilon = \pm 1$, with the following assumptions:

$$\text{(A)} \quad \begin{cases} \text{In Eq. (1.1), } a\in W^{1,\infty}(\mathbb{R}) \text{ is a nonnegative function and}\\ \bullet\ a(x) \geqslant \alpha_0 > 0 \text{ for } |x| > R_1,\ R_1 > 0. \end{cases}$$

$$\text{(B)} \quad \begin{cases} \text{In Eq. (1.2), } b\in W^{1,\infty}(\mathbb{R}) \text{ is a nonnegative function and}\\ \bullet\ b(x) \geqslant \beta_0 > 0 \text{ for } |x| > R_2,\ R_2 > 0. \end{cases}$$

Functions $a(x)$ and $b(x)$ are responsible for the localized effects of the dissipative mechanism in L^2-level. Further, the damping terms present in Eqs. (1.1) and (1.2), mainly in the second case, might be caused by collisions in the absence of lower order effects or by trapped particles.

Eqs. (1.1) and (1.2) when $a, b \equiv 0$ and $\varepsilon = 1$, are the well-known focusing cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2u = 0. \tag{1.3}$$

This equation has been extensively studied in the last few years by many authors, mainly in what concerns the well-posedness questions, see, for instance, [6,10,11] and references therein. Eq. (1.3) has many physical applications, among them we can cite: fluid mechanics, optical fibre technology and analysis of nonlinear wave-packets. The last application arises when it is studied the qualitative theory of special solutions called *solitary standing waves*, that is, solutions of the form $u(x,t) = e^{i\omega t}\phi(x)$ where $\omega > 0$ is the wave-speed and $\phi = \phi_\omega$ is a real function which satisfies $\phi_\omega(x) \to 0$ as $|x| \to +\infty$. In addition, it is well known that Schrödinger equation, specially with cubic interactions, has wide applications in many physical fields such as nonlinear optics, nonlinear plasmas, condensed matter and so on.

In this paper, we establish the global well-posedness related to (1.1) and (1.2). It is known that the behavior of the solution related to the following problem,

$$\begin{cases} iu_t + u_{xx} + g(u) = 0, & \text{in } \mathbb{R}\times(0,+\infty),\\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.4}$$

where $g(u) = \lambda|u|^pu,\ p \geqslant 0$ and $\operatorname{Im}\lambda \neq 0$, is subject to dissipative effects. In fact, since we do not have conserved laws in L^2 and H^1-norms, we could expect a scenario of blow-up in $H^1(\mathbb{R})$ in the following sense: $\|u(t)\|_{H^1(\mathbb{R})} \to +\infty$ as $t \uparrow T_{max}$, if $T_{max} < +\infty$ (see Theorems 2.1 and 2.2). However, in order to overcome this critical fact, we use local well-posedness results due to Kato in [11] (see also [5] and [8]) and a priori estimates.

The main goal of this manuscript is to show exponential and polynomial decay rates related to the cubic Schrödinger equations with localized damping (1.1) and (1.2), respectively. To obtain these results we employ the unique continuation principle, proved by Zhang in [19], considering $\varepsilon = \pm 1$ in the following Schrödinger equation:

$$\begin{cases} iu_t + u_{xx} + \varepsilon|u|^2u = 0, & \text{in } \mathbb{R}\times(0,+\infty),\\ u(x,0) = u_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.5}$$

Combining this result and classical arguments, we are in position to establish the exponential decay rate for Eq. (1.1) given by

$$E_0(t) \leqslant Ce^{-\omega t}, \quad t > 0,$$

where $C > 0$ is a positive constant which depends on $\|u_0\|_{H^1}$, $\omega > 0$ and $E_0(t) = \int_{\mathbb{R}} |u(x,t)|^2\,dx\,dt$ is the energy in L^2-level. If we consider the second equation (1.2), a cubic-polynomial decay rate for the energy in L^2-level is obtained, that is,

$$E_0(t) \leqslant \frac{C}{t^3}, \quad t > 0.$$

Current literature. It is important to point out that the stability results presented in this paper are quite new, since we are concerned with dispersive equations posed on the *whole real line*. In bounded domains, many stability results are given in the literature, particularly when a linear and localized dissipative mechanism is considered ($g(s) = s$), see, for example, [1,2,9,13,15,16] and references therein.

In [17], Rosier and Zhang established exact controllability results in H^s-level for the Schrödinger equation, posed on a bounded domain $\Omega \subset \mathbb{R}^N$,

$$iu_t + \Delta u + \gamma |u|^2 u = 0, \tag{1.6}$$

where $\gamma \in \mathbb{R}$, and either Dirichlet or Neumann boundary conditions are considered (for $s > N/2$, or $0 \leqslant s < N/2$ with $1 \leqslant N < 2s+2$, or $s = 0, 1$ with $N = 2$). As far as we are concerned, the stabilization in L^2-level, for the Schrödinger equation posed on whole real line, is an open and interesting problem.

A pioneer study regarding dispersive equations in the whole real line, was presented by Cavalcanti, Domingos Cavalcanti and Natali in [4]. They showed the exponential decay rates for the Korteweg–de Vries equation with localized damping

$$u_t + uu_x + u_{xxx} + d(x)u = 0, \quad \text{in } \mathbb{R} \times (0, +\infty), \tag{1.7}$$

where $d \in W^{2,\infty}(\mathbb{R})$ is a nonnegative real function satisfying $d(x) \geqslant \alpha_0 > 0$ for $|x| > R_1$, $R_1 > 0$. In addition, according to our best knowledge, to deal with dissipative effects in the *whole line* and assuming that the function a (respectively, b) in hypothesis (A) (respectively, (B)), as in the present paper, have never been treated so far in the literature. This brings new difficulties when establishing the uniform stabilization of the energy, namely, $E_0(t) := \|u(t)\|^2_{L^2(\mathbb{R})}$, $t \geqslant 0$.

