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Fixed point theory for a class of generalized nonexpansive mappings

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ABSTRACT

In this paper we introduce two new classes of generalized nonexpansive mapping and we study both the existence of fixed points and their asymptotic behavior.

1. Introduction

Let C be a nonempty subset of a Banach space X. It is well known that a mapping $T : C \to X$ is said to be nonexpansive whenever $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$.

Among the most important features of nonexpansive mappings are the following facts.

i) If C is closed convex and bounded and $T : C \to C$ is nonexpansive, then there exists a sequence (x_n) in C such that $\|x_n - Tx_n\| \to 0$. Such a sequence is called almost fixed point sequence for T (a.f.p.s. in short).
ii) Even when C is a weakly compact convex subset of X, a nonexpansive self-mapping of C need not have fixed points. Nevertheless, if the norm of X has suitable geometric properties (as for instance uniform convexity, among many others), every nonexpansive self-mapping of every weakly compact convex subset of X has a fixed point. In this case X is said to have the weak fixed point property (WFPP in short).

In a recent paper [1], Suzuki defined a class of generalized nonexpansive mappings as follows.

Definition 1. Let C be a nonempty subset of a Banach space X. We say that a mapping $T : C \to X$ satisfy *condition* (C) on C if for all $x, y \in C$,

$$\frac{1}{2}\|x - Tx\| \leqslant \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leqslant \|x - y\|. \tag{1}$$

Of course, every nonexpansive mapping $T : C \to X$ satisfies condition (C) on C, but in [1] some examples of non continuous mappings satisfying condition (C) are given.

In spite that the class of mapping satisfying condition (C) is broader than the class of nonexpansive mappings, when C is a convex bounded subset of X, every mapping $T : C \to C$ which satisfies condition (C) on C has a.f.p. sequences, that

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is, it shares (i) with nonexpansive mappings (see [1, Lemma 6]), as well as (ii), because for some Banach spaces (see [1, Theorems 4, 5]) mappings satisfying (C) leaving invariant weakly compact convex subsets have fixed points. (See also [4].)

In this paper we define two kind of generalizations of condition (C). This will lead us to some classes of mappings which are wider than those which satisfy condition (C) but preserving their fixed point properties.

2. Notations and preliminaries

Throughout this note we assume that $(X, \|\cdot\|)$ is a real Banach space whose zero vector is 0_X. As it is usual, we will denote by $B[x,r]$ and $S[x,r]$ the closed ball and the sphere of the Banach space $(X, \|\cdot\|)$ with radius r and center $x \in X$, respectively. In particular we will write $B_X := B[0_X, 1]$ and $S_X := S[0_X, 1]$.

We will use $x_n \rightharpoonup x$ to denote that the sequence (x_n) in X is weakly convergent to $x \in X$.

Let C be a nonempty closed and convex subset of X, and let (x_n) be a bounded sequence in X. For $x \in X$ the *asymptotic radius of* (x_n) *at* x is the number

$$r\bigl(x, (x_n)\bigr) := \limsup_{n\to\infty} \|x - x_n\|.$$

The real number

$$r\bigl(C, (x_n)\bigr) := \inf\bigl\{r\bigl(x, (x_n)\bigr): x \in C\bigr\}$$

is called the *asymptotic radius of* (x_n) *relative to* C and finally the set

$$A\bigl(C, (x_n)\bigr) = \bigl\{x \in C: r\bigl(x, (x_n)\bigr) = r\bigl(C, (x_n)\bigr)\bigr\},$$

is called the *asymptotic center of* (x_n) *relative to* C.

It is well known that $A(C, (x_n))$ consists of exactly one point whenever the space X is uniformly convex in every direction (UCED), and that $A(C, (x_n))$ is nonempty and convex when C is weakly compact and convex.

3. A class more general than type (C) mappings

We generalize condition (C) as follows.

Definition 2. Let C be a nonempty subset of a Banach space X. For $\mu \geqslant 1$ we say that a mapping $T : C \to X$ satisfy condition (E_μ) on C if there exists $\mu \geqslant 1$ such that for all $x, y \in C$,

$$\|x - Ty\| \leqslant \mu\|x - Tx\| + \|x - y\|. \tag{2}$$

We say that T satisfies condition (E) on C whenever T satisfies (E_μ) for some $\mu \geqslant 1$.

3.1. It is obvious that *if* $T : C \to X$ *is nonexpansive, then it satisfies condition* (E_1).

The converse is not true, as we will see below (see, for instance Example 1).

3.2. From Lemma 7 in [1] we know that *if* $T : C \to C$ *satisfies condition* (C) *on* C, *then is satisfies condition* (E_3).

There are continuous mappings satisfying condition (E) but failing condition (C), as the following example shows.

Example 1. (See [2, Example 6.3].) In the space $\mathcal{C}([0,1])$ consider the set

$$C := \bigl\{x \in \mathcal{C}\bigl([0,1]\bigr): 0 = x(0) \leqslant x(t) \leqslant x(1) = 1\bigr\}.$$

Take any function $g \in C$ and generate the mapping $F_g : C \to C$

$$F_g x(t) := (g \circ x)(t) = g\bigl(x(t)\bigr).$$

Since each function $x \in C$ takes all the values between zero and one, we have

$$\begin{aligned}\|x - F_g x\|_\infty &= \max\bigl\{\bigl|x(t) - g\bigl(x(t)\bigr)\bigr|: t \in [0,1]\bigr\}\\ &= \max\bigl\{\bigl|\mathrm{Id}(s) - g(s)\bigr|: 0 \leqslant s \leqslant 1\bigr\}\\ &= \|\mathrm{Id} - g\|_\infty,\end{aligned} \tag{3}$$

where Id is the identity function on $[0,1]$. Thus, except for $g = \mathrm{Id}$, any mapping F_g moves each point $x \in C$ to the same distance $\|\mathrm{Id} - g\| > 0$. For $x, y \in C$ we have

$$\|x - F_g y\|_\infty \leqslant \|x - y\| + \|y - F_g y\| = \|x - y\| + \|x - F_g x\|.$$

Thus, F_g satisfies condition (E_1) on C.

