Analytic-numerical solutions of diffusion mathematical models with delays

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Abstract

This work deals with the construction of analytic-numerical solutions of mixed problems for diffusion and reaction-diffusion equations with delay. Using the method of separation of variables, exact theoretical infinite series solutions are derived. Bounds on the truncation errors, when these series are approximated by finite sums, are given, thus providing constructive continuous numerical solutions with prescribed accuracy in bounded domains. In order to improve the computational efficiency of these numerical solutions, polynomial approximations to the initial functions are also considered.

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1. Introduction

Delay differential equations (DDEs) and partial functional differential equations (PFDEs) constitute basic modelling tools in situations where time lags or hereditary characteristics are present, and thus have found wide application in many scientific and technical fields, as in population ecology, control theory, viscoelasticity, materials with thermal memory, etc. (see [1–3] and references therein).

In this paper, we consider mixed problems for the generalized diffusion equation with a delay

\[ u_t(t, x) = a^2 u_{xx}(t, x) + b^2 u_{xx}(t - \tau, x), \quad t > \tau, \quad 0 \leq x \leq l, \]  

and for the delay reaction-diffusion equation

\[ u_t(t, x) = a^2 u_{xx}(t, x) + bu(t - \tau, x), \quad t > \tau, \quad 0 \leq x \leq l, \]  

where \( \tau > 0 \), with the initial condition

\[ u(t, x) = \varphi(t, x), \quad 0 \leq t \leq \tau, \quad 0 \leq x \leq l, \]  

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and Dirichlet boundary conditions
\[ u(t, 0) = u(t, l) = 0, \quad t \geq 0. \] (4)

Eq. (1) without delay, i.e., with \( b = 0 \),
\[ u_t(t, x) = a^2 u_{xx}(t, x), \] (5)

is the classical model for describing diffusion, heat conduction, viscous shear motion and other transport phenomena. It has been a longstanding question the infinite speed of propagation implied by this equation [4–6], and different variations of this model have been proposed to give rise to finite speeds, and thus account for thermal “inertia”, heat waves and delayed responses to thermal disturbances (see [7]).

To be specific, consider heat conduction and the Fourier law in one dimension
\[ q(t, x) = -ku_x(t, x), \] (6)

where \( q \) is the heat flux, \( u \) the temperature, and \( k > 0 \) the thermal conductivity. The classical diffusion model (5) is derived by combining (6) with the energy conservation equation
\[ -q_x(t, x) + Q(t, x) = C \rho u_t(t, x), \]

where \( C \rho \) is the volumetric heat capacity, \( Q \) the volumetric heat source, and \( a^2 = k/C \rho \) is the thermal diffusivity.

The classical model of heat conduction can be successfully applied to conventional technical problems, as it gives accurate macroscopic descriptions for the long-time behavior of systems with large spatial dimensions. However, when the microscale description of heat conduction and fast-transient processes are considered, as in the analysis of ultrafast laser heating on thin film structures or the second sound in helium [8–11], models including phase lags in the heat flux and/or the temperature gradient have to be employed [12,13].

In the dual-phase-lagging (DPL) model by Tzou [13], the Fourier law is replaced by
\[ q(t + \tau_q, x) = -ku_x(t + \tau_u, x), \] (7)

where \( \tau_q \) and \( \tau_u \) are the phase lags of the heat flux and the temperature gradient, respectively. It is common in the applications of the DPL model to use the first-order approximation of (7),
\[ q(t, x) + \tau_q \frac{\partial q}{\partial t}(t + x) \approx -k \left\{ u_x(t, x) + \tau_u \frac{\partial u_x}{\partial t}(t, x) \right\}, \] (8)

and to refer to the equation derived from this approximation as the DPL model [12]. In fact, for \( \tau_u = 0 \), (8) reduces to the well-known Cattaneo–Vernotte model [4–6].

However, retaining the original formulation of the DPL model, as given in (7), and using the appropriate energy conservation equation, it has been shown in [14,15] that, for \( \tau = \tau_q - \tau_u > 0 \), the DPL heat conduction problem is formulated as
\[ u_t(t', x) = \alpha u_{xx}(t' - \tau, x), \quad t' > \tau_q, \] (9)

where \( \alpha \) is the temperature, \( \alpha \) the thermal diffusivity, and \( t' = t + \tau_q \), which is of the form (1) with \( a = 0 \).

