On Reductions of Maximin Machines*

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1. INTRODUCTION

The reduction problem of maximin sequential machines and maximin automata was investigated by the author in [4, 5]. However, the solutions provided there were far from being complete. In the present paper, the same problem is reexamined. Although this time maximin sequential-like machine [6] is used, most of the results are equally applicable to the other two variations of maximin machine.

Three types of equivalence relations are introduced. They are: statewise, compositewise, and distributionwise equivalence. It is shown that the last two concepts coincide. From the first two equivalence relations, two minimal forms are defined. The counterparts of these two minimal forms in the theory of stochastic sequential machines are the reduced [2] and minimal state forms [1].

For stochastic sequential machines, (convex) linear algebra serves as a handy tool. Unfortunately, no similar tools are available for maximin machines in the existing literature. For this reason, the next section is devoted to the development of a new type of algebra, the maximin algebra. It turns out that maximin algebra is a very useful tool for dealing with maximin machines. Although maximin algebra resembles linear algebra and max-product algebra [7] in certain respects, they are almost completely unrelated.

With the aid of maximin algebra, full and effective solutions of the reduction problem are obtained for the two minimal forms considered. Despite the complexity of maximin algebra (as compared to max-product algebra and linear algebra), most of the final results are expressible in rather simple forms. Moreover, almost all of the results have counterparts in the theory of stochastic sequential machines and max-product machines. These counterparts are either known to be true [1–3, 7] or can be shown to be true.

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The paper concludes with a section dealing with nondeterministic and deterministic machines which are special cases of maximin machines. Many of the results for maximin machines are strengthened for these particular cases. In addition, several other interesting results are also obtained.

2. **Maximin Algebra**

A useful tool for dealing with maximin machines is developed in this section. For certain obvious reasons, it is called maximin algebra. Although most of the basic concepts involved in maximin algebra resemble that of linear algebra, their properties differ greatly in many respects.

The role played by maximin algebra in the theory of maximin machines is the same as that played by linear algebra in the theory of stochastic machines [3] and that played by max-product algebra in max-product machines [7]. Like max-product algebra, maximin algebra is almost a complete stranger in the mathematical world. This unfortunate situation is quite a handicap to the study of maximin machines.

It is apparent that the theory developed below is applicable to any mathematical system with the appropriate mathematical structures. The abstraction and study of further properties of maximin algebra may be the topics of another paper. In what follows, we confine ourself only to those concepts and properties of maximin algebra, which are needed in later discussions.

**Definition.** Let $A_1 = (a_{ij})$ and $A_2 = (a_{ij})$ be, respectively, $n \times p$ and $p \times m$ matrices of real numbers. By $A_1 \otimes A_2$ we mean the $n \times m$ matrix $A = (c_{ij})$, where

$$c_{ij} = \max_k \min (a_{ik}, b_{kj}).$$

**Remark.** The above definition is applicable even if $n$ or $m$ is infinite.

**Proposition 2.1.** The operation $\otimes$ is associative, i.e.,

$$(A_1 \otimes A_2) \otimes A_3 = A_1 \otimes (A_2 \otimes A_3).$$

**Notation 1.** $\alpha, \beta,$ and $\gamma$ (with or without subscripts) denote real numbers.

2. $x$ and $y$ (with or without subscripts) denote (finite or infinite) sequences of real numbers. A superscript is used to denote the particular term of the sequence, e.g., $x^k$ denotes the $k$-th term of the sequence $x$.

3. $X$ and $Y$ (with or without subscripts) denote collections of (finite or infinite) sequences of real numbers.
Notation 1. \( \min(x, x) \) denotes the sequence whose \( k \)-th term is \( \min(\alpha, x^k) \).

2. \( \max(x_1, x_2, \ldots, x_n) \) or \( \max_{1 \leq i \leq n} x_i \) denotes the sequence whose \( k \)-th term is \( \max(x_1^k, x_2^k, \ldots, x_n^k) \).

Definition. A maximin combination of \( X = \{x_1, x_2, \ldots, x_n\} \) is an expression of the form

\[
\max_{1 \leq i \leq n} \min(\alpha_i, x_i).
\]  

If \( 0 \leq \alpha_i \leq 1 \) for \( i = 1, 2, \ldots, n \), then (2.1) is a convex maximin combination of \( X \).

Definition. \( x \) is admissible iff the set \( T_x \) of all distinct terms of \( x \) is finite and \( T_x \) can be effectively constructed from \( x \). \( X \) is admissible iff every \( x \) in \( X \) is admissible.

Proposition 2.2. Let

\[
x = \max_{1 \leq i \leq n} \min(\alpha_i, x_i).
\]

If \( T_x \) is finite, then

\[
x = \max_{1 \leq i \leq n} \min(\beta_i, x_i),
\]

where for \( i = 1, 2, \ldots, n \), \( \beta_i \in T_x \).

Proof. Choose \( \beta_i \) to be the largest number in \( T_x \) such that \( \beta_i \leq \alpha_i \).

Proposition 2.3. It is decidable whether or not \( x \) is a (convex) maximin combination of \( X \) provided that \( T_x \) is admissible and \( X \) is finite.

Proof. Follows from Proposition 2.2.

Definition. The (convex) maximin span of \( X \) is the collection of all (convex) maximin combinations of finite subsets of \( X \).

Notation. \( C(X) \) will denote the convex maximin span of \( X \).

Proposition 2.4. (i) \( X \subseteq C(X) \);

(ii) \( C(C(X)) = C(X) \);

(iii) If \( X_1 \subseteq X_2 \), then \( C(X_1) \subseteq C(X_2) \).

Definition. \( Y \) is a convex maximin set iff for every \( y_1, y_2 \in Y \), all convex maximin combinations of \( \{y_1, y_2\} \) are also in \( Y \).
Proposition 2.5. \( Y \) is a convex maximin set iff \( Y = C(Y) \).

