On the normalized eigenvalue problems for nonlinear elliptic operators II

Jing Lin

Department of Mathematics, College of Science, Hefei University of Technology, Hefei, Anhui 230009, PR China

Received 28 July 2006
Available online 31 January 2007
Submitted by D. O’Regan

Abstract

This paper continues our previous research on the following form of normalized eigenvalue problem

\[ Au - C(\lambda, u) = 0, \quad \lambda \geq 0 \text{ and } u \in \partial \mathcal{D}, \]

where the operator \( A \) is maximal monotone on an infinitely dimensional, real reflexive Banach space \( X \) with both \( X \) and its dual space \( X^* \) locally uniformly convex, \( \mathcal{D} \subset X \) is a bounded open set, the operator \( C \) is defined only on \( \mathbb{R}_+ \times \partial \mathcal{D} \) such that the closure of a subset of \( \{ C(\lambda, u)/\|C(\lambda, u)\| \} \) is not equal to the unit sphere of \( X^* \). This research reveals the fact that such eigenvalue problems do not depend on the properties of \( C \) located in \( \mathbb{R}_+ \times \mathcal{D} \). Similar result holds for the bounded, demicontinuous \((S)_+\) operator \( A \). This remarkable discovery is applied to the nonlinear elliptic operators under degenerate and singular conditions. © 2007 Elsevier Inc. All rights reserved.

Keywords: Continuous extension; Maximal monotone; Eigenvalue; \((S)_+\); Nonlinear elliptic equations

1. Introduction and motivation

Let \( X \) and \( Y \) be real Banach spaces, and \( X^* \) the dual space of \( X \). Denote by \( \mathbb{N}_+ \) the set of all positive integers. Let \( \mathcal{D} \) be a subset of \( X \) with its boundary and closure denoted by \( \partial \mathcal{D} \) and \( \overline{\mathcal{D}} \), respectively. Let \( A : X \to 2^Y \) be a mapping with the domain and range denoted by \( D(A) = \{ x \in X : Ax \neq \emptyset \} \) and \( R(A) = \{ Ax : x \in D(A) \} \), respectively. A mapping \( A : X \supset D(A) \to 2^Y \) is “bounded” if it maps bounded subsets of \( D(A) \) into bounded sets. A mapping \( A : X \supset D(A) \to Y \) is “continuous” if it is continuous on \( D(A) \) from the strong topology of \( X \) to the strong topology.

E-mail address: jlin@hfut.edu.cn.

0022-247X/– see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.01.065
of $Y$. It is “demicontinuous” if it is strong-weak continuous on $D(A)$. It is “compact” if it is continuous and maps bounded subsets of $D(A)$ onto relatively compact subsets of $Y$. The symbol “→” (“⇒”) means strong (weak) convergence. The convex hull of a set $E$ is denoted by $\text{co}(E)$, whose closure is denoted by $\text{co}(E)$. An operator $A : X \supset D(A) \to 2^{X^*}$ is said to be “monotone” if the functional
\[
(u - v, x - y) \geq 0 \quad \forall x, y \in D(A), \ u \in Ax, \ v \in Ay.
\]

Let $I : X \to X$ be the identity mapping, $J : X \to 2^{X^*}$ the normalized dual mapping. A monotone operator is “maximal monotone” if $R(A + \lambda J) = X^*$ for every $\lambda > 0$. An operator $A : X \supset D(A) \to X^*$ is said to satisfy condition $(S)_+$ if for every sequence $\{u_n\} \subset D(A)$ such that $u_n \to u_0$ and $\limsup_{n \to \infty} (Au_n, u_n - u_0) \leq 0$ we have $u_n \to u_0$.

The degree theory for mappings of class $(S)_+$ has a long history, Skrypnik constructed this theory in [11] in the separable reflexive Banach spaces, then Browder established this theory in [2] in the general reflexive spaces. See also Berkovits and Mustonen [1] for a different approach in separable spaces. Chang and Chen [3] extended this theory to the multi-valued mapping case. Recently Kartsatos and Skrypnik showed that the solvability of some kind of the nonlinear theory in [11] in the separable reflexive Banach spaces, then Browder established this theory in [1] for a different approach in separable spaces. Chang and Chen [3] extended this theory to the multi-valued mapping case. Recently Kartsatos and Skrypnik showed that the solvability of some kind of the nonlinear elliptic operators under degenerate conditions can be reduced to the abstract eigenvalue problems of densely defined unbounded operators [8]. One of their results is as follows, where the notation $\text{Deg}(\cdot, \cdot, \cdot)$ is the degree function adopted from [10].

**Theorem 1.1.** (See [8, p. 451].) Let $X$ be an infinitely dimensional real reflexive Banach space and let $D$ be a bounded open set in $X$, and $X^*$ the dual space with norm $\| \cdot \|$. Assume that $A : \overline{D} \to X^*$ is a bounded demicontinuous operator which satisfies condition $(S)_+$ and is such that $Au \neq 0, u \in \partial D$, and $\text{Deg}(A, \partial D, 0) \neq 0$. Let $C : \overline{\mathbb{R}}_+ \times \overline{D} \to X^*$ be a compact operator which satisfies conditions:

(i) there exists a positive number $N$ such that the closure of the set
\[
E = \{ C(\lambda, u)/\| C(\lambda, u) \| : \ u \in \overline{D}, \ \lambda \in \overline{\mathbb{R}}_+, \ \| C(\lambda, u) \| > N \}
\]

is not equal to $S_1 = \{ v \in X^* : \| v \| = 1 \}$;

(ii) the following property holds:
\[
\lim_{\lambda \to \infty} m_\lambda = +\infty, \quad \text{where} \quad m_\lambda = \inf_{u \in \partial D} \| C(\lambda, u) \|,
\]

and is such that $C(0, u) \equiv 0$, for $u \in \partial D$. Then there exist $\lambda_0 > 0$ and $u_0 \in \partial D$ such that
\[
Au_0 - C(\lambda_0, u_0) = 0. \tag{1}
\]

A new approach was developed in [8] so as to prove the above theorem, that is, constructing a new continuous mapping $\tilde{C}$ based on values of $C$ on $\overline{\mathbb{R}}_+ \times \partial D$ so that $\inf_{u \in D} \| \tilde{C}(\lambda, u) \| \to \infty$ as $\lambda \to \infty$. A natural question arises: now that this construction does not make use of the properties of $C$ inside $D$, is it sufficient to have $C$ only defined and continuous on $\overline{\mathbb{R}}_+ \times \partial D$ instead of $\overline{\mathbb{R}}_+ \times \overline{D}$ while keeping the conclusion of Theorem 1.1 valid? This problem is interesting and valuable in many applications because it allows $C$ to possess singularities inside $D$, especially a singularity at zero. This work is motivated by the above observation and continues our research [9] in a new kind of eigenvalue problem:

**Problem 1.2.** Solve (1) with $C$ only defined on $\overline{\mathbb{R}}_+ \times \partial D$. 
As well known, the operator \( C \) in the standard eigenvalue problems is usually defined on \( \overline{D} \).