Regarding Eq. (2), it arises as the linearization of some usual models in population dynamics. The delayed logistic equation, also known as the Hutchinson or Wright equation (e.g., [16, p. 4], [17, p. 83]),
\[ N'(t) = rN(t) \left[ 1 - \frac{N(t - \tau)}{K} \right], \] (10)

where \( r \) is the intrinsic rate of growth and \( K \) the carrying capacity, allows for different factors producing lags in population growth to be incorporated through the delayed term \( N(t - \tau) \). When combined with population spread, a diffusion delay differential equation is obtained [18]. Linearization of (10) near the carrying capacity (e.g., [17, p. 84], [19, p. 18]),
\[ n(t) = N(t) - K, \]
yields the linear DDE
\[ n'(t) = -rn(t - \tau), \]
and the incorporation of a diffusion term results in a delayed reaction-diffusion equation of the form (2).

The method of separation of variables is a classical efficient technique to solve linear partial differential mixed problems [20,21], which has also been considered for some classes of PFDEs [22,23]. In [24], the method of separation of variables was used to obtain an exact infinite series solution for the problem (1), (3)–(4), in the case \( a \neq 0 \). In Section 2, it is shown that the case \( a = 0 \) can also be incorporated, and an analogous exact solution for the problem (2), (3)–(4) is derived. In Section 3, bounds on the truncation errors, when these series are approximated by finite sums, are given, thus providing constructive continuous numerical solutions with prescribed accuracy in bounded domains. Finally, in Section 4, with the aim of improving the computational efficiency of these numerical solutions, polynomial approximations to the initial function \( \phi(t, x) \) are considered, showing that it is possible to obtain, in suitable domains, truncated series approximations with truncation error bounds exponentially decaying with the number of terms in the finite sum.

2. Exact infinite series solutions

To obtain exact solutions of the problems (1), (3)–(4) and (2), (3)–(4), we use the method of separation of the variables, and seek solutions of (1) or (2) of the form \( T(t)X(x) \), leading to the separate problems for the time and space variables

\[ \frac{T'(t)}{a^2 T(t) + b^2 T(t - \tau)} = \frac{X''(x)}{X(x)} = -\lambda^2, \]

and

\[ \frac{T'(t) - b T(t - \tau)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2, \]

for (1) and (2), respectively.

Both equations share the same boundary value problem for the space variable, and boundary conditions (3) lead to eigenvalues

\[ \lambda_n = \frac{n\pi}{l}, \]

and eigenfunctions \( X_n(x) = \sin(\lambda_n x) \). For the time variable, similar initial value-DDE problems are obtained,

\[ T_n'(t) = -\lambda_n^2 \left[ a^2 T_n(t) + b^2 T_n(t - \tau) \right], \quad t > \tau, \]  

for Eq. (1), and

\[ T_n'(t) = -\lambda_n^2 a^2 T_n(t) + b T_n(t - \tau), \quad t > \tau, \]  

for Eq. (2), both with the initial condition

\[ T_n(t) = B_n(t), \quad 0 \leq t \leq \tau, \]  

where \( B_n(t) \) are the Fourier coefficients of the initial function \( \varphi \).

\[ B_n(t) = \frac{2}{l} \int_0^l \varphi(t, x) \sin\left(\frac{n\pi x}{l}\right) \, dx. \]

Hence, the proposed formal series solutions of the problems considered are of the form

\[ u(t, x) = \sum_{n=1}^{\infty} T_n(t) \sin(\lambda_n x). \]
Once the problems (11), (13) and (12), (13) are solved, the convergence and regularity properties of the corresponding candidate solutions (14) have to be proven. Problems (11), (13) and (12), (13) are of the general form

\[ T'(t) = \alpha T(t) + \beta T(t - \tau), \quad t > \tau, \]

\[ T(t) = f(t), \quad 0 \leq t \leq \tau. \]