Proposition 2.6. For every \( X \), \( C(X) \) is a convex maximin set.

Proof. Follows from Propositions 2.5 and 2.4(ii).

Notation. In the rest of this section, \( Y \) always denotes a convex maximin set.

Definition. Let \( X \subseteq Y \). \( X \) is a set of generators of \( Y \) iff \( Y = C(X) \). If \( X \) does not contain any proper subset which is itself a set of generators of \( Y \), then \( X \) is a set of vertices of \( Y \).

Proposition 2.7. \( X \) is a set of generators of \( C(X) \).

Proposition 2.8. Let \( X \subseteq Y \). \( X \) is a set of vertices of \( Y \) iff (i) \( Y = C(X) \) and (ii) if \( x \in X \), then \( x \notin C(X - \{x\}) \).

Proposition 2.9. If \( X \) is a finite set of generators of \( Y \), then there exists \( X' \subseteq X \) such that \( X' \) is a set of vertices of \( Y \). Moreover, \( X' \) can be effectively constructed provided \( X \) is admissible.

Proof. Follows from Propositions 2.8, 2.3, and the fact that \( X \) is finite.

Definition. \( Y \) is finitary iff it possesses a set of generators which is finite.

Proposition 2.10. Every finitary convex maximin set has at least one set of vertices.

Proof. Follows from Proposition 2.9.

Remark. Proposition 2.10 is not true in general for arbitrary convex maximin set.

Proposition 2.11. If \( Y \) is finitary, then every set of generators of \( Y \) contains a finite subset which is a set of vertices of \( Y \).

Proof. By Proposition 2.10, there exists a set \( X \) of vertices of \( Y \) which is finite. Let \( X' \) be a set of generators of \( Y \). For every \( x \in X \), \( x \) is a convex maximin combination of a finite subset of \( X' \). Let \( X'' \) be the set of all elements of \( X' \) which appear in one or more of these convex maximin combinations. Clearly, \( X'' \) is finite and is itself a set of generators of \( Y \). The conclusion follows from Proposition 2.9.
**Proposition 2.12.** If \( Y \) is finitary, then every set of vertices of \( Y \) is finite.

**Proof.** Follows from Proposition 2.11.

**Definition.** Let \( X \) be a set of vertices of \( Y \). \( X \) is fundamental iff for every \( x \in X \), there exist no \( y \in Y \) such that \( x \neq y = \min(\alpha, x) \) for some \( 0 \leq \alpha \leq 1 \) and \((X - \{x\}) \cup \{y\}\) is a set of vertices of \( Y \).

**Notation 1.** \( x \leq \alpha \) means \( x_k \leq \alpha \) for all \( k \).

2. \( x_1 \leq x_2 \) means \( x_1^k \leq x_2^k \) for all \( k \).

3. \( x = \alpha \) means \( x_k = \alpha \) for all \( k \).

**Proposition 2.13.** Let \( X \) be a set of vertices of \( Y \). \( X \) is fundamental iff for every \( x \in X \),

\[
x = \max\{\min(\alpha, x), y\}
\]

where \( y \in C(X \setminus \{x\}) \) and \( 0 \leq \alpha \leq 1 \), implies \( x = \min(\alpha, x) \).

**Proof.** If \( x = \max\{\min(\alpha, x), y\} \) where \( y \in C(X \setminus \{x\}) \) and \( 0 \leq \alpha \leq 1 \) but \( x \neq \min(\alpha, x) \), then \((X - \{x\}) \cup \{y\}\) is a set of vertices of \( Y \) where \( y' = \min(\alpha, x) \). Thus \( X \) is not fundamental. Conversely, if \( X \) is not fundamental, then there exists \( x \in X \), \( y' \in Y \) such that \( x \neq y' = \min(\alpha, x) \) for some \( 0 \leq \alpha \leq 1 \) and \((X - \{x\}) \cup \{y'\}\) is a set of vertices of \( Y \). Thus \( x \) is a convex maximin combination of a finite subset \( X' \) of \((X - \{x\}) \cup \{y'\}\). By Proposition 2.8, \( x \in C(X') \), i.e., \( x = \max\{\min(\beta, y'), y\} \) where \( y \in X' \setminus \{y'\} \) and \( 0 \leq \beta \leq 1 \). Let \( \gamma = \min(\alpha, \beta) \), then \( x = \max\{\min(\gamma, x), y\} \). Since

\[
\min(\gamma, x) = \min(\beta, y') \leq y' = \min(\alpha, x) \leq x, \quad x \neq \min(\gamma, x).
\]

**Proposition 2.14.** If \( X \) is a fundamental set of vertices of \( C(X) \) and \( x \in X \), then \( X - \{x\} \) is a fundamental set of vertices of \( C(X \setminus \{x\}) \).

**Proof.** Follows from Propositions 2.8 and 2.13.

**Proposition 2.15.** Let \( X' \) and \( X'' \) be fundamental sets of vertices of \( Y \). If \( \bar{x} \in X' \cap X'' \), then \( C(X' \setminus \{\bar{x}\}) = C(X'' \setminus \{\bar{x}\}) \).