In [18], Shimomura studied the asymptotic behavior in time of small solutions for the initial value problem, given by

$$\begin{cases} iu_t + \dfrac{1}{2}\Delta u + \lambda |u|^{\frac{2}{N}} u = 0, & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.8}$$

where $N = 1, 2, 3$ and $\lambda = \lambda_1 + i\lambda_2$ is a complex constant. He showed that whether $\operatorname{Im}\lambda > 0$, there exists a unique global solution for the initial value problem which decays like $(t\log t)^{-\frac{N}{2}}$ as $t \to +\infty$ in $L^\infty(\mathbb{R}^N)$ for small initial data. In [3], Cavalcanti, Domingos Cavalcanti, Fukuoka and Natali proved the exponential decay rate of the energy, in L^2-level, concerning the following cubic *defocusing* nonlinear Schrödinger equation

$$iu_t + \Delta u - |u|^2 u + i\eta(x)u = 0, \quad \text{in } \mathbb{R}^2 \times (0, +\infty), \tag{1.9}$$

where $\eta \in W^{1,\infty}(\mathbb{R}^2)$ and is a nonnegative function verifying $\eta(x) \geqslant a_0 > 0$ a.e. in $\Omega_R := \{x \in \mathbb{R}^2;\ |x| \geqslant R\}$, $R > 0$. Since η produces a localized dissipative effect in L^2-level, the authors also established in [3], a result of unique continuation in order to obtain the desired exponential stability result.

Recently, an important contribution due to Ohta and Todorova was established in [14]. They proved that the Schrödinger equation with linear full damping

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u + i\lambda u = 0, & \text{in } \mathbb{R}^N \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.10}$$

where $\lambda > 0$, $1 < p < 1 + \frac{4}{N-2}$, and $N \geqslant 3$ has global existence in H^1-norm, considering that the initial data u_0 belongs to an invariant set. They also proved that under certain conditions, a blow-up phenomena can occur in finite time (see Theorem 2 in [14]) if, for instance, $E_1(u_0) < 0$, where E_1 is the energy given by

$$E_1(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}. \tag{1.11}$$

In order to establish the current results, our paper is organized as follows. In Section 2 we prove a well-posedness result concerning Eqs. (1.1) and (1.2). Exponential and polynomial decay rates associated to these equations are presented in Section 3.

Notation. Let us consider $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. We consider the usual Sobolev spaces $W^{m,p}(\mathbb{R})$ given by

$$W^{m,p}(\mathbb{R}) = \left\{u \in L^p(\mathbb{R});\ \partial_x^j u \in L^p(\mathbb{R}),\ j = 1,2,\dots,m\right\}$$

endowed with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{W^{m,p}(\mathbb{R})} = \left(\sum_{j=0}^{m}\int_{\mathbb{R}} \left|\partial_x^j u(x)\right|^p\right)^{\frac{1}{p}}.$$

In particular, when $m = 1$ and $p = 2$, $W^{m,p}(\mathbb{R})$ becomes the Hilbert space $H^1(\mathbb{R})$.

If $z = x + iy \in \mathbb{C}$, $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$ will denote the real and imaginary parts of $z \in \mathbb{C}$, respectively.

2. Well-posedness for Eqs. (1.1) and (1.2)

In order to obtain a well-posedness result regarding Eqs. (1.1) and (1.2), we shall combine the techniques due to Kato, established in [11] (see also [5] and [8]), concerning local well-posedness for Schrödinger-type equations given by

$$\begin{cases} iu_t + u_{xx} + |u|^{\alpha-1}u = 0, & \text{in } \mathbb{R} \times (0,+\infty), \\ u(0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $\alpha \in (1,5)$. The next results summarize our intentions.

Theorem 2.1. *For all $u_0 \in L^2(\mathbb{R})$ there exist $T = T(\|u_0\|_{L^2})$ and a unique mild solution u related to Eqs.* (1.1) *and* (1.2) *such that*

$$u \in C\big([0,T]; L^2(\mathbb{R})\big) \cap L^8\big([0,T]; L^4(\mathbb{R})\big).$$

Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R})$ such that

$$F : V \mapsto C\big(\big[0, T'\big]; L^2(\mathbb{R})\big) \cap L^8\big(\big[0, T'\big]; L^4(\mathbb{R})\big)$$

is Lipschitz.

Proof. See [5] and [8]. □

Theorem 2.2. *For all $u_0 \in H^1(\mathbb{R})$ there exist $T = T(\|u_0\|_{H^1})$ and a unique mild solution u related to Eqs.* (1.1) *and* (1.2) *such that*

$$u \in C\big([0, T]; H^1(\mathbb{R})\big) \cap L^r\big([0, T]; W^{1,\rho}(\mathbb{R})\big),$$

where (r, ρ) is an admissible pair. Moreover, for all $T' < T$ there exists a neighborhood V of u_0 in $H^1(\mathbb{R})$ such that

$$F : V \mapsto C\big(\big[0, T'\big]; H^1(\mathbb{R})\big) \cap L^r\big(\big[0, T'\big]; W^{1,\rho}(\mathbb{R})\big)$$

is Lipschitz.

Proof. See [5] and [8]. □

2.1. Global well-posedness for Eqs. (1.1) and (1.2)

2.1.1. Global well-posedness for Eq. (1.1)

This subsection is devoted to the proof of global well-posedness result associated to Eq. (1.1) with initial data $u_0 \in H^1(\mathbb{R})$. In fact, multiplying the first equation in (1.1) by $\bar{u}$ and integrating over $(0, t) \times \mathbb{R}$, $0 < t < T'$, we have

$$\int_{\mathbb{R}} |u|^2\,dx = -2\int_0^t \int_{\mathbb{R}} a(x)|u|^2\,dx\,ds + \|u_0\|_{L^2}^2. \tag{2.2}$$

Then, we conclude that

$$u \text{ is bounded in } L^\infty\big([0, +\infty); L^2(\mathbb{R})\big). \tag{2.3}$$

Furthermore, from Eq. (1.1) we obtain the following identity:

$$i(u_t\bar{u} - \bar{u}_t u) + u_{xx}\bar{u} + \bar{u}_{xx}u + 2\varepsilon|u|^4 = 0. \tag{2.4}$$