In [24, Lemma 1], using the method of steps (see [25, pp. 45–47], [26, pp. 6–15]), the solution of this problem for the particular initial function \( f(t) = 1 \) was derived, and it was incorporated into an integral representation [26, p. 67] to obtain an explicit expression for an arbitrary differentiable initial function \( f \) [24, Theorem 1], in terms of the gamma and the complementary incomplete gamma functions [27]

\[ \Gamma(k) = \int_0^\infty e^{-s}s^{k-1}ds, \quad \Gamma(k, x) = \int_x^\infty e^{-s}s^{k-1}ds. \]

Writing \( \gamma = \beta / \alpha \), where \( \alpha \neq 0 \) was assumed, and

\[ Q(k, x) = \Gamma(k, x) / \Gamma(k), \]

the expression given in [24, Theorem 1] reads, for \( t \in [m\tau, (m + 1)\tau] \),

\[ T(t) = (f(\tau) + \gamma f(0)) \sum_{k=1}^m (-1)^{k-1} \gamma^{k-1} Q(k, -\alpha(t - k\tau)) \]

\[ + \sum_{k=1}^{m-1} (-1)^{k-1} \gamma^k \int_0^\tau Q(k, -\alpha(t - k\tau - s)) f'(s)ds \]

\[ + (-1)^{m-1} \gamma^m \int_{t-m\tau}^{t-m\tau} Q(m, -\alpha(t - m\tau - s)) f'(s)ds + (-1)^m \gamma^m f(t - m\tau). \]  

Hence, for \( \alpha = -\lambda_n^2 a^2, \beta = -\lambda_n^2 b^2 \), and \( f(t) = B_n(t) \), substituting the expression for \( T_n(t) \) given by (15) into (14), and writing \( c^2 = b^2 / a^2 = \gamma \), for \( \alpha \neq 0 \), gives the following candidate series solution for the problem (1), (3)–(4),

\[ u(t, x) = \sum_1^4, \]

where

\[ \sum_1 = \sum_{n=1}^\infty \sum_{k=1}^m (-1)^{k-1} c^{2(k-1)} (B_n(\tau) + c^2 B_n(0)) \sin(\lambda_n x) Q(k, a^2 \lambda_n^2 (t - k\tau)), \]

\[ \sum_2 = \sum_{n=1}^m \sum_{k=1}^{m-1} (-1)^{k-1} c^{2k} \sin(\lambda_n x) \int_0^\tau B_n'(s) Q(k, a^2 \lambda_n^2 (t - k\tau - s)) ds, \]

\[ \sum_3 = (-1)^{m-1} c^{2m} \sum_{n=1}^\infty \sin(\lambda_n x) \int_{t-m\tau}^{t-m\tau} B_n'(s) Q(m, a^2 \lambda_n^2 (t - m\tau - s)) ds, \]

\[ \sum_4 = (-1)^m c^{2m} \sum_{n=1}^\infty \sin(\lambda_n x) B_n(t - m\tau) = (-1)^m c^{2m} \psi(t - m\tau, x). \]

The convergence and regularity properties of this formal series were proved in [24, Theorem 2], under some regularity conditions on the initial function \( \psi \), thus providing an exact series solution of the problem (1), (3)–(4).

First, we note that the case \( \alpha = 0 \) can also be accommodated after some rewriting. Letting \( P(k, x) = 1 - Q(k, x) \), and using the relation [27, p. 262]

\[ P(k, x) = \frac{x^k \exp(-x)}{\Gamma(k + 1)} M(1, k + 1, x), \]
where $M$ is a confluent hypergeometric function [27, p. 504], that satisfies $M(1, k + 1, 0) = 1$, expression (15) can be written in the form

$$
T(t) = f(\tau) + (\alpha f(\tau) + \beta f(0)) \sum_{k=1}^{m} \frac{\beta^{k-1}(t - k\tau)^k}{k!} e^{\alpha(t - k\tau)} M(1, k + 1, -\alpha(t - k\tau)) + \sum_{k=1}^{m-1} \frac{\beta^{k}}{k!} \int_{0}^{\tau} (t - k\tau - s)^k e^{\alpha(t - k\tau - s)} M(1, k + 1, -\alpha(t - k\tau - s)) f'(s)ds + \frac{\beta^m}{m!} \int_{0}^{t-m\tau} (t - m\tau - s)^m e^{\alpha(t - m\tau - s)} M(1, m + 1, -\alpha(t - m\tau - s)) f'(s)ds.
$$