**Proof.** Let \( x \in X' \setminus \{\bar{x}\} \). Then

\[
x = \max_{0 \leq i \leq n} \min(\alpha_i, x_i^*),
\]

(2.2)

where for \( i = 0, 1, 2, \ldots, n \), \( x_i^* \in X'' \) and \( 0 \leq \alpha_i \leq 1 \). If \( \bar{x} \) does not occur among the \( x_i^* \), then \( x \in C(X \setminus \{\bar{x}\}) \). Therefore, let \( x_0^* = \bar{x} \). For \( i = 1, 2, \ldots, n \),

\[
x_i^* = \max_{0 \leq j \leq m} \min(\beta_{ij}, x_j^*),
\]

(2.3)
where for \( j = 0, 1, 2, \ldots, n \), \( x_j' \in X'' \) and \( 0 \leq \beta_{ij} \leq 1 \). Thus
\[
x = \max \{ \min (\alpha_0, x_j'), \max_{0 \leq j \leq m} \min (\gamma_j, x_j') \}
\]
where
\[
\gamma_j = \max_{1 \leq i \leq n} \min (\alpha_i, \beta_{ij}).
\]
Since \( X' \) is a set of vertices of \( Y \), \( x \) must appear among the \( x_j' \), say \( x_0' \). Moreover, since \( X' \) is fundamental, by Proposition 2.13,
\[
x = \min (\gamma_0, x).
\]
Therefore,
\[
x \leq \gamma_0 = \max_{1 \leq i \leq n} \min (\alpha_i, \beta_{i0}) = \min (\alpha_k, \beta_{k0})
\]
for some \( 1 \leq k \leq n \). Thus \( x \leq \alpha_k \) and \( x \leq \beta_k \). By virtue of (2.2) and (2.3)
\[
x \geq \min (\alpha_k, x_k) \geq \min (\alpha_k, \beta_{k0}, x_0') = x.
\]
Thus \( x = \min (\alpha_k, x_k) \) or \( x \in C(X'' - \{ x \}) \). Hence
\[
C(X' - \{ x \}) \subseteq C(X'' - \{ x \}).
\]
Interchanging the role of \( X' \) and \( X'' \) in the above argument yields
\[
C(X'' - \{ x \}) \subseteq C(X' - \{ x \}).
\]
Therefore,
\[
C(X' - \{ x \}) = C(X'' - \{ x \}).
\]

**Theorem 2.16.** The fundamental set of vertices of \( Y \) is unique provided \( Y \) is finitary.

**Proof.** Let \( X' \) and \( X'' \) be fundamental sets of vertices of \( Y \). By Proposition 2.12, both \( X' \) and \( X'' \) are finite. Let
\[
X' = \{ x_1', x_2', \ldots, x_n' \} \quad \text{and} \quad X'' = \{ x_1'', x_2'', \ldots, x_m'' \}
\]
where \( n \leq m \). The Theorem will be proved by induction on \( m \). If \( m = 1 \), it is trivial. Suppose the Theorem is true for all \( m < p \), we shall show that it is also true for \( m = p \). Let
\[
x_i' = \max_{1 \leq j \leq m} \min (\alpha_{ij}, x_j'),
\]
and
\[
x_j'' = \max_{1 \leq k \leq n} \min (\beta_{jk}, x_k').
\]
Then
\[ x'_i = \max_{1 \leq k \leq n} \min(y_{ik}, x'_k), \]
where
\[ y_{ik} = \max_{1 \leq j \leq m} \min(\alpha_{ij}, \beta_{jk}). \]

Since \( X' \) is fundamental, by Proposition 2.13,
\[ x'_1 = \min(y_{11}, x'_1). \]
Thus
\[ x'_1 \leq \gamma_{11} = \max_{1 \leq j \leq m} \min(\alpha_{1j}, \beta_{j1}) = \min(\alpha_{1j}, \beta_{j1}) \]
for some \( j_1 \). This implies \( x'_1 \leq \alpha_{j_1} \) and \( x'_1 \leq \beta_{j_1} \). Therefore
\[ x''_{j_1} \geq \min(\beta_{j_1}, x'_1) = x'_1. \]

By a similar argument, there exists \( j_2 \) such that \( x''_{j_2} \geq x''_{j_1} \). A sequence \( j_1, j_2, \ldots, j_n \) is obtained by repeating the same argument where
\[ x''_{j_{2k-1}} \leq x''_{j_{2k}} \leq x''_{j_{2k+1}} \leq x''_{j_{2k+2}}. \]

Since \( X'' \) is finite,
\[ x''_{j_{2k-1}} = x''_{j_{2l-1}}, \]
for some \( k \) and \( l \) where \( l > k \). Thus
\[ x''_{j_{2k-1}} = x''_{j_{2k}} = \bar{x}. \]

Consider \( X' - \{\bar{x}\} \) and \( X'' - \{\bar{x}\} \). By Propositions 2.14 and 2.15 and induction hypothesis, \( X' - \{\bar{x}\} = X'' - \{\bar{x}\} \). Thus \( X' = X''. \)

**Notation.** Let \( S \) be an arbitrary set, \(| S | \) will denote the cardinality of \( S \).

**Proposition 2.17.** Let \( Y \) be admissible and \( X = \{x_1, x_2, \ldots, x_n\} \) a set of vertices of \( Y \). There exists a fundamental set \( X' \) of vertices of \( Y \) such that \(| X' | = | X | \). Moreover, \( X' \) can be effectively constructed from \( X \).

**Proof.** We shall define \( x'_i, i = 1, 2, \ldots, n \), inductively as follows: Let
\[ A_1 = \{x : x_1 = \max_{i>1} \{\min(\alpha_i, x_i), \max_{i>1} \min(\alpha_i, x_i)\} \text{ for some } 0 \leq \alpha_i \leq 1, i \neq 1\}, \]
and \( \alpha_i' = \text{g.l.b. } A_i \) where g.l.b. stands for greatest lower bound. Define 
\[ x_i' = \min(\alpha_i', x_i) \]. Suppose \( x_{k-1}' \) has been defined. Let

\[
A_k = \{ x_k = \max[\min(x_i, x_k), \max\min(x_i, x_i'), \max\min(x_i, x_i)] \}
\]
for some \( 0 \leq \alpha_k \leq 1, i \neq k \),

\[ x_k' = \text{g.l.b. } A_k \] and \( x_k' = \min(\alpha_k', x_k) \). Let \( X' = \{ x_1', x_2', \ldots, x_n' \} \). Clearly, \( | X' | = | X | \). By Proposition 2.3, \( X' \) can be effectively constructed from \( X \). Moreover, \( \alpha_k' \in A_k \) for all \( k \). Thus \( X' \) is a set of vertices of \( Y \). It remains to show that \( X' \) is fundamental. Let

