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Resolvent growth and Birkhoff-regularity

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Abstract

We prove a long standing conjecture in the theory of two-point boundary value problems that unconditional basisness implies Birkhoff-regularity. It is a corollary of our two main results: minimal resolvent growth along a sequence of points implies nonvanishing of a regularity determinant, and sparseness of nth-order roots of eigenvalues in small sectors provided that eigen and associated functions of the boundary value problem form an unconditional basis.

Considerations are based on a new direct method, exploiting *almost orthogonality* of Birkhoff's solutions of the equation $l(y) = \lambda y$. This property was discovered earlier by the author.

Keywords: Unconditional basisness; Boundary value problems

To my parents Sarah and Moisei

0. Introduction

0.1. Preliminaries

Set $D = -\imath \mathrm{d}/\mathrm{d}x$ and consider a boundary value problem (bvp) in $L^2(0,1)$, defined by a differential expression

$$l(y) \equiv D^n y + \sum_{k=0}^{n-2} p_k(x) D^k y = \lambda y, \quad 0 \leqslant x \leqslant 1, \; p_k \in L(0,1), \tag{0.1}$$

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and n linearly independent boundary conditions

$$U_J(y) \equiv \sum_{k=0}^{n-1} \bigl(a_{jk} D^k y(0) + b_{jk} D^k y(1)\bigr) = 0, \quad j = 0, \dots, n-1. \tag{0.2}$$

Spectral theory of the operator L, defined by this bvp, is thoroughly investigated during the last hundred years. The bibliography is enormous and we shall refer the reader to the fundamental monographs [3,19,20]. Remind that inverse of L is a finite-dimensional perturbation of Volterra operator

$$L^{-1} f = Vf + \sum_{j=0}^{n-1} (f, h_j) g_j, \tag{0.3}$$

where Vf gives solution to the Cauchy problem for $l(y)$ and zero boundary conditions $D^j y(0) = 0$, $j = 0, \dots, n-1$. Investigation of operators (0.3) was initiated by A.P. Hromov [7].

Question of unconditional basisness plays an important role in the spectral theory. In case of the operator L we shall refer to it as the (UB) problem, assuming in addition that *multiplicities of the eigenvalues* (*evs*) *are uniformly bounded by some constant* M. Sometimes we will also use the term spectrality [3] as is customary in the theory of boundary value problems (bvps). In this paper we give its final solution, Theorem 4, using a new method which works *without* estimates from below for the characteristic determinant. We believe that this result will help in investigation of much more difficult class (0.3) with *nondissipative* V.

0.2. Birkhoff- and Stone-regular bvps

Now let us assume that boundary conditions (0.2) are normalized [24]:

$$U_J(y) \equiv b_J^0 D^j y(0) + b_J^1 D^j y(1) + \cdots = 0, \quad j = 0, \dots, n-1. \tag{0.4}$$

The ellipsis takes place of lower-order terms at 0 and at 1, b_J^0, b_J^1 are column vectors of length r_J, where

$$0 \leqslant r_J \leqslant 2, \quad \sum_{j=0}^{n-1} r_J = n, \quad \operatorname{rank}\bigl(b_J^0 b_J^1\bigr) = r_J. \tag{0.5}$$

Evidently $r_J = 0$ implies absence of order j conditions. For $r_J = 2 (b_J^0 b_J^1)$ equals second-order unit matrix. Below in the Definition 2 we define Birkhoff-regular boundary conditions. They possess a lot of remarkable spectral properties, e.g. estimate of the Green function, asymptotics of evs and eigenfunctions (efs). In 1958 N. Dunford announced J. Schwartz' result that Birkhoff-regular bvps are spectral [2, pp. 217, 221–222]. This statement contained an inaccuracy. The correct formulation asserts that *strong regularity* (briefly $L \in$ (SR), see Definition 3), implies (UB), and it was proved by G.M. Kesel'man [8], V.P. Mihaĭlov [12] as well as in the third volume of N. Dunford and J. Schwartz' monograph [3, Chapter XIX].

In Definition 3 we also call *Birkhoff* but not *strongly regular* boundary conditions *weakly regular* for evident reasons and write $L \in$ (WR). For them A.A. Shkalikov established unconditional basisness with parentheses (two summands in each) [26], further (UBP).

Generally speaking, for $L \in$ (WR) spectrality breaks down unless eigen- and associated functions (eaf) are merged pairwise, see examples of G.M. Kesel'man [8], P. Walker [30], and

J. Locker [10]. For instance, P. Walker considered a second-order bvp with efs $\sin\varrho_k x$, such that characteristic values (cvs) $\varrho_k := \sqrt{\lambda_k}$ are divided into two sequences:

$$2\pi k,\ k = 1, \ldots; \quad 2\pi k + o(1),\ k = 0, \ldots; \quad o(1) \to 0,\ k \to \infty.$$

The efs, corresponding to two *close* ϱ_k, have an angle tending to zero. Summarizing we have

Theorem A *(Kesel'man, Mihaĭlov, Schwartz, Shkalikov).*

$$L \in (\mathrm{SR}) \quad \Rightarrow \quad L \in (\mathrm{UB}), \qquad L \in (\mathrm{WR}) \quad \Rightarrow \quad L \in (\mathrm{UBP}).$$

However, further investigations failed to find even a *single* bvp with the same list of properties if Birkhoff-regularity is violated. Of course, there is a natural candidate for *good* boundary conditions: namely the self-adjoint ones, but they *are* Birkhoff-regular [24, *n* even], [14, *n* odd]. In case of smooth coefficients of $l(y)$ the characteristic determinant $\Delta(\varrho)$, see (1.8), admits further terms of asymptotic expansion. Assuming that necessary amount of them is not vanishing, we come to the notion of Stone-regularity [1,4,6,27], see also books [10,11]. This approach yields spectrum asymptotics, completeness and *upper* polynomial estimate of the Green function. Note that A.A. Shkalikov established unconditional convergence for Stone-regular problems in classes of sufficiently smooth functions [27], see also [28]. Here the smoothness should be enough to suppress possible growth of the resolvent.

For simplest two-point bvps with $l(y) \equiv D^2$ P. Lang and J. Locker [9, p. 554] carried out *a complete classification* of their spectral properties. It is based on the Plücker coordinates p_{ij}, $i < j$, $i, j = 0, \ldots, 3$ (2×2 minor with columns i, j) of the 2×4 matrix of coefficients in the boundary conditions (0.2).