On the other hand, F_g cannot satisfy condition (C) on C: Otherwise, since C is convex and bounded, from [1, Lemma 6], one has $\inf\{\|x - F_g x\|: x \in C\} = 0$, and this contradicts (3).

3.3. Recall that a mapping $T : C \to X$ is said to be *quasi-nonexpansive* provided that it has some fixed point in C and for each fixed point $x_0 \in C$ and for every $y \in C$,

$$\|x_0 - Ty\| \leqslant \|x_0 - y\|. \tag{4}$$

Then, the following result is also obvious.

Proposition 1. *Let $T : C \to X$ be a mapping which satisfies condition* (E) *on C. If T has some fixed point, then T is quasi-nonexpansive.*

The converse is not true.

Example 2. Let $T : [-1, 1] \to [-1, 1]$ given by

$$T(x) = \begin{cases} \frac{x}{1+|x|}\sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

It is easy to check that 0 is the only fixed point of T. Since for all $x \in [-1, 1]$ one has that $|T(x)| \leqslant |x|$, it is obvious that T is quasi-nonexpansive on $[-1, 1]$.

On the other hand, if we take for each positive integer $x_n := \frac{1}{2\pi n+\pi/2}$ and $y_n := \frac{1}{2\pi n}$, then we have

$$\begin{aligned} \frac{|x_n - T(y_n)| - |x_n - y_n|}{|x_n - T(x_n)|} &= \frac{x_n - (y_n - x_n)}{|x_n - T(x_n)|} \\ &= \frac{x_n - (y_n - x_n)}{\frac{x_n^2}{1+x_n}} \\ &= \frac{(1 + x_n)(2 - \frac{y_n}{x_n})}{x_n} \to +\infty. \end{aligned}$$

Consequently, the mapping T does not satisfy condition (E) on $[-1, 1]$.

The following example deals with the converse of Lemma 7 in [1].

Example 3. Let $T : [-2, 1] \to [-2, 1]$ defined as

$$T(x) := \begin{cases} \frac{|x|}{2} & x \in [-2, 1), \\ -1/2 & x = 1. \end{cases}$$

In order to see that T satisfies (E) on $[-2, 1]$ we consider the following (non-trivial) cases.

a) $x \in [-2, 0]$, $y \in [-2, 1]$. Then $|x - T(x)| = \frac{3}{2}|x|$ and we have

$$\left|x - T(y)\right| \leqslant |x| + \frac{1}{2}|y| \leqslant \frac{3}{2}|x| + \frac{1}{2}|x - y| \leqslant \left|x - T(x)\right| + |x - y|.$$

b) $x \in [0, 1)$, $y \in [-2, 1]$. Then $|x - T(x)| = \frac{|x|}{2}$, and we have

$$\left|x - T(y)\right| \leqslant |x| + \frac{1}{2}|y| \leqslant \frac{3}{2}|x| + \frac{1}{2}|x - y| \leqslant 3\left|x - T(x)\right| + |x - y|.$$

c) $x = 1$, $y \in [-2, 1)$. Then $|x - Tx| = \frac{3}{2}$, and we have

$$\left|1 - T(y)\right| = \frac{1}{2} + \frac{1 - |y|}{2} \leqslant \frac{1}{3}\left|1 - T(1)\right| + \frac{1}{2}|1 - y| \leqslant \left|1 - T(1)\right| + |1 - y|.$$

In summary, for all $x, y \in [-2, 1]$,

$$\left|x - T(y)\right| \leqslant 3\left|x - T(x)\right| + |x - y|,$$

that is, T satisfy condition (E_3) on $[-2, 1]$. Notice that every mapping which satisfies condition (C) it satisfies also this last inequality. But this example shows that the converse is not true, because this mapping fails to satisfy condition (C) on $[-2, 1]$. Indeed,

$$\frac{1}{2}\left|\frac{4}{5} - T\left(\frac{4}{5}\right)\right| = \frac{1}{5} \leqslant \left|\frac{4}{5} - 1\right|,$$

while

$$\left|T\left(\frac{4}{5}\right) - T(1)\right| = \frac{2}{5} + \frac{1}{2} > \left|\frac{4}{5} - 1\right|.$$

Finally, since $T(0) = 0$, from Proposition 1, we have that T is quasi-nonexpansive on $[-2, 1]$.

Next we summarize some elementary properties of the mappings which satisfy condition (E).

