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Combinatorial problems in the theory of music

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Abstract

This paper surveys some combinatorial problems that have arisen in Music Theory, or which have been suggested by musical topics. It includes a new look at the enumeration of tone-rows and the problem of finding the number of different ways of playing the piano work 'Klavierstück, no. 11' (1956) by Karlheinz Stockhausen.

1. Introduction

Since classical times, music and mathematics have been traditionally thought of as closely associated. The extent of this association — even its reality — can be hotly debated; but it is certainly true that from time to time problems arise in the theory of music which are of a mathematical nature. Often, a problem of this kind depends on one specific value for a parameter, such as the value 12 for the number of notes into which the octave is divided, and for that one value a solution to the problem may need little more than arithmetic; but the mathematician will naturally extend the problem to general values of the parameter, and this can result in something that is far from being trivial.

In this paper I consider some combinatorial problems that have arisen in this way. Two of them are, basically, old, but will be given an added twist that presents something novel; the third is, as far as I know, completely new.

2. Scales and chords

For the first two problems the basic material with which we shall work is the set of 12 notes which make up the octave, as shown in Fig. 1.

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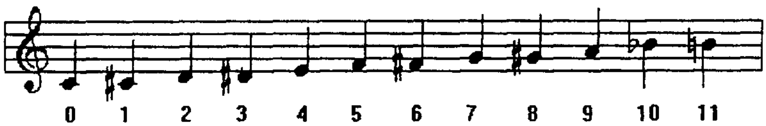

Fig. 1.

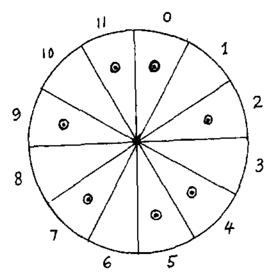

Fig. 2.

This sequence of 12 notes repeats above and below the octave shown, so that, with the numbering indicated, we are essentially working with the integers mod 12. A scale can then be defined as a subset of $(0,1,2,3,4,5,6,7,8,9,10,11)$ arranged in ascending order. A *transposition* of a scale is a subset obtained by a mapping of the form $x \to x+\alpha \pmod{12}$ where α is a constant. We shall regard a scale and a transposition of it to be equivalent. We shall also take no account of which note begins the scale (this is implicit in the definition of a scale as a subset). We ask 'How many inequivalent scales are there of size k, that is, consisting of k notes?'

We can represent this problem by means of a circular birthday cake, divided into 12 sectors. In each sector there may or may not be a single candle. Here the sectors represent the 12 available notes, and the candles indicate those notes which are chosen to be in the scale as in Fig. 2.

Transposition is clearly the same as a rotation of the cake, and we can therefore rephrase the problem as 'How many ways are there of placing k candles to give cakes that are inequivalent under rotation?'

This is a straightforward Pólya-type problem, and the solution is the coefficient of x^k in

$$Z(C_{12}; 1+x)$$

in the notation of [3]. Here C_{12} is the cyclic group of order 12 and $Z(\ )$ denotes the cycle index. The numbers thus obtained are given in the last row of Table 1.

One can ask 'How many of these scales are equivalent to at least one of their transpositions?' These are the scales that Messiaen [2] has called 'Modes de transposition limitée'. In terms of the cakes, they are those that have some kind of rotational

Table 1

Number of notes / Symmetry	0	1	2	3	4	5	6	7	8	9	10	11	12
1		1	5	18	40	66	75	66	40	18	5	1	
2			1		2		3		2		1		
3				1			1			1			
4					1				1				
6							1						
12	1												1
All scales	1	1	6	19	43	66	80	66	43	19	6	1	1

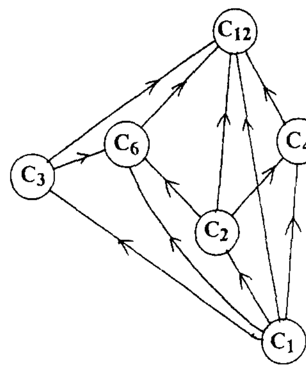

Fig. 3.

symmetry. The more general question would be to ask how many scales have each of the various possible kinds of symmetry.