On the other hand, multiplying the first equation in (1.1) by $-\bar{u}_t$ we obtain

$$\frac{d}{dt}\left[\int_{\mathbb{R}} \left(|u_x|^2 - \frac{1}{2}\varepsilon|u|^4\right) dx\right] = \int_{\mathbb{R}} a(x)\big(i(\bar{u}u_t - \bar{u}_t u)\big)\,dx. \tag{2.5}$$

Then, if we invoke (2.4) we obtain from (2.5), hypothesis (A) and (2.2) that (for the sake of simplicity, consider $\varepsilon = 1$)

$$\frac{d}{dt}\left[\int_{\mathbb{R}} \left(|u_x|^2 - \frac{1}{2}|u|^4\right) dx\right] \leqslant 2\|a\|_{W^{1,\infty}}\|u_0\|_{L^2}^2 + 2\|a\|_{W^{1,\infty}}\int_{\mathbb{R}} |u|^4\,dx + \frac{1}{4}\int_{\mathbb{R}} |u_x|^2\,dx. \tag{2.6}$$

Integrating (2.6) in $t \in [0, T')$, we obtain from Gagliardo–Nirenberg and Gronwall inequalities that

$$\left\|u(t)\right\|_{H^1}^2 \leqslant C\left(T', \|a\|_{W^{1,\infty}}, \|u_0\|_{H^1}\right)e^{\frac{c_0}{4}(1+c_1\|a\|_{W^{1,\infty}})T'}, \tag{2.7}$$

where the constant $c_1 > 0$ does not depend on $T' > 0$ and c_0 is the Gagliardo–Nirenberg constant.

Remark 2.1. It is important to mention that in the defocusing case, that is when $\varepsilon = -1$, a similar bound is obtained since we have the following inequality: $\|u_x(t)\|_{L^2}^2 \leqslant \int_{\mathbb{R}} |u_x|^2 + \frac{1}{2}|u|^4\,dx$.

Now, we are in position to establish the next result.

Theorem 2.3. *Suppose that function a verifies hypothesis* (A) *and consider* $u_0 \in H^1(\mathbb{R})$. *Then, there is a unique mild solution u for the Cauchy problem* (1.1) *which belongs to*

$$C\left([0,T], H^1(\mathbb{R})\right) \cap C^1\left([0,T], H^{-1}(\mathbb{R})\right),$$

for all $T > 0$ *and satisfies the inequality* (2.7). *Moreover, the map*

$$u_0 \in H^1(\mathbb{R}) \mapsto u \in C\left([0,T], H^1(\mathbb{R})\right)$$

is continuous for all $T > 0$.

The next lemma will be useful later. A similar result can be deduced for Eq. (1.2).

Lemma 2.1. *Suppose that* $a \in W^{1,\infty}(\mathbb{R})$ *satisfies assumption* (A) *and consider u a solution related to problem* (1.1) *according to Theorem* 2.1. *Then we have the following estimate*:

$$\int_0^{T'} \int_{|x| \leqslant R} \left|D_x^{1/2} u(x,t)\right|^2 dx\,dt \leqslant C\left(\|u_0\|_{L^2}, T', \|a\|_{L^\infty}\right), \tag{2.8}$$

where $T' > 0$ *was obtained in Theorem* 2.1 *and* $R > 0$.

Proof. The arguments in order to establish this result can be found in Constantin and Saut [7] (see Theorem 3.1 for a more complete explanation) and Linares and Ponce [8] (see p. 108). In fact, from Theorem 2.1 solution $u(\cdot)$ must satisfy the integral equation

$$u(t) = S(t)\left(u_0 + i\int_0^t S(-\tau)\left(\varepsilon|u|^2 u + ia(\cdot)u\right)(\tau)\,d\tau\right), \tag{2.9}$$

where $S(t)$, $t \geqslant 0$, denotes the semigroup related to Schrödinger equation and $\varepsilon = \pm 1$. Let us define

$$I\left(u(x,t)\right) = \left(\int_0^{T'} \int_{|x| \leqslant R} \left|D_x^{1/2} u(x,t)\right|^2 dx\,dt\right)^{1/2},$$

by using a well-known Strichartz type inequality and the smoothing effect in $H^{1/2}$-norm associated to linear Schrödinger equation (see Theorem 4.2 and Corollary 4.2 in Linares and Ponce [8], respectively) we obtain from (2.9),

$$I\big(u(x,t)\big) \leqslant CR\left(\|u_0\|_{L^2} + \sup_{[0,T']}\left\| \int_0^t S(\tau)\big(\varepsilon|u|^2u + ia(\cdot)u\big)(\tau)\,d\tau \right\|_{L^2}\right)$$

$$\leqslant CR\left(\|u_0\|_{L^2} + T'\|a\|_{L^\infty}\|u_0\|_{L^2} + \left(\int_0^{T'} \big\| \big|u(\tau)\big|^2 u(\tau)\big\|_{L^{4/3}}^{8/7}\,d\tau\right)^{7/8}\right). \tag{2.10}$$

The proof can be established by applying the L^2-theory related to Schrödinger equation as obtained in Theorem 2.1 (see Theorem 5.2 in [8]). □

2.1.2. Global well-posedness for Eq. (1.2)

The concept of global well-posedness in this subsection is the same as in the previous subsection, that is, we only need to prove that $T_{max} = +\infty$. We proceed as in Section 2.1. In fact, multiplying Eq. (1.2) by $\bar{u}$ we get, after integrating in $x \in \mathbb{R}$, the following identity:

$$\frac{d}{dt}\int_{\mathbb{R}} |u|^2\,dx + 2\int_{\mathbb{R}} b(x)|u|^4\,dx = 0. \tag{2.11}$$

Then, we integrate in time and using assumptions in (B), we obtain

$$u \text{ is bounded in } L^\infty\big([0,+\infty); L^2(\mathbb{R})\big). \tag{2.12}$$

On the other hand, it is possible to deduce a similar identity as in (2.4) in our case, that is,

$$i(u_t\bar{u} - \bar{u}_t u) + u_{xx}\bar{u} + \bar{u}_{xx}u + 2\varepsilon|u|^4 = 0. \tag{2.13}$$