Proposition 2. *Let $T : C \to X$ be a mapping which satisfies condition (E_α) on C. Then the following statements hold.*

a) *If $TC \subset C$ then for all $x \in C$,*
$$\|x - T^2x\| \leqslant (\alpha + 1)\|x - Tx\|.$$
b) *If $TC \subset C$ then for all $x, y \in C$,*
$$\|Tx - Ty\| \leqslant \alpha\|Tx - T^2x\| + \|Tx - y\|.$$
c) *If $r \in (0, 1)$ then the mappings $T_r : C \to X$ defined as $T_r = rT + (1 - r)I$ (where I is the identity mapping), satisfy the condition (E_α) on C.*

Proof. Taking $y = Tx$ in (2), we have that for all $x \in C$,

$$\|x - T^2x\| \leqslant \alpha\|x - Tx\| + \|x - Tx\| = (\alpha + 1)\|x - Tx\|.$$

Replacing x by Tx in (2), we have that for all $y \in C$,

$$\|Tx - Ty\| \leqslant \alpha\|Tx - T^2x\| + \|Tx - y\|.$$

Let $T_r = rT + (1 - r)I$. Since for every $x \in C$, $x - T_r(x) = r(x - Tx)$ we have, if $x, y \in C$ that

$$\begin{aligned}\|x - T_r(y)\| &\leqslant (1 - r)\|x - y\| + r\|x - Ty\| \\ &\leqslant \|x - y\| + r\alpha\|x - Tx\| \\ &= \alpha\|x - T_r(x)\| + \|x - y\|.\end{aligned}$$

Thus, T_r satisfies condition (E_α). □

Remark 1. A mapping $T : C \to C$ such that there exists a constant $a \in [0, 2)$ such that for every $x \in C$, $\|x - T^2x\| \leqslant a\|x - Tx\|$ is called 2-rotative. (See [2].) Thus, from (a) and (b) of the above proposition, mappings satisfying condition (E_1) can be regarded as somewhat a limiting case of 2-rotative mappings.

Proposition 3 *(Alternative principle). Let C be a bounded subset of X. Let $T : C \to C$ be an arbitrary mapping. Then one at least of the following statements hold.*

a) *There exists an a.f.p.s. for T in C.*
b) *T satisfies condition (E) on C.*

Proof. Suppose that (b) fails, that is that T does not satisfy condition (E) on C for any $\alpha \geqslant 1$. Then, for every positive integer n there exist $x_n, y_n \in C$ such that

$$\|x_n - Ty_n\| > n\|x_n - Tx_n\| + \|x_n - y_n\|.$$

Then, for every positive integer n

$$\frac{\operatorname{diam}(C)}{n} \geqslant \frac{\|x_n - Ty_n\|}{n} > \|x_n - Tx_n\| + \frac{\|x_n - y_n\|}{n}.$$

Letting $n \to \infty$ we obtain that $\|x_n - Tx_n\| \to 0$, that is that (a) holds. □

Thus, when C is a bounded subset of X, all the mappings $T : C \to C$ with positive minimal displacement, that is with $\inf\{\|x - Tx\|: x \in C\} > 0$, trivially satisfy condition (E) on C, in order to get fixed point results for this class of mappings some additional requirement is necessary.

Recall that a Banach space $(X, \|\cdot\|)$ is said to satisfy the Opial condition whenever for every sequence (x_n) with $x_n \rightharpoonup z$ one has that

$$\liminf_{n\to\infty}\|x_n - z\| < \liminf_{n\to\infty}\|x_n - y\|$$

whenever $y \neq z$. For instance the spaces ℓ_p for $1 < p < \infty$ satisfy this condition.

Theorem 1. *Let C be a nonempty subset of a Banach space X. Let $T : C \to X$ be a mapping. If*

a) *There exists an a.f.p.s. (x_n) for T in C such that $x_n \rightharpoonup z \in C$,*
b) *T satisfies condition* (E) *on C, and*
c) *$(X, \|\cdot\|)$ satisfies the Opial condition.*

Then, $Tz = z$.

Proof. From (b), there exists $\alpha \geqslant 1$ such that for every positive integer n one has that

$$\|x_n - Tz\| \leqslant \alpha\|x_n - Tx_n\| + \|x_n - z\|.$$

Since (x_n) is an a.f.p.s. for T,

$$\liminf_{n\to\infty} \|x_n - Tz\| \leqslant \liminf_{n\to\infty} \big[\alpha\|x_n - Tx_n\| + \|x_n - z\|\big] = \liminf_{n\to\infty} \|x_n - z\|.$$

Since $x_n \rightharpoonup z$, if $z \neq Tz$, from the Opial condition we obtain

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - Tz\|,$$

a contradiction. □

Notice that, in this last result, the set C do not need to be bounded nor convex.
On the other hand assumption (b) cannot be removed. To see this, consider the following example.

Example 4. For $0 < r \leqslant 1$, let T_r be the Kakutani mapping $T_r : B_{\ell_2} \to B_{\ell_2}$ defined as $T_r(x) = r(1 - \|x\|)e_1 + S(x)$, where $S(x) = \sum_{i=1}^{\infty} x_i e_{i+1}$ and (e_n) is the standard Schauder basis in ℓ_2. It is straightforward to check that $v_n - T_r(v_n) \to 0_{\ell_2}$ for

$$v_n := \sum_{i=1}^{n^2} \frac{1}{n} e_i,$$

and hence T_r fulfills assumption (a). Moreover, every Hilbert space enjoys the Opial condition, that is (c) is also fulfilled. But T_r is fixed point free, as it is well known, thus it is clear that T_r fails condition (E).