\[ x_k' = \max[\min(\alpha_k', x_k'), y], \]

where \( y \in C(X' - \{ x_k' \}) \). Since \( x_k = \max(x_k', y') \) where \( y' \in C(X_k) \),

\[ X_k = \{ x_1', x_2', \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \}, \]

therefore

\[ x_k = \max[\min(\alpha_k', x_k), y''], \]

where \( y'' \in C(X_k) \). By definition of \( \alpha_k' \), \( \alpha_k' \leq \min(\alpha_k', \alpha_k) \) or \( \alpha_k \geq \alpha_k' \). Thus

\[ x_k' \geq \min(\alpha_k', x_k') = \min(\alpha_k, \alpha_k', x_k) = \min(\alpha_k', x_k) = x_k' \]

or

\[ x_k' = \min(\alpha_k, x_k'). \]

By Proposition 2.13, \( X' \) is fundamental.

**Theorem 2.18.** Every set of vertices of \( Y \) has the same number of elements provided \( Y \) is finitary and admissible.

**Proof.** Follows from Propositions 2.12, 2.17, and Theorem 2.16.

**Proposition 2.19.** Let \( Y \) be finitary and admissible and \( X_1, X_2 \) be sets of generators of \( Y \). If \( | X_1 | > | X_2 | \), then there exists \( x \in X_1 \) such that \( x \in C(X_1 - \{ x \}) \).

**Proof.** By Proposition 2.11, there exist \( X_1' \subseteq X_1 \) and \( X_2' \subseteq X_2 \) where both \( X_1' \) and \( X_2' \) are sets of vertices of \( Y \). By Theorem 2.18, \( | X_1' | = | X_2' | \). Thus \( | X_1' | = | X_2' | \leq | X_2 | < | X_1 | \) or \( X_1' \) is a proper subset of \( X_1 \). The conclusion follows from Proposition 2.8.

**Proposition 2.20.** Let \( X \) be a collection of sequences consisting of 0 and 1. If \( X \) is a set of vertices of \( C(X) \), then \( X \) is fundamental.
Proof. Let \( x \in X \) and

\[
x = \max\{\min(\alpha, x), y\},
\]

where \( 0 \leq \alpha \leq 1 \) and \( y \in C(X - \{x\}) \). Suppose \( \alpha < 1 \) and let \( x_0 = \min(\alpha, x) \).

If \( x_0 = 0 \), then \( y_0 = 0 \). If \( x_0 = 1 \), since \( x_0^k = \alpha < 1 \), therefore \( y_0^k = 1 \).

Thus \( x = y \). A contradiction. Hence \( \alpha = 1 \) and \( x = x_0 \). By Proposition 2.13, \( X \) is fundamental.

**Definition.** \( Y \) is fundamental if and only if every set of vertices of \( Y \) is fundamental.

**Remark.** Not every convex maximin set is fundamental.

**Theorem 2.21.** Let \( Y \) be finitary. \( Y \) is fundamental if and only if \( Y \) possesses one and only one set of vertices.

**Proof.** Follows from Proposition 2.10 and Theorem 2.16.

**Definition.** Let \( X_1, X_2 \subseteq Y \). \( X_1 \) is a basis of \( X_2 \) iff every \( x \in X_2 \) can be expressed uniquely as a convex maximin combination of a unique finite subset of \( X_1 \).

**Proposition 2.22.** If \( X \) is a basis of \( Y \), then \( X \) is a fundamental set of vertices of \( Y \).

**Proof.** Follows from Propositions 2.8 and 2.13.

**Proposition 2.23.** If \( Y \) is finitary and has a basis, then \( Y \) is fundamental.

**Proof.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a basis of \( Y \). By Proposition 2.22, \( X \) if the fundamental set of vertices of \( Y \). Suppose \( Y \) is not fundamental. By Theorem 2.21, \( Y \) has a set \( X' \) of vertices distinct from \( X \). By Proposition 2.17, we may assume, without loss of generality, that

\[
X' = \{x_1, x_2, \ldots, x_{n-1}, x'_n\},
\]

where \( x'_n \neq x_n \). Moreover,

\[
x_n = \min(\alpha, x'_n),
\]

for some \( 0 \leq \alpha \leq 1 \) and

\[
x'_n = \max\{x_n, y\},
\]

where \( y \in C(X - \{x_n\}) \). Thus

\[
x_n = \max\{\min(\alpha, x_n), \min(\alpha, y)\}.
\]
Since $X$ is a basis of $Y$, $\alpha = 1$. Thus $x_n' = x_n$. A contradiction. Hence $Y$ is fundamental.

3. Equivalences

**Definition.** A maximin sequential-like machine (MSLM) may be specified by a quadruple $(U, S, V, p)$ where $U$, $S$, $V$ are finite nonempty sets and $p$ is a function from $S \times U \times V \times S$ into $[0, 1]$.

The sets $U$ and $V$ are, respectively, the input and output alphabets. $S$ is the set of internal states and $p$ is the transition function. In the light of fuzzy sets [7], $p(s, u, v, s')$ may be interpreted as the grade of membership that the MSLM will enter state $s'$ and produce output $v$ given that the present state is $s$ and input $u$ is applied.

The readers should note the difference between the above definition and the one given in [6].

Finite sequences of elements of $U(V)$ will be called input (output) tapes. The collection of all input (output) tapes will be denoted by $U^*(V^*)$. For completeness sake, we shall also consider the empty tape $e$ with the property that $xe = e = ex$ for all tape $x$. The length ($lg$) of the tape $x$ will be denoted by $lg(x)$. By definition, $lg(e) = 0$. Moreover,

$$(U \times V)^* = \{(x, y) : x \in U^*, y \in V^*, lg(x) = lg(y)\}.$$

In what follows, the symbol $M$, with or without subscript, will always denote MSLM. All MSLMs will be assumed to have the same $U$ and $V$ sets. Thus a MSLM will be represented by $(S, p)$ with $U$ and $V$ being suppressed. Moreover, we shall assume that $(U \times V)^*$ is ordered in such a way that $lg(x_1) < lg(x_2)$ implies $(x_1, y_1) < (x_2, y_2)$. The order of $(U \times V)^*$ is the same for all MSLM and will be kept fixed throughout the entire paper.