Then, multiplying the first equation in (1.2) by $-\bar{u}_t$, we deduce the identity

$$\frac{d}{dt}\left[\int_{\mathbb{R}} |u_x|^2 - \frac{1}{2}\varepsilon|u|^4\,dx\right] = \int_{\mathbb{R}} b(x)|u|^2\big(i(\bar{u}_t u + \bar{u}u_t)\big)\,dx. \tag{2.14}$$

Considering $\varepsilon = 1$ and taking (2.13) and (2.14) into account, we obtain

$$\frac{d}{dt}\left[\int_{\mathbb{R}}\left(|u_x|^2 - \frac{1}{2}|u|^4\right)dx\right] \leqslant -2\int_{\mathbb{R}} b(x)\big(\mathrm{Re}\big[(u_x\bar{u})^2\big] + |u|^2|u_x|^2\big)\,dx$$

$$+ 2\int_{\mathbb{R}} b(x)|u|^6\,dx - 2\int_{\mathbb{R}} b'(x)|u|^2\,\mathrm{Re}(u_x\bar{u})\,dx. \tag{2.15}$$

Since $|z_1|^2|z_2|^2 + \mathrm{Re}[(z_1\bar{z}_2)^2] = 2[\mathrm{Re}(z_1\bar{z}_2)]^2 \geqslant 0$, $\forall z_1, z_2 \in \mathbb{C}$, from (2.11), (2.15) and the Gagliardo–Nirenberg inequality, we conclude that

$$\frac{d}{dt}\left[\int_{\mathbb{R}}\left(|u_x|^2 - \frac{1}{2}|u|^4\right)dx\right] \leqslant \left(6c_2\|b\|_{W^{1,\infty}}\|u_0\|_{L^2}^4 + \frac{1}{2}\right)\int_{\mathbb{R}} |u_x|^2\,dx, \tag{2.16}$$

where $c_2 > 0$ is the constant deriving from Gagliardo–Nirenberg inequality. Integrating (2.16) over $t \in [0,T')$ we obtain

$$\|u(t)\|_{H^1}^2 \leqslant 2\left(c_3\|u_0\|_{H^1}^4+\frac{3}{2}\right)\|u_0\|_{H^1}^2+\left(6c_4\|b\|_{W^{1,\infty}}\|u_0\|_{H^1}^4+\frac{1}{2}\right)\int_0^t \|u(s)\|_{H^1}^2\,ds. \tag{2.17}$$

Defining $K_1 = 2(c_3\|u_0\|_{H^1}^4 + \frac{3}{2})\|u_0\|_{H^1}^2$ and $K_2 = (6c_4\|b\|_{W^{1,\infty}}\|u_0\|_{H^1}^4 + \frac{1}{2})$, (2.17) and Gronwall inequality implies that

$$\|u(t)\|_{H^1}^2 \leqslant K_1 e^{K_2 T'}. \tag{2.18}$$

Now, we are able to establish the following result:

Theorem 2.4. *Let $b \in W^{1,\infty}(\mathbb{R})$ satisfy hypothesis* (B) *and consider $u_0 \in H^1(\mathbb{R})$. Then, there is a unique mild solution u for the Cauchy problem* (1.2) *which belongs to*

$$C\big([0,T],H^1(\mathbb{R})\big)\cap C^1\big([0,T],H^{-1}(\mathbb{R})\big),$$

for all $T>0$, and satisfies inequality (2.18)*. Moreover, the map*

$$u_0 \in H^1(\mathbb{R}) \mapsto u \in C\big([0,T],H^1(\mathbb{R})\big)$$

is continuous for all $T>0$.

Remark 2.2. Similar results as obtained in Theorems 2.3, 2.4 and Lemma 2.1 can be established if we consider, in equation (1.2), $\varepsilon = -1$.

3. Exponential and polynomial decay

3.1. Exponential decay concerning Eq. (1.1)

In this subsection we are interested in obtaining exponential decay rate for the energy in $L^2(\mathbb{R})$-norm related to the DCNLS equation (1.1). In order to obtain the desired result, we begin multiplying the first equation in (1.1) by $\bar{u}$ and integrating over $\mathbb{R}$ and then, over $[0,+\infty)$:

$$E_0(t) := \int_{\mathbb{R}} \big|u(x,t)\big|^2\,dx = -2\int_0^t\int_{\mathbb{R}} a(x)\big|u(x,t)\big|^2\,dx\,ds + \|u_0\|_{L^2}^2 \leqslant \|u_0\|_{L^2}^2. \tag{3.1}$$

From Eq. (1.1) we have that $\frac{d}{dt}E_0(t) := \frac{d}{dt}\int_{\mathbb{R}} |u(x,t)|^2\,dx = -2\int_{\mathbb{R}} a(x)|u(x,t)|^2\,dx$ and, consequently, we have the following energy estimate:

$$\int_0^T E_0(t)\,dt \leqslant -\alpha_0^{-1}\left[\int_{\mathbb{R}} |u|^2\,dx\right]_0^T + \underbrace{2\int_0^T \int_{\{x\in\mathbb{R};\,|x|\leqslant R_1\}} |u|^2\,dx\,dt}_{\mathcal{I}}. \tag{3.2}$$

Now, we are in a position to establish the following result:

Theorem 3.1. *Consider the potential $a(\cdot)$ satisfying hypothesis* (A). *For any $L > 0$, there are $c = c(L) > 0$ and $\omega = \omega(L)$ such that*

$$E_0(t) \leqslant ce^{-\omega t},$$

for all $t \geqslant 0$ and for any solution of (1.1) *given in Theorem* 2.3 *and provided that the initial data satisfies $\|u_0\|_{H^1(\mathbb{R})} \leqslant L$.*

Before establishing the above result, we need to prove a preliminary lemma which gives an estimate for the integral equation $\mathcal{I}$. Consider T_0 a positive constant and suppose that the initial data lies in a bounded set of H^1, then we have the following lemma:

Lemma 3.1. *Let u be a mild solution associated to the DCNLS equation* (1.1) *with initial data u_0 belonging to a bounded set of $H^1(\mathbb{R})$. Then, for all $T > T_0$ there exists a positive constant $c_5 > 0$ which depends on T such that the following inequality holds,*

$$\int_0^T \int_{|x| \leqslant R_1} |u|^2 \, dx\, dt \leqslant c_5 \int_0^T \int_{\mathbb{R}} a(x)|u|^2 \, dx\, dt. \tag{3.3}$$