The solution to this problem can be found by using Möbius inversion over the lattice of subsets of the cyclic group C_{12}, which is shown in Fig. 3.

The Möbius function for this very simple lattice is just the classical Möbius function $\mu(m)$. The number of cakes that are invariant under a rotation of $12/k$ sectors, (and hence under C_k), is 2^k. This number may include cakes which are invariant under a smaller rotation. If A_n is the number invariant under C_n but not under any larger group, then by the Möbius inversion formula we have

$$A_n = \sum_{k|n} \mu\left(\frac{n}{k}\right) 2^k.$$

The numbers thus obtained are shown in Table 1. Needless to say, for the case of 12 notes these numbers could be (and have been) obtained by exhaustive enumeration; but for scales of more notes it would be impractical to compute them by trial and error.

We remark that if the notes of a scale are sounded simultaneously the result is a chord. Hence the problem of enumerating chords is similar to that of enumerating scales — sufficiently so that we do not need to consider it further.

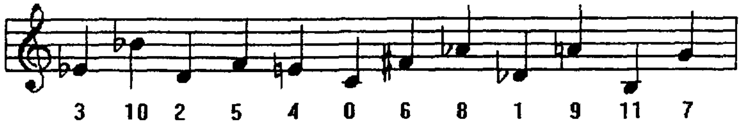

Fig. 4.

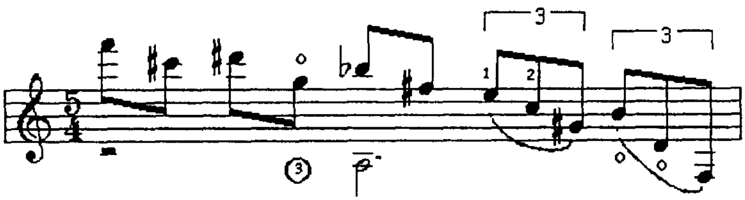

Fig. 5.

3. Tone rows

Dodecaphonic or 12-tone music, very popular earlier this century with what were then regarded as avant-garde composers, is based on the concept of a 'tone row'.

The basic definition of a tone row is that it is a permutation of the 12 notes of the chromatic scale, that is, a permutation of $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$. A typical tone row (from Schönberg's Piano Concerto) is shown in Fig. 4.

The purpose of the tone row is that a composer will use these notes over and over again in strict order. However, for any given tone row there are others that are regarded as equivalent to it. These equivalences are

(1) Transposition $x \to x + \alpha \pmod{12}$ (as above)

(2) Inversion, given by $x \to -x \pmod{12}$ and

(3) the retrograde form, obtained by reading the tone row backwards.

The problem of enumerating tone rows turns out to be another simple problem of the Pólya type. The results obtained can be found in the interesting paper by Reiner [8], and need not detain us. Suffice it to say that the number of tone rows is 9, 985, 920.

Let us now look at a variation of this problem. Fig. 5 shows one bar from the middle of a composition by an obscure Canadian composer [6].

If we are asked 'What is the tone row for this piece?' we cannot answer. For although we know the cyclic order of the notes, we would have to see the beginning of the piece to know on which note the tone row started.

It seems natural therefore (although I know of no previous consideration of it) to consider tone rows that are cyclic permutations of each other as being equivalent. With this extension of the definition of 'equivalent', the enumeration problem becomes one for which Pólya's theorem no longer applies.

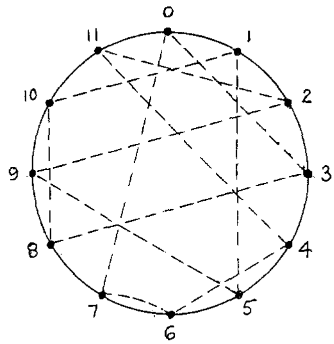

Fig. 6.

Table 2

2	3	4	5	6	7	8	9	10	11	12
1	1	2	4	12	39	202	1219	9468	83435	836017

It is possible to represent a whole equivalence class of tone rows by a single diagram. A typical one is shown in Fig. 6.