and, for $\alpha = 0$, it reduces to

$$
T(t) = f(\tau) + \beta f(0) \sum_{k=1}^{m} \frac{\beta^{k-1}(t - k\tau)^k}{k!} + \sum_{k=1}^{m-1} \frac{\beta^{k}}{k!} \int_{0}^{\tau} (t - k\tau - s)^k f'(s)ds + \frac{\beta^m}{m!} \int_{0}^{t-m\tau} (t - m\tau - s)^m f'(s)ds,
$$

which provides the expressions for $T_n(t)$ to substitute into (14), and so obtain the solution to the problem (1), (3)–(4) for the particular case $a = 0$.

Now we consider the second problem, (2), (3)–(4), and use expression (15) with $\alpha = -\lambda_n^2 a^2$, $\beta = b$, and $f(t) = B_n(t)$. Assuming both $a$ and $b$ to be nonzero, substituting the expression for $T_n(t)$ given by (15) into (14), and writing $c^2 = b^2/a^2 = -b\lambda_n^2 \gamma$, we obtain

$$
u(t, x) = u_1 + u_2 + u_3 + u_4,
$$

where

$$
u_1 = \sum_{n=1}^{\infty} \sum_{k=1}^{m} \left( \frac{b^2}{b\lambda_n^2} \right)^{k-1} \left( B_n(\tau) - \frac{c^2}{b\lambda_n^2} B_n(0) \right) \sin(\lambda_n x) Q \left( k, a^2\lambda_n^2(t - k\tau) \right),
$$

$$
u_2 = -\sum_{n=1}^{\infty} \sum_{k=1}^{m-1} \left( \frac{b^2}{b\lambda_n^2} \right)^{k} \sin(\lambda_n x) \int_{0}^{\tau} B_n'(s) Q \left( k, a^2\lambda_n^2(t - k\tau - s) \right) ds,
$$

$$
u_3 = -\sum_{n=1}^{\infty} \left( \frac{c^2}{b\lambda_n^2} \right)^{m} \sin(\lambda_n x) \int_{0}^{t-m\tau} B_n'(s) Q \left( m, a^2\lambda_n^2(t - m\tau - s) \right) ds,
$$

$$
u_4 = \sum_{n=1}^{\infty} \left( \frac{c^2}{b\lambda_n^2} \right)^{m} \sin(\lambda_n x) B_n(t - m\tau).
$$

We note that $u_4$ can be computed exactly, as it can be expressed in terms of the iterated integrals of the initial function (see Section 4).

The next theorem gives sufficient conditions for (17) to represent a solution of the problem (2), (3)–(4). The proof is similar to that in [24, Theorem 2], but the conditions on the initial function $\varphi$ can be weakened.

**Theorem 1.** Consider the problem (2), (3)–(4) and let $u$ be the candidate solution given in (17). Assume that $\varphi(., x)$ is continuously differentiable for each $x$, and $\varphi(t, .)$ is continuously differentiable for each $t$. Then $u(t, x)$ is continuous in $[\tau, \infty) \times [0, l]$, satisfies that $u_t(t, x)$ and $u_{xx}(t, x)$ are continuous in $(\tau, \infty) \times (0, l)$, and it is an exact solution of the problem (2), (3)–(4).

3. Numerical approximations with a priori error bounds

In this section, we consider the problem of approximating the exact solutions (16) and (17) by partial finite sums with $N$ terms. For every $0 < \delta < \tau$, and $(t, x) \in [m\tau + \delta, (m + 1)\tau] \times [0, l]$, we will obtain bounds on the truncation
errors in terms of \( N \), thus allowing the computation of continuous numerical solutions with the prescribed accuracy in these domains.

Next we show with some detail the derivation of the bounds for the terms in (16), and then will state the final results for both problems.

