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Cycles and Paths through Specified Vertices in k-Connected Graphs

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Let G be a k-connected graph with minimum degree d and at least $2d$ vertices. Then G has a cycle of length at least $2d$ through any specified set of k vertices. A similar result for paths is also given.

INTRODUCTION

The following two theorems appear in Dirac [1, 2].

THEOREM 1. *Let G be a 2-connected graph with minimum degree d. Then G has a cycle of length at least* $\min(|V(G)|, 2d)$.

THEOREM 2. *Let G be a k-connected graph and X be a set of k vertices of G, $k \geqslant 2$. Then G contains a cycle C which contains every vertex of X.*

In this paper we prove the following theorems.

THEOREM 3. *Let G be a k-connected graph, $k \geqslant 2$, with minimum degree d, and with at least $2d$ vertices. Let X be a set of k vertices of G. Then G has a cycle C of length at least $2d$ such that every vertex of X is on C.*

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THEOREM 4. *Let G be a k-connected graph, $k \geqslant 3$, with minimum degree d, and with at least $2d-1$ vertices. Let x and z be distinct vertices of G and Y be a set of $k-1$ vertices of G. Then G has an (x, z)-path P of length at least $2d-2$ such that every vertex of Y is on P.*

We note that the case $k=2$ of Theorem 3 was proven in [5]. The reader who wishes to begin with a simplified version of Theorem 3 is directed to that paper. We shall use notation and technique that was presented in [5]. For the sake of completeness, these will be repeated in this paper. Saito [8] establishes a result relating the length of such a cycle to the length of a longest cycle of G. Versions of Theorems 3 and 4 also appear in [3].

Theorem 3 is best possible if X is an independent set of vertices which collectively have exactly d neighbours. Then no cycle of length exceeding $2d$ can contain all of X. It is also best possible in the sense that there are graphs of connectivity k which contain a set X of $k+1$ vertices with no cycle through every vertex of X. The only hope for a stronger theorem would lie in replacing the hypothesis that G is k-connected by the hypothesis that G is 2-connected and the given vertices of X lie on a common cycle. Such a result for three vertices in 2-connected graphs is given in [6].

NOTATION

As in [5], we find it convenient to prove the main theorems simultaneously. We use the term *trail* for a path or a cycle, which is a restriction of the usual definition of trail. Let $Y=\{y_1, y_2, ..., y_{k-1}\}$ be a set of distinct vertices of G. Let x and z be vertices of G distinct from Y (we allow the possibility that $x=z$). An *$(x, Y, z : m)$-trail* is a cycle of length at least m containing x and Y if $x=z$ or an (x, z)-path of length at least m containing Y if $x \neq z$. Also, we define δ_x^z to be 1 if $x \neq z$ and 0 if $x=z$.

For a trail T the section of T from u to v including u and v is $T[u, v]$. We shall use $T(u, v)$ for $T[u, v]-\{u, v\}$. The case that Y consists of a single vertex y and T is a path of length at least m, occurs frequently and we shall call such a path an *$(x, y, z : m)$-path*. For a subgraph H, a path *through H* is a path P with all internal vertices of P in H.

For a vertex x, $N(x)$ denotes the set of neighbours of x. For a set of vertices S, we write $N(S)$ for the set $\bigcup_{x \in S} N(x)$, and for a subgraph H we write $N(H)$ for $N(V(H))$. For a vertex x, a set of vertices S and subgraphs H and K, $n_K(x)$ denotes the number of neighbours of x in K, $N_K(x)$ denotes the set of neighbours of x in K, $N_K(S)$ is the set $N(S) \cap V(K)$, and $N_K(H)$ is the set $N_K(V(H))$.

We shall use a variation of Lemma 2.1 from [5]. This variation appears as Lemma 3.2.2 of [4]. The proof is repeated here for the sake of completeness.

LEMMA 5. *Let G be a 2-connected graph with at least 4 vertices, let u and v be distinct vertices of G, and let d be an integer. Suppose that every vertex of G, except possibly u and v and one other vertex, has degree at least d. If x is any vertex of G with the degree of x at least 3, then there is a $(u, x, v : d)$-path in G.*

Alternatively, if G is nonseparable on $\nu \leqslant 3$ vertices, then G is K_1, K_2, or K_3 and there is a $(u, x, v : \nu - 1)$-path in G for any $u, v, x \in V(G)$, with $u \neq v$ if $\nu \neq 1$.