The points round the circle represent the 12 notes of the octave, and a tone row is represented by the dotted path going from one note of the tone row to the next, returning to the starting note. Rotation of this diagram corresponds to transposition. The new equivalence corresponds to starting at a different point on the dotted cycle. The retrograde form of a tone row corresponds to traversing the dotted path in the opposite direction. The inverse of a tone row will correspond to the same diagram reflected about some suitable diameter.

The number of inequivalent tone rows (in the new sense) is therefore the number of such diagrams that are inequivalent under the action of the dihedral group D_{12}.

This is precisely the number of superpositions of two graphs, each being a cycle of length 12. Using the superposition theorem, which I introduced in my Ph.D. thesis [4], the required number for the general case of n notes can be written as

$$N(D_n * D_n)$$

using the notation of [5]. In fact this result was specifically given in my thesis as an example of the use of the theorem, though at the time I had no idea that it had anything to do with tone rows! Table 2 gives the numbers for $n \leqslant 12$.

The same result was rediscovered by Golomb and Welsh [1], again as a solution to a purely geometrical problem.

4. Stockhausen's Klavierstück, n. XI

The music of Klavierstück, n. XI by Karlheinz Stockhausen [9] consists of a single large sheet of paper with 19 fragments of music on it. The performer is instructed to choose a fragment and play it; then to choose a *different* fragment and play that; and so on. When whatever random device is being used chooses a fragment that has been played twice already, the performance stops.

We ask, 'How many different performances are possible, and what is their average length?' More generally, we ask the same question for the case when there are n fragments, and when the performance ends with the choice of a fragment that has already been played some given number, r, of times. The length of a performance is defined as the number of fragments it contains.

We can state the problem more mathematically as follows.

A 'performance' is a string formed from the alphabet $(1,2,3,\ldots,n)$ having the following properties.

(1) No two consecutive symbols are the same.

(2) The last symbol occurs $r+1$ times in all.

(3) No other symbol occurs more than r times.

Problem. Find the number of such strings and their average length.

We start by considering strings subject only to property (1) above, and define a generating function

$$\Phi(x_1,x_2,\ldots,x_n) = \sum A_{m_1m_2\ldots m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$$

where $A_{m_1m_2\ldots m_n}$ is the number of strings in which the symbol 'i' occurs m_i times ($i = 1$ to n).

If, in such a string, we replace every occurrence of i by a number (at least one) of repetitions of i, then we shall get a general string over the given alphabet. In fact, all strings can be obtained in this way. The replacement of 'i' by '$ii\ldots i$' corresponds to the substitution

$$x_i = \frac{t_i}{1-t_i}$$

and the generating function for all strings, obtained in this round-about way is therefore

$$\Phi_n\left(\frac{t_1}{1-t_1},\frac{t_2}{1-t_2},\ldots,\frac{t_n}{1-t_n}\right).$$

But this generating function is known to be

$$\frac{1}{1-\sum t_i}.$$

Making the substitution $t_i = x_i/(1+x_i)$ we find that

$$\Phi_n(x_1, x_2, \ldots, x_n) = \frac{1}{1 - \sum \frac{x_i}{1+x_i}} = \frac{1}{1 - \sum (x_i - x_i^2 + x_i^3 \ldots)}$$

$$= \frac{1}{1 - (p_1 - p_2 + p_3 \ldots)},$$

where $p_k = \sum_i x_i^k$ is the power sum symmetric function.

Now choose one symbol, say t, to be the terminating symbol. Strings satisfying properties (1) *and* (2) above are then of the form.

$$\underbrace{??\ldots?}_{A}\,t\,\underbrace{??\ldots?}_{B_1}\,t\,\underbrace{??\ldots?}_{B_2}\,t??\ldots t\,\underbrace{??\ldots?}_{B_r}\,t$$

where A and the B's are strings on $n-1$ symbols satisfying property (1). A may be empty, but the B's must not be. The generating function for these strings is therefore

$$n\Phi_{n-1}(\Phi_{n-1} - 1)^r, \tag{4.1}$$

where the multiplier n is for the ways of choosing t.