**Definition.** Let $M = (S, p)$ be a MSLM. The extended transition function $p^*$ of $M$ is a function from $S \times (U \times V)^* \times S$ into $[0, 1]$ defined inductively on $lg(x)$, $x \in U^*$, as follows:

$$p^*(s', e, e, s'') = \begin{cases} 1 & \text{if } s' = s'' \\ 0 & \text{if } s' \neq s'' \end{cases},$$

$$p^*(s', u, v, s'') = \max_{s \in S} \{\min[p(s', u, v, s), p^*(s, x, y, s'')]\}.$$ 

Moreover, the overall transition function $q^M$ of $M$ is a function from $S \times (U \times V)^*$ into $[0, 1]$ defined as follows:

$$q^M(s, x, y) = \max_{s' \in S} p^*(s, x, y, s').$$
The superscript $M$ is used for identifying the specific MSLM and will be omitted if the context is clear.

For ease of notation, we shall assume that $S = \{s_1, s_2, \ldots, s_n\}$. Moreover,

(i) $P^M(x, y)$ will denote the matrix whose $(i, j)$-entry is $p^*(s_i, x, y, s_j)$,
(ii) $Q^M(x, y)$ will denote the column matrix whose $i$-th row is $q^M(s_i, x, y)$ and
(iii) $E$ will denote the column matrix whose entries are all 1.

**Proposition 3.1.** Let $M = (S, p)$ be a MSLM. Then for every $(x_1, y_1), (x_2, y_2) \in (U \times V)^*$,

(i) $P^M(x_1, y_1) = P^M(x_1, y_2) \otimes P^M(x_2, y_2)$, and
(ii) $Q^M(x_1, y_1) = P^M(x_1, y_2) \otimes Q^M(x_2, y_2) = P^M(x_1, y_2) \otimes E$.

**Proof.** Follows immediately from Proposition 2.1.

**Notation.** 1. $A^M$ will denote the matrix whose columns are $Q^M(x, y)$ arranged in the order of $(U \times V)^*$.
2. For any nonnegative integer $n$, $A_n^M$ will denote the submatrix of $A^M$ consisting of only those columns corresponding to $Q^M(x, y)$ with $\lg(x) \leq n$.
3. $B^M$ and $B_n^M$ will denote, respectively, the matrix obtained from $A^M$ and $A_n^M$ by omitting all columns which are convex maximin combination of previous columns.
4. Let $A$ be a (finite or infinite) matrix. $|A|$ will denote the number of columns of $A$ and $\rho(A)$ will denote the set of distinct rows of $A$.

**Proposition 3.2.** If $B_{n-1}^M = B_n^M$ for some $n$, then $B_m^M = B_{n-1}^M$ for all $m \geq n - 1$.

**Proof.** It suffices to show that $B_{n+1}^M = B_n^M$. Let $(x, y) \in (U \times V)^*$ where $\lg(x) = n$, then $Q^M(x, y)$ is a convex maximin combination of columns of $B_{n-1}^M$. Since

$$Q^M(u, v) = P^M(u, v) \otimes Q^M(x, y),$$

therefore, the conclusion follows from Propositions 2.1 and 3.1.

**Theorem 3.3.** Let $M = (S, p)$ be a MSLM. Then $B^M = B_m^M$ where $m \leq a^{|S|} - 1$ and $a$ is the number of distinct entries in $A^M$.

**Proof.** It is clear that $|B_n^M| \leq |B_{n+1}^M| \leq |B^M| \leq a^{|S|}$ for every $n$. Thus $B_{m-1}^M = B_m^M$ where $m = a^{|S|} - 1$. The conclusion follows from Proposition 3.2.
Proposition 3.4. The matrix $B^M$ can be constructed effectively from $M$.

Proposition 3.5. $C[p(A^M)]$ is admissible and finitary.

Definition. Let $M = (S, p)$ be a MSLM. A state distribution (sd) of $M$ is a function $h$ from $S$ into $[0, 1]$. $h$ is said to be concentrated at $s \in S$ iff $h(s) = 1$ and 0 elsewhere.

The symbol $h$ will also be used to denote the row matrix whose $i$-th row is $h(s_i)$.

Definition. An initialized maximin sequential-like machine (IMSLM) is an ordered pair $(M, h)$ where $M$ is a MSLM and $h$ is a sd of $M$.

Notation. If $h$ is concentrated at $s$, we shall also write $(M, s)$ for $(M, h)$.

Definition. Let $I = (M, h)$ be a IMSLM. The response function $r^I$ or $I$ is a function from $(U \times V)^\infty$ into $[0, 1]$ where

$$r^I(x, y) = \max_{s \in S} \min[h(s), g^M(s, x, y)].$$

If $R^I = h \otimes A^M$, then $R^I$ is a row matrix whose entries are $r^I(x, y)$. Moreover, $R^I$ is a convex maximin combination of $\rho(A^M)$.

Definition. Let $I_1$ and $I_2$ be IMSLM. $I_1$ and $I_2$ are equivalent ($\sim$) iff $r^{I_1} = r^{I_2}$.

Definition. Let $h_1$ and $h_2$ be sd of $M$. $h_1$ and $h_2$ are $M$-equivalent ($\sim^M$) iff $(M, h) \sim (M, h_2)$.

The symbol $M$ in $\sim^M$ will be omitted if the context is clear.

Notation. If $h_1$ is concentrated at $s$, then we shall write $s \sim h_2$ for $h_1 \sim h_2$.