[9] figured out a new technique so as to improve Theorem 4 in [8], where a condition similar to condition (i) in Theorem 1.1 assumed that zero was not in the weak closure of the corresponding set \( E \). We find such an technique can be modified and used to improve the above Theorem 1.1 in this paper. It turns out that the eigenvalue problem (1) do not depend on any properties of \( C \) located in \( \mathbb{R}_+ \times D \) provided \( X^* \) is locally uniformly convex. We will show how this remarkable discovery can be used to improve some results on the solvability of nonlinear elliptic operators under degenerate and singular conditions in Sections 3 and 4. Our main tool, Proposition 2.3, is proved in Section 2.

2. The main proposition

**Lemma 2.1.** Let \( (Y, \| \cdot \|) \) be a real, strictly convex Banach space and \( \eta \in Y \) with \( \| \eta \| = 1 \) such that the norm of \( Y \) is locally uniformly convex at \( \eta \). For any \( p > 0 \) and \( \delta \in (0, 1) \), define

\[
E_\eta(p, \delta) = \{ y \in Y : \| y \| \geq p, \| y \| - \| \eta \| \geq \delta \},
\]

and the mapping \( T \) by

\[
Ty = \frac{y - \| y \| \eta}{\| y \| - \| \eta \|} \quad \text{for} \ y \in D(T) = \{ y \in Y : y \neq \| y \| \eta \}. \tag{3}
\]

Let \( p \) and \( \delta \) be any positive numbers with \( \delta \in (0, 1) \). Then the following statements hold:

- (P1) \( \| Ty \| = \| y \|, \ T(\mu y) = \mu Ty \) and \( \mu E_\eta(p, \delta) = E_\eta(\mu p, \delta) \ \forall y \in D(T), \ \forall \mu > 0 \);
- (P2) \( T \) is continuous and the set \( T(E_\eta(p, \delta)) \) is closed;
- (P3) there exists a number \( \epsilon \in (0, 1) \) depending on \( \delta \) such that
  \[
  \overline{co}(T(E_\eta(\mu, \delta))) \subset T(E_\eta(\mu \epsilon, \delta \epsilon)) \quad \text{for all} \ \mu > 0;
  \]
- (P4) \( T^{-1} \) is continuous on the set \( T(E_\eta(p, \delta)) \).

**Proof.** Property (P1) is obvious and \( T \) is continuous at any \( y \in D(T) \). To show (P2), let \( y_n \in E_\eta(p, \delta) \) be a sequence and \( Ty_n \rightarrow w \). Then \( p \leq \| y_n \| = \| Ty_n \| \| w \| \); on the other hand, because the set \( \{ \| y_n \| / \| y_n \| - \| \eta \| \} \) is bounded, it has a subsequence converging to some \( \alpha \in [\delta, 2] \).

By virtue of definition of \( Ty_n \), we have \( y_n \rightarrow y \equiv \alpha w + \| w \| \eta \) along such a subsequence, which in turn implies that \( \| y \| = \| w \| \geq p \) and \( \| y \| - \| \eta \| = \alpha \), that is, \( y \in E_\eta(p, \delta) \). Thus, \( w = (y - \| w \| \eta) / \alpha = Ty \). This proves (P2).

To show (P3), for any \( \mu > 0 \) and any \( w \in co(T(E_\eta(\mu, \delta))) \), we have

\[
w = \sum_{k=1}^{n} \beta_k Ty_k,
\]

where \( n \) is a positive integer, each \( y_k \in E_\eta(\mu, \delta) \) and \( \beta_k > 0 \) with \( \sum_{k=1}^{n} \beta_k = 1 \). Because the norm of \( Y \) is the locally uniform convexity at \( \eta \), there exists a number \( \epsilon \in (0, 1) \) such that

\[
\| z + \eta \| = 2(1 - \epsilon) \quad \text{provided} \ z \in Y, \ \| z \| = 1 \text{ and} \ \| z - \eta \| \geq \delta. \tag{4}
\]

Let \( d_k \equiv \| y_k / y_k - \eta \| \) and \( \gamma = \sum_{k=1}^{n} \beta_k \| y_k \| / d_k \). Then \( d_k \in [\delta, 2] \) and

\[
\| w \| = \left\| \sum_{k=1}^{n} \beta_k Ty_k \right\| = \left\| \sum_{k=1}^{n} \beta_k \frac{y_k - \gamma y_k}{d_k} \right\| \leq \gamma \left\| \sum_{k=1}^{n} \frac{\beta_k \| y_k \| / d_k}{\gamma} y_k - \eta \right\|.
\]
\[ \gamma \|2\eta\| - \gamma \left( \sum_{k=1}^{n} \frac{\beta_k \| y_k \|/d_k}{\gamma} y_k + \eta \right) \geq 2\gamma - \gamma \left( \sum_{k=1}^{n} \frac{\beta_k \| y_k \|/d_k}{\gamma} (y_k + \eta) \right) \]

\[ \geq 2\gamma - \gamma 2(1 - \epsilon) = 2\epsilon \gamma. \tag{5} \]

Hence,

\[ \|w\| \geq 2\epsilon \sum_{k=1}^{n} \frac{\beta_k \| y_k \|}{d_k} \geq \epsilon \sum_{k=1}^{n} \beta_k \| y_k \| \geq \epsilon \sum_{k=1}^{n} \beta_k \mu = \mu \epsilon, \tag{6} \]

which keeps valid for any \( \mu > 0 \). Define the function

\[ f(\alpha) = \left\| \frac{\alpha w}{\|w\|} + \eta \right\|, \quad \alpha > 0. \]