Further in the book [10] J. Locker performed a thorough investigation of two-point bvps for $l(y) \equiv D^n y$. He classified degeneracy of the polynomial coefficients by the leading exponentials in the characteristic determinant and obtained same results as for classical Stone-regular bvps. However, no claim of nonspectrality has been made, cf. [10, Remark, p. 98].

Thus for irregular two-point bvps criterion of (UB) *was not established.*

0.3. Functional model

In seventies B.S. Pavlov discovered a projection method [22,23], which afterwards solved the (UB) problem for exponentials [15]. Later G.M. Gubreev deeply developed this method and obtained criterion of similarity to normal operator for finite-dimensional perturbations of Volterra dissipative operators [5]. Let us stress, however, that in (0.3) operator V is *not dissipative*.

Another approach relies on the functional model theory, which is deeply explored for dissipative operators [21]. Note a criterion of (UB) of a family of invariant subspaces [29]. Let us state this remarkable result in a simplest form.

Theorem B. *Assume that the differential expression* (0.1) *is formally self-adjoint and L is a dissipative operator. Then $L \in$* (UB) *in the span of eaf $\Leftrightarrow$ uniform minimality of eaf.*

In abstract situation uniform minimality is *much weaker* than (UB). Nevertheless, the former seems to be difficult to verify for operator L.

Hence we conclude that presently it is not possible to solve the (UB) problem via known abstract approaches either.

0.4. Notations

- $[a] := a + O(1/\varrho)$—Birkhoff's symbol;
- fss—fundamental system of solutions;
- $|\Delta \underset{k}{\longleftarrow} d|$—determinant Δ with the kth column replaced by a vector d;
- $A \asymp B$ means a double-sided estimate $C_1 \cdot |A| \leqslant |B| \leqslant C_2 \cdot |A|$ with some absolute constants $C_{1,2}$, which do not depend on A and B;
- mcm—modified characteristic matrix;
- $\#E$—number of elements in the set E;
- ∂S stands for boundary of the set S;
- given a system of elements $\{y_j\}_{j=1}^m$ of a Hilbert space H their Gram matrix is

$$G[y_1, \ldots, y_m] := \big[(y_j, y_k)\big];$$

- $O(\varrho^{-\infty}) = O(\varrho^{-N})$, $\forall N > 0$ with constants depending on N.

Throughout the paper components of matrices and vectors are enumerated beginning from zero. Matrices and their brackets are written in boldface to distinguish from Birkhoff's symbol. Different constants are denoted C, C_1, c, δ and so on. They may vary even during a single computation.

0.5. Paper outline

In the next section we recall background of the spectral theory of bvps, including a new notion of mcm, and describe main results of the paper. The key ingredients of our method are Theorem 8, which ties together minimal resolvent growth with nonvanishing of regularity determinants, and Theorem 17 asserting sparseness of cvs in some small sectors away from the real axis. Theorem 8 is formulated in the Section 2. As a consequence we immediately obtain Theorem 6, asserting Birkhoff-regularity of even-order dissipative differential operators. Then in Section 3 we investigate properties of the mcm and deduce Theorem 8. Further in Section 4 we first establish almost orthogonality of an auxiliary system (4.7) and then prove the sparseness Theorem 17. At last in the Section 5 we establish existence of a sequence of minimal resolvent growth and prove Theorem 4.

1. Green function

1.1. Birkhoff's solutions

Set $\varepsilon_\jmath = \exp(2\pi\imath\jmath/n)$, $\varrho = \lambda^{1/n}$, $|\varrho| = |\lambda|^{1/n}$,

$$\arg\varrho = \arg\lambda/n, \qquad 0 \leqslant \arg\lambda < 2\pi \tag{1.1}$$

and define sectors $S_\nu = \{\varrho \mid \pi\nu/n \leqslant \arg\varrho < \pi(\nu+1)/n\}$. Then $\varrho = \lambda^{1/n} \in S_0 \cup S_1$. Let R_0 be a fixed positive number such that in every sector S_ν there exists a fss $\{y_\jmath(x,\varrho)\}_{j=0}^{n-1}$ of (0.1) with an asymptotics:

$$D^k y_\jmath(x,\varrho) = (\varrho\varepsilon_\jmath)^k \cdot \exp(\imath\varrho\varepsilon_\jmath x)[1], \quad \jmath, k = 0, \ldots, n-1,\ |\varrho| \geqslant R_0. \tag{1.2}$$

Note that for sector S_ν solutions $y_\jmath(x,\varrho)$ decay as $j < p$ and exponentially grow otherwise (for $x > 0$), except maybe for a boundary ray. The number p depends upon the sector's choice, and

Table 1
Values of $p-1$

$\varrho \diagdown n$	$2q$	$2q+1$
$\in S_0$	$q-1$	q
$\in S_1$	$q-1$	$q-1$

values of $p-1$ are presented in Table 1. It will be convenient to use another fss of Eq. (0.1) with canceled growth:

$$z_k(x,\varrho) := \begin{cases} y_k(x,\varrho), & k=0,\dots,p-1, \\ y_k(x,\varrho)/\exp(\imath\varrho\varepsilon_k), & k=p,\dots,n-1, \end{cases} \tag{1.3}$$

$$z_k = O(1), \quad k=0,\dots,n-1,\ 0 \leqslant x \leqslant 1,\ \varrho \in S_\nu. \tag{1.4}$$

1.2. Formula for the Green function

Let $W_\jmath(x,\varrho)$ be the algebraic complement of $D^{n-1}y_\jmath$ in the Wronskian $W(x,\varrho) = |D^k y_\jmath(x,\varrho)|_{\jmath,k=0}^{n-1}$. Setting $\tilde{y}_\jmath(x,\varrho) := W_\jmath/W$ we calculate

$$\tilde{y}_\jmath(x,\varrho) = \frac{1}{n(\varrho\varepsilon_\jmath)^{n-1}} \exp(-\imath\varrho\varepsilon_\jmath x)[1]. \tag{1.5}$$

Introduce the kernel

$$g_0(x,\xi,\varrho) = \imath \cdot \begin{cases} \sum_{k=0}^{p-1} y_k(x,\varrho)\tilde{y}_k(\xi,\varrho), & x > \xi, \\ -\sum_{k=p}^{n-1} y_k(x,\varrho)\tilde{y}_k(\xi,\varrho), & x < \xi. \end{cases} \tag{1.6}$$