Proposition 3.6. Let $h_1$ and $h_2$ be sd of $M$. $h_1 \sim h_2$ iff

$$h_1 \otimes B^M = h_2 \otimes B^M.$$

Definition. Let $M_1 = (S_1, p_1)$ and $M_2 = (S_2, p_2)$ be MSLM.

(i) $M_1$ and $M_2$ are statewise equivalent ($\sim$) iff for every $s' \in S_1$, there exists $s'' \in S_2$ such that $(M_1, s') \sim (M_2, s'')$ and vice versa.

(ii) $M_1$ and $M_2$ are compositewise equivalent ($\simeq$) iff for every $s \in S_1$, there exists a sd $h$ of $M_2$ such that $(M_1, s) \sim (M_2, h)$ and vice versa.

(iii) $M_1$ and $M_2$ are distributionwise equivalent ($\approx$) iff for every sd $h_1$ of $M_1$, there exists a sd $h_2$ of $M_2$ such that $(M_1, h_1) \sim (M_2, h_2)$ and vice versa.
Theorem 3.7. Let $M_1$ and $M_2$ be MSLM.

(i) $M_1 \sim M_2$ iff $\rho(A^{M_1}) = \rho(A^{M_2})$.

(ii) $M_1 \sim M_2$ iff $\rho(A^{M_1}) \subseteq C[\rho(A^{M_2})]$ and $\rho(A^{M_2}) \subseteq C[\rho(A^{M_1})]$.

(iii) $M_1 \sim M_2$ iff $C[\rho(A^{M_1})] = C[\rho(A^{M_2})]$.

Proposition 3.8. If $M_1 \sim M_2$, then $M_1 \simeq M_2$.

Proof. Follows from Propositions 2.4(i) and Theorem 3.7.

Proposition 3.9. $M_1 \simeq M_2$ iff $M_1 \approx M_2$.

Proof. From Theorem 3.7, it is clear that $M_1 \approx M_2$ implies $M_1 \simeq M_2$. The converse follows from Propositions 2.4(ii) and (iii) and Theorem 3.7.

4. Irreducibility and Minimality

Definition. Let $M = (S, p)$ be a MSLM.

(i) $M$ is statewise irreducible iff for every $s', s'' \in S$, $s' \sim s''$ implies $s' = s''$.

(ii) $M$ is compositewise irreducible iff for every $s \in S$ and $sd h$ of $M$, $s \sim h$ implies $h(s) > 0$.

(iii) $M$ is distributionwise irreducible iff for every $sd h_1$ and $h_2$ of $M$, $h_1 \sim h_2$ implies $h_1 = h_2$.

Theorem 4.1. Let $M$ be a MSLM.

(i) $M$ is statewise irreducible iff no two rows of $B^M$ are identical.

(ii) $M$ is compositewise irreducible iff $\rho(B^M)$ is a set of vertices of $C[\rho(B^M)]$, i.e., no row of $B^M$ is a convex maximin combination of the other rows of $B^M$.

(iii) $M$ is distributionwise irreducible iff $\rho(B^M)$ is a basis of $C[\rho(B^M)]$.

Proof. (i) follows from Proposition 3.6. (ii) follows from Propositions 3.6 and 2.8. (iii) follows from Proposition 3.6 and the definition of basis.

Remark. All assertions of Theorem 4.1 are also valid if $B^M$ is replaced by $A^M$.

Proposition 4.2. If $M$ is distributionwise irreducible, then $M$ is compositewise irreducible.

Proposition 4.3. If $M$ is compositewise irreducible, then $M$ is statewise irreducible.
**DEFINITION.** \( M \) is statewise (compositewise) minimal iff \( M \) is not statewise (compositewise) equivalent to a MSLM with a fewer number of states.

**Notation.** If \( M = (S, p) \), then \( |M| = |S| \).

**THEOREM 4.4.** \( M \) is statewise minimal iff \( M \) is statewise irreducible.

**Proof.** Let \( M = (S, p) \). Suppose \( M \) is not statewise minimal, then for some MSLM \( M' \) with \( |M'| < |M| \), \( M' \sim M \). By Theorems 3.7(i) and 4.1(i), \( M \) is not statewise irreducible. Conversely, suppose \( M \) is not statewise irreducible. There exist \( s', s'' \in S \) such that \( s' \sim s'' \) but \( s' \neq s'' \). By rearranging \( S \) if necessary, we may assume that \( s' = s_{n-1} \) and \( s'' = s_n \) where \( n = |S| \). Let \( M' = (S', p') \) where \( S' = S - \{s_n\} \) and for \( i = 1, 2, ..., n-1 \),

\[
p'(s_i, u, v, s_j) = \begin{cases} p(s_i, u, v, s_j) & \text{if } j = 1, 2, ..., n-2 \\ \max[p(s_i, u, v, s_{n-1}), p(s_i, u, v, s_n)] & \text{if } j = n-1. \end{cases}
\]

Since for every \((x, y) \in (U \times V)^*\),

\[
q^M(s_i, x, y) = \begin{cases} q^M(s_i, x, y) & \text{if } i = 1, 2, ..., n - 1 \\ q^M(s_{n-1}, x, y) & \text{if } i = n. \end{cases}
\]

Therefore, \( \rho(A^M) = \rho(A^M') \). By Theorem 3.7(i), \( M \sim M' \). Thus \( M \) is not statewise minimal.

**THEOREM 4.5.** Let \( M \) be a MSLM. There exists an effective procedure for constructing a statewise minimal MSLM which is statewise equivalent to \( M \).

**Proof.** Consider the following procedure:

Step 1. Construct \( B^M \).

Step 2. Are any two rows of \( B^M \) identical? If no, stop. If yes, proceed to Step 3.

Step 3. Construct \( M' \) as given in the proof of Theorem 4.4. Return to Step 1 with \( M' \) replacing \( M \).