Proof. First of all, we observe that from continuous dependence and density arguments it is sufficient to establish the result for strong solutions. Consider $B_{R_1} := \{x \in \mathbb{R};\ |x| \leqslant R_1,\ R_1 > 0\}$. We argue by contradiction. Let us suppose that (3.3) is not true and let $\{u_k(0)\}_{k\in\mathbb{N}}$ be a sequence of initial data where the corresponding solutions $\{u_k\}_{k\in\mathbb{N}}$ of (1.1) with $E_0^k(0)$, defined in (3.1) for all $k \in \mathbb{N}$, which is assumed to be uniformly bounded by a positive constant $L > 0$ in k, verifies

$$\lim_{k\to+\infty} \frac{\int_0^T \|u_k(t)\|^2_{L^2(B_{R_1})}\, dt}{\int_0^T \int_{\mathbb{R}} (a(x)|u_k|^2)\, dx\, dt} = +\infty, \tag{3.4}$$

that is,

$$\lim_{k\to+\infty} \frac{\int_0^T \int_{\mathbb{R}} (a(x)|u_k|^2)\, dx\, dt}{\int_0^T \|u_k(t)\|^2_{L^2(B_{R_1})}\, dt} = 0. \tag{3.5}$$

Since,

$$E_0^k(t) \leqslant E_0^k(0) \leqslant L,$$

we obtain a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, still denoted by $\{u_k\}_{k\in\mathbb{N}}$ from now on, which verifies the convergence:

$$u_k \rightharpoonup u \quad \text{weakly in } L^\infty\big([0,T]; L^2(\mathbb{R})\big). \tag{3.6}$$

From (3.5) and (3.6) we deduce

$$\lim_{k\to+\infty} \int_0^T \int_{\mathbb{R}} a(x)|u_k|^2 \, dx\, dt = 0, \tag{3.7}$$

consequently, from hypothesis (A) we have

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}\setminus B_{R_1}}|u_k|^2\,dx\,dt=0. \tag{3.8}$$

On the other hand, from (2.7) we guarantee that $\{u_k\}_{n\in\mathbb{N}}$ is bounded in $C([0,T];H^1(B_{R_1}))$ for $T>0$ arbitrary but fixed. Further, since $H^1(B_{R_1})$ is compactly embedded in $L^p(B_{R_1})$, $2\leqslant p\leqslant\infty$, we guarantee the existence of a subsequence, still denoted by $\{u_k\}_{k\in\mathbb{N}}$, such that

$$u_k\to u\quad\text{strong in } L^p\big(B_{R_1}\times[0,T]\big), \tag{3.9}$$

where $2\leqslant p<\infty$. Therefore,

$$u_k\to u,\quad\text{a.e. in } B_{R_1}\times(0,T). \tag{3.10}$$

Statements (3.8) and (3.10) enable us to deduce the following convergence:

$$u_k\to\tilde{u},\quad\text{a.e. in } \mathbb{R}\times(0,T), \tag{3.11}$$

where

$$\tilde{u}=\begin{cases}u, & \text{a.e. in } B_{R_1}\times(0,T),\\ 0, & \text{a.e. in } \mathbb{R}\setminus B_{R_1}\times(0,T).\end{cases}$$

In addition, from (2.7) we have

$$u_{x,k}\rightharpoonup u_x\quad\text{weakly in } L^2\big([0,T];L^2(\mathbb{R})\big). \tag{3.12}$$

At this point we will divide the proof into two cases, namely: when $u\neq0$ and $u=0$.

Case (I): $u\neq0$.

Passing to the limit in Eq. (1.1), when $k\to+\infty$, we get that the mild solution u satisfies

$$\begin{cases}iu_t+u_{xx}+\varepsilon|u|^2u=0, & \text{in } C\big([0,T];H^{-1}(\mathbb{R})\big),\\ u=0, & \text{a.e. in } \mathbb{R}\setminus B_{R_1}\times(0,T).\end{cases} \tag{3.13}$$

From Unique Continuation Principle due to Zhang [19], we conclude that $u\equiv0$ a.e. in $B_{R_1}\times(0,T)$. Being $u\equiv0$ a.e. in $\mathbb{R}\setminus B_{R_1}\times(0,T)$ we get $u\equiv0$ a.e. in $\mathbb{R}\times(0,T)$; which contradicts the fact that $u\neq0$.

Case (II): $u=0$.

We denote

$$\nu_k=\|u_k\|_{L^2([0,T];L^2(B_{R_1}))}. \tag{3.14}$$

Then, if we define $v_k=\frac{u_k}{\nu_k}$, we obtain

$$\|v_k\|_{L^2([0,T];L^2(B_{R_1}))}=1,\quad\forall k\in\mathbb{N}. \tag{3.15}$$

Next, we see from (3.2), (3.15) and the energy identity in (2.2) applied to v_k that

$$\left\| v_k(0) \right\|^2_{L^2(\mathbb{R})} = \frac{\|u_{0,k}\|^2_{L^2(\mathbb{R})}}{\nu_k^2} \leqslant c_6 \left(\int_0^T \int_{\mathbb{R}} a(x) \left| v_k(x,t) \right|^2 dx\, dt + 1 \right) \leqslant M, \tag{3.16}$$

which establishes a bound for the initial data $v_k(0)$ in L^2-level. Therefore, from Theorem 2.1 we obtain that v_k satisfies the equation

$$i v_{t,k} + v_{xx,k} + |u_k|^2 v_k + i a(x) v_k = 0, \quad \text{in } \mathcal{D}'\big(\mathbb{R} \times (0,T)\big). \tag{3.17}$$

From (3.4) we have that

$$\lim_{k \to +\infty} \frac{\int_0^T \|v_k(t)\|^2_{L^2(B_{R_1})}\, dt}{\int_0^T \int_{\mathbb{R}} (a(x)|v_k|^2)\, dx\, dt} = +\infty, \tag{3.18}$$

and from (3.15) it comes that

$$\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}} \big(a(x)|v_k|^2\big)\, dx\, dt = 0. \tag{3.19}$$