Then Green function admits a representation

$$G(x,\xi,\varrho) = \frac{(-1)^n \Delta(x,\xi,\varrho)}{n\varrho^{n-1}\Delta(\varrho)}, \tag{1.7}$$

$$\Delta(\varrho) = \det \mathbf{\Delta}(\varrho) = \det\big[V(z_0)\dots V(z_{n-1})\big], \tag{1.8}$$

$$\Delta(x,\xi,\varrho) = i \cdot \begin{vmatrix} z^T & g(x,\xi,\varrho) \\ \mathbf{\Delta}(\varrho) & H(\xi,\varrho) \end{vmatrix}, \tag{1.9}$$

where z^T stands for the row $(z_0(x,\varrho),\dots,z_{n-1}(x,\varrho))$,

$$g(x,\xi,\varrho) = g_0(x,\xi,\varrho)\cdot\big(n\varrho^{n-1}\big)/i, \qquad H(\xi,\varrho) = V_x\big(g(x,\xi,\varrho)\big)$$

and $V_x(\cdot) := (\varrho^{-j}U_j(\cdot))_{j=0}^{n-1}$. The subscript x means action over argument x. $\Delta(\varrho)$ is referred to as the *characteristic determinant*.

1.3. Regularity determinants

Definition 1. Fix some $\varepsilon \in (0,\pi/2n)$. Let $S_\nu(\varepsilon)$ be the sector

$$S_\nu(\varepsilon) = \left\{ \left| \arg\varrho - \frac{(\nu+1/2)\pi}{n} \right| \leqslant \varepsilon \right\}. \tag{1.10}$$

Define the regularity determinants, corresponding to sectors S_ν, via the formula

$$\Theta(S_\nu) := \lim_{\varrho\to\infty} \Delta(\varrho), \quad \varrho \in S_\nu(\varepsilon). \tag{1.11}$$

Let $q = \text{entier}(n/2)$. For $0 \leqslant k \leqslant n-1$ set

$$b^\iota = \left(b^i_J\right)_{j=0}^{n-1}, \qquad B^\iota_k = \left(b^\iota_J \cdot \varepsilon^j_k\right)_{j=0}^{n-1}, \quad \iota = 0, 1. \tag{1.12}$$

It is easy to calculate the limit in (1.11) for the determinant and its matrix

$$\Theta(S_\nu) = \Theta_p\left(b^0, b^1\right), \tag{1.13}$$

$$\mathbf{\Theta}(S_\nu) = \mathbf{\Theta}_p\left(b^0, b^1\right) := \left[B^0_k, k = 0, \ldots, p-1 \,\middle|\, B^1_k, k = p, \ldots, n-1\right]. \tag{1.14}$$

The vertical line | separates columns with superscripts 0 and 1. Recall that $p = p(\nu)$, $\nu = 0, 1$, see Table 1. From (1.13) it is clear that Definition 1 is equivalent to the standard one [24, p. 361].

Definition 2. We shall call boundary conditions (0.4) and the corresponding operator L Birkhoff-regular and write $L \in (R)$, if

$$\Theta(S_0) \neq 0, \qquad \Theta(S_1) \neq 0. \tag{1.15}$$

Definition 3. Birkhoff-regular bvp is *strongly regular*, $L \in$ (SR), if either n is odd or if it is even, $n = 2q$, and the second-order polynomial $F(s) = \det \mathbf{F}(s)$ has two simple roots. Here

$$\mathbf{F}(s) := \left[B^0_0 + s \cdot B^1_0, B^0_k, k = 1, \ldots, q-1 \,\middle|\, s \cdot B^0_q + B^1_q, B^1_k, k = q+1, \ldots, n-1\right].$$

Otherwise we shall call bvp *weakly regular* and write $L \in$ (WR), i.e. for classes of bvps we define (WR) := $(R) \setminus$ (SR).

1.4. Necessity of Birkhoff-regularity

Theorem 4. $L \in$ (UB) $\Rightarrow L \in (R)$.

It was a widely held tacit conjecture though never formulated explicitly. Theorem 4 together with Theorem 0.5 solves the (UB) problem except for $L \in$ (WR). In the latter case we have only (UBP).

Problem 5. *Give necessary and sufficient conditions for* $L \in$ (WR) $\cap$ (UB).

Theorem 6. *Even-order two-point dissipative bvps are Birkhoff-regular.*

This theorem partially answers S.G. Krein's question.

1.5. Modified characteristic matrix

Set

$$u_t = \begin{cases} \tilde{y}_t(\xi, \varrho) \cdot n(\varrho\varepsilon_t)^{n-1} \cdot \mathrm{e}^{\iota\varrho\varepsilon_t} = \mathrm{e}^{\iota\varrho\varepsilon_t(1-\xi)} \cdot [1], & t < p, \\ \tilde{y}_t(\xi, \varrho) \cdot n(\varrho\varepsilon_t)^{n-1} = \mathrm{e}^{\iota\varrho\varepsilon_t(-\xi)} \cdot [1], & t \geqslant p. \end{cases} \tag{1.16}$$

The following formula stems immediately from definitions of z_k and u_t:

$$g(x, \xi, \varrho) = \begin{cases} +\sum_{k<p} \varepsilon_k z_k(x, \varrho) u_k(\xi, \varrho) \mathrm{e}^{-\iota\varrho\varepsilon_k}, & x > \xi, \\ -\sum_{k \geqslant p} \varepsilon_k z_k(x, \varrho) u_k(\xi, \varrho) \mathrm{e}^{+\iota\varrho\varepsilon_k}, & x < \xi. \end{cases} \tag{1.17}$$

Introduce notation

$$\sim(t) = \begin{cases} 1, & t < p, \\ 0, & t \geqslant p. \end{cases}$$

We shall omit index t if it is clear from context. Applying V over variable x to (1.17), we obtain representation

$$H(\xi, \varrho) = \sum_{t=0}^{n-1} (-1)^{(1-\sim)} [B_t^{\sim}] \cdot \varepsilon_t u_t(\xi, \varrho). \tag{1.18}$$

Lemma 7.