Consider \( \sum_1, \sum_2, \) and \( \sum_3 \), in (16), write \( d = \frac{\pi a}{T} \), and let

\[
R_{1,k}^N = \left| \sum_{n=N+1}^\infty \frac{(-c^2)^{k-1}}{\Gamma(k)} (B_n(\tau) + c^2 B_n(0)) \sin \left( \frac{n \pi x}{l} \right) \Gamma(k, n^2 d^2 (t - k \tau)) \right|,
\]

\[
R_{2,k}^N = \left| \sum_{n=N+1}^\infty \frac{(-c^2)^k}{\Gamma(k)} \sin \left( \frac{n \pi x}{l} \right) \int_0^\tau B_n'(s) \Gamma(k, n^2 d^2 (t - k \tau - s)) ds \right|,
\]

\[
R_{3,k}^N = \frac{(-c^2)^m}{\Gamma(m)} \sum_{n=N+1}^\infty \sin \left( \frac{n \pi x}{l} \right) \int_0^{\tau - m \tau} B_n'(s) \Gamma(m, n^2 d^2 (t - m \tau - s)) ds \right|.
\]

In [24, Theorem 2], it was assumed that \( \psi(t, .) \) and \( \varphi(t, .) \) were twice continuously differentiable for each \( t \), and thus we have the bounds on their Fourier coefficients [21, p. 41]

\[
|B_n(t)| \leq \frac{B}{n^2}, \quad |B_n'(t)| \leq \frac{B'}{n^2},
\]

for some positive constants \( B \) and \( B' \), and for all \( t \in [0, \tau] \).

Hence, writing \( B_1 = B + c^2 B \), we have

\[
\left| B_n(\tau) + c^2 B_n(0) \right| \leq \frac{B_1}{n^2},
\]

and

\[
R_{1,k}^N \leq \frac{B_1 c^{2k-2}}{\Gamma(k)} \sum_{n=N+1}^\infty \frac{\Gamma(k, n^2 d^2 (t - k \tau))}{n^2}.
\]

We will use [27, pp. 261–263] that \( \Gamma(k, v) \) is a monotone decreasing function of \( v \), with \( \Gamma(k, 0) = \Gamma(k) \) and \( \lim_{v \to \infty} \Gamma(k, v) = 0 \), and that for a positive integer \( k \), \( \Gamma(k) = k! \Gamma(k-1) = (k-1)! \), and

\[
\Gamma(k, v) = \Gamma(k) e^{-v} \sum_{j=0}^{k-1} \frac{v^j}{j!}.
\]

Thus, we have

\[
\sum_{n=N+1}^\infty \frac{\Gamma(k, n^2 d^2 (t - k \tau))}{n^2} \leq \Gamma(k, (N+1)^2 d^2 (t - k \tau)) \sum_{n=N+1}^\infty \frac{1}{n^2} \leq \Gamma(k, (N+1)^2 d^2 (t - k \tau)) \int_N^\infty \frac{du}{u^2} = \frac{\Gamma(k, (N+1)^2 d^2 (t - k \tau))}{N},
\]

so that

\[
R_{1,k}^N \leq \frac{B_1 c^{2k-2} \Gamma(k, (N+1)^2 d^2 (t - k \tau))}{\Gamma(k) N}.
\]

We note that the immediate bound from (18),

\[
\Gamma(k, N^2 d^2 (t - k \tau)) \leq \Gamma(k) e^{-N^2 d^2 (t - k \tau)} (1 + N^2 d^2 (t - k \tau))^{k-1},
\]

shows that the above bound on \( R_{1,k}^N \) is essentially exponentially decaying with \( N \).

Similarly, for \( R_{2,k}^N \) we have, since

\[
\Gamma(k, n^2 d^2 (t - k \tau - s)) \leq \Gamma(k, n^2 d^2 (t - (k+1) \tau))
\]
Consider the problem \( R_{2,k}^N \leq \frac{\sum_{n=N+1}^{\infty} c^{2k} \int_0^\tau |B_n'(s)| \Gamma(k, n^2 d^2(t - (k + 1)\tau)) ds}{\Gamma(k)} \leq \frac{B'\tau c^{2k}}{\Gamma(k)} \sum_{n=N+1}^{\infty} \frac{\Gamma(k, n^2 d^2(t - (k + 1)\tau))}{n^2} \leq \frac{B'\tau c^{2k}}{\Gamma(k)} \frac{\Gamma(k, (N + 1)^2 d^2(t - (k + 1)\tau))}{N} \).