Since $a(x) \geqslant \alpha_0 > 0$ for $|x| \geqslant R_1$, we obtain from (3.19)

$$\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R} \setminus B_{R_1}} |v_k|^2\, dx\, dt = 0. \tag{3.20}$$

Thus,

$$v_k \to 0 \quad \text{in } L^2\big([0,T]; L^2(\mathbb{R} \setminus B_{R_1})\big). \tag{3.21}$$

From (3.16), (3.17), (3.19) and (3.20) we get a function v which verifies $v_k \rightharpoonup \tilde{v}$ in $L^2([0,T]; L^2(\mathbb{R}))$, where

$$\tilde{v} = \begin{cases} v, & \text{a.e. in } B_{R_1} \times (0,T), \\ 0, & \text{a.e. in } \mathbb{R} \setminus B_{R_1} \times (0,T), \end{cases}$$

and

$$\begin{cases} i v_t + v_{xx} = 0, & \text{in } \mathcal{D}'\big(\mathbb{R} \times [0,T]\big), \\ v = 0, & \text{a.e. in } \mathbb{R} \setminus B_{R_1}. \end{cases} \tag{3.22}$$

Therefore, from Holmgreen's Theorem we conclude that $v \equiv 0$ in $B_{R_1} \times (0,T)$. The proof is not complete. For this purpose we need to use a similar inequality as established in Lemma 2.1. Indeed, from (2.10), (3.16) and the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we have

$$\begin{aligned} I\big(v_k(x,t)\big) &\leqslant CR_1\left(1+T\|a\|_{L^\infty}+\left(\int_0^T \big\|\big(|u_k|^2 v_k\big)(\tau)\big\|_{L^{4/3}}^{8/7}\,d\tau\right)^{7/8}\right)\\ &= CR_1\left(1+T\|a\|_{L^\infty}+\left(\int_0^T \big\|v_k^2\big(|v_k|^2 v_k\big)(\tau)\big\|_{L^{4/3}}^{8/7}\,d\tau\right)^{7/8}\right)\\ &= CR_1\left(1+T\|a\|_{L^\infty}+C\big(\big\|u_k(0)\big\|_{H^1},T\big)\left(\int_0^T \big\|v_k(\tau)\big\|_{L^2}^{12/7}\,d\tau\right)^{7/8}\right)\\ &\leqslant CR_1\big(1+T\|a\|_{L^\infty}+C\big(\big\|u_k(0)\big\|_{H^1},T\big)T^{7/8}\big), \end{aligned} \tag{3.23}$$

for all $T>0$, where $T>0$ does not depend on $k\in\mathbb{N}$. On the other hand, since $\{u_k(0)\}_{k\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R})$ and v_k is bounded in $L^2([0,T];L^2(\mathbb{R}))$, we can conclude from (3.23) that

$$v_k \text{ is bounded in } L^2\big([0,T];H^{1/2}(B_{R_1})\big). \tag{3.24}$$

Considering standard compactness arguments and having in mind that $v\equiv 0$ one has

$$\lim_{k\to+\infty}\int_0^T\int_{B_{R_1}}\big|v_k(x,t)\big|^2\,dx\,dt=0. \tag{3.25}$$

In addition, note that from (3.15) we infer

$$\lim_{k\to\infty}\int_0^T\int_{B_{R_1}}\big|v_k(x,t)\big|^2\,dx\,dt=\lim_{k\to\infty}\|v_k\|_{L^2([0,T];L^2(B_{R_1}))}^2=1, \tag{3.26}$$

which establishes a contradiction. The proof is now completed. $\square$

Proof of Theorem 3.1. Indeed, from (3.2) and (3.3) we deduce that

$$\int_0^T E_0(t)\,dt\leqslant\frac{\alpha_0^{-1}}{2}E_0(0)+c_7\int_0^T\int_{\mathbb{R}}a(x)\big|u(x,t)\big|^2\,dx\,dt,\quad\text{for all } T>T_0. \tag{3.27}$$

Next, by using identity of the energy in L^2-level, namely:

$$E_0(t)-E_0(0)=-2\int_0^t\int_{\mathbb{R}}a(x)\big|u(x,t)\big|^2\,dx\,dt,\quad\text{for all } t\geqslant 0, \tag{3.28}$$

we infer that $E(t)$ is non-increasing, and, furthermore that

$$2\int_0^t a(x)\big|u(x,s)\big|^2\,dx\,ds=E_0(0)-E_0(t),\quad\text{for all } t\geqslant 0. \tag{3.29}$$

Thus, combining (3.27) and (3.29), we have

$$T E_0(T) \leqslant \frac{\alpha_0^{-1}}{2}\left[E_0(T) + 2\int_0^T a(x)\left|u(x,t)\right|^2 dx\,dt\right] + c_7 \int_0^T \int_{\mathbb{R}} a(x)\left|u(x,t)\right|^2 dx\,dt, \tag{3.30}$$

which implies that for $T > T_0$,

$$\left(T - \frac{\alpha_0^{-1}}{2}\right) E_0(T) \leqslant \left(\alpha_0^{-1} + c_4\right) \int_0^T a(x)\left|u(x,t)\right|^2 dx\,dt. \tag{3.31}$$

For $T > \max\{\frac{\alpha_0^{-1}}{2}, T_0\}$ the last inequality yields

$$E_0(T) \leqslant C(T) \int_0^T a(x)\left|u(x,t)\right|^2 dx\,dt. \tag{3.32}$$

Finally, combining (3.29) and (3.32) we obtain

$$E_0(T) \leqslant C(T)\left[\frac{E_0(0) - E_0(T)}{2}\right], \tag{3.33}$$

that is,

$$E_0(T) + \frac{C(T)}{2} E_0(T) \leqslant \frac{C(T)}{2} E_0(0), \tag{3.34}$$

which leads us

$$E_0(T) \leqslant \alpha E_0(0), \quad \text{where } \alpha = \frac{\frac{C(T)}{2}}{1 + \frac{C(T)}{2}}. \tag{3.35}$$

Since in (3.35) we have $\alpha < 1$, we get by employing the semigroup property (see arguments due to Zuazua in [20]), the exponential decay. $\square$

3.2. Polynomial decay concerning Eq. (1.2)

Multiplying (1.2) by $\bar{u}$, we conclude that

$$E_0(t_2) - E_0(t_1) = -2 \int_{t_1}^{t_2} \int_{\mathbb{R}} b(x)\left|u(x,s)\right|^4 dx\,ds, \tag{3.36}$$

where $E_0(t)$ is the energy in L^2-level given by

$$E_0(t) := \int_{\mathbb{R}} \left|u(x,t)\right|^2 dx. \tag{3.37}$$

We have a similar result to the one established in Lemma 3.1 whose proof is analogue with the appropriate modifications. We note that in this case, the constant c_5 cannot depend on $T > 0$ as in Lemma 3.1. This fact is essential in order to apply the approach in [12].

