On the approximation of convex bodies by convex algebraic level surfaces

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Abstract

In this note we consider the problem of the approximation of convex bodies in \( \mathbb{R}^d \) by level surfaces of convex algebraic polynomials. Hammer (1963) [1] verified that any convex body in \( \mathbb{R}^d \) can be approximated by a level surface of a convex algebraic polynomial. In Kroó (2009) [3] a quantitative version of Hammer’s approximation theorem was given by showing that the order of approximation of convex bodies by convex algebraic level surfaces of degree \( n \) is bounded from above by \( c \log n / n \). In this paper we improve further this approximation result by verifying an upper bound of order \( 1 / n \). Moreover, it will be also shown that this bound is sharp, in general.

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Let us consider the problem of the approximation of convex bodies in \( \mathbb{R}^d \) by level surfaces of algebraic polynomials. This problem goes back to Minkowski [4] who first verified that the boundary of any convex body in \( \mathbb{R}^d \) can be approximated arbitrarily well by a level surface of a convex analytic function.

Hammer [1] generalized Minkowski’s result by showing that his approximation result remains valid for approximation by convex algebraic level surfaces. In [3] Hammer’s approximation theorem was quantified by showing that the order of approximation of convex surfaces by convex
algebraic level surfaces of degree \( n \) is bounded from above by \( c \frac{\log n}{n} \) (see Theorem 1 below). It was also conjectured in [3] that the \( \log n \) term in this upper bound cannot be omitted. In this paper we shall show that indeed the boundary of every convex body in \( \mathbb{R}^d \) can be approximated by level surfaces of convex algebraic polynomials of degree at most \( n \) with order \( 1/n \). It will be also shown that in a certain sense this order of approximation is the best possible.

Let us introduce some basic notations used in this paper. We shall denote by \( S^{d-1} \) the unit sphere in \( \mathbb{R}^d \), \( B^d(a, r) := \{ x \in \mathbb{R}^d : |a - x| \leq r \} \) stands for the Euclidian ball centered at \( a \in \mathbb{R}^d \) and radius \( r \). For a compact set \( K \in \mathbb{R}^d \) denote by \( \partial K \) and \( K^0 \) its boundary and interior, respectively. \( d(K) \) stands for the diameter of \( K \). Let \( P_n^d \) be the space of real polynomials of degree \( n \) and total degree less than or equal to \( n \). Given a convex body \( K \in \mathbb{R}^d \) with \( 0 \) in the interior of \( K \) denote

\[
K_a := \{ ax : x \in K \}, \quad a > 0, \quad \varphi_K(x) := \inf \{ a > 0 : x/a \in K \}.
\]

Note that \( \varphi_K(x) \) is the usual Minkowski functional of \( K \) centered at the origin, \( K_a \) is the \( a \)-dilation of \( K \). Also note that \( K = K_1 \).

For any real valued function \( f(x) \) on \( \mathbb{R}^d \) and \( t \in \mathbb{R} \) set \( L_t(f) := \{ x : f(x) \leq t \} \) and let \( \partial L_1(f) = \{ x : f(x) = t \} \) be the \( t \)th level surface of \( f \). For simplicity of notation we denote \( \partial L_1(f) = \partial L(f) \) the usual level surface. In this paper we shall study the rate of approximation of boundaries of convex bodies (i.e., convex surfaces) by level surfaces of convex algebraic polynomials. Let us denote by \( h(A, B) \) the Hausdorff distance between sets \( A \) and \( B \). In case when the sets \( A, B \) are convex and both of them contain \( 0 \) in their interiors we shall also use the following radial distance

\[
\rho(\partial A, \partial B) := \sup \{ |t_1 - t_2| : t_1, t_2 > 0, t_1 x \in \partial A, t_2 x \in \partial B, x \in S^{d-1} \}.
\]

Clearly, \( h(A, B) \leq \rho(\partial A, \partial B) \).

It was proved by Hammer [1] that for any convex body \( K \subset \mathbb{R}^d \), there exist convex polynomials \( p_n \in P_n^d \) such that \( h(\partial K, \partial L(p_n)) \to 0 \) as \( n \to \infty \).

The following theorem proved in [3] gives a quantitative version of the above result of Hammer.