Finally, for \( R_3^N \) we have,

\[
R_3^N \leq \frac{c^{2m}}{\Gamma(m)} \sum_{n=N+1}^{\infty} \int_0^{t-m\tau} \frac{1}{n^4 d^2} \frac{\Gamma(m, n^2 d^2(t - m\tau - s)) ds}{\Gamma(m, v) dv} \leq \frac{B' c^{2m}}{\Gamma(m) d^2} \int_0^\infty \frac{\Gamma(m, v) dv}{v^4} \sum_{n=N+1}^{\infty} \frac{1}{n^4},
\]

and taking into account that \( \int_0^\infty \frac{\Gamma(m, v) dv}{v^4} = \Gamma(m + 1) \), [27, p. 263], we obtain

\[
R_3^N \leq \frac{B' c^{2m} \Gamma(m + 1)}{\Gamma(m) d^2} \sum_{n=N+1}^{\infty} \frac{1}{n^4} \leq \frac{B' mc^{2m}}{d^2} \int_N^\infty \frac{du}{u^4} = \frac{B' mc^{2m}}{3d^2 N^3}.
\]

Therefore, we have the following result.

**Theorem 2.** Consider the problem (1), (3)–(4); let \( u \) be the exact solution given in (16), and let \( u_N \) be the approximation obtained when \( \sum_1, \sum_2, \text{ and } \sum_3 \) in (16) are substituted by the corresponding partial sums with the \( N \) first terms. Then, for \( (t, x) \in [m\tau, (m + 1)\tau] \times [0, 1] \), we have the bound

\[
|u_N(t, x) - u(t, x)| \leq \sum_{k=1}^{m} \frac{B_k c^{2k-2}}{\Gamma(k)} \frac{\Gamma(k, (N + 1)^2 d^2(t - k\tau))}{N} + \sum_{k=1}^{m-1} \frac{B' \tau c^{2k}}{\Gamma(k)} \frac{\Gamma(k, (N + 1)^2 d^2(t - (k + 1)\tau))}{N} + \frac{B' mc^{2m}}{3d^2 N^3}.
\]

Consequently, given \( \delta > 0 \) a prefixed admissible error \( \epsilon > 0 \), \( N \) can be found such that \( |u_N(t, x) - u(t, x)| \leq \epsilon \) holds for \( (t, x) \in [m\tau + \delta, (m + 1)\tau] \times [0, 1] \).

An example of application of these approximations \( u_N \) to compute numerical solutions of the problem (1), (3)–(4), for different values of the parameters \( a \) and \( b \) in (1), is shown in Fig. 1. An example of the error bounds given by Theorem 2 as a function of \( N \) is presented in Fig. 2 (left).

The next theorem gives the analogous result for problem (2), (3)–(4), and its derivation is similar to that of Theorem 2. We will denote by \( B \) and \( B' \) the upper bounds of \( |B_n(t)| \) and \( |B'_n(t)| \) for \( t \in [0, \tau] \), which exist by the conditions on \( \varphi \) assumed in Theorem 1, and write \( B_2 = B(1 + b/d^2) \).

**Theorem 3.** Consider the problem (2), (3)–(4); let \( u \) be the exact solution given in (17), and let \( u_N \) be the approximation obtained when \( u_1, u_2, \text{ and } u_3 \) in (17) are substituted by the corresponding \( N \) first terms partial sums.
Then, for \((t, x) \in [m\tau, (m + 1)\tau] \times [0, l]\), we have

\[
|u_N(t, x) - u(t, x)| \leq \frac{B_2}{d(t - \tau)^{1/2}} e^{-N^2d^2(t-\tau)} + \sum_{k=2}^{m} \frac{b^{k-1}B_2}{(2k-3)d^{2(k-1)}} \Gamma(k, (N+1)^2d^2(t-k\tau)) \Gamma(k)N^{2k-3} \\
+ \sum_{k=1}^{m-1} \frac{b^kB^{k\tau}}{(2k-1)d^{2k}} \Gamma(k, (N+1)^2d^2(t-(k+1)\tau)) \Gamma(k)N^{2k-1} + \frac{mB' b^m}{(2m+1)d^{2m+2}N^{2m+1}}.
\]

Consequently, given \(\delta > 0\) a prefixed admissible error \(\epsilon > 0\), \(N\) can be found such that \(|u_N(t, x) - u(t, x)| \leq \epsilon\) holds for \((t, x) \in [m\tau + \delta, (m + 1)\tau] \times [0, l]\).