$$G(x, \xi, \varrho) = g_0(x, \xi, \varrho) - \frac{2\pi\imath}{n\varrho^{n-1}} \sum_{t,k=0}^{n-1} a_{tk}(\varrho) z_k(x, \varrho) u_t(\xi, \varrho), \tag{1.19}$$

where

$$a_{tk} := \begin{cases} +\frac{\varepsilon_t}{2\pi} \cdot |\Delta \underset{k}{\leftarrow} [B_t^1]|/\Delta, & t < p, \\ -\frac{\varepsilon_t}{2\pi} \cdot |\Delta \underset{k}{\leftarrow} [B_t^0]|/\Delta, & t \geqslant p. \end{cases} \tag{1.20}$$

Proof. Denote Δ_k the kth column of the matrix $\mathbf{\Delta}$. Below we use the standard agreement that $\hat{d}$ means absence of the corresponding column in determinant. Expanding $\Delta(x, \xi, \varrho)$ along the topmost row and replacing H by the sum in (1.18), we obtain

$$\begin{aligned} &G(x, \xi, \varrho) \\ &\quad = g_0(x, \xi, \varrho) + \frac{(-1)^n i}{n\varrho^{n-1}\Delta} \sum_{k=0}^{n-1} (-1)^k z_k(x, \varrho) |\Delta_0 \ldots \hat{\Delta}_k \ldots \Delta_{n-1} H| \\ &\quad = g_0(x, \xi, \varrho) + \frac{-i}{n\varrho^{n-1}\Delta} \sum_{k=0}^{n-1} z_k(x, \varrho) \sum_{t=0}^{n-1} (-1)^{(1-\sim)} \left|\Delta \underset{k}{\leftarrow} [B_t^{\sim}]\right| \cdot \varepsilon_t u_t(\xi, \varrho). \qquad \square \end{aligned}$$

Earlier coefficients (1.20) were introduced in [17,18]. So it would be natural to call the matrix

$$\mathbf{A} = \mathbf{A}(\varrho) = [a_{tk}]_{k,t=0}^{n-1} \tag{1.21}$$

mcm of the bvp (0.1), (0.4) because it differs from the analogous object in [20, p. 135] by another choice of the fss. Namely, in [20] fss is analytic in λ.

2. Resolvent growth

For $L \in$ (UB) holds an estimate

$$\|R_\lambda\| \leqslant \frac{C}{\min(\operatorname{dist}(\lambda, \Lambda), \operatorname{dist}(\lambda, \Lambda)^M)}. \tag{RG}$$

Indeed, assume that f is a finite linear combination of eaf. Taking into account the resolvent expansion near a pole [19, p. 41] and uniform boundedness of evs multiplicities, we see that $\|R_\lambda f\| \leqslant \text{const}\|f\|$ where const equals the rhs of (RG) whence the latter immediately follows.

Obviously (RG) implies an inequality

$$\|R_\lambda\| \leqslant C|\varrho|^{-n} \tag{2.1}$$

for all λ such that

$$\operatorname{dist}(\lambda, \Lambda) \geqslant C|\lambda| \geqslant 1. \tag{2.2}$$

Let us call $\{\tau_m\}_{m=1}^\infty$ a set of *minimal resolvent growth* (mrg) if (2.1) fulfills for $\varrho = \tau_m, \forall m$ with C independent of m.

Theorem 8. *Let L be a differential operator, defined by bvp* (0.1), (0.4). *Fix $\nu \in \{0, 1\}$ and set $\mathbf{D} := \operatorname{diag}(\varepsilon_0, \dots, \varepsilon_{p-1}, -\varepsilon_p, \dots, -\varepsilon_{n-1})/(2\pi)$. Given a mrg sequence $\{\tau_m\}_1^\infty \subset S_\nu(\varepsilon)$, we get that*

$$\exists \lim_{m\to\infty} \mathbf{\Delta}(\tau_m) = \mathbf{\Theta}_p\big(b^0, b^1\big),$$

$$\exists \lim_{m\to\infty} \mathbf{A}(\tau_m) =: \mathbf{A}_\infty(L) = \mathbf{\Theta}_p\big(b^0, b^1\big)^{-1} \cdot \mathbf{\Theta}_p\big(b^1, b^0\big) \cdot \mathbf{D}$$

and all the matrices are invertible.

Thus in order to prove Theorem 4 it is enough to find one mrg sequence in $S_0(\varepsilon)$ and another in $S_1(\varepsilon)$.

Remark 9. Theorem 6 is an immediate corollary of Theorem 8. Indeed, if L is dissipative, then (2.1) is valid for $S_1(\varepsilon)$. Therefore the corresponding regularity determinant $\Theta(S_1)$ is nonzero. It is enough in the even-order case, since then the second determinant is the *same*, compare (1.13) and the value of $p - 1$ in the Table 1.

Remark 10. Recently E.A. Shiryaev found another proof of Theorem 6 [25]. He established self-adjointness of senior terms of even-order dissipative boundary conditions. In the odd-order case he also presented an example where regularity is violated.

Let us add that one nonzero determinant is enough to assert that odd-order dissipative differential operator is *half-regular* in the sense of [13].

3. Limit of mcm

3.1. Almost orthogonality

An *almost orthogonality* property was proved in [16] and asserts that

$$\left\|\sum_{k=0}^{n-1} c_k y_k(x, \varrho)\right\|^2_{L^2(0,1)} \asymp \sum_{k=0}^{n-1} |c_k|^2 \big\|y_k(x, \varrho)\big\|^2_{L^2(0,1)} \tag{3.1}$$

for any coefficients c_k, which may vary with ϱ.

Remark 11. Moreover, the system (1.16) is also almost orthogonal. This is valid because it has an exponential asymptotics and this is the unique ingredient needed for this property.

3.2. Boundedness of mcm

Below R_0 is the positive number from (1.2).