**Theorem 1.** For any \( d \geq 2 \), \( n \in \mathbb{N} \) and any convex body \( K \subset \mathbb{R}^d \) there exist positive polynomials \( p_n \in P_n^d \), convex on \( \mathbb{R}^d \) such that with some absolute constant \( c > 0 \)

\[
h(\partial L(p_n), \partial K) \leq cd(K)d^3 \frac{\log(n + 1)}{n}, \quad n \in \mathbb{N}.
\]

The above theorem gives a domain independent estimate for the rate of approximation by convex algebraic level surfaces but the question of sharpness of this estimate in terms of \( n \) remained open. In this paper we shall prove that in fact an upper bound of order \( 1/n \) holds in the estimation above, i.e., the \( \log n \) factor turns out to be superfluous. On the other hand we shall also show that the \( 1/n \) order of convergence is sharp in the sense that in cannot be better, in general, for nonregular convex bodies. Recall, that a convex body is nonregular if at some point of its boundary it possesses two distinct supporting hyperplanes.

The following statement is the main new result of this paper.

**Theorem 2.** Let \( K \subset \mathbb{R}^d \) be a convex body, and consider any \( a < 1 < b \). Then there exist convex on \( \mathbb{R}^d \) polynomials \( p_n \in P_n^d, n \in \mathbb{N} \) such that
\[
h(\partial L_i(p_n), \partial K_i) = O \left( \frac{1}{n} \right), \quad n \in \mathbb{N}, a \leq t \leq b,
\]
where the constant in (1) is independent of \( t \) and \( n \).

Moreover, if \( K \) is nonregular, then there exists a positive constant \( c = c(K, a, b) \) such that for every polynomial \( p_n \in P_n^d \), convex on \( \mathbb{R}^d \) we have

\[
\sup_{a \leq t \leq b} h(\partial L_i(p_n), \partial K_i) \geq \frac{c}{n}.
\]

The upper bound of Theorem 2 provides the desired \( O(\frac{1}{n}) \) approximation result for simultaneous approximation of convex surfaces \( \partial K_i \) by level surfaces \( \partial L_i(p_n) \) of convex polynomials \( p_n \) for \( a \leq t \leq b \). Moreover, the lower bound of Theorem 2 shows that this simultaneous approximation cannot occur with \( o(\frac{1}{n}) \) if \( K \) is nonregular. Naturally, this raises the question if the \( o(\frac{1}{n}) \) rate of convergence holds in Theorem 2 for every regular convex body. This question remains open at this moment.

The proof of the upper bound in Theorem 2 will be based on a Jackson-type estimate for the approximation of convex functions by multivariate convex polynomials, see Theorem 3 below. The first result in this direction is due to Shvedov [7] who gave an estimation of this type, but in [7] the polynomials were shown to be convex only on some compact domain containing the domain of approximation. However, in order to verify Theorem 2 we need the convexity of polynomials to hold on all of \( \mathbb{R}^d \). This refinement of Shvedov’s result which seems to be new and of independent interest is accomplished in Theorem 3 below.

1. A Jackson-type result in multivariate convex approximation

Let \( D \subset \mathbb{R}^d \) be a compact set. For a function \( f \in C(D) \) denote by

\[
\omega_D(f, h) := \sup \{|f(x) - f(y)| : x, y \in D, |x - y| \leq h\}, \quad h > 0
\]
the standard modulus of continuity of \( f \) on \( D \). By the multivariate version of Jackson theorem (see [9]) for any \( f \in C(D) \) and \( n \in \mathbb{N} \) there exist \( p_n \in P_n^d \) such that

\[
\|f - p_n\|_D \leq c\omega_D(f, 1/n), \quad n \in \mathbb{N},
\]
where \( c > 0 \) depends on \( f \) and \( D \).

Here and in what follows \( \|\cdot\|_D \) stands for the usual supremum norm on \( D \).

Shvedov [7] gave the following extension of the multivariate Jackson theorem for convex functions: given any convex bodies \( D \subset D_1 \subset \mathbb{R}^d \) and a function \( f \in C(D) \), convex on \( D_1 \) there exist polynomials \( p_n \in P_n^d \), convex on \( D_1 \) such that (2) holds with a constant \( c > 0 \) depending only on \( D \). This raises the following natural question: can \( p_n \in P_n^d \) be chosen to be convex on all of \( \mathbb{R}^d \) ? In this respect only a weaker statement was verified in [7]: there exist polynomials \( p_n \in P_n^d \) convex on \( \mathbb{R}^d \) which converge uniformly to \( f \), but no rate of convergence was given. On the other hand in order to verify the upper bound in Theorem 2 we need estimate (2) to hold for polynomials convex on the whole space.

