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Covariant and contravariant points of view in topology with applications to function spaces

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Abstract

The purpose of this paper is to present a way of viewing of basic topology which unifies quite a few results and concepts previously seemed not related (quotient maps, product topology, subspace topology, separation axioms, topologies on function spaces, dimension, metrizability). The basic idea is that in order to investigate an unknown space X, one either maps known spaces to X or maps X to known spaces. Mapping known spaces to X leads to covariant functors. Therefore, it will be part of what we call the covariant point of view. Mapping X to known spaces leads to contravariant functors. It will be part of what we call the contravariant point of view. The covariant approach is an abstraction of the well known methodology of the homotopy theory: to investigate properties of CW complexes one computes their homotopy groups, i.e., one considers maps from spheres to CW complexes. Once some CW complexes are well understood, one can map them to a space X in order to detect its topological properties. The dual to covariant approach, the contravariant approach, is an abstraction of the well known methodology of the shape theory: to investigate topological properties of space X one maps X to CW complexes.

It is explained in the paper that many notions/results can be better understood as analyzed from either covariant or contravariant points of view. Particular attention is given to function spaces. It is shown that the three main topologies on function spaces (the basic covariant topology, the compact-open topology, the pointwise convergence topology) can be introduced in the same manner: they are covariantly induced by functions $f : S \to Map(X, Y)$ so that $adj_X(f)|S \times K$ is continuous for

(a) $K = X$ (the basic covariant topology),

(b) any locally compact K in X (the compact-open topology),

(c) any finite subset K of X (the pointwise convergence topology).

By applying the concept of adjointness of functors, two new topologies on the product $X \times Y$ are introduced. The PC-product $X \times_{PC} Y$ arises as a left adjoint to the pointwise convergence topology, and the CO-product $X \times_{CO} Y$ arises as a left adjoint to the compact-open topology.

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0. Introduction

In [8] the author presented certain results of basic topology from the point of view of Extension Theory. In this paper we broaden the approach of [8]. Namely, extension theory can be viewed as part of the contravariant approach, and it makes sense to ponder its dual, the covariant approach.

Suppose we have a class of known spaces K, and we are faced with an unknown space X. We may choose one of the following strategies:

(1) (Covariant approach) X will be investigated by considering maps $f:K \to X$ from known spaces to X.

(2) (Contravariant approach) X will be investigated by considering maps $f:X \to K$ from X to known spaces.

The covariant approach is widely used in the classical homotopy theory and leads to homotopy/homology groups (see [26]). The contravariant approach is the main-stay of shape theory (see [24]), cohomological dimension theory (see [28]), and leads to cohomology/cohomotopy groups. However, in basic topology the prevalent approach is that of intrinsic definitions/theorems in terms of open sets/covers.

The purpose of this paper is to translate the intrinsic approach of basic topology into covariant/contravariant approaches in an effort to unify various concepts which seemed unrelated up to now (quotient maps, product topology, subspace topology, separation axioms, dimension, metrizability). We believe that it brings better understanding and better results. In particular we present a new way of looking at topologies of function spaces. We show that a certain topology on $Map(X,Y)$ (we call it the basic covariant topology) which played only a marginal role up to now, has exactly the same properties as the compact-open topology but is much more natural and allows functorial proofs. Actually, the same proofs can be used, word for word, for analogous statements about the compact-open topology. Also, both the compact-open topology and the pointwise convergence topology are derived from the basic covariant topology. More generally, it is shown that the three main topologies on function spaces (the basic covariant topology, the compact-open topology, the pointwise convergence topology) can be introduced in the same manner: they are covariantly induced by functions $f:S \to Map(X,Y)$ so that $adj_X(f)|S \times K$ is continuous for

(a) $K = X$ (the basic covariant topology),

(b) any locally compact K in X (the compact-open topology),

(c) any finite subset K of X (the pointwise convergence topology).

One of the main accomplishments of this paper is establishing the connection between the problem of continuity of the evaluation map $eval:Map(X,Y) \times X \to Y$ and the characterization of locally compact spaces Z as those for which $f \times id_Z$ is a quotient map if f is a quotient map. That characterization is a consequence of a result of Whitehead and a result of Michael (see Corollary 5.4 and Theorem 5.5 in this paper).

Our interest in function spaces arises from the fact that, for compactly generated spaces X, the kth Čech cohomology group $H^k(X;G)$ can be defined as the fundamental group $\pi_1(K^X)$, where K is a $K(G,k+1)$, i.e., a CW complex whose only non-trivial homotopy group is $\pi_{k+1}(K)=G$. If $I^n\times X$ is a cw-space for each n as in [5] (this means that all contractible CW complexes are absolute extensors of that space), then for every closed subset A of X, the restriction map $K^X\to K^A$ is a Serre fibration (this is a sophisticated reformulation of the Homotopy Extension Theorem) which leads to the long exact sequence of cohomology (see [6]). Also, if $X\times Y$ is a cw-space for some compact space Y, then (see [6]) one gets a nice geometrical interpretation of the Kunneth Formula. Namely, the homotopy classes $[X\times Y,K]$ are identified with $[X,K^Y]$, and K^Y is shown to be homotopy equivalent to

$$\prod_{n=0}^{k+1} K\bigl(H^{k+1-n}(Y;G),n\bigr).$$

In the above approach to cohomology one may say that we are using an ingredient of Eckmann–Hilton duality (see [9–11]). Namely, cofibrations are dual to Serre fibrations. Let us recall other examples of Eckmann–Hilton duality (see [16]):

(1) Homotopy groups are dual to cohomology groups.
(2) The wedge of spaces is dual to the Cartesian product of spaces.
(3) The suspension operator is dual to the loop space operator.

The modern way to express Eckmann–Hilton duality is by applying the concept of *adjoint functors* (see [19]):

Definition 0.1. Suppose $F:\mathcal{C}\to\mathcal{D}$, $G:\mathcal{D}\to\mathcal{C}$ are two covariant functors such that for each pair of objects X,Y there is a natural equivalence $\eta_{XY}:Mor_{\mathcal{D}}(F(X),Y)\to Mor_{\mathcal{C}}(X,G(Y))$. F is said to be a *left adjoint* to G, G is said to be a *right adjoint* to F, and the functor $\eta:\mathcal{C}^{opp}\times\mathcal{D}\to Sets$ is the *adjugant equivalence* or, simply, *adjugant*.

One of the basic examples of adjoint functors in algebra is the tensor product being left adjoint to Hom. This is expressed as the Adjoint Associativity Theorem (see [18, Theorem 5.10, p. 214]):

$$\mathrm{Hom}_R(M\otimes_R N,P)\sim\mathrm{Hom}_R\bigl(M,\mathrm{Hom}_R(N,P)\bigr).$$

We are interested in the analog of Adjoint Associativity Theorem in the category of sets:

$$Mor_{Sets}\bigl(X,Mor_{Sets}(Y,Z)\bigr)\sim Mor_{Sets}(X\times Y,Z).$$

Let us describe the adjugant equivalence in this case:

Definition 0.2. Given a function $f:A\to Mor_{Sets}(X,Y)$ one defines the *adjoint function* $adj_X(f):A\times X\to Y$ by $adj_X(f)(a,x)=f(a)(x)$. Conversely, given a function $g:A\times X\to Y$ one has the *adjoint function* $adj^X(g):A\to Mor_{Sets}(X,Y)$ defined by $adj^X(g)(a)(x)=g(a,x)$.

If $f : A \to Mor_{Sets}(X, Y)$, then $adj^A(adj_X(f)) : X \to Mor_{Sets}(A, Y)$ will be denoted by $tran(f)$ and called the *transpose* of f.

The basic function $eval : Mor_{Sets}(X, Y) \times X \to Y$ is equal to $adj_X(id)$, where $id : Mor_{Sets}(X, Y) \to Mor_{Sets}(X, Y)$ is the identity function.

Notice that the equivalence $\eta : Mor_{Sets}(X, Mor_{Sets}(Y, Z)) \to Mor_{Sets}(X \times Y, Z)$ is given by $\eta(f) = adj_Y(f)$. Also notice that *tran*:

$$Mor_{Sets}\big(X, Mor_{Sets}(Y, Z)\big) \to Mor_{Sets}\big(Y, Mor_{Sets}(X, Z)\big)$$

is a natural equivalence.

Our terminology is inspired by the terminology of the vector space theory. Indeed, if $Z = \mathbb{R}$ is the field of reals, $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and linear maps are considered instead of general functions, then the transpose function corresponds to taking of the transpose of a matrix. Also notice that $Mor_{Sets}(X, Mor_{Sets}(Y, Z))$ is formally obtained from $Mor_{Sets}(Y, Mor_{Sets}(X, Z))$ by transposing X and Y.

Traditionally, in the topology textbooks, the compact-open topology is introduced out of the blue and then its properties are being proved. In this paper, in contrast, our method is to specify the goal for a useful topology on function spaces from the beginning:

Goal 0.3. We are interested in the set of maps $Map(X, Y)$ from X to Y to be equipped with a topology *Top* (the resulting topological space is denoted by $Map_{Top}(X, Y)$) so that the induced (from the category of sets) function

$$adj_Y : Map\big(X, Map_{Top}(Y, Z)\big) \to Map(X \times Y, Z)$$

is a natural equivalence on some useful subcategory of $\mathcal{TOP}$.

Thus, we are trying to construct a right adjoint to the Cartesian product functor. A more ambitious goal is to require that

$$adj_Y : Map_{Top}\big(X, Map_{Top}(Y, Z)\big) \to Map_{Top}(X \times Y, Z)$$

is a homeomorphism in analogy to the Adjoint Associativity Theorem of algebra.

Dually, given a way of prescribing topology *Top* on function spaces $Map(X, Y)$, so that $Map_{Top}(X, Y)$ is a bifunctor, on can ponder the question of existence of a left adjoint to the functor $G(Z) = Map_{Top}(Y, Z)$ (Y is fixed). This amounts to finding a new topology on the product space $X \times Y$. Historically, besides the product topology, $X \times Y$ was given the topology of a k-space via the functor k to the category of k-spaces. In this paper we invent two new topologies on $X \times Y$: the PC-product $X \times_{PC} Y$ which leads to a left adjoint to the pointwise convergence topology, and the CO-product $X \times_{CO} Y$ which leads to a left adjoint to the compact-open topology. The CO-product is interesting in the sense that, in contrast to other products, it is not commutative. It is commutative in the class of compactly generated spaces, in which case it coincides with the product in that category, i.e., it is equal to $k(X \times Y)$.

Definitions 0.4. Suppose $\eta_{XY} : Mor_{\mathcal{D}}(F(X), Y) \to Mor_{\mathcal{C}}(X, G(Y))$ is an adjugant. In the general theory of adjoint functors (see [19,22]) one considers the induced natural transformations ε, δ: ε from the identity functor on $\mathcal{C}$ to $G \circ F$, δ from $F \circ G$ to the identity functor on $\mathcal{D}$. ε is called the *unit* (or *front adjunction*) and δ is called the counit (or *rear adjunction*). One obtains the morphism $\varepsilon_X : X \to G(F(X))$ by assuming $Y = F(X)$ and defining $\varepsilon_X = \eta_{XY}(id_{F(X)})$. Similarly, one obtains the morphism $\delta_Y : F(G(Y)) \to Y$ by assuming $X = G(Y)$ and defining $\delta_Y = \eta^{-1}_{XY}(id_{G(Y)})$.

1. Covariant and contravariant topologies

Given a set X one can consider the set of all topologies on X. It is well known that one can put a lattice structure on that set. We will find it convenient, instead, to put a category structure on it.

Definition 1.1. Let $\mathcal{TOP}$ be the category whose morphisms are all continuous maps. Given a set X let $\mathcal{TOP}(X)$ be the subcategory of $\mathcal{TOP}$ whose morphisms are all maps equal, as functions, to the identity function on X.

Thus, the objects of $\mathcal{TOP}(X)$ are all possible topological spaces (X, τ) with underlying set equal to X, and $f : (X, \tau_1) \to (X, \tau_2)$ exists iff $\tau_1 \supset \tau_2$.

The following proposition is of much better use to us than the lattice structure on the set of topologies on X:

Proposition 1.2. *The category $\mathcal{TOP}(X)$ has products and coproducts.*

(a) *The product object of a family $\{(X, \tau_s)\}_{s \in S}$ is (X, τ), where τ is the smallest topology containing $\bigcup_{s \in S} \tau_s$. Thus, $\bigcup_{s \in S} \tau_s$ is a sub-basis of τ.*

(b) *The coproduct object of a family $\{(X, \tau_s)\}_{s \in S}$ is (X, τ), where $\tau = \bigcap_{s \in S} \tau_s$. Thus, τ is the largest topology contained in all τ_s, $s \in S$.*

Proof. (a) Let $\bigcup_{s \in S} \tau_s$ be a sub-basis of τ. Clearly, the unique morphism of $\mathcal{TOP}(X)$ $f_s : (X, \tau) \to (X, \tau_s)$ exists for each $s \in S$. If (X, τ') is an object such that $g_s : (X, \tau') \to (X, \tau_s)$ exists for each $s \in S$, then $\tau' \supset \tau_s$, $s \in S$. Hence, $\tau' \supset \tau$ and the unique $f : (X, \tau') \to (X, \tau)$ exists so that $f_s \circ f = g_s$ for $s \in S$.

(b) Let $\bigcap_{s \in S} \tau_s = \tau$. Notice that τ is a topology on X. Clearly, the unique morphism of $\mathcal{TOP}(X)$ $f_s : (X, \tau_s) \to (X, \tau)$ exists for each $s \in S$. If (X, τ') is an object such that $g_s : (X, \tau_s) \to (X, \tau')$ exists for each $s \in S$, then $\tau' \subset \tau_s$, $s \in S$. Hence, $\tau' \subset \tau$ and the unique $f : (X, \tau) \to (X, \tau')$ exists so that $f \circ f_s = g_s$ for $s \in S$. □

Definition 1.3. The topology τ on X is called the *covariant topology* induced by a class of functions $\mathcal{F} = \{f : X_f \to X\}$ if each X_f is a topological space and τ is the largest of all topologies on X under which all $f \in \mathcal{F}$ are continuous.

In view of Proposition 1.2 one has:

Proposition 1.4. *The covariant topology induced by $\mathcal{F}$ exists and consists of all sets U such that $f^{-1}(U)$ is open for each $f \in \mathcal{F}$.*

Proof. Consider all topologies on X under which all $f \in \mathcal{F}$ are continuous. This set of topologies is non-empty as it contains the anti-discrete topology. The product (in $\mathcal{TOP}(X)$) of that set is the covariant topology induced by $\mathcal{F}$. $\square$

The following three examples show that the notion of the covariant topology unifies previously known concepts:

Example 1.5. A surjective function $f : X \to Y$ is a quotient map iff the topology on Y is the covariant topology induced by the single function $\{f\}$.

Example 1.6. Given a set $\{X_s\}_{s\in S}$ of topological spaces, the classical topology on the disjoint union $\coprod_{s\in S} X_s$ is the covariant topology induced by inclusions $i_t : X_t \to \coprod_{s\in S} X_s$, $t \in S$.

Example 1.7. Given a simplicial complex K, the weak topology $|K|_w$ is the covariant topology induced by all inclusions $i_\Delta : |\Delta|_m \to |K|$, where Δ is a simplex in K and $|\Delta|_m$ is $|\Delta|$ equipped with the standard metric topology.

One of the basic classes of topological spaces are Fréchet spaces (see [12, Section 1.6]). We will show that Fréchet spaces can be introduced in a covariant manner:

Proposition 1.8. *X is a Fréchet space iff its topology is the covariant topology induced by a family of functions from* $\{0\} \cup \{1/n \mid n \geqslant 1\} \subset \mathbb{R}$.