Lemma 12. *Consider an integral operator g_0 in $L^2(0,1)$ with kernel* (1.6). *It admits an estimate*

$$\|g_0\| \leqslant C|\varrho|^{-n}, \quad \varrho \in S_\nu(\varepsilon),\ |\varrho| \geqslant R_0. \tag{3.2}$$

Proof. Removing brackets from the asymptotic expressions for the functions $y_k(x,\varrho)$ and $\tilde{y}_k(\xi,\varrho)$, we come to a kernel $G_0(x,\xi,\varrho)$ which naturally extends to $\mathbb{R}$. Obviously the extended kernel coincides with the Green function of the self-adjoint operator D^n in $L^2(\mathbb{R})$ and therefore obeys an analogue of (3.2) in $L^2(\mathbb{R})$. Together with representation

$$g_0(x,\xi,\varrho) = G_0(x,\xi,\varrho) + O\left(\frac{1}{\varrho^n}\right)$$

it completes the proof. □

Lemma 13. *The mcm is uniformly bounded*

$$\sum_{t,k=0}^{n-1} \left|a_{tk}(\varrho)\right|^2 = O(1) \tag{3.3}$$

for $\varrho \in S_\nu(\varepsilon)$, $|\varrho| \geqslant R_0$ such that (2.2) *is valid.*

Proof. Let $P = P(\varrho)$ be an operator in $L^2(0,1)$ with the kernel

$$P(x,\xi,\varrho) = \sum_{t,k=0}^{n-1} a_{tk}(\varrho)\, z_k(x,\varrho) u_t(\xi,\varrho).$$

Under lemma's conditions

$$\|P\| \leqslant C/|\varrho|, \quad |\varrho| \geqslant R_0,\ \varrho \in S_\nu(\varepsilon) \tag{3.4}$$

due to (1.19), (2.1) and (3.2). Our next goal is to establish a relation

$$\|P\| \asymp \sqrt{\sum_{t,k=0}^{n-1} \left|a_{tk}(\varrho)\right|^2 \cdot \frac{1}{|\varrho|^2}}. \tag{3.5}$$

Let $f \in L^2(0,1)$. Then $Pf = \sum_{k=0}^{n-1} d_k z_k(x,\varrho)$. Invoking (3.1), we arrive at the double sided estimate

$$\|Pf\|^2_{L^2(0,1)} \asymp \sum_{k=0}^{n-1} |d_k|^2 \|z_k\|^2_{L^2(0,1)}.$$

Observe that $\|z_k\|^2 \asymp \|u_t\|^2 \asymp \frac{1}{|\varrho|}$, $\varrho \in S_\nu(\varepsilon)$, $|\varrho| \geqslant R_0$ and introduce a sum $T_k = \sum_{t=0}^{n-1} |a_{tk}(\varrho)|^2$. Then (3.5) is equivalent to

$$\sup_{\|f\| \leqslant 1} \sum_{k=0}^{n-1} |d_k|^2 \asymp \frac{1}{|\varrho|} \sum_{k=0}^{n-1} T_k. \tag{3.6}$$

Applying Remark 11, we obtain

$$\sup_{\|f\|\leqslant 1} |d_k|^2 = \left\| \sum_{t=0}^{n-1} a_{tk} u_t(\cdot, \varrho) \right\|_{L^2(0,1)}^2 \asymp T_k \frac{1}{|\varrho|},$$

which yields (3.6). At last compare (3.4) and (3.5) and we are done. □

3.3. Proof of Theorem 8

Proof. In virtue of (1.20) the tth column A_t of the matrix (1.21) satisfies an equation

$$\mathbf{\Delta} A_t = (-1)^{(1-\sim)} \cdot \frac{\varepsilon_t}{2\pi} [B_t^{\sim}], \quad 0 \leqslant t \leqslant n-1. \tag{3.7}$$

Fix $t \in \{0, \dots, n-1\}$. Using compactness of the set of vectors

$$A_t(\varrho), \quad \varrho \in S_\nu(\varepsilon),\ |\varrho| \geqslant R_0,$$

we deduce existence of a limiting vector $\eta_t = \lim_{l\to\infty} A_t(\varrho_{m_l})$ for some subsequence ϱ_{m_l}. Obviously

$$\lim_{l\to\infty} \mathbf{\Delta}(\varrho_{m_l}) = \mathbf{\Theta}_p\big(b^0, b^1\big).$$

Together with (3.7) it shows that

$$R\big(\mathbf{\Theta}_p\big(b^0, b^1\big)\big) \supset \operatorname{span}\big(B_0^1, \dots, B_{p-1}^1, B_p^0, \dots, B_{n-1}^0\big), \tag{3.8}$$

where $R(\cdot)$ stands for the matrix' image. The second relation is trivial

$$R\big(\mathbf{\Theta}_p\big(b^0, b^1\big)\big) \supset \operatorname{span}\big(B_0^0, \dots, B_{p-1}^0, B_p^1, \dots, B_{n-1}^1\big). \tag{3.9}$$

Inclusions (3.8)–(3.9) yield that

$$R\big(\mathbf{\Theta}_p\big(b^0, b^1\big)\big) \supset \operatorname{span} \mathbf{Q}, \quad \mathbf{Q} = \big[\mathbf{Q}^0, \mathbf{Q}^1\big],\ \mathbf{Q}^\iota = \big[B_0^\iota \dots B_{n-1}^\iota\big].$$

Set $\mathbf{\Psi} = [\varepsilon_j^k]_{j,k=0}^{n-1}$. Since a (j,k)th block-entry of the product $\mathbf{Q}^\iota \cdot \mathbf{\Psi}^*$ is an $r_j \times 1$ vector

$$b_j^\iota \sum_{t=0}^{n-1} \varepsilon_j^t \cdot \overline{\varepsilon_k^t} = b_j^\iota \cdot n \cdot \delta_{jk}, \quad \iota = 0, 1;\ j, k = 0, \dots, n-1,$$

we conclude that $\operatorname{rank}(\mathbf{Q}\mathbf{\Psi}^*) = \sum_{j=0}^{n-1} r_j = n$. Hence $\mathbf{Q}$ is a full range matrix, and the same is $\mathbf{\Theta}_p(b^0, b^1)$. □

4. Sparseness of cvs

4.1. Preliminaries

Denote $\Gamma = \{\varrho_j\}_{j=1}^\infty$ the sequence of all distinct cvs $\varrho_j := \lambda_j^{1/n}$, not counting multiplicities. Fix $\nu \in \{0, 1\}$ and set

$$\Gamma_\varepsilon := \Gamma \cap S_\nu(\varepsilon). \tag{4.1}$$

Let $b_\mu(\varrho) = (\varrho - \mu)/(\varrho - \bar{\mu})$ be the Blaschke factor. Draw a hyperbolic circle $K(\varrho_j, \delta) = \{\varrho\colon |b_{\varrho_j}(\varrho)| \leqslant \delta\}$ around every $\varrho_j \in \Gamma$, remove them from $S_\nu(\varepsilon)$ and denote $S_\nu(\varepsilon, \delta)$ the remaining domain. Set $D(\varrho, \delta) := \{|z - \varrho| \leqslant \delta |\operatorname{Im} \varrho|\}$. Below we shall often use relations from [21, Lecture XI, formulas after (9)]

$$K(\varrho,\delta)\supset D(\varrho,\delta), \tag{4.2}$$

$$K(\varrho,\delta)\subset D(\varrho,\delta_1), \quad \delta_1=\frac{2\delta}{1-\delta} \tag{4.3}$$

and δ_1 will always have the value from (4.3).