\( f(\alpha) \) is continuous in \( \alpha \). In the case \( \alpha \in (0, \delta) \), let \( \tau = \sum_{k=1}^{n} \beta_k \| y_k \| \), then \( \gamma \geq \tau / 2 \) and \( 0 < \mu_k \equiv \alpha \tau / (\|w\|d_k) < \delta \epsilon 2\gamma / (\|w\|\delta) \leq 1 \) by (5), thereby

\[ f(\alpha) = \left\| \alpha \sum_{k=1}^{n} \beta_k \frac{y_k}{\|w\|} + \eta \right\| = \left\| \sum_{k=1}^{n} \beta_k \| y_k \| \cdot \frac{\alpha \tau}{\|w\|d_k} \left( \frac{y_k}{\|y_k\|} - \eta \right) + \eta \right\| \leq \sum_{k=1}^{n} \beta_k \left( \mu_k \left( \frac{y_k}{\|y_k\|} - \eta \right) + \eta \right) \leq \max_{k} \left( \mu_k \frac{y_k}{\|y_k\|} + (1 - \mu_k)\eta \right) < 1 \]

by the strict convexity of \( Y \); on the other hand, \( f(2) \geq 1 \), thereby there exists number \( \alpha_w \in [\delta, 2] \) such that \( f(\alpha_w) = 1 \). Because \( f(\alpha) = 1 \) and \( f(0) = 1 \) and \( Y \) is strictly convex, \( f(\alpha) \) cannot be equal to one for any other \( \alpha \). This proves the uniqueness of \( \alpha_w \). Let \( y_w = \alpha_w w + \|w\| \eta \). Then by (6),

\[ \|y_w\| = f(\alpha_w)\|w\| = \|w\| \geq \mu \epsilon. \]

\[ \delta \epsilon \leq \alpha_w = \frac{\|y_w - \|w\|\eta\|}{\|w\|} = \frac{\|y_w\| - \eta}{\|w\|}. \]

Hence, \( w = (y_w - \|w\|\eta) / \alpha_w = T y_w \), with \( y_w \in E_{\eta}(\mu \epsilon, \delta \epsilon) \). In other word, \( w \in T(E_{\eta}(\mu \epsilon, \delta \epsilon)) \) that is a closed subset according to (P2). This proves (P3).

We show that \( T^{-1} \) is single-valued, suppose \( Ty = Tz \) for \( y, z \in D(T) \), then \( \|y\| = \|z\| \) by (P1), we can assume that there exists a number \( \lambda \in (0, 1] \) such that

\[ \frac{y}{\|y\|} - \eta = \lambda \left( \frac{z}{\|z\|} - \eta \right), \quad \text{that is} \quad \frac{y}{\|y\|} = \lambda \frac{z}{\|z\|} + (1 - \lambda)\eta, \tag{7} \]

which implies \( \lambda = 1 \) since \( Y \) is strictly convex, thereby \( y = z \). To show the continuity of \( T^{-1} \) on the closed set \( (E_{\eta}(p, \delta)) \), let \( y_n, y \in E_{\eta}(p, \delta) \) and \( Ty_n \to Ty \). Then \( \|y_n\| = \|Ty_n\| \to \|Ty\| = \|y\| \neq 0 \) and there exists \( \delta > 0 \) such that \( \|y/\|y\| - \eta\| > \delta \).

\[ \frac{1}{\lambda_n} \left( \frac{y_n}{\|y_n\|} - \eta \right) \to \frac{y}{\|y\|} - \eta \quad \text{with} \quad \lambda_n = \frac{\|y\|}{\|y_n\|} \frac{\|y\|}{\|y_n\| - \eta} \in \left[ \frac{\delta}{4}, \frac{4}{\delta} \right] \]

for large \( n \). We can assume \( \lambda_n \to \lambda \) by passing to subsequence if necessary, and

\[ \frac{y_n}{\|y_n\|} \to \lambda \frac{y}{\|y\|} + (1 - \lambda)\eta, \tag{8} \]
the norm of right-hand side of which is one, that is only possible if \( \lambda = 1 \) because of the strict convexity. Hence \( y_n \to y \). This proves (P4).

\[ \text{Remarks 2.2.} \ (6) \text{ implies that there exists a positive number } \epsilon \text{ such that for any } \mu > 0 \text{ and } \delta \in (0, 1), \text{ it holds} \]
\[
\left\| \sum_{k=1}^{n} \beta_k T y_k \right\| \geq \epsilon \sum_{k=1}^{n} \beta_k \| T y_k \| ,
\]
where \( n \) is a positive integer, each \( \beta_k > 0 \) with \( \sum_{k=1}^{n} \beta_k = 1 \) and each \( y_k \in E_\eta(\mu, \delta) \).

Our main results are based on the following proposition, where the product space \( \mathbb{R}_+ \times M \) is equipped with the standard product topology.

**Proposition 2.3.** Let \( D \) be a nonempty proper open subset of the metric space \( M \), and \( Y \) a real, strictly convex Banach space with norm \( \| \cdot \| \). Let \( C : \mathbb{R}_+ \times \partial D \to Y \) be continuous and satisfies the following condition:

(i) there exist a positive number \( N \) and \( \eta \in Y \) with \( \| \eta \| = 1 \) such that \( Y \) is locally uniformly convex at \( \eta \) and that \( \eta \) is not in the closure of the set
\[
E^* = \left\{ \frac{C(\lambda, u)}{\| C(\lambda, u) \|} : u \in \partial D, \ \lambda \in \mathbb{R}_+, \ \| C(\lambda, u) \| \geq N \right\};
\]

(ii) the following property holds:
\[
\lim_{\lambda \to \infty} m_\lambda = +\infty, \quad \text{where } m_\lambda = \inf \left\{ \| C(\lambda, u) \| : u \in \partial D \right\}.
\]

Then there is number \( \epsilon \in (0, 1) \) such that for each \( N \geq N \) and for any closed subset \( B \subset M \) with \( B \cap \partial D \neq \emptyset \), there exists a continuous mapping \( C_N : \mathbb{R}_+ \times (B \cap \overline{D}) \to Y \) having the following properties:

\[
C_N(\lambda, u) = C(\lambda, u) \quad \text{for } (\lambda, u) \in \mathbb{R}_+ \times (B \cap \partial D),
\]
\[
\| C_N(\lambda, u) \| \geq N\epsilon \quad \text{for } (\lambda, u) \in [A_N, \infty) \times (B \cap \overline{D}),
\]
\[
\sup_{(\lambda, u) \in [0, A_N] \times (B \cap \overline{D})} \left\| C_N(\lambda, u) \right\| \leq \sup_{(\lambda, u) \in [0, A_N] \times (B \cap \overline{D})} \| C(\lambda, u) \| ,
\]

where \( A_N \) is a number depending on \( N \) such that \( m_\lambda \geq N \) for \( \lambda \geq A_N \). Furthermore, if \( C \) is compact on \( \mathbb{R}_+ \times \partial D \) and \( B \) is bounded, then the operator \( C_N \) is compact on \( [0, A_N] \times (B \cap \overline{D}) \); if the closure of the set \( E^* \) is compact, then there exists a compact subset \( K \subset \{ y \in Y : \| y \| = 1 \} \) such that \( \eta \notin K \) and
\[
\left\{ \frac{C_N(\lambda, u)}{\| C_N(\lambda, u) \|} : \lambda \geq A_N, \ u \in B \cap \overline{D} \right\} \subset K, \ \forall N > N'.
\]

**Remarks 2.4.** If, in addition, assume that \( C(0, u) = 0 \) for \( u \in \partial D \) in the Proposition 2.3, then we can have \( C_N(0, u) = 0 \) for \( u \in B \cap \overline{D} \).