Corollary 1. *Let C be a nonempty weakly compact subset of a Banach space X. Suppose that $(X, \|\cdot\|)$ satisfies the Opial condition. Let $T : C \to X$ be a mapping which satisfies condition* (E) *on C. Then T has a fixed point in C if and only if T admits an a.f.p.s.*

Proof. If (x_n) is an a.f.p.s. for T in C, since C is weakly compact, for a subsequence (x_{n_k}) of (x_n) one has that $x_{n_k} \rightharpoonup z \in C$. From the above theorem, $Tz = z$, that is T has a fixed point. Conversely, if T has a fixed point, say $z \in C$, then the sequence (x_n) with $x_n \equiv z$ is an a.f.p.s. for T. □

Notice again that in this last result, as well as in the following one, the set C do not need to be convex.

Theorem 2. *Let C be a nonempty compact subset of a Banach space X. Let $T : C \to X$ be a mapping which satisfies condition* (E) *on C. Then T has a fixed point in C if and only if T admits an a.f.p.s.*

Proof. If (x_n) is an a.f.p.s. for T, since C is compact, for a subsequence (x_{n_k}) of (x_n) one has that $x_{n_k} \to z \in C$. Since T satisfies condition (E) on C, there exists $\alpha \geqslant 1$ such that for every positive integer k one has that

$$\|x_{n_k} - Tz\| \leqslant \alpha\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|.$$

Since (x_{n_k}) is an a.f.p.s. for T,

$$\limsup_{k\to\infty} \|x_{n_k} - Tz\| \leqslant \limsup_{k\to\infty} \big[\alpha\|x_{n_k} - Tx_{n_k}\| + \|x_{n_k} - z\|\big] = 0.$$

Then, $\lim_k \|x_{n_k} - Tz\| = 0 = \lim_k \|x_{n_k} - z\|$, and hence $z = Tz$, that is T has a fixed point. □

Remark 2. Let C be a weakly compact convex subset of X, and $T : C \to X$ be a mapping which satisfies condition (E) on C. If (x_n) is an a.f.p.s. for T, then the function $f : C \to [0, \operatorname{diam}(C)]$ given by

$$f(x) := \limsup_{n\to\infty} \|x - x_n\|$$

attains its minimum on C. This is because f is convex, bounded from below and lower semicontinuous on the weakly compact set C. From the above proofs we derive that, in fact, the nonempty set

$$M := \{z \in C\colon f(z) \leqslant f(x)\ \forall x \in C\}$$

is T invariant. Indeed if $z \in M$ then, from condition (E), for some $\alpha \geqslant 1$ and every positive integer n,

$$\|x_n - Tz\| \leqslant \alpha \|x_n - Tx_n\| + \|x_n - z\|.$$

Since (x_n) is an a.f.p.s. for T, we obtain

$$f(Tz) \leqslant f(z),$$

that is, $Tz \in M$. It is also well known that M is closed and convex. Thus, if T satisfy condition (E) over C and it admits an a.f.p.s. (x_n), then the asymptotic center $A(C,(x_n))$ is T-invariant. Then any geometric property of Banach spaces implying that the asymptotic centers of the bounded sequences are singletons yields a fixed point theorem for such a mapping. Thus we have.

Theorem 3. *Let C be a nonempty weakly compact convex subset of a (UCED) Banach space X. Let $T : C \to X$ be a mapping. If*

a) *T satisfies condition (E) on C, and*
b) $\inf\{\|x - Tx\|\colon x \in C\} = 0$.

Then, T has a fixed point.

4. A direct generalization of condition (C)

Definition 2. For $\lambda \in (0,1)$ we say that a mapping $T : C \to X$ satisfy *condition* (C_λ) on C if for all $x, y \in C$ with $\lambda\|x - Tx\| \leqslant \|x - y\|$ one has that $\|Tx - Ty\| \leqslant \|x - y\|$.

Of course, if $\lambda = \frac{1}{2}$ we recapture the class of mappings satisfying condition (C).
Notice that if $0 < \lambda_1 < \lambda_2 < 1$ then the condition (C_{λ_1}) implies condition (C_{λ_2}).
The following example shows that the converse fails.

Example 5. For a given $\lambda \in (0,1)$, let $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} \frac{x}{2} & x \neq 1, \\ \frac{1+\lambda}{2+\lambda} & x = 1. \end{cases}$$

Then the mapping T satisfies condition (C_λ) but it fails condition $(C_{\lambda'})$ whenever $0 < \lambda' < \lambda$. Moreover T satisfies condition (E_μ) for $\mu = (2+\lambda)/2$.

Indeed, if $0 \leqslant x \leqslant \frac{2}{2+\lambda}$, then

$$|T(x) - T(1)| = \frac{1+\lambda}{2+\lambda} - \frac{x}{2} = \frac{1+\lambda}{2+\lambda} + \frac{x}{2} - x \leqslant 1 - x = |x - 1|.$$

If $\frac{2}{2+\lambda} < x < 1$ then,

$$\lambda|x - T(x)| = \frac{2+\lambda}{2}x - x > 1 - x = |x - 1|$$

and

$$\lambda|1 - T(1)| = 1 - \frac{2}{2+\lambda} > 1 - x = |x - 1|.$$

Thus, the mapping T satisfies condition (C_λ) on $[0,1]$.