Lemma 3.2. *Let u be a mild solution associated to the DCNLS equation* (1.2) *with initial data u_0 belonging to a bounded set of $H^1(\mathbb{R})$. Then, there exists $T_0 > 0$ such that if $T > T_0$ the following inequality holds*:

$$\int_0^T \int_{|x| \leqslant R_2} |u|^2 \, dx \, dt \leqslant c_8 \int_0^T \int_{\mathbb{R}} b(x)|u|^4 \, dx \, dt, \tag{3.38}$$

where c_8 is a positive constant which depends on T_0.

Proof. We are going to omit the proof since the arguments are similar to Lemma 3.1. Moreover, a similar result as in Lemma 2.1 makes necessary. $\square$

Before going on, we need to prove a basic result which will be used later.

Lemma 3.3. *Let a, b be real numbers. Then,*

$$a^2 - \theta b^2 \leqslant \theta (a - b)^2,$$

provided that $\theta > 2$.

Proof. Let us define $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = \theta(x - y)^2 - (x^2 - \theta y^2) = (\theta - 1)x^2 + 2\theta y^2 - 2\theta xy$. F is clearly twice differentiable on $\mathbb{R}^2$ and it possesses a unique critical point, namely $(0, 0)$. Since $\theta > 2$, we get $F_{xx}(x, y) = 2(\theta - 1) > 0$, $F_{yy}(x, y) = 4\theta > 0$ and $F_{xx}(x, y)F_{yy}(x, y) - F_{xy}(x, y)^2 = 4\theta(\theta - 2) > 0$. Therefore, we are able to conclude that $(0, 0)$ is an absolute minimum for F. This argument establishes the lemma. $\square$

Next, we present a technical lemma which will be useful to deduce the cubic-polynomial stability result.

Lemma 3.4. *Consider $\beta_0 > 0$ as in hypothesis* (B)*. Then, there is a constant $c_9 > 0$ such that, for all $\sigma > 1$, $\theta > 2$ and $T > S > 0$, we get*

$$\begin{aligned} c_9 \beta_0 \int_S^T E_0^{\sigma+2}(t) \, dt \leqslant{}& -\frac{\theta^2}{2} \left[E_0^\sigma(t) \int_{\mathbb{R}} |u|^2 \, dx \right]_S^T + \theta^2 c_9 \int_S^T E_0^\sigma(t) \big\| \sqrt{b(\cdot)} u(t) \big\|_{L^2(\mathbb{R} \setminus B_r)}^4 \, dt \\ & + \theta c_9 \beta_0 \int_S^T E_0^\sigma(t) \int_{B_{R_2}} |u|^2 \, dx \, dt, \end{aligned} \tag{3.39}$$

where B_r denotes the closed ball with center at origin and radius $r > 0$.

Proof. Multiplying (1.2) by $E_0^\sigma(t)\bar{u}$ and taking the complex conjugate, we obtain after integration over the rectangle $[S, T] \times \mathbb{R}$ that

$$0=\int_S^T\int_{\mathbb{R}}\left(E_0^\sigma(t)\frac{d}{dt}|u|^2+2b(x)|u|^4E_0^\sigma(t)\right)dx\,dt$$

$$=2\int_S^T\int_{\mathbb{R}}E_0^\sigma(t)b(x)|u|^4\,dx\,dt-\int_S^T\int_{\mathbb{R}}\sigma|u|^2E_0^{\sigma-1}(t)E_0'(t)\,dx\,dt+\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T. \tag{3.40}$$

Since $E_0'(t)\leqslant 0$ for all $t\in(0,T)$ and $L^4(B_r)\hookrightarrow L^2(B_r)$, we conclude that

$$c_9\int_S^T E_0^\sigma(t)\left(\int_{B_r}\sqrt{b(x)}|u|^2\,dx\right)^2dt\leqslant\int_S^T\int_{B_r}E_0^\sigma(t)b(x)|u|^4\,dx\,dt\leqslant-\frac{1}{2}\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T, \tag{3.41}$$

where $c_9=c_9(B_r)>0$. Then,

$$c_9\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}}\sqrt{b(x)}|u|^2\,dx-\int_{\mathbb{R}\backslash B_r}\sqrt{b(x)}|u|^2\,dx\right]^2dt\leqslant-\frac{1}{2}\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T. \tag{3.42}$$

From Lemma 3.3 and inequality (3.42) we have

$$c_9\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}}\sqrt{b(x)}|u|^2\,dx\right]^2dt$$

$$\leqslant-\frac{\theta}{2}\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T+\theta c_9\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}\backslash B_r}\sqrt{b(x)}|u|^2\,dx\right]^2dt. \tag{3.43}$$

Thus, since $a^2\leqslant(a+b)^2$, for all $a,b\geqslant 0$, we conclude

$$c_9\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}\backslash B_{R_2}}\sqrt{b(x)}|u|^2\,dx\right]^2dt$$

$$\leqslant-\frac{\theta}{2}\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T+\theta c_9\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}\backslash B_r}\sqrt{b(x)}|u|^2\,dx\right]^2dt. \tag{3.44}$$

