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Eventual Partition of Conserved Quantities in Wave Motion

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Let u be a classical solution to the wave equation in an odd number n of space dimensions, with compact spatial support at each fixed time. Duffin (*J. Math. Anal. Appl.* **32** (1970), 386–391) uses the Paley–Wiener theorem of Fourier analysis to show that, after a finite time, the (conserved) energy of u is partitioned into equal kinetic and potential parts. The wave equation actually has $(n+2)(n+3)/2$ independent conserved quantities, one for each of the standard generators of the conformal group of $(n+1)$-dimensional Minkowski space. Of concern in this paper is the "zeroth inversional quantity" I_0, which is commonly used to improve decay estimates which are obtained using conservation of energy. We use Duffin's method to partition I_0 into seven terms, each of which, after a finite time, is explicitly given as a constant-coefficient quadratic function of the time. Zachmanoglou has shown that under the above assumptions if $n \geqslant 3$, the spatial L^2 norm of u is eventually constant. A consequence of the analysis here is a bound on this constant in terms of the energy and the radius of the support of the Cauchy data of u at a fixed time.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $n \geqslant 1$ and let M be the $(n+1)$-dimensional Minkowski space, with coordinates $t \in \mathbb{R}$ (time) and $x \in \mathbb{R}^n$ (position). The *wave equation* on a function $u = u(x,t): M \to \mathbb{R}$ is $\ddot{u} = \Delta u$, where a dot denotes time differentiation and Δ is the space Laplacian $\sum \partial_j^2$, $\partial_j = \partial/\partial x_j$. As a consequence of its conformal covariance properties, solutions of the wave equation with suitable spatial decay conditions (and certainly solutions with compact spatial support) admit $(n+2)(n+3)/2$ conserved quantities, which were first written down in [4]. These are the integrals over $\mathbb{R}^n$ of the densities

$$\varepsilon = \frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 \qquad \text{(energy)},$$

$$p_j = (\partial_j u)\,\dot{u} \qquad \text{(linear momenta)},$$

$$\lambda_{jk} = x_j p_k - x_k p_j \qquad \text{(angular momenta)},$$

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$$b_j = x_j \varepsilon + t p_j,$$

$$h = t\varepsilon + x \cdot p + \frac{n-1}{2} u\dot{u},$$

$$i_0 = (r^2 + t^2)\,\varepsilon + 2tx \cdot p - \frac{n-1}{2} u^2 + (n-1)\, t u\dot{u},$$

$$i_j = -2tx_j\varepsilon - 2x_j x \cdot p + (r^2 - t^2)\, p_j - (n-1)\, x_j u\dot{u}.$$

Here and below, p is the n-vector $(p_1,..., p_n)$, x is the n-vector $(x_1,..., x_n)$, $r^2 = x \cdot x$, and j and k run from 1 to n. The integrals are conserved because the time derivative of each density is an exact divergence:

$$\dot{\varepsilon} = \nabla \cdot p,$$

$$\dot{p}_j = \nabla \cdot ((\partial_j u)\, \nabla u) + \frac{1}{2} \partial_j(\dot{u}^2 - |\nabla u|^2) \equiv \nabla \cdot \pi_j,$$

$$\dot{\lambda}_{jk} = \nabla \cdot (x_j \pi_k - x_k \pi_j),$$

$$\dot{b}_j = \nabla \cdot (x_j p + t\pi_j),$$

$$\dot{h} = \nabla \tau, \qquad \tau = tp + x \cdot \pi + \frac{n-1}{2} u(\nabla u),$$

$$\dot{i}_0 = \nabla \cdot ((r^2 - t^2)\, p + 2t\tau),$$

$$\dot{i}_j = \nabla \cdot ((r^2 - t^2)\, \pi_j - 2x_j \tau) + \frac{n-1}{2} \partial_j(u^2).$$

For our result concerning $\int_{\mathbb{R}^n} i_0$, we need the following machinery.

DEFINITION 1.1. (See, e.g., [2].) A function $f(z)$ of one complex variable is of *exponential type* $\alpha > 0$ if for each $\varepsilon > 0$, there exists a constant A_ε with $|f(z)| \leqslant A_\varepsilon e^{(\alpha + \varepsilon)|z|}$.

Remark 1.2. If $f(z)$ has exponential type α and $g(z)$ has polynomial growth in z, $g(z) f(z)$ still has exponential type α.

THEOREM 1.3 (Paley–Wiener). (See [2].) *Let $f(z)$ be an entire function of exponential type α, of class L^1 on the real axis. Then the Fourier transform of the restriction of $f(z)$ to the real axis is zero outside $[-\alpha, \alpha]$.*

THEOREM 1.4. (See [2], or the reference to [3] in [5]). *Suppose n is odd and that the Cauchy data $u(\cdot, 0)$, $\dot{u}(\cdot, 0)$ of a solution u to the wave equation have support contained in the closed ball of radius b about the origin. Then for $t \geqslant b$, the kinetic energy $\frac{1}{2}\|\dot{u}(\cdot, t)\|^2$ and the potential energy*

$\frac{1}{2}\|(\nabla u)(\cdot, t)\|^2$ *are both equal to half of the total energy* $\int_{\mathbb{R}^n} \varepsilon$. *(Here* $\| \ \|$ *is the* $L^2(\mathbb{R}^n)$ *norm.)*

From now on, $\| \ \|$ and $\langle \, , \rangle$ will denote $L^2(\mathbb{R}^n)$ norm and inner product, respectively, and we write $v = \dot{u}$. The dependence of, for example, the kinetic energy on time will not be mentioned explicitly: we write $\|v\|^2$ for $\|\dot{u}(\cdot, t)\|^2$.