Our next statement refines Shvedov’s result by showing that estimate (2) holds also for some \( p_n \in P_n^d \) convex on all of \( \mathbb{R}^d \).

**Theorem 3.** Let \( D \subset \mathbb{R}^d \) be a convex body and \( f \in C(D) \) be a convex function on \( D \). Then there exist \( p_n \in P_n^d \), \( n \in \mathbb{N} \) convex on \( \mathbb{R}^d \) so that estimate (2) holds.
Proof of Theorem 3. We may assume that $0 \in D^0$ and $B^d(0, r) \subset D \subset B^d(0, R)$ with some $r, R$ depending on $D$. Then by Shvedov’s result cited above there exist $p_n \in P_n^d$, $n \in \mathbb{N}$ convex on $D_1 := B^d(0, 3R^2/r)$ for which (2) holds with a constant $c > 0$ depending on $D$. In particular, (2) implies that

$$\|p_n\|_{B^d(0,r)} \leq \|p_n\|_D \leq c_1$$

(3)

with some $c_1 > 0$ depending only on $D$ and $f$. By a well-known Markov-type inequality proved by Kellogg [2] for any $p_n \in P_n^d$ and $w \in S^{d-1}$ we have

$$\|D_w p_n\|_{B^d(0,r)} \leq \frac{n^2}{r} \|p_n\|_{B^d(0,r)}$$

where $D_w$ stands for the derivative in direction $w$. Using this estimate for the second derivative $D_w^2$ together with (3) yields

$$\|D_w^2 p_n\|_{B^d(0,r)} \leq c_1 \frac{n^4}{r^2}.$$  

(4)

Now we need to recall another classical polynomial inequality due to Chebyshev: for any univariate polynomial $p_n$ of degree at most $n$ and $d > 0$ we have

$$|p_n(x)| \leq \left(\frac{2|x|}{d}\right)^n \|p_n\|_{[-d,d]}, \quad |x| > d.$$  

Applying this result to $D_w^2 p_n$ with $d = r (D_w^2 p_n$ restricted to lines passing through the origin is a univariate polynomial) and taking into account estimate (4), as well, we obtain

$$|D_w^2 p_n(x)| \leq c_2 n^4 \left(\frac{2|x|}{r}\right)^n, \quad x \in \mathbb{R}^d, |x| > r,$$  

(5)

where $c_2 := c_1/r^2$.

Consider now the polynomial

$$g_n(x) := p_n(x) + \frac{|x|^{2(n+1)}}{nR^{2(n+1)}} \in P_{2n+2}^d.$$  

Since estimate (2) holds on $D$ and $D \subset B^d(0, R)$ we have

$$\|g_n - f\|_D \leq \|f - p_n\|_D + \frac{|x|^{2(n+1)}}{nR^{2(n+1)}} \|B(0,R) \leq c_2 \omega(f', 1/n) + 1/n \leq c_3 \omega(f, 1/n).$$  

Clearly, we may assume that $f$ is not the constant function.

Thus the polynomials $g_n$ provide the needed rate of approximation, and it remains to show now that they are convex on $\mathbb{R}^d$. Clearly, the polynomial $\frac{x^{2(n+1)}}{nR^{2(n+1)}}$ is convex on $\mathbb{R}^d$, and in addition, $p_n$ is also convex on $B^d(0, 3R^2/r)$, i.e., $g_n$ is convex on $B^d(0, 3R^2/r)$. Thus it remains to verify that $g_n$ is convex for $|x| \geq \frac{3R^2}{r}$, as well. Then it will follow that these polynomials are convex on all of $\mathbb{R}^d$. It is easy to check that

$$D_w^2 |x|^{2(n+1)} \geq 2(n+1)|x|^{2n}, \quad w \in S^{d-1}.$$  

Thus taking into account estimate (5) we obtain for $|x| \geq \frac{3R^2}{r}$

$$D_w^2 g_n(x) \geq 2(n+1)\frac{|x|^{2n}}{nR^{2(n+1)}} - c_2 n^4 \left(\frac{2|x|}{r}\right)^n \geq \frac{|x|^{2n}}{R^{2n}} \left(\frac{2}{R^2} - c_2 n^4 \left(\frac{2R^2}{r|x|}\right)^n\right).$$
Theorem 3 to the Minkowski functional \( g \). Thus \( g_n \) are convex on \( \mathbb{R}^d \) for sufficiently large \( n \). This completes the proof of Theorem 3.