In order to find the strings having also property (3) we must extract from the generating function (4.1) those terms for which no power of an x_i exceeds r. This can be done by using a theorem which is implicit in [5].

Theorem 4.1. *If $F(p_1, p_2, p_3, \ldots)$ is any function of the power sums p_i, then the coefficient of*

$$x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots$$

in F is

$$N(h_1^{m_1} h_2^{m_2} h_3^{m_3} \ldots * F(p_1, p_2, p_3 \ldots)), \tag{4.2}$$

where h_i denotes the homogeneous product sum symmetric function, and m_i is the number of a's equal to i.

We sum the left-hand operand of (4.2) over all choices of m_i ($i \leqslant r$), to get a single symmetric function on which we perform the $*$ operation with (4.1) as the right-hand operand. The result of this operation is the required generating function by length for performances. This requires a fair amount of manipulation, but eventually yields the following procedure for finding a reasonably simple formula for the generating function.

(a) Take the function

$$(1 + h_1 t + h_2 t^2 + \cdots + h_r t^r)^{n-1}$$

and write it in terms of the p_i's.

(b) Replace each p_i by $(-1)^{i+1} i\lambda$.

(c) Replace every occurrence of λ^m by $m!\binom{m}{r}$.

Table 3

n	Total number of performances when there are n fragments
2	2
3	114
4	5844
5	380900
6	32817990
7	3679720422
8	524366318504
9	92857556215944
10	20037507147592650
11	5180981746936701530
12	1582222025035216228092
13	563668692910591272692844
14	231745357332413891454727694
15	108930215000782607407231068750
16	58055715904927265592770932599120
17	34827257437680378101378978540586512
18	23363311171682567813792072652701951634
19	17423935148332958167310127282862901334594
20	14370244926747901561521218780905183051217700

(Details of this derivation and much else concerning the Stockhausen problem can be found in the research report [7] by myself and a colleague Lily Yen.)

For the case $r = 2$ the total number of performances assumes a particularly simple form, namely

$$n \sum_i \binom{n-1}{i} \frac{(2i)!\,i(2i-1)}{2^i}$$

from which the number of performances for the original problem ($n = 19$) is found to be

$$1\ 74239\ 35148\ 33295\ 81673\ 10127\ 28286\ 29013\ 34594.$$

Table 3 gives the numbers of performances when there are from 2 to 20 fragments.

The total length of all performances can be found by a modification of the above method. The average length of a performance can then be computed. For the original problem it is 38.0045857. The maximum possible length is, of course, 39.

5. A variation and some asymptotics

The identity of the fragment whose choice for the $(r + 1)$th time terminates the performance is, of course, known to the performer, but not to the listener (since it is not played). Thus two performances, identical to the listener, could be considered distinct in the above treatment by virtue of being terminated by a different choice. Hence, *from the listeners point of view*, the number of distinct performances will be less than the number given above.

This presents us with a slightly different problem, which, in mathematical terms, is that of enumerating strings with the properties

(1) No two consecutive symbols are the same;

(2) No symbol occurs more than r times;

(3) There is at least one symbol other than the last that occurs exactly r times.

This problem can be solved by a variation of the method given above, but I shall not take space to give the details. They can be found in [7]. For the original problem ($r = 2$, $n = 19$) the number of performances drops to

10249 37361 6664 45980 71114 32876 93179 82974

with an average length of 36.9458622 (the maximum possible length being now 38).

Asymptotic expressions for the number of performances and their lengths as $n \to \infty$ can be found quite easily for the case $r = 2$. The behaviour of the averages is of some interest. From the performer's point of view the average length (including the terminating fragment) is $2n + 1/(n-1)$, so that the average approaches $2n$ from above. The average length from the listener's point of view can be shown to approach $2n - 1$ from below. Hence, in either case, the average performance length is almost exactly one less than the maximum possible length.

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