Proof. Recall that X is a Fréchet space if for any $a \in \mathrm{cl}(A)$ there is a sequence $a_n \in A$, $n \geqslant 1$, converging to a. Let $S = \{0\} \cup \{1/n \mid n \geqslant 1\} \subset \mathbb{R}$. Suppose $\mathcal{F} = \{f : S \to X\}$ is a family of functions and τ is the covariant topology induced by $\mathcal{F}$. In order to show that X is a Fréchet space assume $a \in \mathrm{cl}(A) - A$. Since $B = \mathrm{cl}(A) - \{a\}$ is not closed, there is $f \in \mathcal{F}$ with $f^{-1}(B)$ not being closed in S. This can only happen if $0 \notin f^{-1}(B)$ and there is an increasing sequence $\{n(k)\}_{k\geqslant 1}$ with $1/n(k) \in f^{-1}(B)$ for each k. On the other hand, $f^{-1}(\mathrm{cl}(A))$ is closed in S which is possible only if $f(0) = a$. Let $a_k = f(1/n(k))$ for $k \geqslant 1$. Given a neighborhood U of a in X, $f^{-1}(U)$ is open and contains 0. Thus $1/n(k) \in f^{-1}(U)$ for all k large enough, which implies that $a_k \in U$ for all k large enough. Thus, X is a Fréchet space.

Conversely, if X is a Fréchet space, then one can easily check that its topology is the covariant topology induced by $\mathcal{F} = \{f : S \to X \mid f \text{ is continuous}\}$. $\square$

The dual to the notion of the covariant topology is the contravariant topology:

Definition 1.9. The topology τ on X is called the *contravariant topology* induced by the family of functions $\mathcal{F} = \{f : X \to X_f\}$ if each X_f is a topological space and τ is the smallest of all topologies on X under which all $f \in \mathcal{F}$ are continuous.

Notice that the contravariant topology exists and its sub-basis consists of all sets $f^{-1}(U)$, where U is open in X_f for some $f \in \mathcal{F}$.

Example 1.10. If A is a subset of a topological space X, then the subspace topology on A is the contravariant topology induced by the inclusion $i_A : A \to X$.

Proof. The subspace topology on A consists of $A \cap U$, where U is open in X. Notice that $A \cap U = i_A^{-1}(U)$. $\square$

Example 1.11. An injective map $f : X \to Y$ is a homeomorphic embedding iff the topology on X is the contravariant topology induced by $\{f\}$.

Proof. f is a homeomorphic embedding iff it induces a homeomorphism between X and $f(X)$. Thus, U is open in X iff $f(U)$ is open in $f(X)$. By the previous example, $f(U)$ is open in $f(X)$ iff there is an open set U' in X with $f(U) = f(X) \cap U'$. Since f is injective, $f(U) = f(X) \cap U'$ iff $U = f^{-1}(U')$ which completes the proof. $\square$

Example 1.12. The product topology on the Cartesian product $\prod_{s \in S} X_s$ is the contravariant topology induced by projections $\{\pi_t : \prod_{s \in S} X_s \to X_t\}_{t \in S}$.

Proof. Since all projections are continuous when the product topologies are considered, the contravariant topology induced by projections is contained in the product topology. To prove that the product topology is contained in the contravariant topology induced by projections it suffices to show that each element of a sub-basis of the product topology belongs to the contravariant topology. As is well known, the standard sub-basis of the product topology consists of sets $\prod_{s \in S} U_s$ such that there is $t \in S$ with $U_s = X_s$ for $s \neq t$ and U_t is open in X_t. Since $\prod_{s \in S} U_s = \pi_t^{-1}(U_t)$, it must belong to the contravariant topology. $\square$

The basic property of covariant topologies is:

Proposition 1.13. *Suppose the topology of X is the covariant topology induced by a family of functions $\{f_i : X_i \to X\}_{i \in J}$. Then, a function $g : X \to Y$ is continuous iff $g \circ f_i$ is continuous for all $i \in J$.*

Proof. Suppose $g \circ f_i$ is continuous for all $i \in J$. Consider the contravariant topology τ_g on X induced by g and notice that it consists of sets $g^{-1}(U)$, where U is open in Y. Since $f_i^{-1}(g^{-1}(U)) = (g \circ f_i)^{-1}(U)$ is open in X_i if U is open in Y, we conclude that τ_g must be contained in the covariant topology induced by $\{f_i : X_i \to X\}_{i \in J}$. Notice that this statement is identical with the statement that g is continuous. $\square$

Traditionally, locally compact spaces are assumed to be Hausdorff and are defined as spaces which admit an open cover with the closure of each element being compact (see [12, Section 3.3]). It stems from the fact that compact spaces are assumed to be Hausdorff in [12]. In this paper we prefer more general definitions which allow any finite space to be locally compact:

Definition 1.14. A space X is *compact* if any open cover of X has a finite subcover. A space X is *locally compact* if for any neighborhood U of $x_0 \in X$ there is a compact neighborhood C of x_0 in X so that $C \subset U$.

Notice that Kuratowski Theorem (see [12, 3.1.16]) stating that X is compact iff the projection $p_Y : X \times Y \to Y$ is closed for each Y is still valid under Definition 1.14. Also notice that Whitehead Theorem (see [12, 3.3.17]) stating that if Z is locally compact, then $f \times id_Z : X \times Z \to Y \times Z$ is a quotient map for any quotient map $f : X \to Y$, is still valid under Definition 1.14 as the proof of it in [12] uses Kuratowski Theorem.

The basic result regarding covariant topologies is:

Theorem 1.15. *Suppose the topology on X is the covariant topology induced by a class of maps $\{f_s : X_s \to X\}_{s\in S}$ so that $X = \bigcup_{s\in S} f_s(X_s)$. If Z is locally compact, then the covariant topology induced by $\{f_s \times id_Z : X_s \times Z \to X \times Z\}_{s\in S}$ is the product topology on $X \times Z$, where each $X_s \times Z$ is equipped with the product topology.*

Proof. In the case of S being a one-point set, Theorem 1.15 reduces to the well known result of Whitehead stating that if $f : X \to Y$ is a quotient map, then so is $f \times id_Z$, provided Z is locally compact (see [12, Theorem 3.3.17]).

If S is not a one-point set, then consider all topologies τ on X which are identical with the contravariant topology induced by $\{f_s\}$ for some $s \in S$. These topologies form a set, and their intersection is the covariant topology induced by all of $\{f_s : X_s \to X\}_{s\in S}$. Thus, there is a subset T of S such that the topology on X is the covariant topology induced by $\{f_s : X_s \to X\}_{s\in T}$, and $X = \bigcup_{s\in T} f_s(X_s)$. Consider the disjoint union $\coprod_{s\in T} X_s$ with the topology being the covariant topology induced by all inclusions $i_r : X_r \to \coprod_{s\in T} X_s$. Notice that there is a quotient map $\pi : \coprod_{s\in T} X_s \to X$ so that $\pi|X_r = f_r$ for each $r \in T$. Therefore, $\pi \times id_Z$ is a quotient map which means that the product topology on $X \times Z$ is identical with the covariant topology τ_1 induced by $\{f_s \times id_Z : X_s \times Z \to X \times Z\}_{s\in T}$. Notice that the covariant topology τ_2 induced by $\{f_s \times id_Z : X_s \times Z \to X \times Z\}_{s\in S}$ must be contained in τ_1. Also, since each $f_s \times id_Z$ is continuous when product topologies are considered, we infer that the product topology is contained in τ_2. This proves that the product topology on $X \times Z$ and τ_2 are identical. $\square$

The basic property of contravariant topologies is:

Proposition 1.16. *Suppose the topology of X is the contravariant topology induced by a family of functions $\{f_i : X \to X_i\}_{i\in J}$. Then, a function $g : Y \to X$ is continuous iff $f_i \circ g$ is continuous for each $i \in J$.*

Proof. To prove the continuity of g it suffices to show that $g^{-1}(U)$ is open in X for all U belonging to a sub-basis of X. Since the contravariant topology has sets $f_i^{-1}(V)$, V open in X_i, as a sub-basis, and $g^{-1}(f_i^{-1}(V)) = (f_i \circ g)^{-1}(V)$ is open in Y if V is open in X_i, g is continuous. □

2. Basic concepts in topology from covariant/contravariant points of view

Let us assume that the following spaces are well understood:

(1) Anti-discrete spaces (spaces with the smallest topology possible).
(2) Discrete spaces (spaces with the largest topology possible), including the integers $\mathbb{Z}$ and natural numbers $\mathbb{N}$.
(3) S^0 (the 0-dimensional sphere or the simplest discrete space which is not anti-discrete).
(4) The unit interval I with the standard topology.
(5) The real numbers $\mathbb{R}$ with the standard topology. $\mathbb{Q} \subset \mathbb{R}$ are rationals.

It is well known that connected spaces X are precisely those, so that all maps $f : X \to S^0$ are constant. Thus, connectedness is a contravariant property. On the other hand, path connectedness is a covariant property as X is path connected iff any map $f : S^0 \to X$ extends over I. Let us analyze basic concepts of topology from those two points of view.

Being T_0 is a covariant property:

Proposition 2.1. *X is T_0 iff any map $f : A \to X$ from an anti-discrete space A to X is constant.*

Proof. The intrinsic definition of T_0 spaces is that for any two distinct points of X there is an open set containing only one of them. If X is T_0 and $f : A \to X$ is a map, where A is anti-discrete, then $f(A)$ is an anti-discrete subspace of X. If $f(A)$ contained two distinct points, it would have a proper, non-empty open subset contrary to anti-discreteness of $f(A)$.

Suppose every map $f : A \to X$ is constant if A is anti-discrete. Given $x, y \in X$, $x \neq y$, the inclusion $i : \{x, y\} \to X$ is not constant. Therefore, $\{x, y\}$ is not an anti-discrete subspace of X. Thus, there is an open subset U of X containing only one of x, y. □

Anti-discrete spaces are the simplest topological spaces and Proposition 2.1 means that T_0 spaces are precisely those which cannot be detected by the lowest level of intelligence in topology.

If one maps S^0 to X, then all functions are continuous. The only question remaining is which of them are homeomorphic embeddings:

Proposition 2.2. *X is T_1 iff any non-constant map $f : S^0 \to X$ is a homeomorphic embedding.*

Proof. The intrinsic definition of T_1 spaces is that all one-point subspaces are closed. Therefore, all two-point subspaces are discrete, and any non-constant map $f : S^0 \to X$ induces a homeomorphism of S^0 and $f(S^0)$.

Suppose any map $f : S^0 \to X$ is a homeomorphic embedding. Suppose $x_0 \in X$ is a point such that $\{x_0\}$ is not closed. Thus, there is $x_1 \in \mathrm{cl}(x_0) - \{x_0\}$. Consider a bijection $f : S^0 \to \{x_0, x_1\}$. Obviously, f is continuous and it cannot be a homeomorphism. This contradiction proves that X is T_1. $\square$

Thus, T_1 spaces are detected in a covariant manner.

Let us show that the remaining separation properties are all of contravariant nature.

Proposition 2.3. *Suppose X is T_1. Then, X is T_2 (Hausdorff) iff S^0 is an absolute neighborhood extensor of X with respect to finite subspaces.*

Proof. First, let us recall the notion of absolute neighborhood extensor for pairs of topological spaces:

Definition 2.4. A topological pair (Y, B) is an *absolute neighborhood extensor* of (X, A) (notation: $(Y, B) \in ANE(X, A)$) if every map $f : A \to B$ extends to $F : U \to Y$ for some neighborhood U of A in X.

A topological pair (Y, B) is an *absolute extensor* of (X, A) (notation: $(Y, B) \in AE(X, A)$) if every map $f : A \to B$ extends to a map $F : X \to Y$.

$Y \in ANE(X, A)$ (respectively, $Y \in AE(X, A)$) means that $(Y, Y) \in ANE(X, A)$ (respectively, $(Y, Y) \in AE(X, A)$).

$Y \in ANE(X)$ (respectively, $Y \in AE(X)$) means that $(Y, Y) \in ANE(X, A)$ (respectively, $(Y, Y) \in AE(X, A)$) for all closed subsets A of X.

Notice that $X \in AE(I, S^0)$ iff X is path connected.

Being an absolute extensor corresponds to the notion of being an *injective module* in algebra.

Suppose X is Hausdorff. If A is a finite subset of X and $f : A \to S^0$ is a map, then we can find a neighborhood U_a of each $a \in A$ such that $U_a \cap U_b = \emptyset$ for all $a, b \in A$, $a \neq b$. Let $U = \bigcup_{a \in A} U_a$, and let $F : U \to S^0$ be defined by $F(x) = f(a)$ if $x \in U_a$. Notice that F is a continuous extension of f.

Suppose X has the property that $S^0 \in ANE(X, A)$ for all finite subsets of A of X. Suppose $x, y \in X$ and $x \neq y$. Take a bijection $f : \{x, y\} \to S^0$. Since X is T_1, f is continuous, and there is an extension $F : U \to S^0$ of f for some neighborhood U of $\{x, y\}$. Let $V = F^{-1}(f(x))$ and $W = F^{-1}(f(y))$. Then, $x \in V$, $y \in W$, and $V \cap W = \emptyset$. $\square$

Similarly to Proposition 2.3 one can see that regularity is a contravariant property among T_0 spaces.

Proposition 2.5. *Suppose X is T_0. Then, X is $T_{3\frac{1}{2}}$ (Tychonoff) iff the topology of X is the contravariant topology induced by a family of maps*

$$\{f_s : X \to I\}_{s \in S}.$$

Proof. Suppose X is Tychonoff. The intrinsic definition of Tychonoff spaces is that they are T_1 and each point of X has a basis consisting of cozero-sets, i.e., sets of the form $f^{-1}(0,1]$ for some map $f : X \to I$. That means the contravariant topology induced by the set of all maps from X to I coincides with the current topology of X.

Suppose the topology of X is the contravariant topology induced by a family of maps $\{f_s : X \to I\}_{s\in S}$. Given two distinct points x and y in X there is $s \in S$ with $f_s(x) \neq f_s(y)$ (otherwise consider A to be $\{x, y\}$ with the discrete topology, and the inclusion $i : A \to X$ would be continuous as $f_s \circ i$ is continuous for each $s \in S$—see Proposition 1.16), which implies that $\{x, y\}$ is homeomorphic to S^0. Thus, X is T_1. Suppose $x \in X$ and U is a neighborhood of x in X. Since the contravariant topology has sub-basis consisting of all sets $f_s^{-1}(V)$, where V is open in I and $s \in S$, there is a finite set $\{s_1, \ldots, s_k\}$ in S such that $x \in \bigcap_{i=1}^{k} f_{s_i}^{-1}(V_i) \subset U$ for some open sets V_i, $1 \leqslant i \leqslant k$, of I. For each i choose a map $g_i : I \to I$ so that $V_i = g_i^{-1}(0,1]$ and define $h_i = g_i \circ f_{s_i}$. Notice that the average h of all h_i has the property that $x \in h^{-1}(0,1] \subset U$ which proves that X is Tychonoff. □

One can easily generalize the proof of Proposition 2.3 and deduce the following two results:

Proposition 2.6. *Suppose X is T_1. Then, X is T_4 (normal) iff S^0 is an absolute neighborhood extensor of X.*

Proposition 2.7. *Suppose X is T_1. Then, X is collectionwise normal iff all discrete spaces D are absolute neighborhood extensors of X.*

The purest contravariant approximation of compactness is pseudo-compactness (see [12, 3.10]):

X is called *pseudo-compact* if any map $f : X \to \mathbb{R}$ from X to reals is bounded.