On the other hand, since the real function $\varphi(t) = \frac{1+t}{2+t}$ is increasing on $(-2,\infty)$, we have that $\varphi(\lambda') < \varphi(\lambda)$, and therefore

$$\frac{1}{2+\lambda'} < \frac{1+\lambda'}{2+\lambda'} < \frac{1+\lambda}{2+\lambda}. \tag{5}$$

Put $x = \frac{2}{2+\lambda'}$. Then

$$\lambda'|x - T(x)| = \lambda'\left|\frac{2}{2+\lambda'} - \frac{1}{2+\lambda'}\right| = \frac{\lambda'}{2+\lambda'} = 1 - x = |x - 1|,$$

while, bearing in mind (5),

$$\left|T(x)-T(1)\right|=\left|\frac{1}{2+\lambda'}-\frac{1+\lambda}{2+\lambda}\right|=\frac{1+\lambda}{2+\lambda}-\frac{1}{2+\lambda'}>\frac{1+\lambda'}{2+\lambda'}-\frac{1}{2+\lambda'}=|x-1|,$$

which implies that T does not satisfy condition $(\mathrm{C}_{\lambda'})$.

If $\mu=(2+\lambda)/2$, for $x\in[0,1)$ we have

$$\left|x-T(1)\right|\leqslant\mu\left|x-T(x)\right|+|x-1|$$

and

$$\left|1-T(x)\right|\leqslant 1-\frac{x}{2}<\frac{1}{2}+1-x=\mu\left|1-T(1)\right|+|1-x|.$$

Therefore T satisfies (E_μ).

Proposition 4. *Let C be a subset of a Banach space X. If $T:C\to X$ satisfies the condition (C_λ) for some $\lambda\in(0,1)$, then for every $r\in(\lambda,1)$ the mapping $T_r:C\to X$ defined by $T_r(x)=rTx+(1-r)x$ satisfy the condition $(\mathrm{C}_{\lambda/r})$.*

Proof. Suppose that for $x,y\in C$, $\frac{\lambda}{r}\|x-T_r(x)\|\leqslant\|x-y\|$. Since $x-T_r(x)=r(x-Tx)$ it follows that

$$\lambda\|x-Tx\|\leqslant\|x-y\|.$$

Since T satisfies the condition (C_λ) on C we derive that $\|Tx-Ty\|\leqslant\|x-y\|$. Therefore

$$\left\|T_r(x)-T_r(y)\right\|\leqslant r\|Tx-Ty\|+(1-r)\|x-y\|\leqslant\|x-y\|.\qquad\square$$

The class of mappings satisfying condition (C_λ) on a convex bounded subset C of X shares with the class of nonexpansive mappings the existence of almost fixed point sequences. The proof is closely modeled on Lemma 6 of [1].

Theorem 4. *Let C be a bounded convex subset of a Banach space X. Assume that $T:C\to C$ satisfies condition (C_λ) on C for some $\lambda\in(0,1)$. For $r\in[\lambda,1)$ define a sequence (x_n) in C by tacking $x_1\in C$ and*

$$x_{n+1}=rT(x_n)+(1-r)x_n$$

for $n\geqslant 1$.

Then (x_n) is an a.f.p.s. for T, that is, the mappings T_r are asymptotically regular.

Proof. For $n\geqslant 1$ one has

$$\lambda\|x_n-Tx_n\|\leqslant r\|x_n-Tx_n\|=\|x_n-x_{n+1}\|.$$

From the condition (C_λ) we derive that $\|Tx_n-Tx_{n+1}\|\leqslant\|x_n-x_{n+1}\|$, and we can apply Lemma 3 in [1] (see also Propositions 1 and 2 of [3]), to conclude that $\|x_n-Tx_n\|\to 0$. $\square$

Lemma 1. *Let C be a subset of a Banach space X. Let $T:C\to X$ be a mapping satisfying condition (C_λ) for some $\lambda\in(0,1)$. Let (x_n) be a bounded a.f.p.s. for T. Then*

$$\limsup_{n\to\infty}\|x_n-Ty\|\leqslant\limsup_{n\to\infty}\|x_n-y\| \tag{6}$$

holds for all $y\in C$ with $\liminf_n\|x_n-y\|>0$.

Remark 3. It is obvious that we can replace lim sup with lim inf in (6).

Proof of Lemma 1. Fix $y\in C$ and put $\varepsilon:=(1/2)\liminf_n\|x_n-y\|>0$. We note

$$\lambda\|x_n-Tx_n\|\leqslant\|x_n-Tx_n\|<\varepsilon<\|x_n-y\|$$

for sufficiently large $n\in\mathbb{N}$. Since T satisfies (C_λ), we have $\|Tx_n-Ty\|\leqslant\|x_n-y\|$. So

$$\begin{aligned}\|x_n-Ty\|&\leqslant\|x_n-Tx_n\|+\|Tx_n-Ty\|\\&\leqslant\|x_n-Tx_n\|+\|x_n-y\|.\end{aligned}$$

Taking $n\to\infty$, we obtain (6). $\square$

Remark 4. In Proposition 1 we have seen that every mapping satisfying condition (E) is quasi-nonexpansive provided that it has a fixed point. This result holds also for type (C_λ) mappings with $0 < \lambda < 1$. (See Proposition 2 in [1].)

On the other hand, notice that if C is a bounded convex subset of X and $T : C \to C$ has positive minimal displacement then, from Proposition 3, it satisfies condition (E) on C, but from the above theorem it cannot satisfy condition (C_λ) for every $\lambda \in (0, 1)$. On the other hand, from Lemma 7 in [1], if a mapping $T : C \to C$ satisfies the condition (C) (that is $(C_{1/2})$) on C then it satisfies condition (E) on C with $\alpha = 3$.