Proof. If $\nu < 4$, the result is trivial, and we include these cases in the statement of the lemma for ease in reference to the lemma. Now, suppose that $\nu = 4$. Then $G \cong K_4$ or $G \cong K_4 - uv$, and the lemma holds in these cases.

At several points of this proof, we shall need to choose a vertex subject to certain restrictions. Top consolidate some of the cases, we introduce the following notation. For a set of vertices X and a vertex y, let $f(X, y) = \{y\}$ if $y \in X$ and $f(X, y) = X$ if $y \notin X$. Thus, $y' \in f(X, y)$ indicates that $y' = y$, if $y \in X$, but otherwise any vertex of X will suffice.

If $\nu \geqslant 4$ and $d \leqslant 2$ then, by Menger's theorem, there is a (u, w, v)-path P_w, for any vertex w of G. Let $w \in f(V(G) - \{u, v\}, x)$. Then P_w is a $(u, x, v : 2)$-path.

For $\nu > 4$, we shall assume that the lemma holds for any graph on fewer vertices, or when d is replaced by an integer d', where $d' < d$. We may also assume, without loss of generality, that $x \neq u$. We shall prove the lemma by considering two cases depending on the separability of $G - u$.

Case (i). Suppose that $G - u$ is 2-connected. Let $u' \in f(N(u) - \{v\}, x)$. By the induction hypothesis, there is a $(u', x, v : d - 1)$-path P in $G - u$. Then $uu'P$ is a $(u, x, v : d)$-path in G.

Case (ii). Suppose that $G - u$ is separable. It is possible that some vertex w has degree one in $G - u$. If $w \neq v$, then the graph H constructed by contracting the edge uw to a new vertex u' has a $(u', x, v : d)$-path and, hence, G has a $(u, x, v : d)$-path. The case $N_G(v) = \{u, v'\}$ is handled similarly, by contracting vv'. Thus we may assume that every endblock of $G - u$ has at least three vertices.

Let B be an endblock of $G - u$ with cutvertex b, such that v is not an internal vertex of B and such that there is a (b, x, v)-path P_1 or an (x, b, v)-path P_2 in $G - u$. Let $u' \in f(N(u) \cap (V(B) - b), x)$ and $x' \in f(V(B), x)$. By the induction hypothesis, there is a $(u', x', b : d - 1)$-path Q in B. Then $uu'QP_i[b, v]$ is a $(u, x, v : d)$-path, where $i = 1$ or 2, depending on which of P_1 and P_2 exists. Thus in any graph satisfying the given conditions, we have shown the existence of a $(u, x, v : d)$-path, completing the proof of the lemma.

We shall prove Theorems 3 and 4 simultaneously in the form:

THEOREM 6. *Let G be a k-connected graph, $k \geqslant 3$, with minimum degree $d \geqslant k$. Let Y denote a set of $k-1$ distinct vertices of G. Let x and z be vertices of $G-Y$. Suppose that G has at least $2d-\delta_x^z$ vertices. Then G contains an $(x, Y, z : 2d-2\delta_x^z)$-trail.*

Proof of Theorem 6. We proceed by induction on k. For the case $k=3$, it was shown in [5] that G contains an $(x, Y-\{y\}, z : 2d-2\delta_x^z)$-trail for any vertex $y \in Y$. Let T be a longest (x, z)-trail which contains at least $k-2$ of the vertices of Y. The length of T must be at least $2d-2\delta_x^z$. If T contains all of Y then we are done, so we may assume that some vertex v of Y is not on T.

Let H be the component of $G-V(T)$ which contains v. Label the vertices of $X=\{x, z\} \cup Y-v$ along T as $x_1, x_2, ..., x_k$, with $x=x_1$ and $z=x_k$. Let $T_i=T[x_i, x_{i+1}]$. We shall frequently make use of the following observation.

Observation. Suppose that we can find vertices α and β on $T[x_i, x_{i+2}]$, $i \neq k-1$ if $x \neq z$, with neighbours α' and β' in H such that there is an $(\alpha', v, \beta' : m)$-path P in H. Then $T'=T[x, \alpha] \cup T[\beta, z] \cup \{\alpha\alpha', \beta\beta'\} \cup P$ is an $(x, Y-\{y_j\}, z)$-trail for some $y_j \in Y$. By the maximality of T, $T[\alpha, \beta]$ must have length at least $m+2$.