The following result summarizes well known characterizations of Hausdorff compact spaces in terms which are contravariant in spirit:

Theorem 2.8. *Suppose X is Hausdorff. The following conditions are equivalent*:

(1) *X is compact,*
(2) *X is regular and any map $f : X \to Y$ from X to a Hausdorff space is closed,*
(3) *X is regular and $f(X)$ is closed in Y for any map f from X to a Hausdorff space Y,*
(4) *X is regular and f is a homeomorphic embedding for any injective map f from X to a Hausdorff space Y.*

Proof. Before proceeding with the proof of Proposition 2.8 let us notice that conditions (2)–(4) can be viewed is duals to one of the following characterizations of T_1 spaces (see Proposition 2.2):

(a) any map $f : S^0 \to Z$ is closed,
(b) $f(S^0)$ is closed for any map f from S^0 to Z,
(c) f is a homeomorphic embedding for any injective map f from S^0 to Z.

Also notice that if one drops the assumption of regularity in either (3) or (4), then one gets larger classes than compact Hausdorff spaces. Indeed, Alexandroff and Urysohn [1] introduced the concept of being H-closed:

(i) Suppose X is a Hausdorff space. X is called *H-closed* if for any injective map $f : X \to Y$ from X to a Hausdorff space Y, $f(X)$ is closed in Y.

Katětov [21] proved that images of H-closed spaces are H-closed. This means that H-closed spaces satisfy condition (3) of Theorem 2.8 if regularity is disregarded.

Similarly, Parchomienko [25] and Katětov [21] introduced the concept of being H-minimal:

(ii) Suppose X is a Hausdorff space. X is called *H-minimal* if any injective map $f : X \to Y$ from X to a Hausdorff space Y is a homeomorphic embedding.

Theorem 2.8 can be deduced from results in [1,21,25,27]. However, since those papers are rather old, we will provide a direct proof of Theorem 2.8 for the convenience of the reader.

Notice that (1) $\Rightarrow$ (2), (3), (4) are well known (see [12, 3.1.12–3.1.13]).

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Suppose $\{U_s\}_{s\in S}$ is an open cover of X. Since X is regular, it suffices to show that $\{\mathrm{cl}(U_s)\}_{s\in S}$ has a finite subcover. Choose $\infty \notin X$ and create a topology on $Y = X \cup \{\infty\}$ so that X is open in Y and $\{\infty\} \cup (X - \bigcup_{s\in F} \mathrm{cl}(U_s))$, F being finite in S, form a basis of neighborhoods of ∞. Notice that Y is Hausdorff. Indeed, if $x_0 \in U_t$ for some $t \in S$, then $U_t \cap (\{\infty\} \cup (X - \mathrm{cl}(U_t))) = \emptyset$. Thus, x_0 and ∞ have disjoint neighborhoods. Clearly, any pair of different points of X have disjoint neighborhoods in Y. Also notice that Y can be made regular if our original cover $\{U_s\}_{s\in S}$ is enlarged so that it has the following property: if $x \in U_s$, then there is $t \in S$ with $x \in U_t \subset \mathrm{cl}_X(U_t) \subset U_s$.

Since X is closed in Y, then $\emptyset = X - \bigcup_{s\in F} \mathrm{cl}(U_s)$ for some finite subset F of S, and we are done.

(4) $\Rightarrow$ (1) Begin as in the proof of (3) $\Rightarrow$ (1). As above, Y is Hausdorff. Pick any $x_0 \in X$ and consider the quotient space Z obtained from Y by identifying x_0 and ∞. Notice that Z is Hausdorff. Let $q : Y \to Z$ be the quotient map and let $i : X \to Y$ be the inclusion. Since $f = i \circ q$ is bijective, it must be a homeomorphism. Hence, $f(U_t)$, where $x_0 \in U_t$, is open in Z. Now, $q^{-1}(f(U_t)) = U_t \cup \{\infty\}$ is open and there is a finite set $F \subset S$ with

$$X - \bigcup_{s\in F} \mathrm{cl}(U_s) \subset U_t.$$

Notice that $\{\mathrm{cl}(U_s)\}_{s\in G}$, $G = F \cup \{t\}$, covers X. □

It is interesting to note that Stone [27] and Katětov [21] characterized compact Hausdorff spaces as those Hausdorff spaces with all closed subsets being H-closed.

The following well known result of Tamano (see [12, Theorem 5.1.38]) can be interpreted that paracompactness is a contravariant property:

Theorem 2.9 (Tamano). *$X \in T_2$ is paracompact iff $X \times C$ is normal for all compact Hausdorff spaces C.*

The following metrizability criterion proved by the author in [8] means that metrizability is a contravariant property:

Theorem 2.10. *$X \in T_0$ is metrizable iff the topology of X is the contravariant topology induced by a set of maps $\{f_s : X \to I\}_{s\in S}$ such that*

$$\sum_{s\in S} f_s = 1.$$

Theorem 2.10 was improved in [8] as follows:

Theorem 2.11. *$X \in T_0$ is metrizable iff there is a set of maps $\{f_s : X \to I\}_{s\in S}$ such that*

$$\sum_{s\in S} f_s = 1 \quad \textit{and} \quad \bigl\{f_s^{-1}(0,1]\bigr\}_{s\in S}$$

is a basis of X.

Theorem 2.11 implies the well known metrizability criteria, Kuratowski–Wojdysławski Theorem, and Arens–Eells Theorem (see [8]).

Completeness in the sense of Čech is a covariant property:

Proposition 2.12. *Suppose X is a Tychonoff space. Then, X is complete in the sense of Čech iff any map $f : A \to X$ from a subset A of a Tychonoff space Y extends over a G_δ subset of Y.*

Proof. Use [12, Theorem 3.9.1]. □

Being of covering dimension n is a contravariant property (see [13,20]):

Theorem 2.13 (Hurewicz–Wallman). $\dim(X) \leqslant n$ *iff* $S^n \in AE(X)$.

Theorem 2.13 explains why the covering dimension is the most widely used of all theories of dimension.

In [8] the author proved the following generalization of Tietze–Urysohn Theorem and Urysohn Lemma:

Theorem 2.14. *Suppose $Y \neq \{point\}$ is a Hausdorff space. Then, the following conditions are equivalent*:

(1) $Cone(Y) \in AE(X)$,
(2) $(Cone(Y), Y) \in AE(X)$,
(3) $Y \in ANE(X)$.

Traditionally, the $Cone(Y)$ of Y is understood as the quotient space $Y \times I/Y \times \{0\}$. That would mean that the topology of the cone is introduced in a covariant manner. If one wants to map spaces to the cone, then as seen in Proposition 1.13, it is better to introduce a topology on the cone in a contravariant manner. Notice that there are two natural functions: the projection $p_I : Cone(X) \to I$ and the projection $p_X : Cone(X) - pt \to X$. These two functions define a contravariant topology on the cone $Cone(Y)$ which is equivalent to the one introduced in [8]. Thus, for general spaces Y, one has two kinds of cones: the covariant cone and the contravariant cone. In the case of a metric space Y, the covariant cone may not be metrizable but the contravariant cone is metrizable (use Theorem 2.10). Theorem 2.14 deals with contravariant cones.

In homotopy theory (see [22,29]), of fundamental importance is the notion of a k-space (also known as a compactly generated space or a Kelly space). It is usually assumed that k-spaces are Hausdorff and their defining property is that A is closed in X iff $A \cap C$ is closed in C for every compact subset C of X. Let us extend the definition of a k-space to all topological spaces (i.e., not necessarily Hausdorff). Our definition is of covariant nature. First, let us define a functor on the category $\mathcal{TOP}$ of all topological spaces:

Definition 2.15. Given a space X consider the class $\mathcal{F} = \{f : C_f \to X\}$ of all maps from locally compact spaces to X. X equipped with the covariant topology induced by $\mathcal{F}$ is denoted by kX.

If *Top* is a topology on X, then the resulting topology on kX is denoted by k*Top*.

Notice that $id_X : kX \to X$ is continuous.

Definition 2.16. X is called a *k-space* (or a *compactly generated space* or a *Kelly space*) if $id_X : kX \to X$ is a homeomorphism.

Here is a useful characterization of k-spaces which implies that our definition coincides with the classical one:

Proposition 2.17. *X is a k-space iff the topology of X is the covariant topology induced by a class of functions*

$$\{f_s : C_s \to X\}_{s \in S}$$

from locally compact spaces to X.

Proof. If X is a k-space, then its topology is induced by a class of maps from locally compact spaces to X (it follows directly from the definition). Suppose the topology of X is the covariant topology induced by a class of functions $\{f_s : C_s \to X\}_{s \in S}$ from locally compact spaces to X. In particular, each f_s is continuous which means that the topology on X induced by all maps from locally compact spaces to X must coincide with the current topology on X. $\square$

The philosophical meaning of Proposition 2.17 is that k-spaces coincide with those spaces whose topological properties are detectable by maps from locally compact spaces.

Corollary 2.18. *Locally compact spaces are k-spaces. A Hausdorff space X is a k-space iff the following conditions are equivalent*:
(1) *A is closed in X,*
(2) *$A \cap C$ is closed in C for each compact subset C of X.*

Proof. If X is locally compact, then $id_X : X \to X$ induces the topology on X. Use Proposition 2.17 for S consisting of one point.

Notice that (2) $\Rightarrow$ (1) means that the family of inclusions $i_C : C \to X$, C being a compact subset in X, induces the current topology on X. By Proposition 2.17, X is a k-space as Hausdorff compact spaces are locally compact. □

Here is the fundamental property of the functor $k : \mathcal{TOP} \to k\mathcal{TOP}$ to the category of all k-spaces:

Proposition 2.19. *$f = id_X : kX \to X$ is universal in the following sense*: *for any map $g : Z \to X$ from a k-space Z there is a unique map $h : Z \to kX$ with $g = f \circ h$. In particular, $k : \mathcal{TOP} \to k\mathcal{TOP}$ is a right adjoint to the inclusion functor $i : k\mathcal{TOP} \to \mathcal{TOP}$.*

Proof. The first part amounts to proving that $g : Z \to kX$ is continuous if $g : Z \to X$ is continuous and Z is a k-space. First, notice that it is so if Z is locally compact by the definition of kX. Second, to prove that $g : Z \to kX$ is continuous we need, in view of Theorem 1.13, to show that for any map $r : C \to Z$, C being locally compact, $g \circ r$ is continuous. Thus, the general case is reduced to the case of Z being locally compact.

The first part of Proposition 2.19 means that assigning $g : Z \to kX$ to $g : i(Z) \to X$ is a bijection. This clearly establishes a natural equivalence of $Map(Z, kX)$ and $Map(i(Z), X)$ for all spaces X and all k-spaces Z. Thus, k is a right adjoint to the inclusion functor $i : k\mathcal{TOP} \to \mathcal{TOP}$. □

The following result is well known in the case of Hausdorff spaces (see [12, Theorem 3.3.27]). To us it underscores the importance of Theorem 1.15:

Corollary 2.20. *If X is a k-space and Y is locally compact, then $X \times Y$ is a k-space.*

Proof. Suppose the topology of X is the covariant topology induced by a class of functions $\{f_s : C_s \to X\}_{s \in S}$ from locally compact spaces to X. By Theorem 1.15, the product topology on $X \times Y$ is the covariant topology induced by the class of functions $\{f_s : C_s \times Y \to X \times Y\}_{s \in S}$ from locally compact spaces to X. Since the product of two locally compact spaces is locally compact, Proposition 2.17 implies that $X \times Y$ is a k-space. □

In analogy to k-spaces one can introduce a new functor on $\mathcal{TOP}$ which will be useful in investigating of the pointwise convergence topology:

Definition 2.21. Given a space X consider the class $\mathcal{F} = \{f : C_f \to X\}$ of all maps from finite topological spaces to X. X equipped with the covariant topology induced by $\mathcal{F}$ is denoted by fX.

Notice that $id_X : fX \to X$ is continuous and $id_X : fX \to kX$ is continuous by Propositions 2.17 and 2.19.

The following proposition can be easily proved by following the proof of Corollary 2.18:

Proposition 2.22. *The following conditions are equivalent*:

(1) *A is closed in fX,*

(2) *$A \cap F$ is closed in F for each finite subset F of X.*

3. Function spaces

Understanding the variety of topologies on function spaces, and understanding the origins of the compact-open topology has been one of the biggest problems in basic topology this author has encountered. The purpose of this section is to show how the covariant/contravariant approaches help in this task.

Suppose one would like to give a contravariant topology to the space $Map(X, Y)$ of all continuous maps from X to Y. The question arises: Are there any natural functions $Map(X, Y) \to Z$? As far as the author knows, the only natural functions from $Map(X, Y)$ to known spaces are evaluations at a point of X:

Definition 3.1. Given $x \in X$ one defines $e_x : Map(X, Y) \to Y$ by $e_x(f) = f(x)$.

Proposition 3.2. *Suppose X and Y are topological spaces. Then, the contravariant topology on $Map(X, Y)$ induced by $\{e_x\}_{x \in X}$ coincides with the pointwise convergence topology on $Map(X, Y)$.*

Proof. The intrinsic definition of the pointwise convergence topology is that its sub-basis consists of all sets $P(x, U) = \{f \in Map(X, Y) \mid f(x) \in U\}$, where $x \in X$ and U is an open subset of Y. Notice that $P(x, U) = e_x^{-1}(U)$, which proves Proposition 3.2. $\square$

Let us turn our attention to covariant topologies on $Map(X, Y)$.

Let $S = \{0\} \cup \{1/n \mid n \geqslant 1\} \subset \mathbb{R}$. If Y has a metric, then one has a concept of uniform convergence of a sequence of functions. The uniform convergence topology on $Map(X, Y)$ has sets

$$U(g, \varepsilon) = \Bigl(h \in Map(X, Y) \Bigm| \sup_{x \in X} \rho\bigl(g(x), h(x)\bigr) < \varepsilon\Bigr)$$

as a basis. Notice that one can define a function $\mu : Map(X, Y) \times Map(X, Y) \to \mathbb{R} \cup \{\infty\}$ by $\mu(f, g) = \sup_{x \in X} \rho(g(x), h(x))$. This function leads to a metric

$$d(f, g) = \frac{\mu(f, g)}{1 + \mu(f, g)}$$

($d(f,g)=1$ if $\mu(f,g)=\infty$) which induces the same topology as the uniform convergence topology. As in Proposition 1.8 (metric spaces are Fréchet spaces) one gets the following:

Proposition 3.3. *Suppose Y has a metric ρ and consider all the functions*

$$\left\{f_s:\{0\}\cup\{1/n \mid n\geqslant 1\}\to \mathit{Map}(X,Y)\right\}_{s\in S}$$

such that $f_s(1/n)$ converges uniformly to $f_s(0)$ for each $s\in S$. Then, the covariant topology on $\mathit{Map}(X,Y)$ induced by $\{f_s\}_{s\in S}$ coincides with the uniform convergence topology on $\mathit{Map}(X,Y)$.

Now, suppose Y has no metric. Are there any natural functions $f:Z\to \mathit{Map}(X,Y)$? The practice in topology (especially in homotopy theory) is to switch immediately to the function $\mathit{adj}_X(f):Z\times X\to Y$ (see Definition 0.2). Thus, one may view f to be natural if $\mathit{adj}_X(f)$ is continuous. This leads to the following definition:

Definition 3.4. Consider the class of all functions $\{f_i:S_i\to \mathit{Map}(X,Y)\}_{i\in J}$ such that $\mathit{adj}_X(f_i):S_i\times X\to Y$ is continuous for each $i\in J$, where $S_i\times X$ is considered with the product topology. The covariant topology on $\mathit{Map}(X,Y)$ induced by $\{f_i:S_i\to \mathit{Map}(X,Y)\}_{i\in J}$ is called the *basic covariant topology* and the resulting topological space is denoted by $\mathit{Map}_{\mathit{Cov}}(X,Y)$.