**Proof.** Because \( \eta \) is not in the closure of \( E^* \), there exists number \( \delta \in (0, 1) \) such that
\[
\left\| \frac{C(\lambda, u)}{\| C(\lambda, u) \|} - \eta \right\| \geq \delta, \quad \text{provided } (\lambda, u) \in \mathbb{R}_+ \times \partial D \text{ and } \| C(\lambda, u) \| \geq N'.
\]
Let the set \( E_\eta(p, \delta) \) and the mapping \( T \) be defined as that of Lemma 2.1. According to (P3) of Lemma 2.1, there exists a number \( \epsilon \in (0, 1) \) depending on \( \delta \) such that
\[
\overline{\text{co}}(T(E_\eta(N, \delta))) \subset T(E_\eta(N\epsilon, \delta\epsilon)) \quad \forall N > 0.
\] (15)
Now for each number \( N \geq N' \), by condition (ii), there exists a number \( \Lambda_N \geq 0 \) such that \( m_\lambda \geq N \) for \( \lambda \geq \Lambda_N \), which consolidating with (14) implies
\[
\{ C(\lambda, u): \lambda \geq \Lambda_N, u \in \partial D \} \subset E_\eta(N, \delta).
\] (16)
Let \( B \subset \mathcal{M} \) be a closed subset such that \( B \cap \partial D \neq \emptyset \). (16) implies
\[
C([\Lambda_N, \infty) \times (B \cap \partial D)) \subset E_\eta(N, \delta).
\] (17)
By (P2) of Lemma 2.1, \( T \) is continuous on the closed set \( E_\eta(p, \delta) \), therefore \( TC \) is continuous on the closed subset \( \{ \Lambda_N \} \times (B \cap \partial D) \) and has a continuous extension \( h \) on \( \{ \Lambda_N \} \times (B \cap \partial D) \) by the Dugundji theorem [5, p. 163] such that
\[
h(\Lambda_N, u) = TC(\Lambda_N, u) \quad \text{for} \ u \in B \cap \partial D,
\] (18)
\[
h(\{ \Lambda_N \} \times (B \cap \partial D)) \subset \text{co}\{ TC(\{ \Lambda_N \} \times (B \cap \partial D)) \},
\] (19)
on which \( T^{-1} \) is continuous by (17), (15) and (P4) of Lemma 2.1. In the case \( \lambda \geq \Lambda_N \), define \( g \) by
\[
g(\lambda, u) = \begin{cases} h(\lambda, u) & \text{for} \ \lambda = \Lambda_N \ \text{and} \ u \in B \cap \partial D, \\ TC(\lambda, u) & \text{for} \ (\lambda, u) \in (\Lambda_N, \infty) \times (B \cap \partial D). \end{cases}
\] (20)
g is well-defined since \( h(\Lambda_N, u) = TC(\Lambda_N, u) \) for \( u \in B \cap \partial D \), and \( g \) is continuous on a closed subset, thereby can be continuously extended to \( [\Lambda_N, \infty) \times (B \cap \partial D) \) satisfying
\[
g([\Lambda_N, \infty) \times (B \cap \partial D)) \subset \text{co}\{ TC([\Lambda_N, \infty) \times (B \cap \partial D)) \} \subset T(E_\eta(N\epsilon, \delta\epsilon))
\] (21)
because of (19) and (17) as well as (15). In the case \( \lambda \in [0, \Lambda_N] \), define
\[
\hat{g}(\lambda, u) = \begin{cases} T^{-1}h(\lambda, u) & \text{for} \ \lambda = \Lambda_N \ \text{and} \ u \in (B \cap \partial D), \\ C(\lambda, u) & \text{for} \ (\lambda, u) \in [0, \Lambda_N) \times (B \cap \partial D). \end{cases}
\] (22)
If \( C(0, u) = 0 \) for \( u \in \partial D \), then add the condition \( \hat{g}(0, u) = 0 \) for \( u \in B \cap \partial D \) for Remark 2.4. \( \hat{g} \) is well-defined since \( T^{-1}h(\Lambda_N, u) = C(\Lambda_N, u) \) for \( u \in \partial D \), and \( \hat{g} \) is continuous on a closed subset, and can be continuously extended to \( [0, \Lambda_N] \times (B \cap \partial D) \) satisfying
\[
\hat{g}([0, \Lambda_N] \times (B \cap \partial D)) \subset \text{co}\{ C([0, \Lambda_N] \times (B \cap \partial D)) \cup T^{-1}\text{co}\{ TC([\Lambda_N, \infty) \times (B \cap \partial D)) \} \}
\] (23)
by (22) and (19). Combine \( g \) and \( \hat{g} \) and define
\[
C_N(\lambda, u) = \begin{cases} T^{-1}g(\lambda, u) & \text{for} \ (\lambda, u) \in [\Lambda_N, \infty) \times (B \cap \partial D), \\ \hat{g}(\lambda, u) & \text{for} \ (\lambda, u) \in [0, \Lambda_N) \times (B \cap \partial D). \end{cases}
\] (24)
Then \( C_N \) is well-defined on \( \{ \Lambda_N \} \times (B \cap \partial D) \) and continuous on \( \mathbb{F}_+ \times (B \cap \partial D) \), which leads to (10). (21) and (24) lead to
\[
C_N([\Lambda_N, \infty) \times (B \cap \partial D)) \subset E_\eta(N\epsilon, \delta\epsilon),
\] (25)
which implies (11). In the right-hand side of (23), let \( z \in T^{-1}\text{co}\{ TC([\Lambda_N] \times (B \cap \partial D)) \} \). Because \( T \) and \( T^{-1} \) are norm-preserving, \( ||z|| \leq \sup_{u \in B \cap \partial D} ||C(\Lambda_N, u)|| \), therefore (23) implies (12).
Furthermore, assume that $C$ is compact on $\overline{\mathbb{R}}_+ \times \partial D$ and the above $B$ is bounded. Then the set $C([0, \Lambda N] \times (B \cap \partial D))$ is a relatively compact set, so is the set $T^{-1} \text{co}(TC([\Lambda N] \times (B \cap \partial D)))$ by the Mazur’s theorem [5, p. 603]. Therefore the set on the right-hand side of (23) is pre-compact, that is, $\hat{g}$ maps the set $[0, \Lambda N] \times (B \cap \partial D)$ into a compact set. Hence, $C_N$ is a compact operator on $[0, \Lambda N] \times (B \cap \partial D)$.