2. Proof of the upper bound in Theorem 2

We may assume without loss of generality that 0 is in the interior of \( K \). We can apply now Theorem 3 to the Minkowski functional \( \varphi_K(x) \). This functional is homogeneous, i.e., \( \varphi_K(tx) = t \varphi_K(x), t > 0 \), and semi-additive, \( \varphi_K(x + y) \leq \varphi_K(x) + \varphi_K(y), x, y \in \mathbb{R}^d \), see [6]. Thus, in particular, for any \( x, y \in \mathbb{R}^d \)

\[
|\varphi_K(x) - \varphi_K(y)| \leq \varphi_K(x - y) = |x - y| \varphi_K\left(\frac{x - y}{|x - y|}\right) \leq M_K |x - y|,
\]

where \( M_K := \sup_{x \in S^{d-1}} \varphi_K(x) \). Thus the Minkowski functional \( \varphi_K \) satisfies the Lip1 property with constant \( M_K \), on \( \mathbb{R}^d \), i.e., for any compact set \( D \subset \mathbb{R}^d \) we have \( \omega_D(\varphi_K, h) \leq M_K h \).

Applying now Theorem 3 to the convex Lip1 function \( \varphi_K \) with \( D := K_{2b} \), \( b > 1 \) yields that there exist polynomials \( p_n \in P^d_n \), convex on \( \mathbb{R}^d \) such that

\[
\| \varphi_K - p_n \|_{K_{2b}} \leq \frac{c_K}{n} \tag{6}
\]

with some \( c_K > 0 \) depending only on \( K \). Since \( \varphi_K(0) = 0 \) it follows that \( 0 \in L^0_t \) for any \( a \leq t \) and \( n > c_K/a \).

Consider now an arbitrary \( x \in \partial K_t \), i.e., \( \varphi_K(x) = t \), where \( a \leq t \leq b \). Set \( \delta_t := c_K/(tn) \). Then clearly \( x(1 \pm \delta_t) \in K_{2b} \) for \( n \geq n_0(K, a, b) \). Thus by (6)

\[
|\varphi_K(x(1 \pm \delta_t)) - p_n(x(1 \pm \delta_t))| \leq \frac{c_K}{n}.
\]

Therefore

\[
p_n(x(1 + \delta_t)) \geq \varphi_K(x(1 + \delta_t)) - \frac{c_K}{n} = t(1 + \delta_t) - \frac{c_K}{n} = t.
\]

Similarly, \( p_n(x(1 - \delta_t)) \leq t \). Thus for some \( s \in (1 - \delta_t, 1 + \delta_t) \) we have \( p_n(sx) = t \), i.e., \( sx \in \partial L_t(p_n) \), where we evidently have \( |sx - x| \leq |x| |s - 1| \leq |x| \delta_t \leq c(a, b, K)/n \). Since \( x \in \partial K_t \) was chosen arbitrarily and both \( K_t \) and \( L_t(p_n) \) are convex sets containing 0 in their interior this yields the upper bound (1).

3. Proof of the lower bound in Theorem 2

Clearly, it suffices to prove the lower bound for \( d = 2 \), since a lower bound in the uniform norm for a two-dimensional cross-section of \( K \) yields the same lower bound on all of \( K \). Thus without loss of generality we may assume that \( K \subset \mathbb{R}^2 \) with 0 in \( K^0 \) and the point \( y := (0, 1) \in \partial K \) is a nonregular point of the boundary of convex body \( K \).