Basic covariant topology was considered in [3] (under the name of the greatest proper topology) only in the context of comparison to the compact-open topology. Namely, it was shown there that there exists a Tychonoff space X such that $\mathit{Map}(X,I)$ considered with the compact-open topology is not identical with $\mathit{Map}(X,I)$ equipped with the basic covariant topology (see [3, Theorem 5.3]).

Notice that for any $\mathit{Top}\subset \mathit{Cov}$ the function adj^Y maps $\mathit{Map}(X\times Y,Z)$ to $\mathit{Map}(X, \mathit{Map}_{\mathit{Top}}(Y,Z))$. The natural way to proceed is to analyze if $\mathit{adj}^Y:\mathit{Map}(X\times Y,Z)\to \mathit{Map}(X,\mathit{Map}_{\mathit{Top}}(Y,Z))$ is a natural equivalence if Y is being fixed. This amounts to analyzing if $G(Z)=\mathit{Map}_{\mathit{Top}}(Y,Z)$ is a right adjoint to $F(X)=X\times Y$ with adj^Y being an adjugant (see Definition 0.4). As explained in Section 0 (see Definition 0.4), the crucial cases are $X=\mathit{Map}_{\mathit{Top}}(Y,Z)$ and $Z=X\times Z$. If $X=\mathit{Map}_{\mathit{Top}}(Y,Z)$, then one needs to consider the function $(\mathit{adj}^Y)^{-1}(\mathit{id}_X)=\mathit{adj}_Z(\mathit{id}_X)$ which is simply the evaluation function $\mathit{eval}:\mathit{Map}(Y,Z)\times Y\to Z$ (see Definition 0.2). If one wants adj^Y to be a bijection, then eval needs to be continuous. The converse also holds:

Proposition 3.5. *Let $\mathit{Top}\subset \mathit{Cov}$ be a topology on $\mathit{Map}(Y,Z)$. If $\mathit{eval}:\mathit{Map}_{\mathit{Top}}(Y,Z)\times Y\to Z$ is continuous, then*

$$\mathit{adj}^Y:\mathit{Map}(X\times Y,Z)\to \mathit{Map}\big(X,\mathit{Map}_{\mathit{Top}}(Y,Z)\big)$$

is a bijection.

Proof. It suffices to prove that adj^Y is onto. Suppose $f:X\to \mathit{Map}_{\mathit{Top}}(Y,Z)$ is continuous. Notice that $\mathit{adj}_Y(f)=\mathit{eval}\circ(f\times \mathit{id}_Y)$ is continuous. □

Thus, one needs to consider the question of continuity of the evaluation function. The beauty of covariant topologies is that Theorem 1.15 has an immediate partial answer:

Corollary 3.6. *$eval|Map_{Cov}(X,Y)\times K$ is continuous if K is a locally compact subset of X. If X is locally compact, then $f:S\to Map_{Cov}(X,Y)$ is continuous if and only if $adj_X(f):S\times X\to Y$ is continuous.*

Proof. Let $\{f_i:S_i\to Map(X,Y)\}_{i\in J}$ be the class of all functions such that $adj_X(f_i):S_i\times X\to Y$ is continuous for each $i\in J$. By Theorem 1.15, the product topology on $Map_{Cov}(X,Y)\times K$ coincides with the covariant topology induced by

$$\left\{f_i\times id_K:S_i\times K\to Map(X,Y)\times K\right\}_{i\in J}.$$

According to Proposition 1.13, $eval|Map_{Cov}(X,Y)\times K$ is continuous iff $eval\circ(f_i\times id_K)$ is continuous for each $i\in J$. Since $eval\circ(f_i\times id_K)=adj_X(f_i)|S_i\times K$, it is continuous for all locally compact subsets K of X which proves the first part of Corollary 3.6. Notice that $adj_X(f)=eval\circ(f\times id_X)$. If X is locally compact, then $eval:Map_{Cov}(X,Y)\times X\to Y$ is continuous which implies that $adj_X(f)$ is continuous. □

Corollary 3.6 was deduced with the help of Theorem 1.15 which, as can be seen from its proof, is equivalent to the Whitehead Theorem (the special case of Theorem 1.15 where S consists of one point). Observe that Corollary 3.6 implies the Whitehead Theorem. Indeed, if $f:X\to Y$ is a surjective quotient map and Z is locally compact, then in order to show that $f\times id_Z$ is a quotient map we need to prove that for any function $g:Y\times Z\to T$, the continuity of $g\circ(f\times id_Z)$ implies the continuity of g. Now, $adj^Z(g\circ(f\times id_Z))=adj^Z(g)\circ f$ is continuous which implies that $adj^Z(g)$ is continuous as f is a quotient map. By Corollary 3.6, g is continuous.

We are ready to improve Proposition 3.5:

Proposition 3.7.

$$adj^Y:Map_{Cov}(X\times Y,Z)\to Map_{Cov}\big(X,Map_{Cov}(Y,Z)\big)$$

is continuous and is a homeomorphism if Y is locally compact.

Proof. We need to prove continuity of adj^Y. It suffices, in view of Proposition 1.13, to show that if $f:S\to Map(X\times Y,Z)$ is such that $adj_{X\times Y}(f):S\times X\times Y\to Z$ is continuous, then $adj^Y\circ f:S\to Map_{Cov}(X,Map_{Cov}(Y,Z))$ is continuous. This will be guaranteed if $adj_X(adj^Y\circ f):S\times X\to Map_{Cov}(Y,Z)$ is continuous, which, in turn, is guaranteed to be continuous if $adj_Y(adj_X(adj^Y\circ f)):S\times X\times Y\to Z$ is continuous. Clearly, the last function equals $adj_{X\times Y}(f)$ which was assumed to be continuous.

Now, assume Y is locally compact. To prove the existence and continuity of the inverse of adj^Y it suffices to show that if

$$f:S\to Map_{Cov}\big(X,Map_{Cov}(Y,Z)\big)$$

is such that $adj_X(f): S \times X \to Map_{Cov}(Y, Z)$ is continuous, then there is continuous $g: S \to Map_{Cov}(X \times Y, Z)$ with $adj^Y(g) = f$. Indeed, S being one-point space corresponds to adj^Y being surjective, and the general case of S proves, in view of Proposition 1.13, that $(adj^Y)^{-1}$ is continuous. Since Y is locally compact, Corollary 3.6 implies that $adj_Y(adj_X(f)): S \times X \times Y \to Z$ is continuous. This, in turn, means that

$$g = adj^{X \times Y}\big(adj_Y\big(adj_X(f)\big)\big): S \to Map_{Cov}(X \times Y, Z)$$

is continuous which completes the proof. □

Let us show that the basic covariant topology preserves certain separability properties:

Proposition 3.8. *If $Y \in T_i$ for some $i \in \{0, 1, 2\}$, then $Map_{Cov}(X, Y) \in T_i$.*

Proof. Suppose Y is T_0, A is anti-discrete, and $f: A \to Map_{Cov}(X, Y)$ is a map. By Corollary 3.6, $adj_X(f)|A \times \{x\}$ is continuous for each $x \in X$. Since Y is T_0, $adj_X(f)|A \times \{x\}$ is constant for each $x \in X$ which means that f is constant. By Proposition 2.1, $Map_{Cov}(X, Y)$ is T_0.

Suppose Y is T_1 and $f: S^0 \to Map_{Cov}(X, Y)$ is not constant. There is $x \in X$ such that $f(0)(x) \neq f(1)(x)$. Let $S = f(S^0)$ and let $i: S \to Map_{Cov}(X, Y)$ be the inclusion. Since $adj_X(i)|S \times \{x\}$ is continuous (see Corollary 3.6), it establishes a homeomorphism between S and $\{f(0)(x), f(1)(x)\}$. Since Y is T_1, $\{f(0)(x), f(1)(x)\}$ is discrete, and that means S is discrete. By Proposition 2.2, $Map_{Cov}(X, Y)$ is T_1.

Suppose Y is Hausdorff and f, g are two different elements of $Map(X, Y)$. There is $x \in X$ with $f(x) \neq g(x)$, and, by Corollary 3.6, $\alpha = eval|Map_{Cov}(X, Y) \times \{x\}$ is continuous. Choose two disjoint neighborhoods of U of $f(x)$ and V of $g(x)$. Notice that $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ when projected onto $Map_{Cov}(X, Y)$ give two disjoint neighborhoods of f and g. □

Problem 3.9. Suppose Y is regular (respectively, Tychonoff). Is $Map_{Cov}(X, Y)$ regular (respectively, Tychonoff)?

Our next two results show that the basic covariant topology possesses similar properties to those of compact-open topology:

Proposition 3.10. *$Map_{Cov}(X, Y)$ is a contravariant functor from the point of view of X* (*if Y is fixed*), *and is a covariant functor from the point of view of Y* (*if X is fixed*).

Proof. Suppose Y is fixed and $f: X \to Z$ is a map. Then, one has a natural function $f^*: Map_{Cov}(Z, Y) \to Map_{Cov}(X, Y)$ ($f^*(g) = g \circ f$) which we would like to be continuous. According to Proposition 1.13, to prove continuity of f^* one needs to show that if $g: S \to Map(Z, Y)$ is such that $adj_Z(g)$ is continuous, then $f^* \circ g$ is continuous. Notice that $adj_Z(f^* \circ g) = adj_Z(g) \circ (id_S \times g)$ is continuous which implies the continuity of $f^* \circ g$. Also notice that if $f: X \to Z$ and $g: Z \to T$, then $(g \circ f)^* = f^* \circ g^*$ which completes the proof of $Map_{Cov}(X, Y)$ being a functor if Y is fixed.

Suppose X is fixed. If $f:Y\to Z$ is a map, then one has a natural function $f_*:Map_{Cov}(X,Y)\to Map_{Cov}(X,Z)$ $(f_*(g)=f\circ g)$ which we need to prove to be continuous. According to Proposition 1.13, to prove continuity of f_* one needs to show that if $g:S\to Map_{Cov}(X,Y)$ is such that $adj_X(g)$ is continuous, then $f_*\circ g$ is continuous. Notice that $adj_X(f_*\circ g)=f\circ adj_X(g)$ is continuous which implies the continuity of $f_*\circ g$. Also notice that if $f:Y\to Z$ and $g:Z\to T$, then $(g\circ f)_*=g_*\circ f_*$ which completes the proof of $Map_{Cov}(X,Y)$ being a functor if X is fixed. $\square$

Proposition 3.11. *If X is homotopy equivalent to Z, then $Map_{Cov}(X,Y)$ is homotopy equivalent to $Map_{Cov}(Z,Y)$. If Y is homotopy equivalent to Z, then $Map_{Cov}(X,Y)$ is homotopy equivalent to $Map_{Cov}(X,Z)$.*

Proof. First notice that if $f:X\to X$ is homotopic to the identity id_X, then f^* is homotopic to the identity. Indeed, let $H:X\times I\to X$ be a map such that $H(x,0)=f(x)$ and $H(x,1)=x$ for each $x\in X$. Define $G:Map_{Cov}(X,Y)\times I\to Map_{Cov}(X,Y)$ as follows:

$$G(\alpha,t)(x)=\alpha\big(H(x,t)\big)\quad \text{if } \alpha\in Map_{Cov}(X,Y),\ x\in X,\ \text{and } t\in I.$$

Notice that $G(\alpha,0)=f^*(\alpha)$ and $G(\alpha,1)=\alpha$ for each $\alpha\in Map_{Cov}(X,Y)$. Thus, it suffices to show that G is continuous. In view of Theorem 1.15 we need to show that given $\beta:S\to Map_{Cov}(X,Y)$ with $adj_X(\beta)$ continuous, then $G\circ(\beta\times id_I)$ is continuous. Now,

$$adj_X\big(G\circ(\beta\times id_I)\big)(s,t,x)=G\big(\beta(s),t\big)(x)=\beta(s)\big(H(x,t)\big)$$
$$\text{for each } (s,t,x)\in S\times I\times X$$

which means that $adj_X(G\circ(\beta\times id_I))$ is the composition of $F:S\times I\times X\to S\times X$, $F(s,t,x)=(s,H(x,t))$, and $adj_X(\beta):S\times X\to Y$. Thus, $adj_X(G\circ(\beta\times id_I))$ is continuous, and $G\circ(\beta\times id_I)$ is continuous.

If $f:X\to Z$ and $g:Z\to X$ are two maps such that $f\circ g$ is homotopic to id_Z and $g\circ f$ is homotopic to id_X, then $g^*\circ f^*=(f\circ g)^*$ is homotopic to the identity on $Map_{Cov}(Z,Y)$ and $f^*\circ g^*=(g\circ f)^*$ is homotopic to the identity on $Map_{Cov}(X,Y)$ which proves the first part of Proposition 3.11.

Notice that if $f:Y\to Y$ is homotopic to the identity id_Y, then f_* is homotopic to the identity. Indeed, let $H:Y\times I\to Y$ be a map such that $H(y,0)=f(y)$ and $H(y,1)=y$ for each $y\in Y$. Define $G:Map_{Cov}(X,Y)\times I\to Map_{Cov}(X,Y)$ as follows:

$$G(\alpha,t)(x)=H\big(\alpha(x),t\big)\quad \text{if } \alpha\in Map(X,Y),\ x\in X,\ \text{and } t\in I.$$

Notice that $G(\alpha,0)=f_*(\alpha)$ and $G(\alpha,1)=\alpha$ for each $\alpha\in Map(X,Y)$. Thus, it suffices to show that G is continuous. In view of Theorem 1.15 we need to show that given $\beta:S\to Map_{Cov}(X,Y)$ with $adj_X(\beta)$ continuous, then $G\circ(\beta\times id_I)$ is continuous. Now,

$$adj_X\big(G\circ(\beta\times id_I)\big)(s,t,x)=G\big(\beta(s),t\big)(x)=H\big(\beta(s)(x),t\big)$$
$$\text{for each } (s,t,x)\in S\times I\times X$$

which means that $adj_X(G\circ(\beta\times id_I))$ is the composition of $adj_X(\beta)\times id_I:S\times X\times I\to Y\times I$ and H. Thus, $adj_X(G\circ(\beta\times id_I))$ is continuous, and $G\circ(\beta\times id_I)$ is continuous.

If $f:Y \to Z$ and $g:Z \to y$ are two maps such that $f \circ g$ is homotopic to id_Z and $g \circ f$ is homotopic to id_Y, then $g_* \circ f_* = (g \circ f)_*$ is homotopic to the identity on $Map_{Cov}(X, Y)$ and $f_* \circ g_* = (f \circ g)^*$ is homotopic to the identity on $Map_{Cov}(X, Z)$ which proves the second part of Proposition 3.11. $\square$

4. Compact-open and pointwise convergence topologies

The idea of the functor $k:\mathcal{TOP} \to k\,\mathcal{TOP}$ is that topological spaces are investigated by mappings from locally compact spaces (see Definition 2.15). Similarly, one can investigate topological spaces by maps from finite topological spaces (see Definition 2.21). This leads to two additional topologies on function spaces:

Definition 4.1. Consider the class of all maps $\{f_i:C_i \to X\}_{i\in J}$ from locally compact spaces to X. $Map(X, Y)$ equipped with the contravariant topology induced by

$$\left\{f_i^*:Map(X, Y) \to Map_{Cov}(C_i, Y)\right\}_{i\in J}$$

is denoted by $Map_{CO}(X, Y)$.