Case 1. Suppose that H is nonseparable.

Let $W=N_T(H)=N(H)-V(H)$, and label the vertices of W along T as $u_1, u_2, ..., u_r$. Let

$$W_2=\{u_i \in W : |N_H(\{u_i, u_{i+1}\})| \geqslant 2\} \quad \text{and} \quad W_1=W-W_2.$$

Let

$$W_{2,0}=\{u_i \in W_2 : |V(T(u_i, u_{i+1})) \cap X|=0\},$$
$$W_{2,1}=\{u_i \in W_2 : |V(T(u_i, u_{i+1})) \cap X|=1\} \quad \text{and}$$
$$W_{2,2}=W_2-W_{2,0}-W_{2,1}.$$

For an index I, $w_I=|W_I|$.

If T is a cycle, it is possible that $x_1=x_k \in N(H)$. In this case, $u_1=u_r=x_1$. Otherwise, u_r is considered to be in the set W_1. Figure 1 displays a trail T, component H of $G-V(T)$ and labels indicating the set to which each vertex of $N_T(H)$ belongs.

By Menger's Theorem, there are k paths from v to T, pairwise disjoint except at v. There is at least one segment T_i which contains at least two endvertices of these paths. Thus, we may assume that $u_j, u_{j+1} \in T[x_i, x_{i+1}]$,

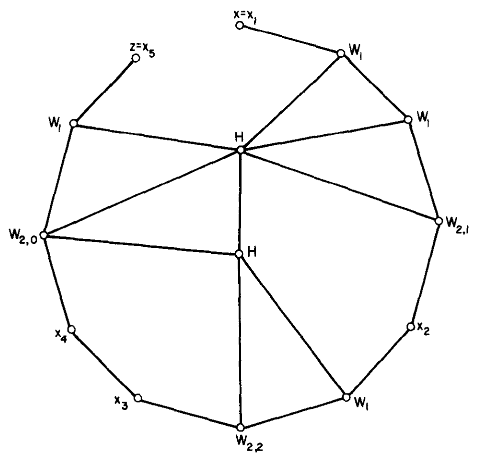

FIGURE 1

for some i, j, and that there is a (u_j, v, u_{j+1})-path internally disjoint from T. Let v_1, v_2 be neighbours in H of u_j, u_{j+1}, respectively, with $v_1 \neq v_2$, if possible. Let P be a longest (v_1, v, v_2)-path in H. If $v_1 = v_2$, then $v = v_1$.

Let y' be a vertex of H with the largest number of neighbours on T, and let z' be a vertex of H with the smallest number of neighbours on T. Set $p = \max\{0, d - n_T(y')\}$. If α_1 and α_2 are distinct vertices of H then, by Lemma 5, H contains an $(\alpha_1, v, \alpha_2 : p)$-path. Also H has $h \geqslant d - n_T(z') + 1$ vertices. By Menger's Theorem, there are at least $\min(k, h)$ disjoint edges from H to T. Therefore, W_2 has at least $\min(k, h) - \delta_x^z$ vertices.

Let $T' = T[x_1, u_j]\, u_j v_1 P v_2 u_{j+1}\, T[u_{j+1}, x_k]$. We proceed to determine lower bounds on the length of T'.

Suppose that $w_{2,0} + w_{2,1} \geqslant 2$. Let u_α and u_β be distinct vertices of $W_{2,0} \cup W_{2,1}$. Then, $|T'[u_\alpha, u_{\alpha+1}]| \geqslant 2 + p$ and $|T'[u_\beta, u_{\beta+1}]| \geqslant 2 + p$. It is possible that $\alpha = j$, but then $u_j v_1 P v_2 u_{j+1}$ has length at least $p + 2$. If $u_\gamma \in W_1$, in the case $x = z$, or $u_\gamma \in W_1 - \{u_r\}$, in the case $x \neq z$, then $|T'[u_\gamma, u_{\gamma+1}]| \geqslant 2$. If $u_\eta \in W_{2,2}$, then $|T'[u_\eta, u_{\eta+1}]| \geqslant 3$. Therefore,

$$\begin{aligned}
|T'| &\geqslant 2(w_1 - \delta_x^z) + (2 + p)(w_{2,0} + w_{2,1}) + 3w_{2,2} \\
&\geqslant 2w_1 + 2(w_{2,0} + w_{2,1}) + p(w_{2,0} + w_{2,1} - 2) + 2p + 3w_{2,2} - 2\delta_x^z \\
&\geqslant 2w + w_{2,2} + p(w_{2,0} + w_{2,1} - 2) + 2(d - n_T(y')) - 2\delta_x^z \\
&\geqslant 2d - 2\delta_x^z.
\end{aligned}$$