Since \( |M| \) is finite, the procedure must terminate in a finite number of steps. By Proposition 3.4, \( B^M \) can be effectively constructed. Thus the procedure is effective. By Theorem 4.4, the resulting MSLM is the desired MSLM.

**THEOREM 4.6.** \( M \) is compositewise minimal iff \( M \) is compositewise irreducible.

**Proof.** Let \( M = (S, p) \). Suppose \( M \) is not compositewise minimal, then for some MSLM \( M' \) with \( |M'| < |M| \), \( M' \sim M \). By Theorems 3.7(iii), 4.1(ii), and Propositions 3.9, 2.19, and 3.5, \( M \) is not compositewise irreducible.
Conversely, suppose $M$ is not compositewise irreducible. There exist $s' \in S$ and $s d h$ of $M$ such that $s' \sim h$ but $h(s') = 0$. By rearranging $S$ if necessary, we may assume that $s' = s_n$ where $n = |S|$. Since $s_n \sim h$ and $h(s_n) = 0$,

$$q^M(s_n, x, y) = \max_{1 \leq i \leq n} \min[h(s_i), q^M(s_i, x, y)],$$

for all $(x, y) \in (U \times V)^*$. Let $M' = (S', p')$ where $S' = S - \{s_n\}$ and for $i, j = 1, 2, \ldots, n - 1$,

$$p'(s_i, u, v, s_j) = \max\{p(s_i, u, v, s_j), \min[h(s_i), p(s_i, u, v, s_n)]\}.$$ 

Since for every $(x, y) \in (U \times V)^*$,

$$q^M(s_i, x, y) = \begin{cases} q^M(s_i, x, y) & \text{if } i = 1, 2, \ldots, n - 1 \\ \max_{1 \leq i \leq n} \min[h(s_i), q^M(s_i, x, y)] & \text{if } i = n. \end{cases}$$

Therefore $C[p(A^M)] = C[p(A^{M'})]$. By Theorem 3.7(iii) and Proposition 3.9, $M' \simeq M$. Thus $M$ is not compositewise minimal.

**Theorem 4.7.** Let $M$ be a MSLM. There exists an effective procedure for constructing a compositewise minimal MSLM which is compositewise equivalent to $M$.

**Proof.** Consider the following procedure:

Step 1. Construct $B^M$.

Step 2. Is there any row of $B^M$ which is a convex maximin combination of the other rows of $B^M$? If no, stop. If yes, proceed to Step 3.

Step 3. Construct $M'$ as given in the proof of Theorem 4.6. Return to Step 1 with $M'$ replacing $M$.

Since $|M|$ is finite, the procedure must terminate in a finite number of steps. By Proposition 3.4, $B^M$ can be constructed effectively. By Propositions 2.3 and 3.5 and the fact that $|M|$ is finite, Step 2 can be carried out effectively. Thus the procedure is effective. By Theorem 4.6, the resulting MSLM is the desired MSLM.

**Proposition 4.8.** Let $\rho(B^{M_1})$ and $\rho(B^{M_2})$ be, respectively, fundamental set of vertices of $C[\rho(B^{M_1})]$ and $C[\rho(B^{M_2})]$. $M_1 \simeq M_2$ iff $M_1 \sim M_2$.

**Proof.** Follows from Theorems 3.7(iii), 2.16 and Proposition 3.9, 3.7(i) and (ii).

**Definition.** $M$ is fundamental iff $C[\rho(B^M)]$ is fundamental.
PROPOSITION 4.9. If $M_1$, $M_2$ are compositewise minimal, $M_1$ fundamental and $M_1 \sim M_2$, then $M_2$ is fundamental and $M_1 \sim M_2$.

Proof. Follows from Theorems 2.16, 4.1(ii), 4.6, and Proposition 4.8.

PROPOSITION 4.10. If $M_1$ is distributionwise irreducible, $M_2$ is compositewise minimal and $M_1 \sim M_2$, then $M_1 \sim M_2$.

Proof. Follows from Propositions 2.23 and 4.9.

PROPOSITION 4.11. If $M_1$, $M_2$ are statewise minimal and $M_1 \sim M_2$, then for every $(x, y) \in (U \times V)^*$,

$$P^{M_1}(x, y) \otimes B^{M_1} = P^{M_2}(x, y) \otimes B^{M_2},$$

after an appropriate rearrangement of states.

Proof. From Theorems 3.7(i), 4.1(i), and 4.4, $A^{M_1} = A^{M_2}$ after an appropriate rearrangement of states.

DEFINITION. $M_1$ and $M_2$ are isomorphic (=) iff they are equal up to a permutation of states.

DEFINITION. Let $M$ be statewise (compositewise) minimal. $M$ is statewise (compositewise) simple iff there exist no statewise (compositewise) minimal MSLM which is statewise (compositewise) equivalent to $M$ but not isomorphic to $M$.

Notation. Let $M$ be a MSLM.

1. $\rho(M) = \bigcup_{u \in U} \bigcup_{v \in V} \rho[P^M(u, v)]$

2. $\bar{\rho}(M) = \{h \otimes B^M : h \in \rho(M)\}$.

THEOREM 4.12. Let $M$ be statewise minimal. $M$ is statewise simple iff $\rho(B^M)$ is a basis of $\bar{\rho}(M)$.

Proof. Let $\rho(B^M)$ be a basis of $\bar{\rho}(M)$ and $M'$ a statewise minimal MSLM such that $M' \sim M$. By Proposition 4.11, for every $u \in U$, $v \in V$,

$$P^{M'}(u, v) = P^M(u, v),$$

after an appropriate rearrangement of states. Thus $M' \equiv M$. Hence $M$ is statewise simple. Conversely, suppose $\rho(B^M)$ is not a basis of $\bar{\rho}(M)$. There exist $h \in \rho(M)$ such that

$$h \otimes B^M = h' \otimes B^M,$$

(4.1)
for some \( h' \neq h \). Let \( h \) be the \( i \)-th row of the matrix \( P^M(u, v) \). Construct \( M' \) from \( M \) by replacing the \( i \)-th row of the matrix \( P^M(u, v) \) by \( h' \) and leaving the rest unaltered. By virtue of (4.1), \( A^M = A^{M'} \). By Theorems 3.7(i), 4.1(i), and 4.4, \( M' \) is statewise minimal and \( M' \sim M \). However, \( M' \neq M \). Thus \( M \) is not statewise simple.