Consider the class of all maps $\{f_i:F_i \to X\}_{i\in J}$ from finite topological spaces to X. $Map(X, Y)$ equipped with the contravariant topology induced by

$$\left\{f_i^*:Map(X, Y) \to Map_{Cov}(F_i, Y)\right\}_{i\in J}$$

is denoted by $Map_{PC}(X, Y)$.

Notice that $PC \subset CO \subset Cov$. Obviously, both $Map_{CO}(X, Y)$ and $Map_{PC}(X, Y)$ define a bifunctor as $Map_{Cov}(X, Y)$ defines a bifunctor (see Proposition 3.10).

Proposition 4.2. *The following conditions are equivalent*:
(1) $g:S \to Map_{PC}(X, Y)$ *is continuous*,
(2) $adj_X(g)|S \times F$ *is continuous for each finite subspace* F *of* X.

Proof. (1) $\Rightarrow$ (2) Suppose $g:S \to Map_{PC}(X, Y)$ is continuous and F is a finite subspace of X. By Corollary 3.6, $eval:Map_{Cov}(F, Y) \times F \to Y$ is continuous. Let $i:F \to X$ be the inclusion. Notice that $adj_X(g)|S \times F = eval \circ (i^*(g) \times id_F)$ is continuous if g and $eval$ are continuous.

(2) $\Rightarrow$ (1) Suppose $g:S \to Map_{PC}(X, Y)$ is a function such that $adj_X(g)|S \times F$ is continuous for each finite subspace F of X. To show continuity of g we need to prove that for any map $h:K \to X$ from a finite topological space K, the function $h^*(g):S \to Map_{Cov}(K, Y)$ is continuous. By Corollary 3.6 this amounts to continuity of $adj_K(h^*(g)):S \times K \to Y$. The last function is the composition of $S \times K \to S \times h(K) \to Y$, where the first function is $id_S \times h$ and the second function is $adj_X(g)|S \times h(K)$. Since both functions are continuous, so is their composition. $\square$

Let us show that $Map_{PC}(X, Y)$ is the classical pointwise convergence topology:

Proposition 4.3. *The following topological spaces are identical*:

(a) *$Map(X, Y)$ equipped with the pointwise convergence topology,*
(b) *$Map(X, Y)$ equipped with the covariant topology induced by the class of all functions $\{f_i : S_i \to Map(X, Y)\}_{i \in J}$ such that $adj_X(f_i)|S_i \times F$ is continuous for each $i \in J$ and for each finite subset F of X,*
(c) *$Map_{PC}(X, Y)$.*

Proof. The equivalence of (b) and (c) follows from Proposition 4.2. Let *Top* be the pointwise convergence topology on $Map(X, Y)$. To show the continuity of $id : Map_{PC}(X, Y) \to Map_{Top}(X, Y)$ it suffices to show, in view of Propositions 1.16 and 3.2, that for any $x \in X$ the function $e_x : Map_{PC}(X, Y) \to Y$ (see Definition 3.1) is continuous. This follows from Proposition 4.2 as e_x is the composition of the natural homeomorphism $Map_{PC}(X, Y) \to Map_{PC}(X, Y) \times \{x\}$ and the evaluation $eval|Map_{PC}(X, Y) \times \{x\}$.

To show continuity of $id : Map_{Top}(X, Y) \to Map_{PC}(X, Y)$ we need to prove that

$$adj_X(id) = eval|Map_{Top}(X, Y) \times F$$

is continuous for each finite subset F of X (see Proposition 4.2). Suppose U is open in Y, $x_0 \in F$, and $f(x_0) \in U$ for some $f \in Map(X, Y)$. The set $V = (f|F)^{-1}(U)$ is finite and is open in F. Let $W = \bigcap_{x \in V} P(x, U)$. Since V is finite, W is open in $Map_{Top}(X, Y)$, $f \in W$, $x_0 \in V$, and $eval(W \times V) \subset U$ which proves that $eval|Map_{Top}(X, Y) \times F$ is continuous. □

Our next observation is that all three topologies *Cov*, *CO*, and *PC* have the same finite subspaces:

Proposition 4.4. *Let F be a finite topological space. The following conditions are equivalent*:

(1) *$g : F \to Map_{Cov}(X, Y)$ is continuous,*
(2) *$g : F \to Map_{CO}(X, Y)$ is continuous,*
(3) *$g : F \to Map_{PC}(X, Y)$ is continuous,*
(4) *$adj_X(g) : F \times X \to Y$ is continuous.*

Proof. Let $adj_X(g) = h$. Since $PC \subset CO \subset Cov$, then (1) ⇒ (2) ⇒ (3). Also, (4) ⇒ (1) by the definition of the basic covariant topology (see Definition 3.4). Thus, it suffices to show that (3) ⇒ (4). Suppose U is open in Y and $g(a)(x) \in U$ for some $(a, x) \in F \times X$. By Proposition 4.2, $h = adj_X(g)|F \times \{x\}$ is continuous. In particular, the set $V = \{b \in F \mid g(b)(x) \in U\}$ is open in F. Let $W = \bigcap_{b \in V} g(b)^{-1}(U)$. Notice that W is open in X as V is finite and $adj_X(g)(V \times W) \subset U$. Thus, $adj_X(g) : F \times X \to Y$ is continuous. □

We are ready to construct a left adjoint to the functor $G(Z) = Map_{PC}(Y, Z)$ (Y is fixed):

Definition 4.5. Given two topological spaces X and Y their *PC-product* $X \times_{PC} Y$ is the set $X \times Y$ equipped with the covariant topology induced by the family of all inclusions

$\{i_j : X_j \times Y_j \to X \times Y\}_{j\in J}$, where $X_i \times Y_i$ is given the product topology and either X_j is a finite subspace of X or Y_j is a finite subspace of Y.

Notice that $X \times_{PC} Y$ is naturally homeomorphic to $Y \times_{PC} X$. Also notice that if X and Y are two k-spaces such that $X \times Y$ (with the product topology) is not a k-space (see Corollary 5.7), then $X \times_{PC} Y \neq X \times Y$. Indeed, by Corollary 2.20 and Proposition 2.19 the topology on $X \times_{PC} Y$ contains $k(X \times Y)$ which is stronger than the product topology.

Here is our main result concerning the pointwise convergence topology:

Theorem 4.6.

$$adj^Y : Map_{PC}(X \times_{PC} Y, Z) \to Map_{PC}\big(X, Map_{PC}(Y, Z)\big)$$

is a homeomorphism.

Proof. First, let us show that

$$adj^Y : Map(X \times_{PC} Y, Z) \to Map\big(X, Map_{PC}(Y, Z)\big)$$

is a bijection. Suppose $f : X \times_{PC} Y \to Z$ is a map. Given $x \in X$, the inclusion $\{x\} \times Y \to X \times_{PC} Y$ is continuous. Hence, $f|\{x\} \times Y$ is continuous and $adj^Y(f)$ is a function on X with values in $Map(Y, Z)$. Also, since $f|X \times F$ is continuous for each finite subspace of Y, $adj^Y(f) : X \to Map_{PC}(Y, Z)$ is continuous by Proposition 4.2. Now, suppose $g : X \to Map_{PC}(Y, Z)$ is continuous. By Proposition 4.2, $adj_Y(g)|X \times F$ is continuous for each finite subspace of Y. By Proposition 4.4, $adj_Y(g)|F \times Y$ is continuous for each finite subspace F of X. By the definition of the PC-product and by Proposition 1.13, $adj_Y(g) : X \times_{PC} Y \to Z$ is continuous.

To prove the continuity of adj^Y and the continuity of its inverse it suffices to show that for any function $g : S \to Map_{PC}(X \times_{PC} Y, Z)$, g is continuous iff $adj^Y \circ g : S \to Map_{PC}(X, Map_{PC}(Y, Z))$ is continuous. As will be shown shortly, this is equivalent to the following:

Claim. $X_1 \times_{PC} (X_2 \times_{PC} X_3) = (X_1 \times_{PC} X_2) \times_{PC} X_3$.

Proof of Claim. Let P be the Cartesian product $X_1 \times X_2 \times X_3$ equipped with the covariant topology induced by all inclusions $\{j_i : Y_1 \times Y_2 \times Y_3 \to X_1 \times X_2 \times X_3\}_{j\in J}$, where $Y_1 \times Y_2 \times Y_3$ is given the product topology, so that at most one Y_s, $s = 1, 2, 3$, is infinite and if Y_s is infinite, then $Y_s = X_s$. We will show that $Z = X_1 \times_{PC} (X_2 \times_{PC} X_3)$. The proof of $Z = (X_1 \times_{PC} X_2) \times_{PC} X_3$ is similar.

To prove continuity of $id : Z \to X_1 \times_{PC} (X_2 \times_{PC} X_3)$ we need, by Proposition 1.13, to show that for any $Y_1 \times Y_2 \times Y_3 \subset X_1 \times X_2 \times X_3$ so that at most one Y_s, $s = 1, 2, 3$, is infinite and if Y_s is infinite, then $Y_s = X_s$, the inclusion $Y_1 \times Y_2 \times Y_3 \to X_1 \times_{PC} (X_2 \times_{PC} X_3)$ is continuous. For example, if $Y_3 = X_3$, then each of maps in the sequence

$$\begin{aligned} Y_1 \times Y_2 \times Y_3 &\to Y_1 \times_{PC} (X_2 \times_{PC} X_3) \to Y_1 \times_{PC} (X_2 \times_{PC} X_3) \\ &\to X_1 \times_{PC} (X_2 \times_{PC} X_3) \end{aligned}$$

is continuous. To prove continuity of $id : X_1 \times_{PC} (X_2 \times_{PC} X_3) \to Z$ the most difficult case is to prove the continuity of the inclusion $Y_1 \times (X_2 \times_{PC} X_3) \to Z$ if Y_1 is a finite subspace of X_1. However, Y_1 is locally compact and another application of Theorem 1.15 helps to overcome that case. Indeed, $Y_1 \times (X_2 \times_{PC} X_3)$ is equipped with the covariant topology induced by all inclusions $Y_1 \times Y_2 \times Y_3 \to Y_1 \times X_2 \times X_3$, where $Y_1 \times Y_2 \times Y_3$ is given the product topology and either $Y_2 \times Y_3$ is finite, Y_2 is finite and $Y_3 = X_3$, or Y_3 is finite and $Y_2 = X_2$. $\square$

Let us show how to use Claim to prove that for any function $g : S \to Map_{PC}(X \times_{PC} Y, Z)$, g is continuous iff $adj^Y \circ g : S \to Map_{PC}(X, Map_{PC}(Y, Z))$ is continuous. g is continuous iff $adj_{X \times_{PC} Y}(g) : S \times_{PC} (X \times_{PC} Y) \to Z$ is continuous. $h = adj^Y \circ g$ is continuous iff $adj_X(h) : S \times_{PC} X \to Map_{PC}(Y, Z)$ is continuous which is equivalent to continuity of $adj_Y(adj_X(h)) : (S \times_{PC} X) \times_{PC} Y \to Z$. $\square$

Proposition 4.7. *The following conditions are equivalent*:

(1) $g : S \to Map_{CO}(X, Y)$ *is continuous,*
(2) $adj_X(g) \circ (id_S \times h) : S \times C \to Y$ *is continuous for each map* $h : C \to X$ *so that* C *is locally compact.*

Proof. (1) ⇒ (2) Suppose $g : S \to Map_{PC}(X, Y)$ is continuous and $h : C \to X$ is a map so that C is locally compact. By Corollary 3.6, $eval : Map_{Cov}(C, Y) \times C \to Y$ is continuous. Notice that $adj_X(g) \circ (id_S \times h) = eval \circ (h^*(g) \times id_C)$ is continuous.

(2) ⇒ (1) Suppose $g : S \to Map_{PC}(X, Y)$ is a function such that $adj_X(g) \circ (id_S \times h) : S \times C \to Y$ is continuous for each map $h : C \to X$ so that C is locally compact. To show continuity of g we need to prove that, for any map $h : K \to X$ from a locally compact topological space K, the function $h^*(g) : S \to Map_{Cov}(K, Y)$ is continuous. By Corollary 3.6 this amounts to continuity of $adj_K(h^*(g)) : S \times K \to Y$. The last function is precisely $adj_X(g) \circ (id_S \times h)$. $\square$

Let us show that $Map_{CO}(X, Y)$ is the classical compact-open topology if X is Hausdorff:

Proposition 4.8. *Suppose* X *is Hausdorff. The following topological spaces are identical*:

(a) $Map(X, Y)$ *equipped with the compact-open topology,*
(b) $Map(X, Y)$ *equipped with the covariant topology induced by the class of all functions* $\{f_i : S_i \to Map(X, Y)\}_{i \in J}$ *such that* $adj_X(f_i)|S_i \times C$ *is continuous for each* $i \in J$ *and for each compact subset* C *of* X*,*
(c) $Map_{CO}(X, Y)$.

Proof. Let *Top* be the covariant topology on $Map(X, Y)$ induced by the class of all functions $\{f_i : S_i \to Map(X, Y)\}_{i \in J}$ such that $adj_X(f_i)|S_i \times C$ is continuous for each $i \in J$ and for each compact subset C of X. Suppose $h : K \to X$ is a map and K is locally compact. Notice that $adj_X(f_i) \circ (id_{S_i} \times h) : S \times K \to Y$ is continuous. Indeed, one may reduce the general case to the case of K being compact and, as $C = h(K)$ is compact, $adj_X(f_i) \circ (id_{S_i} \times h) = (adj_X(f_i)|S_i \times C) \circ (id_{S_i} \times h)$ is continuous. By Proposition 4.7,

$Map_{CO}(X,Y) = Map_{Top}(X,Y)$. Let *CoTop* be the classical compact-open topology on $Map(X,Y)$. Thus, its sub-basis consists of all $P(K,U) = \{f \in Map(X,Y) \mid f(K) \subset U\}$, where K is a compact subset of X and U is open in Y. If $g : S \to Map(X,Y)$ is such that $adj_X(g)|S \times K$ is continuous and K is compact, then $g^{-1}(P(K,U))$ is open in S. Indeed, given $s \in g^{-1}(P(K,U))$, then $s \times K \subset adj_X(g)^{-1}(U)$ which implies the existence of a neighborhood V of s in S such that $V \times K \subset adj_X(g)^{-1}(U)$. Thus, $V \subset g^{-1}(U)$ which proves that $CoTop \subset Top$. Conversely, $eval : Map_{CoTop}(X,Y) \times X \to Y$ has the property that its restriction to $Map_{CoTop}(X,Y) \times K$ is continuous for each locally compact subset K of X. Indeed, it can be factored as $Map_{CoTop}(X,Y) \times K \to Map_{CoTop}(K,Y) \times K \to Y$, where each map is continuous (see [12, Theorem 3.4.3]). Thus, $id : Map_{CoTop}(X,Y) \to Map_{Top}(X,Y)$ is continuous which proves Proposition 4.8. $\square$

We are ready to construct a left adjoint to the functor $G(Z) = Map_{CO}(Y,Z)$ (Y is fixed):

Definition 4.9. Given two topological spaces X and Y their *CO-product* $X \times_{CO} Y$ is the set $X \times Y$ equipped with the covariant topology induced by the family of all maps $\{f_j \times g_j : X_j \times Y_j \to X \times Y\}_{j \in J}$, where $X_i \times Y_i$ is given the product topology and either X_j is a finite subspace of X, f_j is the inclusion, and $g_j = id_Y$ or Y_j is a locally compact space and $f_j = id_X$.