A property of the nonexpansive mappings which is shared with (C_λ)-type mappings concerns to the structure of the fixed points set. Namely, we have the following statement which can be proved adapting the proof of Lemma 4 in [1]: *Let T be a mapping on a closed subset C of a Banach space X. Assume that T satisfies condition (C_λ) for some $\lambda \in (0, 1)$. Then $F(T)$ is closed. Moreover, if X is strictly convex and C is convex, then $F(T)$ is also convex.*

Finally, we do not know if condition (C_λ) for $\lambda \neq \frac{1}{2}$ implies condition (E). Nevertheless this fact holds for Lipschitzian mappings.

Proposition 5. *Let $T : C \to X$ be a Lipschitzian mapping with Lipschitz constant $Lip(T)$ satisfying condition (C_λ) for some $\lambda \in (0, 1)$. Then, T satisfies condition (E_μ) for $\mu = \max\{1, 1 + \lambda(Lip(T) - 1)\}$.*

Proof. If $Lip(T) \leqslant 1$ then T satisfies (E_1) obviously. If $Lip(T) > 1$ we have that $\mu > 1$. Suppose that T fails to satisfy (E_μ). Then there exists $x, y \in C$ such that

$$\|x - Ty\| > \mu\|x - Tx\| + \|x - y\|. \tag{7}$$

Consequently we have

$$\mu\|x - Tx\| + \|x - y\| < \|x - Tx\| + Lip(T)\|x - y\|,$$

that is

$$(\mu - 1)\|x - Tx\| < \big(Lip(T) - 1\big)\|x - y\|.$$

Since $\mu - 1 = \lambda(Lip(T) - 1)$ we obtain

$$\lambda\|x - Tx\| < \|x - y\|,$$

and therefore, by using that T satisfies condition (C_λ) one has that $\|Tx - Ty\| \leqslant \|x - y\|$. Thus, we derive that

$$\|x - Ty\| \leqslant \|x - Tx\| + \|Tx - Ty\| \leqslant \|x - Tx\| + \|x - y\|$$

which contradicts (7). □

Remark 5. The Kakutani mappings T_r considered in Example 4 are Lipschitzian with $Lip(T_r) = \sqrt{1 + r^2}$. Since in this example we showed that T_r does not have condition (E), by the above proposition we know that T_r fails to satisfy condition (C_λ) for every $\lambda \in (0, 1)$, in spite of the fact that $Lip(T_r)$ is as close to 1 as we wish.

Definition 4. Given a mapping $T : C \to X$, we say that $I - T$ is *strongly demiclosed at* 0_X if for every sequence (x_n) in C strongly convergent to $z \in C$ and such that $x_n - Tx_n \to 0_X$ we have that $z = Tz$.

This is a weaker version of the well-known demiclosedness principle, in which the weak convergence has been replaced by the strong convergence. Notice that for every continuous mapping (in particular for every Lipschitzian mapping) $T : C \to X$, $I - T$ is strongly demiclosed at 0_X. On the other hand, Example 5 shows that there are not continuous mappings satisfying the above property.

Proposition 6. *Let C be a nonempty subset of a Banach space X. If $T : C \to X$ satisfies condition (E) on C, then $I - T$ is strongly demiclosed at 0_X.*

Proof. Suppose that (x_n) is an a.f.p.s. for T in C such that $x_n \to z \in C$. From condition (E),

$$\|x_n - Tz\| \leqslant \alpha\|x_n - Tx_n\| + \|x_n - z\|$$

and taking limits as n goes to infinity we derive that $x_n \to Tz$. Hence $Tz = z$. □

The converse is not true; see Example 2.

Lemma 2. *Let T be a mapping on a convex subset C of a Banach space X. Assume that T satisfies condition (C_λ) for some $\lambda \in (0, 1)$. Assume also that there exist $x, y \in C$ and sequences (x_n) and (y_n) in C such that $x \neq y$, $\lim_n x_n = x$, $\lim_n y_n = y$, $\lim_n \|Tx_n - x_n\| = 0$ and $\lim_n \|Ty_n - y_n\| = 0$. Then x and y are fixed points of T.*

Proof. Let $\varepsilon \in (0, 1)$ be arbitrary. Put a convex subset D of C by

$$D = \{u \in C\colon \|u - x\| = \varepsilon\|x - y\|,\ \|u - y\| = (1 - \varepsilon)\|x - y\|\}.$$

By Lemma 1, D is invariant under T. Define a sequence (u_n) in D by $u_1 \in D$ and $u_{n+1} = \lambda T u_n + (1 - \lambda)u_n$. By Theorem 4, (u_n) is an a.f.p.s. for T. Using Lemma 1 again, we have

$$\begin{aligned}\|x - Tx\| &\leqslant \limsup_{n\to\infty}\big(\|x - u_n\| + \|u_n - Tx\|\big)\\ &\leqslant \varepsilon\|x - y\| + \limsup_{n\to\infty}\|u_n - x\|\\ &= 2\varepsilon\|x - y\|.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|x - Tx\| = 0$. Thus x is a fixed point of T. Similarly we can prove that y is a fixed point of T. □

The following lemma is deduced by Lemma 2.