Now, we are able to use hypothesis (B) in order to obtain

$$c_9\beta_0\int_S^T E_0^\sigma(t)\left[\int_{\mathbb{R}\backslash B_{R_2}}|u|^2\,dx\right]^2dt$$

$$\leqslant-\frac{\theta}{2}\left[E_0^\sigma(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T+\theta c_9\int_S^T E_0^\sigma(t)\left\|\sqrt{b(\cdot)}u(t)\right\|_{L^2(\mathbb{R}\backslash B_r)}^4dt. \tag{3.45}$$

Using Lemma 3.3 once more, we conclude from (3.45) that

$$c_9\beta_0\int_S^T E_0^{\sigma+2}(t)\,dt \leqslant -\frac{\theta^2}{2}\left[E_0^{\sigma}(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T + \theta^2 c_9\int_S^T E_0^{\sigma}(t)\left\|\sqrt{b(\cdot)}u(t)\right\|_{L^2(\mathbb{R}\backslash B_r)}^4 dt + \theta c_9\beta_0\int_S^T E_0^{\sigma}(t)\int_{B_{R_2}}|u|^2\,dx\,dt. \qquad \square \tag{3.46}$$

Considering the result established in Lemma 3.4, employing Lemma 3.2, Young inequality and identity (3.36) we get

$$\begin{aligned} c_9\beta_0\int_S^T E_0^{\sigma+2}(t)\,dt &\leqslant -\frac{\theta^2}{2}\left[E_0^{\sigma}(t)\int_{\mathbb{R}}|u|^2\,dx\right]_S^T + \theta^2 c_9\int_S^T E_0^{\sigma}(t)\left\|\sqrt{b(\cdot)}u(t)\right\|_{L^2(\mathbb{R}\backslash B_r)}^4 dt + c_{10}\beta_0 E_0(S) \\ &\leqslant \left(c_{10}\beta_0 + \frac{\theta^2}{2}\right)E_0(S) + \theta^2 c_9\|b\|_{L^\infty}^2\int_S^T E_0^{\sigma+2}(t)\,dt, \end{aligned} \tag{3.47}$$

where $c_{10} = c_{10}(c_8, c_9, \theta, \bar{L}) > 0$, c_8 is the positive constant given in Lemma 3.2 and $\bar{L} > 0$ must satisfy $\|u_0\|_{H^1} \leqslant \bar{L}$.

Considering $\sigma = 2$ in inequality (3.47) we deduce

$$c_9\big(\beta_0 - \theta^2\|b\|_{L^\infty}^2\big)\int_S^T E_0^4(t)\,dt \leqslant \left(c_{10}\beta_0 + \frac{\theta^2}{2}\right)E_0(S). \tag{3.48}$$

Choosing $\beta_0 > 0$ such that $\beta_0 - \theta^2\|b\|_{L^\infty}^2 = c_{11} > 0$ we obtain the inequality

$$\int_S^T E_0^4(t)\,dt \leqslant c_{12}E_0(S), \quad \text{for all } S \geqslant 1, \tag{3.49}$$

where $c_{12} = c_{12}(c_9, c_{10}, c_{11}, \beta_0, \|b\|_{L^\infty}, \theta) > 0$. Therefore, from Lemma 9.1 in Komornik's book [12] and inequality (3.49) we get a constant $C > 0$ depending on $\|u_0\|_{H^1}$ (where $\|u_0\|_{H^1} \leqslant \bar{L}$) and $T_0 > 0$ which implies the cubic-polynomial stability.

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References

[1] E. Bisognin, V. Bisognin, O. Vera, Stabilization of solutions to higher-order nonlinear Schrödinger equation with localized damping, Electron. J. Differential Equations 06 (2007) 1–18.

[2] E. Bisognin, V. Bisognin, G. Perla Menzala, Exponential stabilization of a coupled system of Korteweg–de Vries equations with localized damping, Adv. Differential Equations 8 (2003) 443–469.

[3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, R. Fukuoka, F. Natali, Exponential stability for the 2-D defocusing Schrödinger equation with locally distributed damping, Differential Integral Equations 22 (2009) 617–636.
[4] M.M. Cavalcanti, V.N. Domingos Cavalcanti, F. Natali, Exponential decay rates for the damped Korteweg–de Vries type equation, preprint, 2008.
[5] T. Cazenave, Semilinear Schrödinger Equations, Courant Lect. Notes Math., vol. 10, 2003.
[6] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation, Nonlinear Anal. 14 (1990) 807–836.
[7] A. Constantin, J.-C. Saut, Local smoothing properties of dispersive equations, J. Amer. Math. Soc. 1 (1988) 413–439.
[8] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, Springer, New York, 2008.
[9] F. Linares, A.F. Pazoto, On the exponential decay of the critical generalized Korteweg–de Vries equation with localized damping, Proc. Amer. Math. Soc. 135 (2007) 1515–1522.
[10] J. Ginibre, G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, J. Math. Pures Appl. 64 (1985) 363–401.
[11] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 46 (1987) 113–129.
[12] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, Masson/John Wiley, Paris, 1994.
[13] C.P. Massarolo, G. Perla Menzala, A.F. Pazoto, On the uniform decay for the Korteweg–de Vries equation with weak damping, Math. Methods Appl. Sci. 30 (2007) 1419–1435.
[14] M. Ohta, G. Todorova, Remarks on global existence and blowup for damped nonlinear Schrödinger equations, Discrete Contin. Dyn. Syst. Ser. A 23 (2009) 1313–1325.
[15] A.F. Pazoto, Unique continuation and decay for the Korteweg–de Vries equation with localized damping, ESAIM Control Optim. Calc. Var. 11 (2005) 473–486.
[16] G.P. Menzala, C.F. Vasconcellos, E. Zuazua, Stabilization of the Korteweg–de Vries equation with localized damping, Quart. Appl. Math. 60 (2002) 111–129 (in English).
[17] L. Rosier, B.-Y. Zhang, Exact boundary controllability of the nonlinear Schrödinger equation, J. Differential Equations 246 (2009) 4129–4153.
[18] A. Shimomura, Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities, Comm. Partial Differential Equations 31 (2006) 1407–1423.
[19] B.-Y. Zhang, Unique continuation properties for the nonlinear Schrödinger equation, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 191–206.
[20] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Comm. Partial Differential Equations 15 (1990) 205–235.