4. Polynomial approximations to the initial function

In this section, we will focus on problem (1), (3)–(4), although similar results could be derived for (2), (3)–(4).

In the bounds given in Theorem 2 for the truncation errors, we note that the only term not including an exponentially decaying expression in \(N^2\) is the contribution of \(\sum_3\), for which the bound on \(R_3^N\) is \(O(N^{-3})\). Although it is to be expected that the precision of the finite sums approximation \(u_N\) is much higher than implied by the bounds of Theorem 2, we would like to ensure that the prefixed accuracy is fulfilled with as little computational effort as possible.

Thus, we consider here the approximation of the initial function \(\varphi\), or to be precise of its derivative \(\varphi_t\), by a polynomial of degree \(L\) in \(t\).
\[ P_{L, \varphi_i}(t, x) = \sum_{k=0}^{L} t^k f_k(x). \]  

The aim is to use this polynomial approximation in \( \sum_3 \), so that part of this expression could be computed analytically, while the rest could be approximated with exponential type error bounds.

First, consider an initial function whose derivative is the polynomial

\[ P_L(t, x) = \sum_{k=0}^{L} t^k f_k(x), \]

and let \( P_{n, L}(t) \) be its Fourier coefficients

\[ P_{n, L}(t) = \frac{2}{l} \int_0^l P_L(t, x) \sin \left( \frac{n \pi x}{l} \right) \, dx. \]

We note that if \( g_{n,k} \) are the Fourier coefficients of \( f_k(x) \),

\[ g_{n,k} = \frac{2}{l} \int_0^l f_k(x) \sin \left( \frac{n \pi x}{l} \right) \, dx, \]

then \( P_{n, L}(t) = \sum_{k=0}^{L} t^k g_{n,k}. \)

Hence, letting aside the constant \((-1)^{m-1} c_2^m\), \( \sum_3 \) in (16) can be expressed in the form

\[
\sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{l} \right) \int_0^{t-m} P_{n, L}(s) Q(m, n^2 d^2 (t-m \tau - s)) \, ds \\
= \sum_{k=0}^{L} \sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{l} \right) g_{n,k} \int_0^{t-m} s^k Q(m, n^2 d^2 (t-m \tau - s)) \, ds \\
= v_1(t, x) + v_2(t, x) + v_3(t, x),
\]

where

\[
v_1(t, x) = \sum_{k=0}^{L} \frac{(t-m \tau)^{k+1}}{k+1} f_k(x),
\]

\[
v_2(t, x) = \sum_{k=0}^{L} \sum_{j=1}^{k+1} \frac{(-1)^{k+j} (t-m \tau)^j}{(k+1) \Gamma(m)} \binom{k+1}{j} \Gamma(k+m-j+1) \sum_{n=1}^{\infty} \sin(\lambda_n x) \frac{g_{n,k}}{(n^2 d^2)^{k+1-j}},
\]

and

\[
v_3(t, x) = \sum_{k=0}^{L} \sum_{j=0}^{k+1} \frac{(-1)^{k+j+1} (t-m \tau)^j}{(k+1) \Gamma(m)} \binom{k+1}{j} \sum_{n=1}^{\infty} \sin(\lambda_n x) \frac{g_{n,k} \Gamma(k+m-j+1, n^2 d^2 (t-m \tau))}{(n^2 d^2)^{k+1-j+1}}.
\]

The first two terms, \( v_1 \) and \( v_2 \), can in fact be computed exactly, so that only the infinite sum in \( v_3 \) has to be approximated by the corresponding \( N \)-term finite sum.

The computation of \( v_1 \) is clear, as it is a finite sum. Regarding \( v_2 \), using the \( i \)-iterated integrals of \( f_k \),

\[ F_k^{(i)}(x) = \int_0^x du_i \int_0^{u_{i-1}} du_{i-1} \cdots \int_0^{u_2} f_k(u_1) du_1, \]

it can be expressed in the equivalent form

\[
v_2(t, x) = \sum_{k=0}^{L} \sum_{j=0}^{k+1} \frac{2(-1)^{k+j} (t-m \tau)^j \Gamma(k+m-j+1) \Gamma(k+1)}{(k+1) \Gamma(m) a^{2(k+1-j)m}} \binom{k+1}{j}.
\]
Theorem 2 can be given.