Notice that $X \times_{CO} Y$ defines a bifunctor on $\mathcal{TOP}$ which coincides with the regular product if X is finite or Y is locally compact. Indeed, by Proposition 1.13 the identity function $X \times_{CO} Y \to X \times Y$ is continuous for all X and Y. If either X is finite or Y is locally compact, then, by definition, the identity function $X \times Y \to X \times_{CO} Y$ is continuous.

The following example shows that CO-product is different from both the PC-product and the ordinary product:

Example 4.10. There exist spaces X and Y such that the symmetry $sym : X \times_{CO} Y \to Y \times_{CO} X$, $sym(x,y) = (y,x)$, is not continuous.

Proof. Let $Y = I$ with the standard topology and let $X = I$ with the new topology: any closed set is either countable or equal to X. Consider $A = \{(t,t) \in Y \times X\}$. We plan to show that A with the topology induced from $Y \times_{CO} X$ is discrete but A with the topology induced from $X \times_{CO} Y$ is not discrete which would prove that *sym* is not continuous.

Since Y is locally compact, the identity map $X \times Y \to X \times_{CO} Y$ is continuous (the domain is equipped with the product topology). Thus, any neighborhood of a point $(t,t) \in X \times_{CO} Y$ contains a product neighborhood $U \times V$. Since $X - U$ is countable, $U \cap V \neq \{t\}$ and (t,t) is not an isolated point of A.

Suppose $B \subset A$. We plan to show that B is closed as a subset of $Y \times_{CO} X$. Suppose $f : C \to Y$, $g : D \to X$, C is a finite subspace of Y, f is the inclusion, and $g = id_X$. Notice that $F = (C \times X) \cap B$ is a finite, hence closed, subspace of $Y \times X$. Since $(f \times g)^{-1}(B) = (f \times g)^{-1}(F)$, it is a closed subset of $C \times D$. Suppose $f = id_Y$, $g : D \to X$ is continuous and D is a locally compact topological space. We need to show that $E = (f \times g)^{-1}(B)$ is a closed subset of $Y \times D$. Suppose $(c,d) \notin E$ and choose a compact neighborhood Z of d in D. Notice that $g(Z)$ is finite, and so is $F = (Y \times g(Z)) \cap B$. Now, $(Y \times Z) \cap E = (Y \times Z) \cap (f \times g)^{-1}(F)$ is a closed subset of $Y \times Z$ which does not contain (c,d). Choose neighborhoods U of c in Y and V of d in D so that $(U \times V) \cap (Y \times Z) \cap E = \emptyset$ and notice that $U \times (V \cap Z)$ is a neighborhood of (c,d) missing E. Thus, E is closed which proves that B is closed in $Y \times_{CO} X$. $\square$

Here is our main result concerning the compact-open topology:

Theorem 4.11.

$$adj^Y : Map_{CO}(X \times_{CO} Y, Z) \to Map_{CO}\big(X, Map_{CO}(Y,Z)\big)$$

is a homeomorphism.

Proof. First, let us show that

$$adj^Y : Map(X \times_{CO} Y, Z) \to Map\big(X, Map_{CO}(Y,Z)\big)$$

is a bijection. Suppose $f : X \times_{CO} Y \to Z$ is a map. Given $x \in X$, the inclusion $\{x\} \times Y \to X \times_{CO} Y$ is continuous. Hence, $f|\{x\} \times Y$ is continuous and $adj^Y(f)$ is a function on X with values in $Map(Y,Z)$. Also, since $f \circ (id_X \times h) : X \times C \to Z$ is continuous for any locally compact space C and any map $h : C \to Y$, $adj^Y(f) : X \to Map_{CO}(Y,Z)$ is continuous by Proposition 4.7. Now, suppose $g : X \to Map_{CO}(Y,Z)$ is continuous and let $h = adj_Y(g) : X \times Y \to Z$. To show the continuity of $h : X \times_{CO} Y \to Z$ it suffices to prove that $h \circ (a \times b) : A \times B \to Z$ is continuous for any map $a \times b : A \times B \to X \times Y$ such that either A is finite or B is locally compact (see Proposition 1.13). Suppose A is finite. By Proposition 4.4, $adj_Y(g \circ a) : A \times Y \to Z$ is continuous. Notice that $h \circ (a \times b) = adj_Y(g \circ a) \circ (id_A \times b)$. Suppose B is locally compact. By Proposition 4.7, $adj_Y(g) \circ (id_X \times b)$ is continuous. Notice that $h \circ (a \times b) = adj_Y(g) \circ (id_X \times b) \circ (a \times id_B)$.

To prove the continuity of adj^Y and the continuity of its inverse it suffices to show that for any function $g : S \to Map_{CO}(X \times_{CO} Y, Z)$, g is continuous iff $adj^Y \circ g : S \to Map_{CO}(X, Map_{CO}(Y,Z))$ is continuous. As will be shown shortly, this is equivalent to the following:

Claim. $X_1 \times_{CO} (X_2 \times_{CO} X_3) = (X_1 \times_{CO} X_2) \times_{CO} X_3$.

Proof of Claim. For simplicity let us denote $X_1 \times_{CO} (X_2 \times_{CO} X_3)$ by L, $(X_1 \times_{CO} X_2) \times_{CO} X_3$ by R, and $X_1 \times X_2 \times X_3$ by P.

Case 1. X_1 is finite. In this case $L = X_1 \times (X_2 \times_{CO} X_3)$ and $R = (X_1 \times X_2) \times_{CO} X_3$. We plan to show that both L and R have covariant topologies induced by essentially the same maps. Since X_1 is locally compact, we can apply Theorem 1.15 and conclude that the topology of L is induced by maps which have one of the following forms:

(a) $f \times g \times h : X_1 \times X_2 \times C \to P$, where $f = id_{X_1}$, $g = id_{X_2}$, and C is locally compact,
(b) $f \times g \times h : X_1 \times F \times X_3 \to P$, where $f = id_{X_1}$, $h = id_{X_3}$, F is a finite subspace of X_2, and g is the inclusion.

By the definition of the CO-product the topology of R is induced by maps which have one of the following forms:

(c) $f \times g \times h : X_1 \times X_2 \times C \to P$, where $f = id_{X_1}$, $g = id_{X_2}$, and C is locally compact,

(d) $g \times h : F \times X_3 \to P$, where $h = id_{X_3}$, F is a finite subspace of $X_1 \times X_2$, and g is the inclusion.

Clearly, the classes of maps described in (a) and (c) are identical. Also, class (b) is contained in class (d). Notice that any finite subset F of $X_1 \times X_2$ is contained in a finite subspace $F_1 \times F_2$ of $X_1 \times X_2$. That is sufficient to conclude that the covariant topology induced by the union of classes (a) and (b) coincides with the covariant topology induced by the union of classes (c) and (d).

Case 2. X_3 is locally compact. In this case $L = X_1 \times_{CO} (X_2 \times X_3)$ and $R = (X_1 \times_{CO} X_2) \times X_3$. We plan to show that both L and R have covariant topologies induced by essentially the same maps. Since X_3 is locally compact, we can apply Theorem 1.15 and conclude that the topology of R is induced by maps which have one of the following forms:

(a) $f \times g \times h : F \times X_2 \times X_3 \to P$, where $g = id_{X_1}$, $h = id_{X_3}$, F is a finite subspace of X_1, and f is the inclusion,

(b) $f \times g \times h : X_1 \times C \times X_3 \to P$, where $f = id_{X_1}$, $h = id_{X_3}$, and C is locally compact.

By the definition of the CO-product the topology of L is induced by maps which have one of the following forms:

(c) $f \times g \times h : F \times X_2 \times X_3 \to P$, where $g = id_{X_2}$, $h = id_{X_3}$, F is a finite subspace of X_1, and f is the inclusion,

(d) $g \times h : X_1 \times C \to P$, where $g = id_{X_1}$, C is locally compact and $h : C \to X_2 \times X_3$.

Clearly, the classes of maps described in (a) and (c) are identical. Also, class (b) is contained in class (d) as $C \times X_3$ is locally compact if C is locally compact. Notice that any map $h : C \to X_2 \times X_3$ can be factored as the composition of a map from C to $C \times X_3$ and a map from $C \times X_3$ to $X_2 \times X_3$. That is sufficient to conclude that the covariant topology induced by the union of classes (a) and (b) coincides with the covariant topology induced by the union of classes (c) and (d).

General case: To prove continuity of $id : L \to R$ we will employ Proposition 1.13. Suppose F is a finite subspace of X_1. By case 1, $F \times (X_2 \times_{CO} X_3) = (F \times_{CO} X_2) \times_{CO} X_3$ and by the functoriality of the CO-product the inclusion $F \times (X_2 \times_{CO} X_3) \to (X_1 \times_{CO} X_2) \times_{CO} X_3$ is continuous. Suppose $f : C \to X_2 \times_{CO} X_3$ is continuous and C is locally compact. Let $g : C \to X_2$ and $h : C \to X_3$ be functions such that $f(c) = (g(c), h(c))$ for each $c \in C$. Since $id : X_2 \times_{CO} X_3 \to X_2 \times X_3$ is continuous, both g and h are continuous. Since $X_2 \times_{CO} C = X_2 \times C$, $id \times h : X_2 \times C \to X_2 \times_{CO} X_3$ is continuous. Let $j : C \to X_2 \times C$ be defined by $j(c) = (g(c), c)$. Since $id \times ((id \times h) \circ j) : X_1 \times_{CO} C \to X_1 \times_{CO} (X_2 \times_{CO} C)$ is continuous, $id \times ((id \times h) \circ j) = id \times f$, and $X_1 \times_{CO} (X_2 \times_{CO} C) = (X_1 \times_{CO} X_2) \times_{CO} C$ (by case 2), $id \times f : X_1 \times C \to (X_1 \times_{CO} X_2) \times_{CO} X_3$ is continuous.

To prove continuity of $id : R \to L$ suppose $f : C \to X_3$ is a map and C is locally compact. By case 2, $(X_1 \times_{CO} X_2) \times_{CO} C = X_1 \times_{CO} (X_2 \times_{CO} C)$, so $id \times f : (X_1 \times_{CO} X_2) \times_{CO} C \to X_1 \times_{CO} (X_2 \times_{CO} X_3)$ is continuous. The last remaining case is that of F being a finite subspace of $X_1 \times_{CO} X_2$. We need to show that the inclusion $F \times X_3 \to L$ is continuous. Choose finite subspace $F_1 \times F_2$ of $X_1 \times_{CO} X_2$ containing F. By case 1,

$F_1 \times F_2 \times X_3 = F_1 \times_{CO} (F_2 \times_{CO} X_3)$ and by the functoriality of the CO-product the inclusion $F_1 \times_{CO} (F_2 \times_{CO} X_3) \to X_1 \times_{CO} (X_2 \times_{CO} X_3)$ is continuous. $\square$

Let us show how to use Claim to prove that for any function $g : S \to Map_{CO}(X \times_{CO} Y, Z)$, g is continuous iff $adj^Y \circ g : S \to Map_{CO}(X, Map_{CO}(Y, Z))$ is continuous. g is continuous iff $adj_{X \times_{CO} Y}(g) : S \times_{CO} (X \times_{CO} Y) \to Z$ is continuous. $h = adj^Y \circ g$ is continuous iff $adj_X(h) : S \times_{CO} X \to Map_{CO}(Y, Z)$ is continuous which is equivalent to continuity of $adj_Y(adj_X(h)) : (S \times_{CO} X) \times_{CO} Y \to Z$. $\square$

By applying the functor k (see Definition 2.15), one gets three additional topologies on function spaces: $kCov, kCO, kCO$.

We are ready to construct a left adjoint to the functor $G(Z) = Map_{kCov}(kY, Z)$ (Y is fixed) on the category $k\mathcal{TOP}$ of k-spaces:

Definition 4.12. Given two topological spaces X and Y their *k-product* $X \times_k Y$ is $k(X \times Y)$.

If both X and Y are k-spaces, then their k-product and CO-product are identical:

Proposition 4.13. $X \times_k Y = kX \times_k kY = kX \times_{CO} kY$.

Proof. All arrows in this proof represent the identity function on the Cartesian product $X \times Y$. Since $kX \times kY \to X \times Y$ is continuous, so is $kX \times_k kY \to X \times_k Y$. To check continuity of $X \times_k Y \to kX \times_k kY$ suppose $f : C \to X \times Y$ is a map and C is locally compact. By projecting $X \times Y$ onto X and Y we get two maps $g : C \to X$ and $h : C \to Y$. Since $g : C \to kX$ and $h : C \to kY$ are maps, so is $f : C \to kX \times kY$ and so is $f : C \to kX \times_k kY$. Thus, $X \times_k Y \to kX \times_k kY$ is a map. Also, $id_{kX} \times h : kX \times C \to kX \times kY$ is a map which means that $id_{kX} \times h : kX \times C \to kX \times_{CO} kY$ is a map. By composing the last map with $j : C \to kX \times C$, $j(c) = (g(c), c)$, one gets that $f : C \to kX \times_{CO} kY$ is a map. Thus, $X \times_k Y \to kX \times_{CO} kY$ is a map.

Notice that $kX \times_{CO} kY$ is $kX \times kY$ equipped with the covariant topology induced by all maps $f \times g : C \times D \to X \times Y$ such that either C is finite and $g = id_{kY}$ or D is locally compact and $f = id_{kX}$. In particular, $C \times D$ is a k-space. Hence, $kX \times_{CO} kY \to kX \times_k kY$ is continuous which completes the proof. $\square$

Theorem 4.14. *Suppose $f : S \to Map(kX, Y)$ is a function. The following conditions are equivalent*:

(1) *$f : kS \to Map_{CO}(kX, Y)$ is continuous,*
(2) *$adj_X(f) : S \times_k X \to Y$ is continuous,*
(3) *$f : kS \to Map_{Cov}(kX, Y)$ is continuous.*

Proof. The equivalence of (1) and (2) follows from Theorem 4.11 and Proposition 4.13.

(2) $\Rightarrow$ (3) We need to show that, for any map $g : C \to S$ with C being locally compact, $f \circ g$ is continuous. Since $g \times id_X : C \times kX \to k(S \times X)$ is continuous (see Theorem 2.14

and use Theorem 1.15), then so is $adj_X(f) \circ (g \times id_X): C \times kX \to Y$. Hence, $f \circ g: C \to Map_{Cov}(kX, Y)$ is continuous.

(3) ⇒ (1) follows from the fact that $CO \subset Cov$. □

By applying Theorem 4.14 in the case of S being locally compact one gets:

Corollary 4.15. *$Map_{kCov}(kX, Y) = Map_{kCO}(kX, Y)$. $Map_{kCov}(kX, Y)$ is $Map(kX, Y)$ equipped with the covariant topology induced by all functions $f: C \to Map(kX, Y)$ such that C is locally compact and $adj_{kX}(f): C \times kX \to Y$ is continuous.*

The following result generalizes Theorem 3 on p. 183 of [22]:

Theorem 4.16.

$$adj^Y: Map_{kCov}(X \times_k Y, Z) \to Map_{kCov}\big(kX, Map_{kCov}(kY, Z)\big)$$

is a homeomorphism.