Lemma 3. *Let T be a mapping on a convex subset C of a Banach space X. Assume that T satisfies condition (C_λ) for some $\lambda \in (0, 1)$. Assume also that T has a fixed point. Then x is a fixed point of T provided (x_n) is a sequence satisfying $\lim_n x_n = x$ and $\lim_n \|x_n - Tx_n\| = 0$.*

Theorem 5. *Let C be a nonempty weakly compact convex subset of a Banach space X. Let $T : C \to C$ be a mapping. If*

a) *T satisfies condition (C_λ) on C,*
b) *$(X, \|\cdot\|)$ satisfies the Opial condition, and*
c) *$I - T$ is strongly demiclosed at 0_X.*

Then, $Tz = z$.

Proof. From Theorem 4 there exists an a.f.p.s. (x_n) for T. Without loss of generality, we may suppose that $x_n \rightharpoonup z \in C$. If (x_n) admits any subsequence strongly converging to z, then from (c), $Tz = z$. Otherwise, by Lemma 1

$$\liminf_{n\to\infty}\|x_n - Tz\| \leqslant \liminf_{n\to\infty}\|x_n - z\|,$$

and from the Opial condition this implies that $Tz = z$. □

Theorem 6. *Let T be a mapping on a locally weakly compact convex subset C of a Banach space X. Assume that X satisfies the Opial condition, T satisfies condition (C_λ) for some $\lambda \in (0, 1)$ and T has a fixed point. Define a sequence (x_n) in C by $x_1 \in C$ and*

$$x_{n+1} = \mu T x_n + (1 - \mu)x_n \tag{8}$$

for $n \in \mathbb{N}$, where μ is a real number belonging to $[\lambda, 1)$. Then (x_n) converges weakly to a fixed point of T.

Proof. Since T is quasi-nonexpansive,

$$\{x \in C\colon \|x - z\| \leqslant r\}$$

is weakly compact, convex and T-invariant, where z is a fixed point of T and $r > 0$. So, without loss of generality, we may assume C is weakly compact. By Theorem 4, (x_n) is an a.f.p.s. for T. We consider the following two cases:

- (x_n) has a cluster point.
- (x_n) has no cluster points.

In the first case, let $y \in C$ be a cluster point of (x_n). Then there exists a subsequence (x_{n_j}) of (x_n) such that (x_{n_j}) converges strongly to y. By Lemma 3, y is a fixed point of T. Since T is quasi-nonexpansive, $(\|x_n - y\|)$ is a nonincreasing sequence. So (x_n) itself converges strongly to y. In the second case, arguing by contradiction, we assume that (x_n) does not converge weakly. Since C is weakly compact, we can choose subsequences (x_{n_j}) and (x_{n_k}) of (x_n) converging weakly to distinct points $y, z \in C$, respectively. Since X satisfies the Opial condition, by Lemma 1 y and z are fixed points of T. Also, $(\|x_n - y\|)$ and $(\|x_n - z\|)$ are nonincreasing. Using the Opial condition again, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - y\| &= \lim_{j\to\infty}\|x_{n_j} - y\| < \lim_{j\to\infty}\|x_{n_j} - z\| = \lim_{n\to\infty}\|x_n - z\|\\ &= \lim_{k\to\infty}\|x_{n_k} - z\| < \lim_{k\to\infty}\|x_{n_k} - y\| = \lim_{n\to\infty}\|x_n - y\|.\end{aligned}$$

This is a contradiction. Therefore we obtain the desired result. □

Theorem 7. *Let C be a nonempty weakly compact convex subset of a (UCED) Banach space X. Let $T : C \to C$ be a mapping. If*

a) *T satisfies condition (C_λ) on C, and*
b) *$I - T$ is strongly demiclosed at 0_X.*

Then, T has a fixed point.

Proof. From Theorem 4 there exists an a.f.p.s. (x_n) for T. Define a continuous convex function $f : C \to [0, +\infty)$ by $f(x) = \limsup_n \|x - x_n\|$ for all $x \in C$. Since C is weakly compact and convex, and f is weakly lower semicontinuous there exists $x_0 \in C$ such that $f(x_0) = \min\{f(x)\colon x \in C\}$ in other words, the sequence (x_n) admits an asymptotic center $x_0 \in C$. Let $K := \{x \in C\colon f(x) \leqslant f(x_0)\}$. Since X is (UCED) then it is well known that $K = \{x_0\}$. Finally let us see that $T(x_0) \in K$, and hence $x_0 = Tx_0$. Indeed, if (x_n) admits any subsequence strongly converging to x_0, then from (b), $Tx_0 = x_0$. Otherwise, by Lemma 1

$$f(Tx_0) = \limsup_{n\to\infty} \|x_n - Tx_0\| \leqslant \limsup_{n\to\infty} \|x_n - x_0\| = f(x_0).$$

Consequently, $Tx_0 \in K$. □

Remark 6. If T satisfies condition $(C_{1/2})$ on C, then from Lemma 7 in [1] along with Proposition 6, $I - T$ is strongly demiclosed at 0_X. Therefore, Theorems 5 and 7 allow us to recapture Theorems 4 and 5 in [1] respectively.

By Corollary 1 and Theorems 2, 3 and 4 (or by Proposition 6 and Theorems 5 and 7), we obtain the following theorem. We note that this theorem is a real generalization of Theorems 4 and 5 in [1] because of Example 5.

Theorem 8. *Let C be a convex subset of a Banach space X. Let $T : C \to C$ be a mapping satisfying (E) and (C_λ) for some $\lambda \in (0, 1)$. Assume either of the following holds.*

a) *C is weakly compact and $(X, \|\cdot\|)$ satisfies the Opial condition.*
b) *C is compact.*
c) *C is weakly compact and X is (UCED).*

Then, T has a fixed point.