Recall a definition of a sparse sequence.

Definition 14. Let P be a sequence of points in $\mathbb{C}_+$. Then P is sparse, $P\in(S)$, if for some $\delta>0$ $K(\varrho,\delta)\cap K(\mu,\delta)=\emptyset$, $\varrho\neq\mu$, $\varrho,\mu\in P$. Equivalently each circle $K(\varrho,\delta)$ contains $\leqslant 1$ element from P. In addition P is N-sparse, $P\in(NS)$ if for some $\delta>0$

$$\#\big(P\cap K(\varrho,\delta)\big)\leqslant N, \quad \forall\varrho\in\mathbb{C}_+.$$

4.2. System $\{\omega_{lm}\}$

In what follows we will need functions

$$\omega_{lm}(x,\varrho):=\frac{1}{m!}\frac{d^m}{d\varrho^m}\omega_{l0}(x,\varrho), \quad l=0,\dots,n-1;\ \omega_{l0}(x,\varrho):=z_l(x,\varrho).$$

Define by $\chi_\pm$ an indicator function of the half-axis $\mathbb{R}_\pm$. Introduce functions

$$W_{lm}(x,\xi)=\frac{1}{m!}(\imath\varepsilon_l x)^m\begin{cases}\exp(+\imath\xi\varepsilon_l x)\cdot\chi_+, & 0\leqslant l<p,\\ \exp(-\imath\xi\varepsilon_l x)\cdot\chi_-, & p\leqslant l\leqslant n-1.\end{cases}$$

Let $\varrho=r\xi\in S_\nu(\varepsilon)$, $r=|\varrho|\geqslant R_0$. Then ξ belongs to a closed arc $\mathrm{arc}_{\nu\varepsilon}$ of opening 2ε and $\|W_{lm}\|_{L^2(\mathbb{R})}\asymp 1$, $\xi\in\mathrm{arc}_{\nu\varepsilon}$.

Lemma 15. *The following relation holds true*

$$\big\|\omega_{lm}(x,\varrho)\big\|^2=\frac{[1]}{r^{2m+1}}\cdot\big\|W_{lm}(\cdot,\xi)\big\|^2_{L^2(\mathbb{R})}\asymp\frac{1}{r^{2m+1}}, \quad \varrho\in S_\nu(\varepsilon),\ r\geqslant R_0. \tag{4.4}$$

Proof. For the sake of definiteness let $\nu=0$. Fix $\delta>0$ small enough such that $\forall\varrho\in S_\nu(\varepsilon)$ the circle $D(\varrho,\delta)$ lies in a sector $S_\nu(\varepsilon_1)$, $0<\varepsilon_1<\frac{\pi}{2n}$ which is strictly contained in S_ν. Observe that $\omega_{lm}(x,\varrho)$ is a sum of the main term $\exp(\imath\varepsilon_l\varrho x)$ and remainder $\exp(\imath\varepsilon_l\varrho x)[0]$. Rewrite this as a representation

$$\omega_{lm}(x,\varrho)=\frac{(i\varepsilon_l x)^m}{m!}\exp(\imath\varepsilon_l\varrho x)+\frac{1}{2\pi\imath}\int\limits_{\partial D(\varrho,\delta)}\frac{\exp(\imath\varepsilon_l zx)\cdot[0](x,z)}{(z-\varrho)^{m+1}}\,dz. \tag{4.5}$$

The $L^2(0,1)$-norm of the remainder is

$$O\left(\frac{1}{|\varrho|}\cdot(\operatorname{Im}\varrho)^{-m}\right)=O\big(\varrho^{-m-1}\big). \tag{4.6}$$

The $L^2(0,1)$-norm of the first summand is

$$r^{-m-1/2}\cdot\big\|W_l m(\cdot,\xi)\big\|_{L^2(\mathbb{R})}+O(\varrho^{-\infty})$$

and we are done. □

Lemma 16. *The system*

$$\{\omega_{lm}(x,\varrho)\}_{l=0,m=0}^{n-1,N-1}, \quad \varrho \in S_\nu(\varepsilon),\ r \geqslant R_0 \tag{4.7}$$

is almost orthogonal in $L^2(0,1)$.

Proof. Actually we need to prove that Gram matrix of the normalized system (4.7) is positive definite uniformly with respect to ϱ. For the sake of definiteness let $\nu = 0$. Consider a scalar product

$$(\omega_{lm}, \omega_{jk})_{L^2(0,1)}. \tag{4.8}$$

First assume that $l, j < p$ and substitute representation (4.5) for both functions in (4.8). Scalar product of main terms equals $r^{-(m+k+1)} \cdot (W_{lm}, W_{jk})_{L^2(\mathbb{R})} + O(\varrho^{-\infty})$. Next consider scalar product of the remainder in (4.5) with the main term $\frac{(i\varepsilon_j x)^k}{k!}\exp(\imath\varrho\varepsilon_j x)$. Remainder's norm is $O(r^{-(m+1)})$, see (4.6), whereas the exponential's norm is $\asymp r^{-(k+1/2)}$. Hence impact of remainder terms to (4.8) is $O(r^{-(k+m+3/2)})$.