Suppose that $w_{2,0} + w_{2,1} = 1$. Then $k \geqslant h$ and $w_{2,2} \geqslant h - 1 - \delta_x^z \geqslant d - n_T(z') - \delta_x^z$. We note that if $n_T(z') = w$ then every vertex of H is adjacent to every vertex of W. In this case, $w_{2,0}+w_{2,1}>1$. Thus, $N_T(z')<w$ and

$$\begin{aligned}|T'| &\geqslant 2(w_1-\delta_x^z)+(2+p)(w_{2,0}+w_{2,1})+3w_{2,2}\\ &\geqslant 2w_1+2(w_{2,0}+w_{2,1})+p+3w_{2,2}-2\delta_x^z\\ &=2w+(d-n_T(y'))+(d-n_T(z'))-3\delta_x^z\\ &\geqslant 2d-2\delta_x^z.\end{aligned}$$

Therefore, we may assume that $w_{2,0}+w_{2,1}=0$. If $x\neq z$, there is no vertex of W_2 in $T[x_{k-1}, x_k]$. Hence, $w_2\leqslant(k-1)/2$, $h\leqslant k/2$, $n_T(z')\geqslant d-h+1\geqslant(2d-k+2)/2$, and $|T'|\geqslant 2(w_1-\delta_x^z)+3w_{2,2}$.

We now obtain lower bounds on the number of neighbours in W_1 for each vertex of H. Clearly, each such vertex has at least $d-h+1-w_2$ neighbours in W_1, and no two vertices have a common neighbour in $W_1-\{u_r\}$. Thus, $w_1\geqslant h(d-h-w_2)$. Note that $d-h+1-w_{2,2}\geqslant d-k/2+1-(k-1)/2=d-k+3/2\geqslant 3/2$. Suppose that $h>1$, then

$$\begin{aligned}|T'| &\geqslant 2(w_1-\delta_x^z)+3w_{2,2}\\ &\geqslant 2h(d-h-w_{2,2})+3w_{2,2}-2\delta_x^z\\ &=2(h-1)(d-h-w_{2,2})+2d-2h+w_{2,2}-2\delta_x^z\\ &\geqslant 2d-2\delta_x^z.\end{aligned}$$

If $h=1$, then $w_1\geqslant d$ and $|T'|\geqslant 2d-2\delta_x^z$. This concludes the case in which H is nonseparable.

Case 2. Now let us suppose that H is separable. We note that we may choose endblocks B_1 and B_2 of H having cutvertices b_1 and b_2, respectively, such that there is a (b_1, v, b_2)-path or there is a (v, b_1, b_2)-path in H. In the second alternative, $v\in V(B_1)$.

There are k paths $P_1, P_2, ..., P_k$ from v to $V(T)$, pairwise disjoint except at v, with endvertices $p_1, p_2, ..., p_k$ on T. Some two of these paths meet, say, T_i as we have noted in Case 1. We set T' as in that case.

We first prove that if B_1-b_1 has a neighbour t_1 on T_m, then B_2-b_2 has no neighbour on $T[x_{m-1}, x_{m+2}]-\{t_1\}$ (with the obvious modifications if $m=1$ or $k-1$). We then show that one of these endblocks has neighbours on at most $(k-2)/2$ of the segments T_i, $1\leqslant i\leqslant k-1$.

Suppose that B_1-b_1 has a neighbour t_1 on T_m and that B_2-b_2 has a neighbour $t_2\neq t_1$ on T_m or T_{m+1}. Let s_j be a neighbour of t_j in B_j-b_j, $j=1,2$. Let y_j be the vertex of B_j-b_j with the largest number of neighbours on T. Then there is an (s_1, v, s_2)-path of length $p\geqslant 2d-n_T(y_1)-n_T(y_2)$. By the maximality of T, $|T[t_1, t_2]|\geqslant 2+p$. Also, every neighbour of y_j on T, except for t_1 and the last neighbour of y_j on T, is followed by at least two edges of T or by a path through H. Thus, if i, $i-1 \neq m$, or if t_2 is on T_m, $|T'| \geqslant (2+p) + 2(n_T(y_j) - 1 - \delta_x^z)$. Therefore $|T'| \geqslant 2d - 2\delta_x^z$.