The following theorem tells that if C is a closed interval of $\mathbb{R}$ and T satisfies (C_λ) for some $\lambda \in [0, 3/4]$, then T has a fixed point.

Theorem 9. *Let C be a closed interval of $\mathbb{R}$. Let T be a mapping on C satisfying condition $(C_{3/4})$. Then T has a fixed point.*

Proof. Define a sequence (x_n) in C by $x_1 \in C$ and (8), where $\mu \in [3/4, 1)$. We consider the following two cases:

- (x_n) has at least two cluster points.
- (x_n) has only one cluster point.

In the first case, by Lemma 2, cluster points are fixed points of T. So, by the proof of Theorem 6, (x_n) converges to their cluster points. Since $\mathbb{R}$ is Hausdorff, this is a contradiction. Therefore the first case cannot be possible. In the second case, we note that (x_n) converges to some $y \in C$. Arguing by contradiction, we assume that T has no fixed points. In particular, $Ty \neq y$. Put $y_a = (1 - a)y + aTy$ and define a mapping S on some subset of $\mathbb{R}$ by

$$Ty_a = y_{Sa}$$

for all $a \in \mathbb{R}$ with $y_a \in C$. Put $\ell := |y - Ty| > 0$. Then we note

$$|y_a - y_b| = |a - b|\ell.$$

We also note if $a \neq 0$, then $|y - Ty_a| \leqslant |y - y_a|$ holds by Lemma 1. Thus

$$|Sa| \leqslant |a| \tag{9}$$

holds for every $a \in \mathbb{R}$ with $a \neq 0$ and $y_a \in C$. Put

$$A = \big\{a \in \mathbb{R}\colon y_a \in C,\ |Ty - Ty_a| \leqslant |y - y_a|\big\}.$$

Since $(3/4)|y - Ty| = |y - y_{3/4}|$, we have $3/4 \in A$. It follows from $3/4 \in A$ and (9) that $-3/4 \leqslant S(3/4) \leqslant 3/4$ and $|1 - S(3/4)| \leqslant 3/4$. So $1/4 \leqslant S(3/4) \leqslant 3/4$ holds. We have

$$(3/4)|y_{3/4} - Ty_{3/4}| = (3/4)\big|3/4 - S(3/4)\big|\ell \leqslant (3/4)(3/4 - 1/4)\ell = (3/8)\ell = |y_{3/4} - y_{3/8}|$$

and hence $|Ty_{3/4} - Ty_{3/8}| \leqslant |y_{3/4} - y_{3/8}|$. So

$$-1/8 = -|3/4 - 3/8| + 1/4 \leqslant -|3/4 - 3/8| + S(3/4) \leqslant S(3/8) \leqslant 3/8.$$

If $3/8 \in A$, then

$$|y - Ty| \leqslant |y - Ty_{3/8}| + |Ty_{3/8} - Ty| \leqslant 2|y - y_{3/8}| = (3/4)|y - Ty| < |y - Ty|,$$

which is a contradiction. So $3/8 \notin A$. From the assumption, $(3/4)|y_{3/8} - Ty_{3/8}| > |y_{3/8} - y|$. We have

$$(3/8)\ell = (3/4)\big(3/8 - (-1/8)\big)\ell \geqslant (3/4)|y_{3/8} - Ty_{3/8}| > |y_{3/8} - y| = (3/8)\ell,$$

which is a contradiction. Therefore we obtain the desired result. □

The constant 3/4 is best possible to assure the existence of fixed points.

Example 6. Put $X = \mathbb{R}$ and $C = [-1/4, 1]$. Define a mapping T on C by

$$Tx = \begin{cases} 1 & x = 0, \\ -(1/3)x & x \in [-1/4, 0) \cup (0, 3/4), \\ 1 - x & x \in [3/4, 1]. \end{cases}$$

Then T satisfies

$$(3/4)|x - Tx| < |x - y| \quad \Longrightarrow \quad |Tx - Ty| \leqslant |x - y|.$$

However, T does not have a fixed point.

Proof. Put $C_1 = [-1/4, 0)$, $C_2 = \{0\}$, $C_3 = (0, 3/4)$ and $C_4 = [3/4, 1]$. It is easy to check $|Tx - Ty| \leqslant |x - y|$ in the case where

$$(x, y) \in (C_1 \cup C_3)^2 \cup (C_2 \cup C_4)^2 \cup (C_1 \cup C_4)^2.$$

In the case where $(x, y) \in (C_1 \cup C_3) \times C_2$, we have

$$(3/4)\min\big\{|x - Tx|, |y - Ty|\big\} \geqslant |x - y|. \tag{10}$$

Finally we consider the case where $(x, y) \in C_3 \times C_4$. If $2x < y$, we have

$$|Tx - Ty| = 1 - y + (1/3)x = \big(1 - (4/3)y\big) + (2/3)(2x - y) + y - x < |x - y|.$$

If $2x \geqslant y$, we have

$$(3/4)|x - Tx| = x \geqslant x + (y - 2x) = |x - y|$$

and

$$(3/4)|y - Ty| = (y - 3/4) + (1/2)y \geqslant (1/2)y + (1/2)(y - 2x) = |x - y|.$$

Hence we obtain (10). It is obvious that T does not have a fixed point. □

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