Let \( u \) be the exact solution of the problem for given in (16).

Then, for every \( t \), we have

\[
\phi \text{ obtained when in which shows that for the truncation error in this approximation, bounds similar to those obtained in Theorem 4.}
\]

Consider now the approximation of \( v_3 \) by the \( N \)-term finite sum. We assume that there is a constant \( g > 0 \) such that

\[
|g_{n,k}| \leq \frac{g}{n}, \quad n \geq 1, \quad 0 \leq k \leq L,
\]

which follows from the bounds on the Fourier coefficients of \( \varphi_t \) when we consider a polynomial approximation to this function. Then, for every \( t \in [m \tau, (m + 1) \tau] \), and \( 0 \leq j \leq k + 1 \), the expression

\[
\sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{l} \right) g_{n,k} \frac{\Gamma(k + m - j + 1, n^2 d^2(t - m \tau))}{(n^2 d^2)^{k+1-j}}
\]

is bounded by

\[
g \Gamma(k + m - j + 1, (N + 1)^2 d^2(t - m \tau)) \cdot \frac{(2k - j + 3d^{2(k+1-j)}N^{2(k-j)+3}}{(2k - j + 3d^{2(k+1-j)}N^{2(k-j)+3}}
\]

which shows that for the truncation error in this approximation, bounds similar to those obtained in Theorem 2 for \( \sum_1 \) and \( \sum_2 \) in (16) can be given.

Our next result shows that the error of the polynomial approximation remains under control.

**Theorem 4.** Let \( u \) be the exact solution of the problem (1), (3)–(4) given in (16), and let \( u_P \) be the approximation obtained when in \( \sum_3 \) in (16), the Fourier coefficients \( B'_n(t) \) is substituted by those of a polynomial approximation to \( \varphi_t \) as given in (19). Then, if

\[
|\varphi(t, x) - P_{L,\varphi}(t, x)| < \varepsilon \quad \forall (t, x) \in [0, \tau] \times [0, l],
\]

we have

\[
|u - u_P| \leq \frac{4\varepsilon \sqrt{m \tau}}{d}.
\]
Finally, the next theorem provides bounds for the total error when $u_P$ is computed with approximating finite sums.

**Theorem 5.** Let $u$ and $u_P$ be as in Theorem 4, and denote by $u_{P,N}$ the approximation to $u_P$ obtained when $\sum_1^1$ and $\sum_2^2$ in (16), and $v_3$ in (20) are substituted by the corresponding partial sums with the $N$ first terms. Then, for $(t, x) \in [m\tau, (m + 1)\tau] \times [0, l]$, we have the bound

$$
|u_{P,N}(t, x) - u(t, x)| \leq \sum_{k=1}^{m} \frac{B_1 e^{2k-2}}{\Gamma(k)} \frac{\Gamma(k, (N + 1)^2 d^2(t - k\tau))}{N} + \sum_{k=1}^{m-1} \frac{B'\tau c^{2k}}{\Gamma(k)} \frac{\Gamma(k, (N + 1)^2 d^2(t - (k + 1)\tau))}{N} + \sum_{k=0}^{L} \frac{gc^{2m}2^{k+1}}{\Gamma(m)(k+1)d^{2(k+1)}} \frac{\Gamma(k + m + 1, (N + 1)^2 d^2(t - m\tau))}{N^{2k+3}} + \frac{4c^2\sqrt{m\tau}}{d^2} \varepsilon.
$$

An example of the error bounds given by Theorem 5 as a function of $N$ is shown in Fig. 2 (right). Note the higher precision assured by these bounds in comparison with those given by Theorem 2 (left figure). A common approximating polynomial was used for all values of $N$, with $\varepsilon$ in Theorem 4 set to obtain a uniform bound of $10^{-150}$ in (21), which is therefore a lower limit for the error bounds here (see figure). It should be clear, however, that given $\delta > 0$ and a prefixed admissible error, a suitable approximating polynomial and a value of $N$ can be found such that the precision of the approximate solution $u_{P,N}$ fulfills the a priori requirements uniformly for $(t, x) \in [m\tau + \delta, (m + 1)\tau] \times [0, l]$.

**References**