To show (13), let $K$ be the closure of $E^*$. $K$ is compact by the assumption. Now for any number $N \geq N, \lambda \geq \Lambda N$ and $u \in B \cap \partial D$, we have

$$
C_N(\lambda, u) = T^{-1}g(\lambda, u),
$$

where $g(\lambda, u) = \sum_{k=1}^{n} \beta_k Ty_k$, $n$ is a positive integer, $\beta_k > 0$ with $\sum_{k=1}^{n} \beta_k = 1$, $y_k = C(\lambda_k, u_k)$ with $(\lambda_k, u_k) \in ([\Lambda N, \infty) \times (B \cap \partial D)$ for $k = 1, \ldots, n$. By virtue of Remarks 2.2, we have

$$
y_k \in E_\eta(N, \delta) \quad \text{and} \quad \frac{y_k}{\|y_k\|} \in E_\eta(1, \delta),
$$

$$
\frac{C_N(\lambda, u)}{\|C_N(\lambda, u)\|} = \frac{T^{-1}g(\lambda, u)}{\|T^{-1}g(\lambda, u)\|} = \frac{T^{-1}g(\lambda, u)}{\|g(\lambda, u)\|} = \frac{\sum_{i=1}^{n} \beta_i T y_i}{\|\sum_{j=1}^{n} \beta_j T y_j\|} \frac{T^{-1}\left(\sum_{k=1}^{n} \left(\frac{\beta_k}{\sum_{i=1}^{n} \beta_i T y_i} \right) T\left(\frac{y_k}{\|y_k\|}\right)\right)}{\|g(\lambda, u)\|} \in \left\{v y: v \in [1, \epsilon^{-1}], y \in T^{-1}(\overline{\partial D}(K \cap E_\eta(1, \delta)))\right\},
$$

the right-hand side of which is a well-defined pre-compact set since $K$ is compact and $T^{-1}$ is continuous on the set $\overline{\partial D}(K \cap E_\eta(1, \delta)) \subset \overline{\partial D}(E_\eta(1, \delta)) \subset T(E_\eta(\epsilon, \delta \epsilon))$.

On the other hand, we have

$$
\frac{C_N(\lambda, u)}{\|C_N(\lambda, u)\|} \in E_\eta(1, \delta \epsilon)
$$

because of (25). Hence, there exists a compact set $K \subset E_\eta(1, \delta \epsilon)$ such that (13) holds, where $\delta$ and $\epsilon$ depend on $E^*$. $\square$

3. Applications on eigenvalue problems of $(S)_+$ operators

In this section $X$ is a real infinitely dimensional Banach space with norm $\|\cdot\|_X$, and $X^*$ is its dual space with norm $\|\cdot\|$. We will use Proposition 2.3 to improve Theorem 1.1 and Theorem 6 in [8].

**Theorem 3.1.** Theorem 1.1 keeps valid with the operator $C: \overline{\mathbb{R}}_+ \times \overline{D} \to X^*$ and the set $E$ replaced by $C: \overline{\mathbb{R}}_+ \times \partial D \to X^*$ and

$$
\left\{\frac{C(\lambda, u)}{\|C(\lambda, u)\|}: u \in \partial D, \lambda \in \overline{\mathbb{R}}_+, \|C(\lambda, u)\| \geq N\right\}
$$

respectively, provided $X^*$ is locally uniformly convex.

**Proof.** By the assumption $Y$ is locally uniformly convex, and all conditions of Proposition 2.3 hold with $M = X$ and $Y = X^*$, therefore there exists a number $\epsilon \in (0, 1)$ such that for any $N \geq N$ and $B = \overline{D}$, the operator $C$ has a continuous extension $C_N: \overline{\mathbb{R}}_+ \times \overline{D} \to X^*$ having the
properties (10)–(12) and Remark 2.4, and $C_N$ is compact on $[0, \Lambda_N] \times \overline{D}$ with $\Lambda_N$ defined as that of Proposition 2.3. Now choose a number $N$ such that

$$N \epsilon > \max \left\{ N, \sup_{u \in \overline{D}} \| Au \| \right\}.$$  

Then (11) implies

$$Au - C_N(\Lambda_N, u) \neq 0 \text{ for } u \in \overline{D}.$$  

(27)

Consider the operators $A_t$ defined by

$$A_t u \equiv Au - C_N(t \Lambda_N, u) \text{ for } u \in \overline{D} \text{ and } t \in [0, 1].$$

Because $A$ satisfies condition $(S)_+$ and $C_N$ is compact on $[0, \Lambda_N] \times \overline{D}$ by Proposition 2.3, the operators $A_t$ satisfies condition $\alpha(t)$ for every $t \in [0, 1]$, that is, if whenever sequences $u_n \in \partial D$, $t_n \in [0, 1]$ are such that $u_n \rightharpoonup u_0$, $A_{t_n}(u_n) \rightharpoonup 0$ and $\lim_{n \to \infty} \langle A_{t_n}(u_n), u_n - u_0 \rangle = 0$, then $u_n$ converges strongly to $u_0$. By (27),

$$\text{Deg}(A_1, \overline{D}, 0) = 0;$$

on the other hand, by the assumptions,

$$\text{Deg}(A_0, \overline{D}, 0) = \text{Deg}(A, \overline{D}, 0) \neq 0,$$

we conclude that there must exist $t_0 \in (0, 1)$ and $u_0 \in \partial D$ such that $Au_0 - C_N(t_0 \Lambda_N, u_0) = 0$. Let $\lambda_0 = t_0 \Lambda_N$, this completes the proof. 