Note that whenever \( h(\partial L_t(p_n), \partial K_t) < h_0(a, b, K) \) it follows that \( 0 \in L^0_t(p_n), 0 < a \leq t \leq b \). Moreover we clearly also have for the convex sets \( L_t(p_n), K_t \)

\[
\varphi(\partial L_t(p_n), \partial K_t) \leq c(a, b, K) h(\partial L_t(p_n), \partial K_t), \quad 0 < a \leq t \leq b.
\]
Let us show that there exist \( c_1, c_2 > 0 \) such that for every \( p_n \in P_n^2 \) convex on \( \mathbb{R}^2 \) we have in view of the above inequality

\[
|p_n(x) - \varphi_K(x)| \leq c_1 \sup_{a \leq t \leq b} \rho(\partial L_t(p_n), \partial K_t) \leq c_0 \sup_{a \leq t \leq b} h(\partial L_t(p_n), \partial K_t), \quad x \in B^2(y, c_2). \tag{7}
\]

Here and in what follows \( c_j \) denote positive constants depending only on \( K, a, b \).

Set

\[
\epsilon := \rho(\partial L_a(p_n), \partial K_a), \quad M_K := \sup_{x \in S^{d-1}} \varphi_K(x).
\]

First we claim that \( L_a(p_n) \subset K_{a+2M_K} \). Indeed, if this was not the case we would have for some \( x \in \partial K_{a+2M_K} \) and \( t > 1 \) that \( tx \in \partial L_a(p_n) \). Clearly, \( t_1 x \in \partial K_a \), where \( t_1 := \frac{a}{a+2M_K} \). Then using the fact that \( \varphi_K(x) = a + 2M_K \) for \( x \in \partial K_{a+2M_K} \) we have

\[
\epsilon \geq |t x - t_1 x| \geq \frac{\varphi_K(x)}{M_K} (t - t_1) \geq \frac{\varphi_K(x)}{M_K} (1 - t_1) = 2\epsilon,
\]
a contradiction. Hence \( L_a(p_n) \subset K_{a+2M_K} \). Similarly, \( K_{b-2M_K} \subset L_b(p_n) \). Assume now that

\[
\epsilon < \frac{1}{4M_K} \min\{1 - a, b - 1\}.
\]

Since our goal is to show that \( \sup_{a \leq t \leq b} h(\partial L_t(p_n), \partial K_t) \geq c/n \) we can make the above assumption without loss of generality. Then

\[
L_a(p_n) \subset K_{a+2M_K} \subset K_{(a+1)/2} \subset K_{(b+1)/2} \subset K_{b-2M_K} \subset L_b(p_n).
\]

Hence \( B^2(y, c_2) \subset L_c(p_n) \) and \( B^2(y, c_2) \subset L_b(p_n) \) for some \( c_2 \) depending only on \( K, a, b \). (Here \( D^c \) denotes the complement of the set \( D^c \).) Thus for any \( x \in B^2(y, c_2) \) we have that \( p_n(x) > a, p_n(x) \leq b \) which means that \( x \in \partial L_t(p_n) \) with some \( a \leq t \leq b \). Moreover, there exists a \( \xi > 0 \) such that \( \xi x \in \partial K_t \) yielding that

\[
|x||1 - \xi| \leq \rho(\partial L_t(p_n), \partial K_t). \tag{8}
\]

In addition, by \( \xi x \in \partial K_t \) we have \( \varphi_K(\xi x) = t = p_n(x) \). Hence using that the functional \( \varphi_K \) is \( \text{Lip}1 \) with constant \( M_K \) we have by (8) that for any \( x \in B^2(y, c_2) \) there exists \( a \leq t \leq b \) such that

\[
|p_n(x) - \varphi_K(x)| = |\varphi_K(\xi x) - \varphi_K(x)| \leq M_K |x||1 - \xi| \leq M_K \rho(\partial L_t(p_n), \partial K_t).
\]

This completes the proof of estimate (7).

We shall also need the following lemma providing a Stechkin-type estimate for the derivatives of univariate polynomials.

**Lemma 1.** Let \( f \) be a \( \text{Lip}1 \) function on \( I := [-1, 1] \) and assume that \( p_n \in P_n^1 \) satisfy

\[
\|f - p_n\|_I \leq \frac{1}{n}, \quad n \in \mathbb{N}. \tag{9}
\]