The cases $m = i$ and $m = i - 1$ are symmetric. Thus we consider $m = i$. We may assume that when we found k paths from v to T, at most one of these paths meets each T_j, for $j \neq i$, $i+1$. Also, $t_1 \in T_i$ and $t_2 \in T_{i+1}$. Since there are two paths from v to T meeting T_i, there is a vertex $q \neq t_1$ on T_i such that there are paths through H from v to T meeting T_i at t_1 and q, with these paths being disjoint except at v. If q precedes t_1 on T_i, then $|T'| \geqslant 2d - 2\delta_x^z$. Thus we may assume that t_1 precedes q on T_i, and that $N(q) \cap V(B_1 - b_1) = \varnothing$.

If there are also two paths through H from v to T_{i+1}, disjoint except at v, and meeting T_{i+1} at t_2 and q^*, with no neighbours of y_2 between t_2 and q^*, then the segment of T between t_2 and q^* must be of length at least $d - n_T(y_2) + 2$ and $|T'[t_1, q]| \geqslant d - n_T(y_1) + 2$. Again, any two neighbours of y_i must be separated by at least 2 edges of T and, hence, of T'. Therefore,

$$\begin{aligned}|T'| &\geqslant (d - n_T(y_1) + 2) + (d - n_T(y_2) + 2)\\ &\quad + 2(\max\{n_T(y_1), n_T(y_2)\} - 2 - \delta_x^z)\\ &\geqslant 2d - 2\delta_x^z.\end{aligned}$$

We have already seen that there are at least two paths through H from v meeting T_i. Suppose that there are three paths through H, disjoint except at v, from v meeting T_i at t_1, q, and q'. Then, without loss of generality, the order of the ends of these paths on T_i is t_1, q, q'. Then there are paths through H from v to t_2 and one of q or q' such that v is the only common vertex of these paths. Then $|T[q, t_2]| \geqslant 2 + d - n_T(y_2)$ and there is a (t_1, v, q)-path through H of length at least $d - n_T(y_1) + 2$. Again, $|T'| \geqslant 2d - 2\delta_x^z$.

Hence, for any set of k paths $P_1, ..., P_k$ from v to T, disjoint except at v, exactly two meet T_i and exactly one meets T_j, for $j \neq i$. Suppose that $N_{T[x_2, x_{k-1}]}(B_2 - b_2) \neq \varnothing$. Let p_i' denote the penultimate vertex of P_i, for $i = 1, 2, ..., k$. If $v = p_i'$ for some i, some slight care is necessary with respect to multiplicities. By Perfect's theorem [7], there are k paths from v to $\{p_1', p_2', ..., p_k'\} \cup \{b_2\}$, disjoint except at v, with one of the paths ending at b_2. We may thus construct k paths from v to T so that one, say P_j, ends at p_j, with $p_j \in V(T[x_2, x_{k-1}]) \cap N(B_2 - b_2)$ and uses at least $d - n_T(y_2)$ edges of $B_2 - b_2$. Let p_j^1 be the first vertex of $T[p_{j-1}, p_j] - \{p_{j-1}\}$ that is a neighbour of $B_2 - b_2$, and let p_j^2 be the last vertex of $T[p_j, p_{j+1}] - \{p_{j+1}\}$ that is a neighbour of $B_2 - b_2$. Then $|T[p_{j-1}, p_j^1]| \geqslant 2 + d - n_T(y_2)$ and $|T[p_j^2, p_{j+1}]| \geqslant 2 + d - n_T(y_2)$. But then $|T'| \geqslant 2d - 2\delta_x^z$.

Therefore, we may assume that $N_T(B_2-b_2)\subset V(T_1)\cup V(T_{k-1})$. If B_2-b_2 has neighbours on both T_1 and T_{k-1} then, as in the previous paragraph, each of $T_1\cup T_2$ and $T_{k-2}\cup T_{k-1}$ contains a segment of length at least $d-n_T(y_2)+2$. Again, $|T'|\geqslant 2d-2\delta_x^z$.