We are going to apply Theorem 3.1 to improve Theorem 6 in [8]. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^2$. Let $q$ be a number with $q > n$, and $X = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and $D$ a bounded open set of $X$ with $0 \in D$. Assume that the function $a_{ij}: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, \ldots, n$, and the function $C: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions:

$(A_1)$ the function $a_{ij}(x, u, p)$ is defined and continuous for $x \in \overline{\Omega}$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$;

$(A_2)$ there exists a positive nondecreasing function $\mu: \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x, u, p)\xi_i \xi_j \geq \mu(|u| + |p|) \sum_{i=1}^{n} \xi_i^2$$

for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \in \overline{\Omega}$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$.

$(C_1)$ $C(\lambda, x, u(x), \nabla u(x))$ are defined and continuous for $(\lambda, x, u(x)) \in \overline{\Omega} \times \mathbb{R} \times \partial D$, moreover, $C(0, x, u(x), \nabla u(x)) \equiv 0$ for all $(x, u(x)) \in \mathbb{R} \times \partial D$;

$(C_2)$ there exist a function $f(\lambda)$ and a continuous function $C'(x, u(x), \nabla u(x))$ for $(x, u(x)) \in \mathbb{R} \times \partial D$ such that $f(\lambda) \to \infty$ and

$$\lim_{\lambda \to \infty} \frac{1}{f(\lambda)} C(\lambda, x, u(x), \nabla u(x)) = C'(x, u(x), \nabla u(x)),$$

(28)

where the last limit is uniform w.r.t. $(x, u(x)) \in \Omega \times \partial D$;
there exists a nonnegative continuous function $C''(x, r), (x, r) \in \Omega \times \mathbb{R}^+$, such that for an arbitrary number $R \in \mathbb{R}$, we have

$$|C'(x, u(x), \nabla u(x))| \geq C''(x, R) \quad \text{and} \quad \int_{\Omega} C''(x, R) \, dx > 0. \quad (29)$$

for all $x \in \Omega$ and $u \in \partial \mathcal{D}$.

**Theorem 3.2.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with boundary $\partial \Omega$ of class $C^2$, $q$ be a number with $q > n$, and $\mathcal{D}$ be a bounded open set of $W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ with $0 \in \mathcal{D}$. Assume that the functions $a_{ij}(x, u, p)$, $C(\lambda, x, u(x), \nabla u(x))$ satisfy the conditions $(A_1)$, $(A_2)$ and $(C_1)$–$(C_3)$, respectively. Then there exist $\lambda_0 > 0$, $u_0 \in \partial \mathcal{D}$ satisfying the eigenvalue problem

$$\sum_{i, j=1}^n a_{ij}(x, u_0(x), \nabla u_0) \frac{\partial^2 u_0(x)}{\partial x_i \partial x_j} - C(\lambda_0, x, u_0(x), \nabla u_0(x)) = 0,$$

for $x \in \Omega$, $u_0(x) = 0$, $x \in \partial \Omega$.

**Proof.** As well known, the Sobolev space $W^{2,q}(\Omega)$ is uniformly convex, so is $X$. Please note that the conditions $(A_1)$ and $(A_2)$ on $a_{ij}$ are exactly the same as $(A_1^{(1)})$ and $(A_2^{(1)})$ of Theorem 6 in [8, p. 547], while the conditions $(C_1)$, $(C_2)$ and $(C_3)$ are different from that of Theorem 6 in [8] because the operator $C(\lambda, x, u(x), \nabla u(x))$ in our case is not defined for $u \in \mathcal{D}$. However, the proof of Theorem 6 in [8] only makes use of the properties of $C(\lambda, x, u(x), \nabla u(x))$ with $(\lambda, x, u(x)) \in \mathbb{R}^+ \times \overline{\Omega} \times \partial \mathcal{D}$. Such a proof keeps valid if we restrict $(\lambda, x, u(x)) \in \mathbb{R}^+ \times \overline{\Omega} \times \partial \mathcal{D}$, which matches the conditions of Theorem 3.1. Thus, our proof can follow that of Theorem 6 in [8] line by line provided that

- the operator $C: \mathbb{R}^+ \times \overline{\mathcal{D}} \to [X^{(1)}]^*$ in [8, (37), p. 458] is replaced by $C: \mathbb{R}^+ \times \partial \mathcal{D} \to [X^{(1)}]^*$, where $X^{(1)} = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$;
- each $u \in \overline{\mathcal{D}}$ in [8, p. 460] are replaced by $u \in \partial \mathcal{D}$;
- “Theorem 1” at the last third line on p. 460 in [8] is replaced by “Theorem 3.1.”

Theorem 3.2 improves Theorem 6 in [8], for example, the former allows $C$ to possess a singularity at $0 \in \mathcal{D}$, while the latter does not.

4. Applications on eigenvalue problems of maximal monotone operators

In this section, $X$ denotes an infinitely dimensional, real reflexive, locally uniformly convex Banach space with the locally uniformly convex dual space $X^*$. In this settings $X^*$ is strictly convex (Proposition 2.7 in [4, p. 49]). Denote by $B_N$ the open ball $\{y \in X^*: \|y\| < N\}$ for $N > 0$. We need the multi-valued version of Definition 3 in [8, p. 452].

**Definition 4.1.** We say that the operator $A: X \supset D(A) \to 2^Y$ satisfies condition $(A_\infty)$ on the bounded set $F \subset D(A)$ if there is no sequence $\{y_n\} \subset A(F)$ such that

$$\|y_n\| \to \infty \quad \text{and} \quad \frac{y_n}{\|y_n\|} \text{ converges.}$$
Theorem 4.2. Let $\mathcal{D} \subset X$ be a bounded open set and $A : X \supset D(A) \to 2^{X^*}$ a maximal monotone operator satisfying condition $(A_\infty)$ on $D(A) \cap \overline{\mathcal{D}}$. Assume that $(J + A)^{-1}$ is compact, $0 \in D(A) \cap \mathcal{D}$, $A0 \equiv 0$, and $0 \notin A(D(A) \cap \partial \mathcal{D}) \neq 0$. Let $C : \overline{\mathcal{D}}_+ \times (D(A) \cap \partial \mathcal{D}) \to X^*$ be a bounded continuous operator with $C(0, u) = 0$ for $u \in D(A) \cap \partial \mathcal{D}$. Assume that

(i') the closure of the set

$$E' = \left\{ \frac{C(\lambda, u)}{\|C(\lambda, u)\|} : u \in D(A) \cap \partial \mathcal{D}, \lambda \in \overline{\mathcal{D}}_+, \|C(\lambda, u)\| \geq N' \right\}$$

is compact;

(ii') $\lim_{\lambda \to +\infty} m'_\lambda = +\infty$, where $m'_\lambda = \inf \{\|C(\lambda, u)\| : u \in D(A) \cap \partial \mathcal{D}\}$.