**Proposition 4.13.** Let \( M \) be statewise minimal. If \( M \) is distributionwise irreducible, then \( M \) is statewise simple.

**Proof.** Follows from Theorem 4.1(iii) and 4.12.

**Theorem 4.14.** Let \( M \) be compositewise minimal and fundamental. \( M \) is compositewise simple iff \( \rho(B^M) \) is a basis of \( \tilde{\rho}(M) \).

**Proof.** Let \( \rho(B^M) \) be a basis of \( \tilde{\rho}(M) \) and \( M' \) a compositewise minimal MSLM such that \( M' \sim M \). By Proposition 4.3 and Theorems 4.4 and 4.6, both \( M \) and \( M' \) are statewise minimal. Moreover, by Proposition 4.9, \( M' \sim M \). The rest of the proof is similar to Theorem 4.12.

**Proposition 4.15.** Let \( M \) be compositewise minimal. If \( M \) is distributionwise irreducible, then \( M \) is compositewise simple.

**Proof.** Follows from Theorems 4.1(iii), 4.14, and Proposition 2.23.

5. **Nondeterministic and Deterministic Case**

**Definition.** A nondeterministic sequential-like machine (NSLM) is a MSLM \((S, \rho)\) where the range of \( \rho \) is a subset of \( \{0, 1\} \). If for every \( s \in S, u \in U \), there exist uniquely \( v \in V \) and \( s' \in S \) such that \( \rho(s, u, v, s') = 1 \), then it is a deterministic sequential-like machine (DSLM).

**Proposition 5.1.** Let \( M \) be a DSLM. Then for every \( s \in S, x \in U^* \), there exists uniquely \( y \in V^* \) with \( \lg(x) = \lg(y) \) such that \( q^M(s, x, y) = 1 \).

Due to the special characters of NSLM and DSLM, several of the results given in Section 4 can be strengthened.

**Proposition 5.2.** Let \( M \) be a NSLM. \( M \) is statewise minimal iff \( M \) is not statewise equivalent to any NSLM with a fewer number of states.

**Proof.** Follows from the proof of Theorem 4.4.

**Proposition 5.3.** Let \( M \) be a NSLM. There exists an effective procedure for constructing a statewise minimal NSLM which is statewise equivalent to \( M \).
PROPOSITION 5.4. Let $M$ be a NSLM. $M$ is compositewise minimal iff $M$ is not compositewise equivalent to any NSLM with a fewer number of states.

Proof. Follows from Proposition 2.2 and the proof of Theorem 4.6.

PROPOSITION 5.5. Let $M$ be a NSLM. There exists an effective procedure for constructing a compositewise minimal NSLM which is compositewise equivalent to $M$.

PROPOSITION 5.6. Let $M$ be a DSLM. $M$ is statewise minimal iff $M$ is not statewise equivalent to any DSLM with a fewer number of states.

Proof. Follows from the proof of Theorem 4.4.

PROPOSITION 5.7. Let $M$ be a DSLM. There exists an effective procedure for constructing a statewise minimal DSLM which is statewise equivalent to $M$.

THEOREM 5.8. Let $M$ be a DSLM. $M$ is compositewise minimal iff $M$ is statewise minimal.

Proof. Suppose $M$ is not compositewise minimal. There exists a row of $B^M$, say the $n$-th row where $n = |M|$, which is a convex maximin combination of the other rows of $B^M$. By virtue of Proposition 2.2, we may assume, without loss of generality, that

$$x_n = \max_{1 \leq i \leq m} x_i,$$

where $m < n$, after an appropriate rearrangement of states. Here, $x_i$ denotes the $i$-th row of $B^M$. Since $x_i^k = 0$ implies $x_i^k = 0$ for $i = 1, 2, \ldots, m$, therefore by Proposition 5.1, $x_n = x_i$ for $i = 1, 2, \ldots, m$. Thus $M$ is not statewise minimal. The converse is trivial.

PROPOSITION 5.9. Let $M_1$ and $M_2$ be compositewise minimal NSLM. $M_1 \sim M_2$ iff $M_1 \sim M_2$.

Proof. Follows from Propositions 2.20 and 4.8.

DEFINITION. Let $M$ be a statewise (compositewise) minimal NSLM. $M$ is statewise (compositewise) ND-simple iff there exists no statewise (compositewise) minimal NSLM which is statewise (compositewise) equivalent to $M$ but not isomorphic to $M$.

DEFINITION. Let $M$ be a statewise minimal DSLM. $M$ is statewise $D$-simple iff there exists no statewise minimal DSLM which is statewise equivalent to $M$ but not isomorphic to $M$. 

THEOREM 5.10. Let $M$ be a statewise (compositewise) minimal NSLM. $M$ is statewise (compositewise) ND-simple iff for every $x_0 \in \rho(M)$, there exists uniquely $L \subseteq \rho(B^M)$ such that

$$x_0 = \max_{x \in L} x.$$

Proof. Similar to Theorems 4.12 and 4.14. For the compositewise case, Proposition 5.9 is needed.

THEOREM 5.11. If $M$ is a statewise (compositewise) minimal DSLM, then $M$ is a statewise (compositewise) ND-simple.

Proof. Follows from Theorem 5.10 and the fact that every row of any matrix $P^M(u, v)$ has at most one nonzero entry and this nonzero entry is a 1.

COROLLARY. If $M$ is a statewise minimal DSLM, then $M$ is statewise D-simple.

Remark. This is another proof of a well-known theorem concerning deterministic machines.

REFERENCES