Now, without loss of generality, we may assume that $N_T(B_2-b_2)\subset V(T_1)$. But $|N_T(B_2-b_2)|\geqslant k-1\geqslant 2$. Therefore, $T_1\cup T_2$ contains two segments of length at least $d-n_T(y_2)+2$, and $|T'|\geqslant 2d-2\delta_x^z$.

Thus, we may assume that if B_1-b_1 has a neighbour t on T_m, for some m, then B_2-b_2 has no neighbour on T_m, T_{m-1}, or T_{m+1} distinct from t.

Let m_3 denote the number of indices m for which T_m contains a neighbour of both B_1-b_1 and B_2-b_2. For such a segment T_m, $|N(B_1\cup B_2-\{b_1,b_2\})\cap V(T_m)|=1$. Let m_j denote the number of indices m for which T_m contains a neighbour of B_j-b_j, $j=1,2$, but not both B_1-b_1 and B_2-b_2. If $m_j=0$, for $j=1$ or $j=2$, then $S=\{b_j\}\cup(N_T(B_1-b_1)\cap N_T(B_2-b_2))$ is a set of vertices which disconnects G. But $|S|\leqslant 1+k/2<k$. Therefore, $m_1\geqslant 1$, and $m_2\geqslant 1$. Now, since $k\geqslant 2$, $(m_1+1)+(m_2+1)+2m_3\leqslant k$.

Thus one of B_1-b_1 or B_2-b_2 has neighbours on at most $(k-2)/2$ of the intervals T_m. Without loss of generality, suppose that $m_1+m_2\leqslant m_2+m_3$, and thus that B_1-b_1 has neighbours on at most $(k-2)/2$ of the intervals T_m. Let $m_2^*=|N_T(B_2-b_2)-N_T(B_1-b_1)|$. Since G is k-connected, $m_2^*\geqslant 2$. We note that v might be in B_1-b_1. Set $\mu=1$ if $v=b_1$ and $d_{B_1}(v)=2$ and $|V(B_1)|>3$, otherwise set $\mu=0$. The case $\mu=1$ is the case in which Lemma 5 cannot be applied to B_1 to give a long path containing v. If $\mu=1$, then for any distinct vertices α and β in B_1, there is an $(\alpha, v, \beta : 2)$-path in B_1.

Now suppose that z_1 is a vertex of B_1-b_1 with the least number of neighbours on T and let h be the number of vertices of B_1. There are at least $\min(k, h)$ disjoint paths from B_1 to T, all except possibly one having length one. Let $q_1, q_2, \ldots$ denote the ends of these paths. We note that one of the q_j may not be in $N_T(B_1-b_1)$.

Let $J=\{j:\{q_j, q_{j+1}\}\subset V(T_m)$ for some $m\}$. If $|J|\geqslant 2+\mu$ then there are distinct integers m and j, such that neither $T[q_m, q_{m+1}]$ nor $T[q_j, q_{j+1}]$ contains a vertex of X, but each has length at least $d-n_T(y_1)+2$. If $\mu=0$ and $v\in B_1$, there is a $(q_m, v, q_{m+1} : d-n_T(y_1)+2)$-path through H. We may assume that $N_H(T(q_m, q_{m+1}))=N_H(T(q_j, q_{j+1}))=\varnothing$. For both values of μ, each of $T'[q_m, q_{m+1}]$ and $T'[q_j, q_{j+1}]$ has length at least $d-n_T(y_1)+2$. Also, any two neighbours of y_1 on T are separated by at least two edges of T. Therefore, $|T'|\geqslant 2d-2\delta_x^z$.

If $h-\mu\geqslant(k+4)/2$, then $|J|\geqslant 2+\mu$.

If $(k+3)/2\geqslant h-\mu\geqslant(k+2)/2$ and $|J|\leqslant 1+\mu$, then $m_1+m_3=(k-2)/2$ and $|J|=1+\mu$. If $\mu=1$, there is a path P from some some T_m through H and back to T_m, with $|P|\geqslant 4$ and $v\in V(P)$. But then,

$$\begin{aligned}|T'| &\geqslant (d-n_T(y_1)+2)+\mu\,|P|+2(n_T(y_1)-1-\mu-m_3)\\ &\quad +2m_2^*+3m_3-3\delta_x^z\\ &\geqslant d+n_T(y_1)+2\mu-3\delta_x^z+m_2^*+m_2+m_3\\ &\geqslant d+(d-h+1)+2\mu-3\delta_x^z+m_2^*+m_2+m_3\\ &\geqslant 2d-2\delta_x^z+\mu+1-\frac{k+3}{2}+\frac{k-2}{2}\\ &\geqslant 2d-2\delta_x^z-\frac{1}{2}.\end{aligned}$$

Hence, we may assume that $h-\mu\leqslant(k+1)/2$, $|J|\leqslant 1+\mu$, and $n_T(z_1)\geqslant d-h+1$.