Then there exist $\lambda_0 > 0$ and $u_0 \in D(A) \cap \partial \mathcal{D}$ such that $Au_0 - C(\lambda_0, u_0) \equiv 0$.

Proof. In this settings $J$ is single-valued, bounded and bicontinuous, and the operator $(J + A)^{-1} : X^* \to D(A)$ is compact. Because $D(J + A) = D(A)$ and $0 \in D(A) \cap \mathcal{D}$, $\mathcal{G} \equiv (J + A)(D(A) \cap \mathcal{D})$ is an open subset of $X^*$ with $0 \in \mathcal{G}$, and

$$\partial \mathcal{G} \subset (J + A)(D(A) \cap \partial \mathcal{D}) \quad \text{and} \quad \mathcal{G} \subset (J + A)(D(A) \cap \overline{\mathcal{D}}).$$

Please note that the set $\mathcal{G}$ may be unbounded. Define $\hat{C} : \overline{\mathcal{D}}_+ \times \partial \mathcal{G} \to X^*$ by $\hat{C}(\lambda, v) = C(\lambda, (J + A)^{-1}v)$. By assumptions, $\hat{C}$ is jointly continuous and maps any bounded subset of $\overline{\mathcal{D}}_+ \times \partial \mathcal{G}$ into a relatively compact subset, that is, $\hat{C}$ is a compact operator; and $\hat{C}(0, v) = 0$ for $v \in \partial \mathcal{G}$. Because $E'$ is relatively compact, so is

$$E^* = \left\{ \frac{\hat{C}(\lambda, v)}{\|\hat{C}(\lambda, v)\|} : v \in \partial \mathcal{G}, \lambda \in \overline{\mathcal{D}}_+, \|\hat{C}(\lambda, v)\| \geq N' \right\}.$$ 

Because $X^*$ is infinitely dimensional, $E^*$ cannot be equal to the unit sphere of $X^*$ according to Lemma B of [6, p. 126]; furthermore, because $X^*$ is locally uniformly convex, condition (i*) of Proposition 2.3 holds with $Y = X^*$. By condition (ii')

$$m_\lambda := \inf \{\|\hat{C}(\lambda, v)\| : v \in \partial \mathcal{G}\} \geq m'_\lambda \to +\infty \quad \text{as} \quad \lambda \to +\infty.$$ 

That is, conditions (ii) in Theorem 1.1 holds. Let $\{N_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $N_k \to +\infty$ as $k \to +\infty$. Now we can apply Proposition 2.3 with $\mathcal{M}, Y, \mathcal{D}$ and $C$ replaced by $X^*, X^*, \mathcal{G}$ and $\hat{C}$, respectively. Because $\hat{C}([0, A_{N_k}] \times \partial \mathcal{G}) \subset C([0, A_{N_k}] \times (D(A) \cap \partial \mathcal{D}))$ and $C$ is a bounded mapping and $\partial \mathcal{D}$ is a bounded set, we conclude that $A_{N_k} \to +\infty$ as $k \to +\infty$, otherwise $m_\lambda$ will not approach infinite, and that for each $k$, there exists a positive number $M_k$ such that

$$J(\overline{\mathcal{D}}) \subset B_{M_k}/2 \quad \text{and} \quad \hat{C}([0, A_{N_k}] \times \partial \mathcal{G}) \subset B_{M_k}/2.$$ 

Now we apply Proposition 2.3 to $\hat{C}$: there exists a number $\epsilon \in (0, 1)$ such that for each $N_k$ and $B_{M_k}$ for $k = 1, 2, \ldots$, there is a continuous mapping $C_{N_k} : \overline{\mathcal{D}}_+ \times (\overline{\mathcal{G}} \cap B_{M_k}) \to X^*$ having properties:

$$C_{N_k}(\lambda, v) = \hat{C}(\lambda, v) \quad \text{for} \quad (\lambda, v) \in \overline{\mathcal{D}}_+ \times (\partial \mathcal{G} \cap B_{M_k}),$$

$$\|C_{N_k}(\lambda, v)\| \geq \epsilon N_k \quad \text{for} \quad (\lambda, v) \in [A_{N_k}, +\infty) \times (\overline{\mathcal{G}} \cap B_{M_k}),$$

$$\sup_{(\lambda, v) \in [0, A_{N_k}] \times (\overline{\mathcal{G}} \cap B_{M_k})} \|C_{N_k}(\lambda, v)\| \leq \sup_{(\lambda, v) \in [0, A_{N_k}] \times \partial \mathcal{G}} \|\hat{C}(\lambda, v)\|.$$
Furthermore, $C_{N_k}$ is compact on $[0, \Lambda_{N_k}] \times (\overline{G} \cap \overline{B}_{M_k})$. Because $E^*$ is relatively compact, by Proposition 2.3 there exists a compact set $K \subset X^*$ depending only on $\varepsilon$ and $E^*$ such that $\{C_{N_k}(\lambda, v)/\|C_{N_k}(\lambda, v)\|: \lambda \geq \Lambda_{N_k}, v \in \overline{G} \cap \overline{B}_{M_k}\} \subset K$ for all $k$. By Remark 2.4,

$$C_{N_k}(0, v) = 0 \quad \text{for} \quad v \in \overline{G} \cap \overline{B}_{M_k}. \quad (37)$$