Then for any \( \delta > 0 \) we have

\[
\|p_n'\|_{[-1+\delta, 1-\delta]} \leq c(\delta), \quad \|p_n''\|_{[-1+\delta, 1-\delta]} \leq c(\delta)n, \quad n \in \mathbb{N}, \tag{10}
\]

where \( c(\delta) > 0 \) depends only on \( \delta \).
**Proof.** By the well-known Stechkin inequality (see [8], or [9], p. 267) and (9)
\[
\|p_n\|_{[-1+\delta/2,1-\delta/2]} \leq c(\delta)n\omega_1 \left(p_n, \frac{1}{n}\right) \leq c(\delta)n\omega_1 \left(f - p_n, \frac{1}{n}\right) + c(\delta)n\omega_1 \left(f, \frac{1}{n}\right)
\]
\[
\leq 2c(\delta)n\|f - p_n\|_I + c(\delta) \leq 3c(\delta).
\]
Using now the classical Bernstein inequality we have
\[
\|p_n''\|_{[-1+\delta/2,1-\delta/2]} \leq c(\delta)n\|p_n'\|_{[-1+\delta/2,1-\delta/2]} \leq 3c(\delta)n.
\]
The last two estimates complete the proof of the lemma. □

For an arbitrary \(p_n \in P^2_n\) set
\[
\epsilon(p_n) := \sup_{a \leq t \leq b} \rho(\partial L_t(p_n), \partial K_t).
\]
(11)

In order to verify the lower bound of Theorem 2 in view of (7) it suffices to show that whenever \(p_n \in P^2_n\) is convex on \(\mathbb{R}^2\) we have \(\epsilon(p_n) \geq c/n\) with some positive constant depending only on \(K, a, b\). Therefore without loss of generality we may assume that \(\epsilon(p_n) < 1/n\).

Then by (7) for any \(x \in B^2(y, c_2)\) we have
\[
|p_n(x) - \varphi_K(x)| \leq c_1 \epsilon(p_n) \leq \frac{c_1}{n}.
\]
(12)

Applying estimate (10) of Lemma 1 for the Lip1 function \(\varphi_K\) and \(p_n\) satisfying (12) we have for the smaller disc \(B_\eta := B^2(y, c_2\eta), 0 < \eta < 1\)
\[
|p_n|, \left|\frac{\partial p_n}{\partial x}\right|, \left|\frac{\partial p_n}{\partial y}\right| \leq c_3, \quad \left|\frac{\partial^2 p_n}{\partial x \partial y}\right|, \left|\frac{\partial^2 p_n}{\partial x^2}\right|, \left|\frac{\partial^2 p_n}{\partial y^2}\right| \leq c_3 n, \quad x \in B^*.
\]
(13)

Here and in what follows we denote by \(c_j\) positive constants depending only on \(a, b, K\).

Furthermore, we may assume that the disc \(B_\eta\) is sufficiently small so that in this disc the boundary of \(K\) is given by some positive convex univariate function \(y = f(x), f(0) = 1,\) and \(\partial L_1(p_n) = \partial L(p_n)\) is given by some positive convex function \(y = g(x), |x| \leq x_0\). Hence, in particular,
\[
p_n(x, g(x)) \equiv 1, \quad |x| \leq x_0.
\]
(14)

Moreover, since \(y := (0, 1) \in \partial K\) is a nonregular point of the boundary we have that
\[
f_+(0) := A > B := f'_-(0),
\]
(15)

where \(f^+_L, f^-_L\) are the right and left derivatives of \(f\), respectively.

Choose an arbitrary \(0 < \delta < 1\) such that the function \(y = f(x) - \delta\) intersects the interior of disc \(B_\eta\). Consider any \((x, y) \in B_\eta\) with \(y < f(x) - \delta\). Clearly \((x, y) \in \partial K_t\) with some \(0 < t < t_0 < 1\) where \(t_0\) depends only on \(K\) and \(\delta\). In addition, \(\delta\) can be chosen sufficiently small (and depending only on \(a\)) so that \(a \leq t < t_0\), i.e., \((x, y) \in \partial K_t\) with some \(a \leq t < t_0\). Thus by (11) and (12) there exists \(t_1 > 0\) such that \(t_1(x, y) \in \partial L_t(p_n)\) i.e., \(p_n(t_1 x, t_1 y) = t\) and \(|1 - t_1| \leq c_4 \epsilon(p_n) < c_4/n\). Using that by (13) \(p_n\) has bounded derivatives in \(B_\eta\) we obtain
\[
|p_n(x, y) - t| = |p_n(x, y) - p_n(t_1 x, t_1 y)| \leq c_5/n.
\]
(16)