Proof. First notice that $Map(kX, Map_{kCO}(kY, Z)) = Map(kX, Map_{CO}(kY, Z))$ as kX is a k-space. By Theorem 4.11 and Proposition 4.13, $adj^Y: Map_{CO}(X \times_k Y, Z) \to Map_{CO}(kX, Map_{kCO}(kY, Z))$ is a homeomorphism. By applying the functor k one gets Theorem 4.16. □

Since, by Corollary 4.15, the spaces $Map_{Cov}(kX, Y)$ and $Map_{CO}(kX, Y)$ have the same compact Hausdorff subspaces, one deduces the following result from the corresponding theorem for compact-open topologies (see [12, 3.4.20]):

Ascoli Type Theorem 4.17. *Suppose $F \subset Map_{Cov}(X, Y)$, X is a Hausdorff k-space, and Y is regular. Then,* $\mathrm{cl}(F)$ *is compact in $Map_{Cov}(X, Y)$ iff F consists of equicontinuous functions and* $\mathrm{cl}(eval(F \times \{x\}))$ *is compact in Y for each $x \in X$.*

Proof. Strictly speaking, Theorem 3.4.20 in [12] deals with F being closed in $Map_{CO}(X, Y)$. However, its proof clearly works for the analog of Theorem 4.17 with the basic covariant topology being replaced by the compact-open topology. □

5. Evaluation map

This section is devoted to issues related to the continuity of the evaluation function. First, let us point out that $eval: Map_{CO}(\mathbb{Q}, \mathbb{R}) \times \mathbb{Q} \to \mathbb{R}$ is not continuous (see [14, Theorem 3]). We are ready for a generalization of this observation. The interesting aspect of this generalization is that its proof is modeled on the proof of Tamano Theorem (see [12, 5.1.38]).

Proposition 5.1. *Suppose X is a paracompact space. X is locally compact iff $eval: Map_{kCov}(X, I) \times X \to I$ is continuous.*

Proof. Suppose X is locally compact. By Corollary 3.6 one gets the continuity of *eval*.

Suppose $eval\colon Map_{kCov}(X, I) \times X \to I$ is continuous. Pick $x_0 \in X$ and let $c\colon X \to I$ be the constant map $c \equiv 0$. Choose a neighborhood N of c in $eval\colon Map_{kCov}(X, I) \times X \to I$ and choose a neighborhood V of x_0 in X such that $eval(N \times V) \subset [0, 1)$ (use continuity of *eval*). Let βX be the Čech–Stone compactification of X and assume there is x_1 in the closure C of V in βX which is not in $\mathrm{cl}_X(V)$. Since $C \times X$ is normal and $\{x_1\} \times X$, $\{(x, x) \mid x \in \mathrm{cl}_X(V)\}$ are disjoint closed subsets, there is a map $\alpha\colon C \times X \to I$ such that $\alpha(x_1 \times X) = \{0\}$ and $\alpha(x, x) = 1$ for $x \in \mathrm{cl}_X(V)$. Since $\gamma = adj^X(\alpha)\colon C \to Map_{kCov}(X, I)$ is continuous and $\gamma(x_1) = c$, $\gamma^{-1}(N)$ is a neighborhood of x_1. This neighborhood must intersect V. Thus, there is $x_2 \in \gamma^{-1}(N) \cap V$ which leads to a contradiction: $1 = \alpha(x_2, x_2) = \gamma(x_2)(x_2) = eval(\gamma(x_2), x_2) \in eval(N \times V) \subset [0, 1)$. Thus, X is locally compact. □

Here is a generalization of the previous result:

Theorem 5.2. *Suppose X is a regular* (*respectively, Tychonoff*) *space. X is locally compact iff $eval\colon Map_{Cov}(X, Y) \times X \to Y$* (*respectively, $eval\colon Map_{kCov}(X, Y) \times X \to Y$*) *is continuous for each T_1 space Y.*

The proof of Theorem 5.2 is modeled on the proof of Proposition 5.1 with the help of the following:

Proposition 5.3. *Suppose X is a regular* (*respectively, Tychonoff*) *space. There is a T_1 space Y containing I with the following property*: *given an open set V of X and given a Hausdorff* (*respectively, compact Hausdorff*) *space C containing $\mathrm{cl}_X(V)$ as a proper dense set, there is, for some $x_1 \in C - \mathrm{cl}_X(V)$, a map $\alpha\colon C \times X \to Y$ so that $\alpha(\{x_1\} \times X) = \{0\}$ and $\alpha(x, x) = 1$ for all $x \in \mathrm{cl}_X(V)$.*

Proof. Let us discuss a basic construction first. Let S be the set of all triples (A, V, x) such that

(1) $V \neq \emptyset$ is an open subset of X,
(2) $\mathrm{cl}_X(V)$ is a proper dense subset of Hausdorff (respectively, compact Hausdorff) space A,
(3) $x \in A - \mathrm{cl}_X(V)$,
(4) for any open subset V of X and any Hausdorff (respectively, compact Hausdorff) space B containing $\mathrm{cl}_X(V)$ as a proper dense set, there is a triple $(A, V, x) \in S$ and a homeomorphism $h\colon A \to B$ such that $h|\,\mathrm{cl}_X(V)$ is the identity.

Suppose Z is a T_1 space containing I. For each $s = (A, V, x) \in S$ let $f_s\colon D_s = \{x\} \times X \cup \{(a, a) \mid a \in \mathrm{cl}_X(V)\} \to Z$ be defined by $f_s(A) = \{0\}$ and $f_s(a, a) = 1$ if $a \in \mathrm{cl}_X(V)$. Think of D_s as being the subset of a copy P_s of $A \times X$ so that $P_s \cap P_t = \emptyset$ if $s \neq t$. Notice that D_s is the union of two disjoint closed subsets of a T_1 space P_s. Let $\mu(Z)$ be the adjunction space $Z \cup_f \coprod_{s\in S} P_s$, where $f\colon \coprod_{s\in S} D_s \to Z$ is defined by $f|D_s = f_s$ for each $s \in S$. Thus, $\mu(Z)$ is the quotient space of the disjoint union of Z and $\coprod_{s\in S} P_s$, where

$d \in D_s$ is identified with $f_s(d)$. Notice that $\mu(Z)$ is T_1. Also, notice that there is a natural inclusion $i_Z : Z \to \mu(Z)$ which is a homeomorphic imbedding.

Let $Z_0 = I$ and $Z_{n+1} = \mu(Z_n)$ for $n \geqslant 0$. Let $Y = \bigcup_{n=0}^{\infty} Z_n$ and let $i_n : Z_n \to Y$ be the inclusion for each n. Y is equipped with the covariant topology induced by $\{i_n\}_{n \geqslant 0}$. $\square$

Proof of Theorem 5.2. Suppose $eval : Map_{Cov}(X, Y) \times X \to Y$ is continuous for each T_1 space Y. Suppose W is a neighborhood of $x_0 \in X$. Choose Y as in Proposition 5.3 and choose neighborhoods N of the constant map $c \equiv 0$ (0 belongs to Y) and a neighborhood V of x_0 in W so that $eval(N \times V) \subset Y - \{1\}$ and $\mathrm{cl}_X(V) \subset W$. If $\mathrm{cl}_X(V)$ is not compact, then (see Theorem 2.8) one can find a Hausdorff space C containing $\mathrm{cl}_X(V)$ as a proper dense subset. Pick $x_1 \in C - \mathrm{cl}_X(V)$ and choose a map $\alpha : C \times X \to Y$ such that $\alpha(\{x_1\} \times X) = \{0\}$ and $\alpha(x, x) = 1$ for all $x \in \mathrm{cl}_X(V)$. Since $\gamma = adj^X(\alpha) : C \to Map_{Cov}(X, Y)$ is continuous and $\gamma(x_1) = c$, $\gamma^{-1}(N)$ is a neighborhood of x_1. This neighborhood must intersect V. Thus, there is $x_2 \in \gamma^{-1}(N) \cap V$ which leads to a contradiction: $1 = \alpha(x_2, x_2) = \gamma(x_2)(x_2) = eval(\gamma(x_2), x_2) \in eval(N \times V) \subset Y - \{1\}$. Thus, $\mathrm{cl}_X(V)$ is compact and X is locally compact.

In the case of X being Tychonoff one can assume C is compact as we can simply put $C = \beta(\mathrm{cl}_X(V))$. If C is compact, then $\gamma = adj^X(\alpha) : C \to Map_{kCov}(X, Y)$ is continuous and the rest of the proof is the same as the case of X being only regular. $\square$

Now, we can recover Michael's [23] result:

Corollary 5.4 (Michael). *Suppose X is regular. If $f \times id_X$ is quotient for every quotient map $f : Y \to Z$, then X is locally compact.*

Proof. As in Corollary 3.6, spaces X with the property that $f \times id_X$ is quotient for every quotient map f, have the property that $eval : Map_{Cov}(X, Y) \times X \to Y$ is continuous for each Y. By Theorem 5.2, X must be locally compact. $\square$

Combining Corollary 5.4 of Michael with Whitehead Theorem (the special case of Theorem 1.15 where S consists of one point) one gets a global, non-intrinsic characterization of regular, locally compact spaces:

Theorem 5.5 (Michael–Whitehead). *Suppose X is regular. Then, X is locally compact iff $f \times id_X$ is quotient for every quotient map $f : Y \to Z$.*

Notice the similarity to Kuratowski's characterization of compact Hausdorff spaces (see [12, 3.1.16]):

Theorem 5.6 (Kuratowski). *Suppose X is Hausdorff. Then, X is compact iff the projection $p_Y : X \times Y \to Y$ is a closed map for each Y.*

Theorem 5.2 also implies another result of Michael [23]:

Corollary 5.7 (Michael). *Suppose X is Tychonoff. If $X \times Y$ is a k-space for every k-space Y, then X is locally compact.*

Proof. Suppose X is not locally compact. By Theorem 5.2, there is a space Y such that $eval: Map_{kCov}(X, Y) \times X \to Y$ is not continuous. If $Map_{kCov}(X, Y) \times X$ was a k-space, then, by Theorem 4.14, $eval: Map_{kCov}(X, Y) \times X \to Y$ would be continuous as $id: Map_{kCov}(X, Y) \to Map_{Cov}(X, Y)$ is continuous. $\square$

6. Reflexive spaces

The purpose of this section is to discuss the topological analog of the natural equivalence

$$tran: Mor_{Sets}\big(X, Mor_{Sets}(Y, Z)\big) \to Mor_{Sets}\big(Y, Mor_{Sets}(X, Z)\big)$$

(see Definition 0.2). We plan to restrict ourselves to the basic set of function spaces topologies $\{PC, CO, Cov, kPC, kCO, kCov\}$. Thus, given $Top \in \{PC, CO, Cov, kPC, kCO, kCov\}$ we plan to discuss cases under which

$$tran: Map\big(X, Map_{Top}(Y, Z)\big) \to Map\big(Y, Map_{Top}(X, Z)\big)$$

is a bijection. Also, we plan to discuss cases under which

$$tran: Map_{Top}\big(X, Map_{Top}(Y, Z)\big) \to Map_{Top}\big(Y, Map_{Top}(X, Z)\big)$$

is a homeomorphism.

From the categorical point of view, we are discussing cases under which the contravariant functor $F(X) = Map_{Top}(X, Z)$ (Z is fixed) is self-adjoint. As seen in Section 0 (see Definition 0.4), if $tran: Map(X, Map_{Top}(Y, Z)) \to Map(Y, Map_{Top}(X, Z))$ is a bijection for all X and Y, then the most important case is that of $Y = Map_{Top}(X, Z)$ and one focuses attention on $tran^{-1}(id_Y)$. Denote $tran^{-1}(id_Y)$ by i_X and notice that $i_X(x)(f) = f(x)$ for all $x \in X$. Thus, $i_X(x) = e_x$ is the evaluation function at x for all $x \in X$. Obviously, we need i_X to be continuous if we have any hope of $tran$ to be a bijection.

Definition 6.1. Suppose a space Z is given ($Z = \mathbb{R}$ is a basic example). The *dual space* X^* is $Map(X, Z)$.

X^* may be equipped with any topology Top among $\{PC, CO, Cov, kPC, kCO, kCov\}$. The choice of topology is emphasized by notation X^*_{Top}. Since evaluation at $x \in X$, $e_x: X^* \to Z$, is continuous in the case of the smallest topology $Top = PC$ (see Propositions 4.2 and 3.2), then it is continuous in all cases and one has a natural function

$$i_X: X \to X^{**}$$

defined by $i_X(x) = e_x$.

One may say that i_X is the topological analog of the canonical homomorphism $i_V : V \to V^{**}$ from linear algebra or functional analysis (V could be a vector space over a field Z, a module over the ring Y, or a topological vector space).

Proposition 6.2. *Suppose Z is T_0 and Top $\in \{PC, CO, Cov, kPC, kCO, kCov\}$. The following conditions are equivalent*:
(a) $i_X : X \to X^{**}$ *is injective,*
(b) *X equipped with the contravariant topology induced by all of Map(X, Z) is T_0.*

Proof. (a) $\Rightarrow$ (b) Suppose $x, y \in X$, $x \neq y$. Since $e_x \neq e_y$, there is $f : X \to Z$ so that $f(x) \neq f(y)$. If U is an open set in Z containing precisely one point of $\{f(x), f(y)\}$, then $f^{-1}(U)$ contains precisely one point of $\{x, y\}$.

(b) $\Rightarrow$ (a) Suppose $x, y \in X$, $x \neq y$. Since X equipped with the contravariant topology induced by all of $Map(X, Z)$ is T_0, there is a sequence $f_1, \ldots, f_n$ of elements of $Map(X, Y)$ such that $U = \bigcap_{i=1}^{n} f_i^{-1}(U_i)$ contains precisely one element of $\{x, y\}$ for some open sets U_i, $i \leqslant n$, of Z. In particular, $f_i(x) \neq f_i(y)$ for some $i \leqslant n$. Thus, $e_x(f_i) \neq e_y(f_i)$ and $i_X(x) \neq i_X(y)$. $\square$

The following result explains why functionally Hausdorff spaces are useful when dealing with function spaces:

Proposition 6.3. *Suppose $Z = \mathbb{R}$. The following conditions are equivalent*:
(a) *X is functionally Hausdorff,*
(b) $i_X : X \to X^{**}$ *is injective.*

Proof. X being functionally Hausdorff means that for any pair of different points $x, y \in X$ there is a map $f : X \to \mathbb{R}$ with $f(x) \neq f(y)$. i_X being injective means that for any pair of different points $x, y \in X$ there is a map $f : X \to \mathbb{R}$ such that $f(x) = e_x(f) \neq e_y(f) = f(y)$. Thus, condition (a) is equivalent to condition (b). $\square$

Proposition 6.4. *Suppose Z is fixed. The following conditions are equivalent*:
(a) $i_X : X \to i_X(X) \subset X^{**}_{PC}$ *is an open function,*
(b) *the contravariant topology on X induced by all of Map(X, Z) is identical with the current topology.*

Proof. (a) $\Rightarrow$ (b) Suppose U is open in X and $x \in U$. Since $i_X(U)$ is open in $i_X(X)$, there is a sequence $f_1, \ldots, f_n$ of elements of $Map(X, Z)$ such that

$$i_X(x) \in \left(\bigcap_{j=1}^{n} P(f_j, U_j)\right) \cap i_X(X) \subset i_X(U)$$

for some open sets U_j of Z, $j \leqslant n$. Notice that $e_y \in P(f, V)$ iff $y \in f^{-1}(V)$. Thus, $x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset U$. This proves that the contravariant topology induced by all of $Map(X, Z)$ contains the current topology on X. Clearly, as all $f \in Map(X, Z)$ are continuous, the current topology contains all the open sets of the contravariant topology which means that the two topologies are identical.