Let j_2 denote the number of indices j such that B_1-b_1 has at most one neighbour on T_j. Let

$$j_3=\tfrac{1}{2}\,|\{q_j:\{q_j,q_{j+1}\}\subset V(T_m)\text{ or }\{q_{j-1},q_j\}\subset V(T_m)\text{ for some }m\}|.$$

The only possible values of j_3 at this point are 0, 1, 3/2, and 2. If $\mu=0$ then $j_3\leqslant 1$ and $j_2+j_3\leqslant(k-2)/2$. Hence, $j_2+2j_3\leqslant k/2$, and $n_T(z_1)-j_2-2j_3\geqslant d-h+1-k/2\geqslant 1/2$. Similarly, if $\mu=1$ and $j_3=0$, or if $\mu=1$, $j_3=1$, and $h\leqslant(k+1)/2$, or if $\mu=1$, $j_3>1$, and $h\leqslant(k-1)/2$, then $n_T(z_1)-j_2-2j_3\geqslant 1$.

In all these cases,

$$\begin{aligned}|N_T(B_1-b_1)| &\geqslant (n_T(z_1)-j_2-2j_3)(h-1)+j_2+2j_3\\ &\geqslant (n_T(z_1)-j_2-2j_3)(h-2)+n_T(z_1)\\ &\geqslant (h-2)+(d-h+1)\\ &\geqslant d-1.\end{aligned}$$

Then,

$$\begin{aligned}|T'| &\geqslant 2\,|N_T(B_1-b_1)|+2m_2^*-2\delta_x^z\\ &\geqslant 2d-2\delta_x^z.\end{aligned}$$

It remains to consider the cases $\mu=1$ that are not covered above. If $j_3=1$, we may assume that $(k+2)/2\leqslant h\leqslant(k+3)/2$. Let z' denote the vertex of $B_1-\{v\}-N(v)$ with the fewest neighbours on T. Since B_1 is not complete, $n_T(z')\geqslant d-h+2$. Then,

$$\begin{aligned}|N_T(B_1-b_1)| &\geqslant (n_T(z')-j_2-2j_3)(h-3)+j_2+2j_3\\ &\leqslant (n_T(z')-j_2-2j_3)(h-4)+n_T(z')\\ &\geqslant (h-4)+(d-h+2)\\ &\geqslant d-2\end{aligned}$$

Again,

$$|T'| \geqslant 2\,|N_T(B_1 - b_1)| + 2m_2^* - 2\delta_x^z$$
$$\geqslant 2d - 2\delta_x^z.$$

If $j_3 > 1$, we may assume that $k/2 \leqslant h \leqslant (k+3)/2$. Then $m_2 + m_3 \geqslant m_1 + m_3 \geqslant h - 3 \geqslant (k-6)/2$, $m_2 \geqslant m_1 \geqslant 1$, $n_T(y_1) \geqslant d - h + 2 \geqslant d - (k-1)/2$. There is a path P from some T_m through B_1 and back to T_m, with $|P| \geqslant 4$ and $v \in V(P)$, and a path Q from some T_λ through B_1 and back to T_λ with length at least $d - n(y_1) + 2$. Then

$$|T'| \geqslant |Q| + |P| + 2(n_T(y_1) - 2 - m_3) + 2m_2 + 3m_3 - 3\delta_x^z$$
$$\geqslant (d - n_T(y_1) + 2) + 4 + 2(n_T(y_1) - 2 - m_3) + 2m_2 + 3m_3 - 3\delta_x^z$$
$$\geqslant d + n_T(y_1) + 2 + (m_2 + m_3) - 2\delta_x^z$$
$$\geqslant 2d - \frac{k-1}{2} + 2 + \frac{k-6}{2} - 2\delta_x^z$$
$$\geqslant 2d - 2\delta_x^z - \frac{1}{2}.$$

Thus in all cases we have $|T'| \geqslant 2d - 2\delta_x^z$. This completes the proof of Theorem 6.

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