Moreover, by the convexity of \(p_n\)
\[
p_n(x, y) \geq \frac{\partial p_n}{\partial y}(x, g(x))(y - g(x)) + p_n(x, g(x)),
\]
where the expression on the right-hand side is the equation of the tangent line to the graph of \( z = p_n(x, y) \) at \((x, g(x))\) in the direction \( y \). Hence and by (16) and (14) using that \(|y - g(x)| \leq c_6\) for any \(|x| \leq x_0\)

\[
c_6 \left| \frac{\partial p_n}{\partial y}(x, g(x)) \right| \geq \left| \frac{\partial p_n}{\partial y}(x, g(x))(y - g(x)) \right| \geq p_n(x, g(x)) - p_n(x, y) = 1 - p_n(x, y) \geq 1 - t - c_5/n > 1 - t_0 - c_5/n.
\]

Thus for any \(|x| \leq x_0, n \geq n_0\)

\[
\left| \frac{\partial p_n}{\partial y}(x, g(x)) \right| \geq c_7 > 0. \tag{17}
\]

Differentiating relation (14) with respect to \( x \) yields

\[
\frac{\partial p_n}{\partial x}(x, g(x)) + \frac{\partial p_n}{\partial y}(x, g(x))g'(x) = 0. \tag{18}
\]

Applying (13) and (17) in (18) yields that

\[
|g'(x)| \leq c_8, \quad |x| \leq x_0. \tag{19}
\]

Furthermore, second differentiation of (18) implies

\[
2 \frac{\partial^2 p_n}{\partial x \partial y}g' + \frac{\partial^2 p_n}{\partial x^2} + \frac{\partial^2 p_n}{\partial y^2}(g')^2 + \frac{\partial p_n}{\partial y}g'' = 0.
\]

Using in the last relation estimates (13), (19) and (17) we arrive at

\[
|g''(x)| \leq c_9 n, \quad |x| \leq x_0. \tag{20}
\]

For the convex differentiable function \( g \) we have (note that in view of (17) and the Implicit Function theorem \( g \) is differentiable for \(|x| \leq x_0\))

\[
g'(x) \geq \frac{g(x) - g(0)}{x}, \quad g'(-x) \leq \frac{g(-x) - g(0)}{-x}, \quad 0 < x \leq x_0. \tag{21}
\]

Similarly, for the convex function \( f \)

\[
A = f'_+(0) \leq \frac{f(x) - f(0)}{x} \quad B = f'_-(0) \geq \frac{f(0) - f(-x)}{x}, \quad 0 < x \leq x_0. \tag{22}
\]

Recall that \( f \) and \( g \) are functions representing \( \partial K \) and \( \partial L(p_n) \), respectively. Therefore we clearly obtain by (11) that

\[
|f(x) - g(x)| \leq c_{10} \epsilon(p_n), \quad |x| \leq x_0. \tag{23}
\]

Hence using (21)–(23) yields

\[
g'(x) - g'(-x) \geq \frac{g(x) - g(0)}{x} - \frac{g(-x) - g(0)}{-x} = \frac{f(x) - f(0)}{x} - \frac{f(-x) - f(0)}{-x} - 4c_{10} \frac{\epsilon(p_n)}{x} \geq A - B - 4c_{10} \frac{\epsilon(p_n)}{x}. \tag{24}
\]

On the other hand by (20)

\[
|g'(-x) - g'(x)| \leq 2c_9 n |x|. \tag{25}
\]
Combining (24) and (25) and setting $x := \frac{A-B}{4c_9n}$ we arrive at
\[
\frac{A-B}{2} = 2c_9nx \geq A - B - 4c_{10} \frac{\epsilon(p_n)}{x} = A - B - c_{11}n\epsilon(p_n).
\]
Obviously, the last estimate implies that
\[
\epsilon(p_n) \geq \frac{A - B}{2c_{11}n}
\]
which is the desired lower bound of Theorem 2.