For $l, j \geqslant p$ (4.8) is estimated along the same lines whereas for all other cases it is $O(\varrho^{-\infty})$. Thus taking into account (4.4) we see that Gram matrices of the normalized system (4.7) and of the normalized system $\{W_{lm}\}_{l-0,m=0}^{n-1,N-1}$ coincide upto [0]. The latter system is almost orthogonal uniformly with respect to $\xi \in \mathrm{arc}_{\nu\varepsilon}$ which completes the proof. $\square$

4.3. Spectrum sparseness

Theorem 17. *For sufficiently small δ in Definition* 14 *the cvs are n-sparse in the sector $S_\nu(\varepsilon)$, i.e. $\Gamma_\varepsilon \in (nS)$.*

Proof. Choose some circle $K(\varrho,\delta)$ intersecting Γ_ε. Assume on the contrary that $\#(K(\varrho,\delta) \cap \Gamma_\varepsilon) > n$. Take any $N = n+1$ cvs from this intersection and enumerate them $\{\varrho_j\}_{j=1}^N$. They are distinct because we chose Γ not counting multiplicities. Obviously $|\varrho_j| \asymp |\varrho|$, $j = 1,\dots,N$. Denote respective normalized efs $u_1,\dots,u_N$. Expanding $u_j(x)$ over the system (1.3) and applying its almost orthogonality we get

$$u_j(x) = \sum_{l=0}^{n-1} d_{jl}\omega_{l0}(x,\varrho_j), \qquad \sum_{l=0}^{n-1}|d_{jl}|^2 \cdot \frac{1}{|\varrho_j|} \asymp 1, \quad j = 1,\dots,N. \tag{4.9}$$

Consider linear combination $u(x) = \sum_{j=1}^N c_j u_j(x)$ with *normalized* coefficients: $\sum_{j=1}^N |c_j|^2 = 1$. Then (UB) yields

$$\|u\|^2 \asymp 1. \tag{4.10}$$

Expand the function $\omega_{l0}(x,\varrho_j)$ in Taylor series centered at ϱ with respect to the second argument and substitute the result into (4.9). Thus we come to the representation

$$u(x) = \sum_{l=0}^{n-1}\sum_{m=0}^{\infty} a_{ml}\cdot\omega_{lm}(x,\varrho), \tag{4.11}$$

$$a_{ml} = \sum_{j=1}^{N} c_j \cdot d_{jl} \cdot (\varrho_j - \varrho)^m. \tag{4.12}$$

Denote u^N the double sum as in (4.11) with summation over m taken from 0 to $N-1$. In [17, Chapter 4, Lemma 6.1] we established an estimate

$$\|u\| \asymp \|u^N\|. \tag{4.13}$$

In fact there we assumed that ϱ lies in a strip $|\operatorname{Im}\varrho| \leqslant C$ but the case in question may be considered along the same lines and even simpler, e.g. there $I^0 = \emptyset$ whence the summand $u_1(x)$ [17, Chapter 4, (6.4)] disappears.

Fix $0 < \varepsilon_1 < \pi$. Due to almost orthogonality of the system (4.7) and formulas (4.4), (4.13) we arrive at

$$\|u\|^2 \asymp \sum_{l=0}^{n-1}\sum_{m=0}^{N-1} |a_{ml}|^2 \cdot \frac{1}{|\varrho|^{2m+1}}, \quad \varepsilon_1 \leqslant |\arg\lambda| \leqslant \pi - \varepsilon_1. \tag{4.14}$$

Now observe that the row $(a_{00}, \ldots, a_{0,n-1})$ coincides with linear combination of matrix $d = [d_{jl}]_{j=1,\,l=0}^{N\,n-1}$ rows. Since $N > n$ they are linearly dependent. Therefore for appropriate normalized coefficients c_j in (4.12) $a_{0l} = 0$, $l = 0, \ldots, n-1$. Further

$$|a_{ml}|^2 \leqslant \sum_{j=1}^{N} |d_{jl}|^2 \cdot |\delta_1 \cdot \operatorname{Im}\varrho|^{2m} \leqslant \sum_{j=1}^{N} |d_{jl}|^2 \cdot \delta_1^{2m} \cdot |\varrho|^{2m}.$$

Substituting it into (4.14) and taking into account (4.9), we obtain

$$\|u\|^2 \leqslant C \cdot \sum_{l=0}^{n-1}\sum_{m=1}^{N-1}\sum_{j=1}^{N} |d_{jl}|^2 \cdot \delta_1^{2m} \cdot \frac{|\varrho|^{2m}}{|\varrho|^{2m+1}} \leqslant (N+1)C\delta_1^2 \tag{4.15}$$

which contradicts (4.10) if δ_1 is sufficiently small. □

4.4. Estimate off cvs

Lemma 18. *Let $0 < \varepsilon < \pi/2n$. Then $S_\nu(\varepsilon, \delta)$ is a set of mrg.*

Proof. We need to verify the estimate (2.2). Let for definiteness $\nu = 0$. We will use the identity

$$|\varrho^n - \varrho_j^n| = \prod_{m=0}^{n-1} |\varrho\varepsilon_m - \varrho_j|. \tag{4.16}$$

Case $m = 0$. Then $|b_\varrho(\varrho_j)| \equiv |b_{\varrho_j}(\varrho)| > \delta$ because $\varrho \in S_0(\varepsilon, \delta)$. So $\varrho_j \notin K_\varrho(\delta)$ and all the more $\varrho_j \notin D(\varrho, \delta)$. Therefore $|\varrho_j - \varrho| > \delta|\operatorname{Im}\varrho|$.

Case $0 < m \leqslant n-1$. Let us demonstrate that

$$|\varrho\varepsilon_m - \varrho_j| \geqslant c|\operatorname{Im}\varrho|, \quad m = 1, \ldots, n-1.$$

Assume for simplicity $m = 1$. Other m may be considered in a similar way. Recall that $\varrho\varepsilon_1 \notin S_0 \cup S_1$, whereas ϱ_j belongs to this union due to the choice of the root's branch (1.1). Suppose $n > 2$. Then

$$|\varrho\varepsilon_1 - \varrho_j| \geqslant \operatorname{dist}(\varrho\varepsilon_1, \partial S_1) = \operatorname{dist}(\varrho, \mathbb{R}_+) \geqslant |\operatorname{Im}\varrho|.$$

If $n = 2$ then $\varrho\varepsilon_1 = -\varrho$, $S_0 \cup S_1 = \mathbb{C}_+$ and $\operatorname{dist}(-\varrho, \mathbb{C}_+) = |\operatorname{Im}\varrho|$.