(b) $\Rightarrow$ (a) Suppose U is open in X and $x \in U$. Choose $f_j \in Map(X, Z)$ so that $x \in \bigcap_{j=1}^{n} f_j^{-1}(U_j) \subset U$ for some open sets U_j of Z, $j \leqslant n$. Notice that $i_X(x) \in (\bigcap_{j=1}^{n} P(f_j, U_j)) \cap i_X(X) \subset i_X(U)$, which proves that $i_X(U)$ is open in $i_X(X)$. $\square$

Proposition 6.5. *Fix Z. $i_X : X \to X_{PC}^{**}$ is continuous for any space X.*

Proof. It suffices to show that $i_X^{-1}(P(f, U))$ is open for any $f \in X^* = Map(X, Z)$ and any open set U of Z. Notice that $i_X^{-1}(P(f, U)) = f^{-1}(U)$. $\square$

Here is another characterization of Tychonoff spaces:

Proposition 6.6. *Suppose $Z = \mathbb{R}$ and X is a T_0 space. The following conditions are equivalent*:

(a) *X is Tychonoff,*

(b) *$i_X : X \to X_{PC}^{**}$ is a homeomorphic embedding.*

Proof. Since X_{PC}^{**} is Tychonoff (as X_{PC}^{*} is Tychonoff for any space X), it is clear that (b) implies (a).

(a) $\Rightarrow$ (b) Notice that X being Tychonoff is the same (in view of X being T_0) as its topology being the contravariant topology induced by all of $Map(X, \mathbb{R})$ (see Proposition 2.5). By Proposition 6.4, $i_X : X \to i_X(X) \subset X_{PC}^{**}$ is an open function. By Proposition 6.5, i_X is continuous. $\square$

Let us investigate continuity of $i_X : X \to X_{Top}^{**}$ for $Top \neq PC$.

Definition 6.7. Suppose $Top \in \{CO, Cov, kPC, kCO, kCov\}$. X is said to be *Top-reflexive* provided $i_X : X \to X_{Top}^{**}$ is continuous for any space Z.

Here is the main property of reflexive spaces:

Proposition 6.8. *Let $Top \in \{CO, Cov, kPC, kCO, kCov\}$. If X is a Top-reflexive space, then $tran : Map(Y, Map_{Top}(X, Z)) \to Map(X, Map_{Top}(Y, Z))$ is well defined. If both X and Y are Top-reflexive spaces, then*

$$tran : Map\big(Y, Map_{Top}(X, Z)\big) \to Map\big(X, Map_{Top}(Y, Z)\big)$$

is a bijection.

Proof. Fix Z. Suppose $f : Y \to X_{Top}^{*}$ is continuous. Notice that $tran(f)$ is the composition of $i_X : X \to X_{Top}^{**}$ and $f^* : X_{Top}^{**} \to Y_{Top}^{*}$. $\square$

Corollaries 3.6 and 3.16 imply that k-spaces are *Top*-reflexive for all major topologies *Top*:

Corollary 6.9. *If X is a k-space, then it is Top-reflexive for $Top \in \{CO, Cov, kCO, kCov\}$.*

Proof. Fix Z. First, consider X to be locally compact. By Corollary 3.6,

$$eval : X^*_{Top} \times X \to Z$$

is continuous. Let $Y = X^*_{Top}$. Notice that $i_X = adj^Y(eval)$. Hence $i_X : X \to Map_{Cov}(Y, Z)$ is continuous and $i_X : X \to Map_{kCov}(Y, Z)$ is continuous. Notice that $id : Map_{kCov}(Y, Z) \to X^{**}_{Top}$ is continuous as $kCov$ is the largest topology among basic function topologies.

In the general case it suffices to show that for any map $g : C \to X$, C being locally compact, $i_X \circ g : C \to X^{**}_{Top}$ is continuous. Notice that $i_X \circ g = g^{**} \circ i_C$. □

Proposition 6.10. *If X is Top-reflexive for $Top \in \{kPC, kCO, kCov\}$, then X is a k-space.*

Proof. Put $Z = X$ and let $a = id_X \in X^*_{Top}$. Consider $r = e_a : X^{**}_{Top} \to Z$. Notice that r is continuous and $r \circ i_X = id_X$. Thus, X is a retract of a k-space and must be a k-space ($k(i_X) \circ r$ is the topological inverse of $id_X : kX \to X$). □

Problem 6.11. Suppose X is CO-reflexive. Is X a k-space?

Proposition 6.12. *The following conditions are equivalent*:

(a) *X is CO-reflexive,*

(b) *$sym : X \times_{CO} Y \to Y \times_{CO} X$ is continuous for all spaces Y,*

(c) *$Y \times X = Y \times_{CO} X$ for all locally compact spaces Y.*

Proof. (a) ⇒ (b) Suppose X is CO-reflexive and put $Z = Y \times_{CO} X$. By Theorem 4.11, $f = adj^X(id_Z) : Y \to Map_{CO}(X, Z)$ is continuous. By Proposition 6.8, $tran(f) : X \to Map_{CO}(Y, Z)$ is continuous and, by Theorem 4.11, $sym = adj_Y(tran(f)) : X \times_{CO} Y \to Z$ is continuous which proves (a) ⇒ (b).

(b) ⇒ (c) Notice that $X \times_{CO} Y = X \times Y$ if Y is locally compact. Since $sym : X \times Y \to Y \times_{CO} Y$ is continuous, so is $id : Y \times X \to Y \times_{CO} X$. Since $id : Y \times_{CO} X \to Y \times X$ is continuous for all spaces, we are done.

(c) ⇒ (a) Fix Z. To prove the continuity of $i_X : X \to X^{**}_{CO}$ we plan to use Proposition 1.16 as the compact-open topology was defined in a contravariant manner. Suppose $f : Y \to X^*_{CO}$ is continuous and Y is locally compact. By Theorem 4.11, $adj_X(f) : Y \times_{CO} X \to Z$ is continuous. Hence, $adj_X(f) : Y \times X \to Z$ is continuous and so is $adj^Y(adj_X(f)) : X \to Y^*_{Cov}$. We need to show that $i_X \circ f^* : X \to Y^*_{Cov}$ is continuous. However, $i_X \circ f^* = adj^Y(adj_X(f))$. □

Corollary 6.13. *If X is CO-reflexive, then $Map_{kCO}(X, Z) = Map_{kCov}(X, Z)$ for all spaces Z. If X is Cov-reflexive and $Map_{kCO}(X, Z) = Map_{kCov}(X, Z)$ for all spaces Z, then X is CO-reflexive.*

Proof. To prove the first part of Corollary 6.13 notice that if Y is locally compact and $f : Y \to Map_{CO}(X, Z)$ is continuous, then $f : Y \to Map_{Cov}(X, Z)$ is continuous. Indeed, by Proposition 6.12 and Theorem 4.11, $adj_X(f)$ is continuous which is sufficient for the continuity of $f : Y \to Map_{Cov}(X, Z)$ by the definition of the basic covariant topology.

To prove the second part of Corollary 6.13 assume that Y is locally compact and put $Z = Y \times_{CO} X$. We need to prove that $id : Y \times X \to Z$ is continuous (see Proposition 6.12). Obviously, $id : Y \times_{CO} X \to Y \times X$ is continuous and so is $f = adj^X(id) : Y \to Map_{CO}(X, Z)$ (see Definition 4.12). Since $Map_{kCO}(X, Z) = Map_{kCov}(X, Z)$, we infer that $f : Y \to Map_{Cov}(X, Z)$ is continuous. By Proposition 6.8, $tran(f) : X \to Map_{Cov}(Y, Z)$ is continuous and by Corollary 3.6 $id = adj_Y(tran(f)) : X \times Y \to Z$ is continuous. By Proposition 6.12, X is CO-reflexive. □

Let us improve Proposition 6.8 in the case of compact-open topology:

Proposition 6.14. *If X is a CO-reflexive space, then $tran : Map_{CO}(Y, Map_{CO}(X, Z)) \to Map_{CO}(X, Map_{CO}(Y, Z))$ is continuous. If both X and Y are CO-reflexive spaces, then $tran : Map_{CO}(Y, Map_{CO}(X, Z)) \to Map_{CO}(X, Map_{CO}(Y, Z))$ is a homeomorphism.*

Proof. Suppose X is a CO-reflexive space. By Theorem 4.11,

$$adj^Y : Map_{CO}(X \times_{CO} Y, Z) \to Map_{CO}\big(X, Map_{CO}(Y, Z)\big)$$

is a homeomorphism and

$$adj^X : Map_{CO}(Y \times_{CO} X, Z) \to Map_{CO}\big(Y, Map_{CO}(X, Z)\big)$$

is a homeomorphism. By Proposition 6.12, $sym : X \times_{CO} Y \to Y \times_{CO} X$ is continuous (a homeomorphism if Y is CO-reflexive). Hence, $(adj^Y) \circ sym^* = tran \circ (adj^X)$ is continuous which proves that $tran$ is continuous (a homeomorphism if Y is CO-reflexive). □

Here is another corollary of Proposition 6.12:

Corollary 6.15. *If X and Y are CO-reflexive, then $X \times_{CO} Y$ is CO-reflexive.*

Proof. We need to prove that $sym : (X \times_{CO} Y) \times_{CO} Z \to Z \times_{CO} (X \times_{CO} Y)$ is continuous for each space Z. Here is a sequence of identities (arising from the associativity of the CO-product as seen in Claim of the proof of Theorem 4.11) and symmetries which accomplish that goal:

$$\begin{aligned}(X \times_{CO} Y) \times_{CO} Z &= X \times_{CO} (Y \times_{CO} Z) \to X \times_{CO} (Z \times_{CO} Y)\\ &= (X \times_{CO} Z) \times_{CO} Y\\ &= (X \times_{CO} Z) \times_{CO} Y \to (Z \times_{CO} X) \times_{CO} Y\\ &= Z \times_{CO} (X \times_{CO} Y). \qquad \square\end{aligned}$$

7. Local connectivity

The purpose of this section is to show how covariant topologies can be used to define other functors in a similar manner to the way $k : \mathcal{TOP} \to k\mathcal{TOP}$ was defined.

Definition 7.1. Let $lc\mathcal{TOP}$ (respectively, $lpc\mathcal{TOP}$) be the full category of $\mathcal{TOP}$ whose objects are all locally connected (respectively, locally path connected) spaces.

Proposition 7.2. *Suppose $\mathcal{F} = \{f_s : X_s \to X\}_{s\in S}$ is a class of functions defined on topological spaces. Consider X with the covariant topology induced by $\mathcal{F}$. If each X_s is path connected* (*respectively, locally path connected*)*, then so is X.*

Proof. Consider an open neighborhood U of $x \in X$. Let A be the union of all connected (respectively, path connected) subsets of U containing x. Thus, A is the component (respectively, path component) of x in U. We need to show that A is open which amounts to proving that $f_s^{-1}(A)$ is open in X_s for all $s \in S$. If $f_s^{-1}(A)$ is not empty, then the component (respectively, path component) in $f_s^{-1}(U)$ of any of its element is contained in $f_s^{-1}(A)$. Thus, $f_s^{-1}(A)$ is open. $\square$

Definition 7.3. Functors $lc : \mathcal{TOP} \to lc\mathcal{TOP}$ and $lpc : \mathcal{TOP} \to lpc\mathcal{TOP}$ are defined as follows: lcX (respectively, $lpcX$) is X equipped with the covariant topology induced by all maps $f : Y \to X$ so that Y is locally connected (respectively, locally path connected).

Notice that both $id_X : lcX \to X$ and $id_X : lpcX \to X$ are continuous and it is clear that $lc : \mathcal{TOP} \to lc\mathcal{TOP}$ is a right adjoint to the inclusion functor $i : lc\mathcal{TOP} \to \mathcal{TOP}$. Similarly, $lpc : \mathcal{TOP} \to lpc\mathcal{TOP}$ is a right adjoint to the inclusion functor $i : lpc\mathcal{TOP} \to \mathcal{TOP}$.

8. Fuks topology

In his work on formalization of the Eckmann–Hilton duality Fuks [16] considered covariant functors on the category of pointed functionally Hausdorff k-spaces and defined the dual DS to S by assigning to Y the space of natural transformations from S to the functor Σ_Y. Σ_Y assigns to X the smash product $k(X \wedge Y)$ of X and Y with the k-topology. This required an introduction of topology on any set of natural transformations from a functor S to a functor T. The purpose of this section is to show that the Fuks topology is a contravariant topology.

Proposition 8.1. *Suppose S and T are two covariant functors on the category of pointed functionally Hausdorff k-spaces. If X is a set of natural transformations from S to T, then the Fuks topology on X is the contravariant topology induced by functions $t_Y : X \to Map_{kCO}(S(Y), T(Y))$, $t_Y(\phi) = \phi_Y : S(Y) \to T(Y)$.*

Proof. Suppose $\phi_0 \in X$ and $A \subset X$. Fuks [16] defines the closure operator cl^F (our notation) on subsets of X by declaring $\phi_0 \in \mathrm{cl}^F(A)$ iff $t_Y(\phi_0) \in \mathrm{cl}(t_Y(A))$ for each Y. Let cl be the closure operator stemming from the contravariant topology induced by $\{t_Y\}$. We need to show that $\mathrm{cl}^F = \mathrm{cl}$.

Suppose $\phi_0 \in \mathrm{cl}(A) - \mathrm{cl}^F(A)$. Since $\phi_0 \notin \mathrm{cl}^F(A)$, there is Y such that $t_Y(\phi_0) \notin \mathrm{cl}(t_Y(A))$. Thus, $t_Y^{-1}(Map_{kCO}(S(Y), T(Y)) - \mathrm{cl}(t_Y(A)))$ is a neighborhood of ϕ_0 (in the contravariant topology) missing A which contradicts $\phi_0 \in \mathrm{cl}(A)$.

Suppose $\phi_0 \in \mathrm{cl}^F(A) - \mathrm{cl}(A)$. Since $\phi_0 \notin \mathrm{cl}(A)$, there exists a sequence of spaces $Y(1), \ldots, Y(k)$ and a sequence of open subsets $U(i)$ in $Map_{kCO}(S(Y(i)), T(Y(i)))$, $1 \leqslant i \leqslant k$, so that $V = \bigcap_{i=1}^{k} t_{Y(i)}^{-1}(U(i))$ is a neighborhood of ϕ_0 (in the contravariant topology) disjoint with A. Let $Y = k(\prod_{i=1}^{k} Y(i))$. Notice that Y is a pointed functionally Hausdorff space. Notice that for each i there is the projection $p(i): Y \to Y(i)$ and the inclusion $j(i): Y(i) \to Y$. Define $\alpha(i): Map_{kCO}(S(Y), T(Y)) \to Map_{kCO}(S(Y(i)), T(Y(i)))$ by $\alpha(i)(f) = T(p(i)) \circ f \circ S(j(i))$. Notice that each $\alpha(i)$ is continuous. Also notice that for any natural transformation $\phi: S \to T$ one has $\alpha(i)(\phi_Y) = \phi_{Y(i)}$. Define $U = \bigcap_{i=1}^{k} \alpha(i)^{-1}(U(i))$ and notice that U is a neighborhood of $t_Y(\phi_0)$ in $Map_{kCO}(S(Y), T(Y))$. Now, $\phi_0 \in \mathrm{cl}^F(A)$ implies that $t_Y(\phi_0) \in \mathrm{cl}(t_Y(A))$. Thus, $U \cap t_Y(A)$ is not empty and contains $t_Y(\psi)$, $\psi \in A$. Since $t_{Y(i)}(\psi) = \alpha(i)(t_Y(\phi)) \in U(i)$ for each i, $\psi \in V$ which contradicts $V \cap A = \emptyset$. $\square$

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