The lower bound in Theorem 2 shows that when the convex body is not regular, in general, we cannot approximate it by convex $n$th degree algebraic level surfaces faster than $1/n$. The situation is quite different if we approximate by algebraic level surfaces, which are not necessarily convex. In fact, when approximating convex bodies by arbitrary algebraic level surfaces (not necessarily convex), even exponential rate of convergence can occur! This phenomena is exhibited in the next example. □

**Remark.** Let $d = 2$ and consider the convex nonregular set $K := \{(x, y) \in \mathbb{R}^2 : |x| + y^2 \leq 1\}$. We shall construct polynomials $p_n \in P_n^2$ such that
\[
h(\partial L(p_n), \partial K) \leq e^{-c\sqrt{n}}, \quad n \in \mathbb{N}
\]
with some absolute constant $c > 0$. We shall need a famous result due to D. Newman according to which there exist rational functions $r_n^* = \frac{g_n^*}{q_n^*}$ with $q_n^*, g_n^* \in P_n^1$ being even polynomials such that for every $n \in \mathbb{N}$
\[
|x - r_n(x)| \leq e^{-c\sqrt{n}}, \quad x \in [-1, 1]
\]
with some absolute constant $c > 0$ (see [5]).

Using the substitution $x = 2t/(1+t^2)$ we can easily obtain from above estimate the following weighted version which holds on the whole real line: for some $r_n$
\[
|x - r_n(x)| \leq (1 + x^2)e^{-c\sqrt{n}}, \quad x \in \mathbb{R}^1. \quad (26)
\]
In particular, we have by (26)
\[
|x - r_n(x)| \leq 10e^{-c\sqrt{n}} := \delta_n, \quad |x| \leq 3. \quad (27)
\]
Set now
\[
r_1(x) := r_n(x) + (x/2)^{2n} + \delta_n := \frac{g^*}{q^*}, \quad (28)
\]
where $g^*, q^* \in P_{3n}^1$ are even polynomials. Clearly, we have by (26) and (28)
\[
|x - r_1(x)| \leq 2e^{-c\sqrt{n}} + \delta_n + (1/2)^{2n} \leq e^{-c_1\sqrt{n}} := \delta_n^*, \quad |x| \leq 1. \quad (29)
\]
Moreover (28) and (27) yield that for $1 < |x| \leq 3$
\[
r_1(x) > r_n(x) + \delta_n = |x| + (r_n(x) - |x|) + \delta_n \geq |x| > 1.
\]
In addition, when $|x| > 3$ and $n > 3$ (28) and (26) imply that
\[
r_1(x) \geq (x/2)^{2n} + r_n(x) - |x| + |x| \geq (x/2)^{2n} - (1 + x^2) > 1.
\]
Thus combining the last two estimates we have that $r_1(x) > 1$ for any $|x| > 1, n > 3$. 

Set now
\[ p_n(x, y) := y^2 q^*(x) - q^*(x) + g^*(x) + 1 \in P_{3n+2}^2. \]

Clearly, \( \partial L(p_n) = \{(x, y) \in \mathbb{R}^2 : y^2 = 1 - r_1(x)\}. \) Moreover, since \( r_1(x) > 1 \) for \( |x| > 1, n > 3 \) it follows that
\[ \partial L(p_n) \subset \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}. \] (30)

In addition, by (29) if \( |x| \leq 1 - \delta_n^* \) then
\[ r_1(x) \leq |x| + \delta_n^* \leq 1. \] (31)

By symmetry it suffices to consider now \( x, y > 0. \) By (31) \( a := (x, \sqrt{1 - r_1(x)}) \in \partial L(p_n) \) if \( 0 \leq x \leq 1 - \delta_n^* \), \( b := (x, \sqrt{1 - x}) \in \partial K. \) Thus by (29)
\[ |a - b| \leq |\sqrt{1 - r_1(x)} - \sqrt{1 - x}| \leq \frac{|r_1(x) - x|}{\sqrt{1 - x}} \leq \sqrt{\delta_n^*}. \]

In view of (30) it remains now to consider such \( 1 \geq x \geq 1 - \delta_n^* \) for which \( c := (x, y) \in \partial L(p_n) \) with some proper \( y > 0. \) Clearly, we must have \( r_1(x) \leq 1 \) and hence by (29)
\[ y = \sqrt{1 - r_1(x)} = \sqrt{1 - x} + (x - r_1(x)) \leq \sqrt{2\delta_n^*}. \]

Thus the point \( c \) is at most distance \( \sqrt{3}\delta_n^* \) from the point \( (1, 0) \in \partial K. \) This leads to the desired estimate
\[ h(\partial L(p_n), \partial K) \leq 3\sqrt{\delta_n^*} \leq e^{-c\sqrt{n}}. \]

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References