Substitute the estimates above into (4.16), taking into account that $|\operatorname{Im}\varrho| \geqslant \sin(\frac{\pi}{2n} - \varepsilon)|\varrho|$, and we arrive at (2.2). □

5. Proof of Theorem 4

Step 1. For the sake of definiteness let $\nu = 0$. The case $\nu = 1$ may be considered similarly. Define

$$\mathcal{D} = S_0(\varepsilon) \cap \{r \leqslant |\varrho| \leqslant r + |\delta| r\}.$$

Set $\omega = (2\varepsilon)/M$. We will choose the integer M later. Dissect $S_0(\varepsilon)$ by M rays into sectors of opening ω. They divide $\mathcal{D}$ into M quadrilaterals Π_k, $k = 0, \dots, M-1$. Since Π_k has vertices

$$V_1 = r \cdot \exp(i\theta), \qquad V_2 = r \cdot \exp\bigl(i(\theta+\omega)\bigr), \qquad V_3 = (r + \delta r) \cdot \exp\bigl(i(\theta+\omega)\bigr),$$
$$V_4 = (r+\delta r) \cdot \exp(i\theta), \quad \theta = \frac{\pi}{2n} - \varepsilon + \omega k,$$

its diameter coincides with $|V_3 - V_1| = |V_4 - V_2|$. Obviously

$$|V_3 - V_1| \leqslant |V_3 - V_2| + \text{arc-length between } V_1 \text{ and } V_2 \leqslant \delta r + \omega r \leqslant 2\delta r,$$

if we take $M \geqslant \frac{2\varepsilon}{\delta}$, say $M = \operatorname{entier}(\frac{2\varepsilon}{\delta}) + 1$.

Let Q be the center of the interval $[V_1, V_3]$. Taking $\delta_2 = 2\delta/\sin(\frac{\pi}{2n} - \varepsilon)$, we cover Π_k with the circle $D(Q, \delta_2)$, which in turn contains in $K(Q, \delta_2)$. Decreasing δ we make δ_2 as small as is needed in Theorem 17. Thus $K(Q, \delta_2)$ contains no more than n cvs from Γ_ε. Assuming $2\varepsilon + \delta < 1$, we conclude that

$$\#(\Gamma \cap \mathcal{D}) \leqslant Mn < \frac{n}{\delta}.$$

Step 2. *Areas' estimates.* Reenumerate cvs in $\mathcal{D}$:

$$\varrho_1, \dots, \varrho_N, \quad N \leqslant N_0 := \operatorname{entier}\left(\frac{n}{\delta}\right) + 1$$

and set $K = \bigcup_{j=1}^N D(\varrho_j, \frac{\delta}{N_0})$. Let us prove a relative area's estimate

$$|K|/|\mathcal{D}| \leqslant C\delta^2/\varepsilon. \tag{5.1}$$

From one hand $|\mathcal{D}| = \varepsilon \cdot (r^2(1+\delta)^2 - r^2) > \varepsilon \cdot 2r^2\delta$. From the other hand

$$|K| \leqslant \sum_{j=1}^N \left| D\left(\varrho_j, \frac{\delta}{N_0}\right)\right| \leqslant N\pi \frac{\delta^2}{N_0^2} \cdot \bigl(r(1+\delta)\bigr)^2 \leqslant \pi\delta^2(1+\delta)^2 r^2/N_0$$

whence (5.1) follows.

Step 3. We have also to take into account the area of the circles $K(\varrho_j, \delta)$, intersecting $\mathcal{D}$, such that ϱ_j lie outside $S_0(\varepsilon)$. Replacing them with greater ones $D(\varrho_j, \delta_1)$, we see that their area attains maximum if they are moved in a parallel way to $\mathbb{R}$ into $S_0(\varepsilon)$ such that new circles are tangent to one of the boundary rays of $S_0(\varepsilon)$. Consider for instance one of the rays, say $Ray_1 = \{\arg \varrho = \frac{\pi}{2n} + \varepsilon\}$. Then respective intersections are inside the angle

$$Ang_1 = \left\{\frac{\pi}{2n} + \varepsilon - \nu \leqslant \arg z \leqslant \frac{\pi}{2n} + \varepsilon\right\}.$$

Let us show that its opening $\nu \leqslant c\delta_1$. Draw a circle $D(\varrho, \delta_1)$ in $S_0(\varepsilon)$, tangent to the ray Ray_1 at some point A and also tangent to the ray

$$Ray_2 = \left\{\arg z = \frac{\pi}{2n} + \varepsilon - \nu\right\}.$$

Obviously the bisector of Ang_1 passes through the circle's center. Radius $|\overrightarrow{\varrho A}|$ is perpendicular to Ray_1. Thus $\delta_1|\operatorname{Im}\varrho| = |\overrightarrow{\varrho A}| = |\varrho| \cdot \sin\nu/2$, whence

$$\sin\nu/2 = \delta_1 \sin\left(\frac{\pi}{2n} + \varepsilon - \nu/2\right).$$

Then the desired estimate of ν follows after elementary calculations.

The intersection with circles $K(\varrho_j, \delta)$, $\varrho_j \notin S_0(\varepsilon)$ near the other boundary ray of $S_0(\varepsilon)$ is estimated along the same lines and is contained within the angle

$$Ang_0 = \left\{\frac{\pi}{2n} - \varepsilon \leqslant \arg z \leqslant \frac{\pi}{2n} - \varepsilon + \nu\right\}.$$

Step 4. Now let us show that *mrg* domain $\mathcal{D} \cap S_0(\varepsilon, \delta)$ is not empty.

Set $Ang = Ang_0 \cup Ang_1$. Note that

$$|\mathcal{D} \cap Ang|/|\mathcal{D}| \leqslant \frac{2\nu}{2\varepsilon} \leqslant \frac{c\delta_1}{\varepsilon}. \tag{5.2}$$

Summing up the rhs of inequalities (5.1)–(5.2), we come to an expression which can be made as small as needed for sufficiently small δ. Hence, $\mathcal{D} \cap S_0(\varepsilon, \delta) \neq \emptyset$.

Step 5. At last, setting $r_m = (1+\delta)^m$, $m = 1, 2, \ldots$, we get a sequence of domains $\mathcal{D}_m$ and thus a sequence of *mrg* points $\tau_m \in \mathcal{D}_m \cap S_\nu(\varepsilon, \delta)$, tending to infinity. It suffices to apply Theorem 8 and Theorem 4 is proved.

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