Suppose there exists a sequence $\{v_k\}$ with $v_k \in \overline{G} \cap \overline{B}_{M_k}$ and $\lambda_k \geq \Lambda_{N_k}$ such that

$$v_k - C_{N_k}(\lambda_k, v_k) - J(J + A)^{-1}v_k = 0. \quad (38)$$

Let $y_k = (J + A)^{-1}v_k$ and $w_k \in A y_k$ be such that $v_k = J y_k + w_k$, then $w_k = C_{N_k}(\lambda_k, v_k)$. Because of this and (35)

$$\|w_k\| = \|C_{N_k}(\lambda_k, v_k)\| \geq \varepsilon N_k \to \infty \quad \text{as} \quad k \to \infty,$$

and $\{w_k/\|w_k\|\} = \{C_{N_k}(\lambda_k, v_k)/\|C_{N_k}(\lambda_k, v_k)\|\} \subset K$ has a convergent subsequence, which is contrary to the property $(A_\infty)$ of $A$. We conclude that for some numbers $N \in \{N_k\}$ and $M = M_N$ there exists a compact operator $\tilde{C} : \overline{B} \times (\overline{G} \cap \overline{B}_M) \to X^*$ such that

$$v - \tilde{C}(\lambda, v) - J(J + A)^{-1}v \neq 0 \quad \text{for} \quad \lambda \geq \Lambda \quad \text{and} \quad v \in \overline{G} \cap \overline{B}_M. \quad (39)$$

where $\Lambda$ is a number such that $m_{\lambda} \geq N$ for $\lambda \geq \Lambda$. Consider the family of operators

$$I - \tilde{C}(\lambda, \cdot) - J(J + A)^{-1} : \overline{G} \cap \overline{B}_M \to X^*, \quad 0 \leq t \leq 1,$$

which is a compact displacement of the identity. The Leray–Schauder degree

$$\text{deg}(I - \tilde{C}(\lambda, \cdot) - J(J + A)^{-1}, \overline{G} \cap \overline{B}_M, 0) = 0.$$ 

We will show that

$$\text{deg}(I - J(J + A)^{-1}, \overline{G} \cap \overline{B}_M, 0) > 0.$$ 

Claim that $v - J(J + A)^{-1}v \neq 0$ for $v \in \partial(\overline{G} \cap \overline{B}_M)$. In fact,

$$\partial(\overline{G} \cap \overline{B}_M) \subset \partial \overline{G} \cup (\partial \overline{G} \cap \partial \overline{B}_M).$$

Let $z = (J + A)^{-1}v$; in the case of $v \in \partial \overline{G}$, $z \in D(A) \cap \partial D$ by (30), then $v - J(J + A)^{-1}v = 0$ would imply that $0 = v - J z \in A z$, contrary to the assumption $0 \notin A(D(A) \cap \partial D)$; in the case that $v \in \overline{G} \cap \partial \overline{B}_M$, we would have $z \in D(A) \cap \overline{D}$, then $0 = \|v - J(J + A)^{-1}v\| \geq M - \|J z\| > 0$, which is a contradiction. This proves the claim. Because $0 \in \overline{G} \cap \overline{B}_M$ and $0 - J(J + A)^{-1}0 = 0$ by (33). Hence,

$$\text{deg}(I - \tilde{C}(0, \cdot) - J(J + A)^{-1}, \overline{G} \cap \overline{B}_M, 0) = \text{deg}(I - J(J + A)^{-1}, \overline{G} \cap \overline{B}_M, 0) > 0.$$ 

We conclude that there exists $\tilde{t} \in (0, 1)$ and $\tilde{y} \in \partial(\overline{G} \cap \overline{B}_M)$ such that

$$0 = \tilde{y} - \tilde{C}(\tilde{t} A, \tilde{y}) - J(J + A)^{-1} \tilde{y}.$$ 

It is easy to see that $\tilde{y} \notin \overline{G} \cap \partial \overline{B}_M$, otherwise $\tilde{z} = (J + A)^{-1} \tilde{y} \in \overline{D}$ and

$$\partial \overline{B}_M \ni \tilde{y} = \tilde{C}(\tilde{t} A, \tilde{y}) - J \tilde{z} \in \tilde{C}([0, A] \times (\overline{G} \cap \overline{B}_M)) - J(\overline{D}) \subset \overline{B}_M$$

by (33) and (36), which leads to a contradiction. Thereby $\tilde{y} \in \partial \overline{G}$. By (10)

$$0 = \tilde{y} - \tilde{C}(\tilde{t} A, \tilde{y}) - J(J + A)^{-1} \tilde{y}.$$ 

Let $\lambda_0 = \tilde{t} A$ and $u_0 = (J + A)^{-1} \tilde{y}$. Then $\lambda_0 > 0$ and $u_0 \in D(A) \cap \partial D$ by (30) and $0 \in Au_0 - C(\lambda_0, u_0)$. This finishes the proof. \qed
Remarks 4.3. The merits of this theorem are that $C(\lambda, \cdot)$ is densely defined and that no assumptions on $C$ located in the set $\mathbb{R}_+ \times \mathcal{D}$. In the proof we only need the set $E^*$ defined in (31) to be relatively compact so as to ensure condition (i*) of Proposition 2.3.

The following corollary is an improvement of Theorem 11 from [7] by removing any conditions on $C$ located in $\mathcal{D}$.

**Corollary 4.4.** Let $\mathcal{D} \subset X$ be a bounded open subset and $A : X \supset D(A) \rightarrow 2^{X^*}$ a maximal monotone operator with $A(D(A) \cap \overline{\mathcal{D}})$ being bounded. Assume that $(J + A)^{-1}$ is compact and that $0 \in D(A) \cap \mathcal{D}$, $0 \in A(0)$ and $0 \notin A(D(A) \cap \partial \mathcal{D}) \neq \emptyset$. Let $C : D(A) \cap \partial \mathcal{D} \rightarrow X^*$ be bounded and continuous. Assume that there exists number $\alpha > 0$ such that $\|Cx\| \geq \alpha$ for $x \in D(A) \cap \partial \mathcal{D}$. Then there exists $(\lambda_0, x_0) \in (0, \infty) \times (D(A) \cap \partial \mathcal{D})$ such that $Ax_0 - \lambda_0 Cx_0 \ni 0$.

**Proof.** In this case $C(\lambda, x)$ in Theorem 4.2 is equal to $\lambda Cx$. By assumptions $A$ satisfies condition (A$_{\infty}$) on $D(A) \cap \overline{\mathcal{D}}$. Define the operator $\tilde{C}(\lambda, v) \equiv \lambda C(A + J)^{-1}v$ for $\lambda \geq 0$ and $v \in \partial \mathcal{G}$ with $\mathcal{G} \equiv (A + J)(D(A) \cap \mathcal{D})$. Then $C(A + J)^{-1}$ is compact on $\partial \mathcal{G}$. It is clear that condition (ii') of Theorem 4.2 holds and that $E^*$ defined by (31) is relatively compact, thereby condition (i*) of Proposition 2.3 holds by virtue of Remark 4.3. Then the conclusion follows by Theorem 4.2. \(\square\)

Corollary 4.4 can also be modified into the $m$-accretive version so as to extend Theorem 2.4 in [6] provided $X$ is locally uniformly convex.

Acknowledgments

The author expresses his thanks to Professor Kartsatos for his comments during this research, and to the reviewers and editors for their helpful suggestions and comments.

References