THEOREM 1.5 (Zachmanoglou [5]). *Under the assumptions of Theorem* 1.4 *with* $n \geqslant 3$, $\|u\|^2$ *is constant for* $t \geqslant b$.

Let C be the eventual constant value of $\|u\|^2$ guaranteed by Theorem 1.5, and let E, H, and I_0 be the conserved quantities $\int_{\mathbb{R}^n} \varepsilon$, $\int_{\mathbb{R}^n} h$, $\int_{\mathbb{R}^n} i_0$, respectively. Our theorem states that each of the seven terms in

$$I_0 = \frac{1}{2}\|rv\|^2 + \frac{1}{2}\|r\nabla u\|^2 + \frac{1}{2}t^2\|v\|^2 + \frac{1}{2}t^2\|\nabla u\|^2 + 2t\langle vx, \nabla u\rangle - \frac{n-1}{2}\|u\|^2 + (n-1)\,t\langle u, v\rangle$$

is given by a constant-coefficient quadratic function of t for $t \geqslant b$.

THEOREM 1.6. *Under the assumptions of Theorem* 1.4 *with* $n \geqslant 3$, $t \geqslant b$ *implies that*

$$\frac{1}{2}\|rv\|^2 = \frac{1}{2}Et^2 - Ht + \frac{1}{2}I_0, \tag{1.1}$$

$$\frac{1}{2}\|r\nabla u\|^2 = \frac{1}{2}Et^2 - Ht + \frac{1}{2}I_0 + \frac{n-1}{2}C, \tag{1.2}$$

$$\frac{1}{2}t^2\|v\|^2 = \frac{1}{2}Et^2, \tag{1.3}$$

$$\frac{1}{2}t^2\|\nabla u\|^2 = \frac{1}{2}Et^2, \tag{1.4}$$

$$2t\langle vx, \nabla u\rangle = -2Et^2 + 2Ht, \tag{1.5}$$

$$-\frac{n-1}{2}\|u\|^2 = -\frac{n-1}{2}C, \tag{1.6}$$

$$(n-1)\,t\langle u, v\rangle = 0. \tag{1.7}$$

Equations (1.3)–(1.7) are consequences of Theorems 1.4 and 1.5. Indeed, (1.6) is Theorem 1.5, (1.7) follows from $(d/dt)\|u\|^2 = 2\langle u, v\rangle$, (1.5) is essentially the definition of H, and (1.3) and (1.4) are Theorem 1.4. Equation

(1.1) requires an imitation of Duffin's proof of Theorem 1.4, which we carry out in Section 2. Equation (1.2) then follows.

An interesting application of Theorem 1.6 is a bound on C. The Cauchy data of u at time b are supported in the ball of radius $2b$; thus $\frac{1}{2}\|rv\|^2$ and $\frac{1}{2}\|r\nabla u\|^2$ lie between 0 and $2b^2E$ at time b. Their difference satisfies

$$\frac{n-1}{2}C = \frac{1}{2}\|r\nabla u\|^2 - \frac{1}{2}\|rv\|^2 \leqslant 2b^2E,$$

and we have

COROLLARY 1.7. *Under the assumptions of Theorem* 1.4 *with* $n \geqslant 3$,

$$C \leqslant \frac{4}{n-1}b^2E.$$

2. PROOF OF THEOREM 1.6

Let $\hat{}$ denote Fourier transformation in the x variables: if f is a C^∞ function of x and t, say Schwartz class in x for fixed t,

$$\hat{f}(\xi, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x, t)\, dx,$$

where dx is Lebesgue measure. (In what follows, we will sometimes suppress the dependence on t.) We have the identities

$$\left(\frac{\partial f}{\partial x_j}\right)^{\wedge} = \frac{1}{i}\xi_j \hat{f}, \tag{2.1}$$

$$(x_j f)^{\wedge} = \frac{1}{i}\frac{\partial \hat{f}}{\partial \xi_j}. \tag{2.2}$$

Taking the Fourier transform of the wave equation, we have

$$\ddot{\hat{u}}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t). \tag{2.3}$$

(The compact support assumption permits us to interchange $\hat{}$ and $\dot{}$.) The explicit solution of (2.3) is (setting $\rho = |\xi|$)

$$\hat{u}(\xi, t) = F(\xi)\cos\rho t + G(\xi)\frac{\sin\rho t}{\rho}, \tag{2.4}$$

where $F(\xi) = \hat{u}(\xi, 0)$, $G(\xi) = \hat{v}(\xi, 0)$. The *second kinetic moment* is

$$K^{(2)} = \tfrac{1}{2}\|rv\|^2 = \tfrac{1}{2}\|vx\|^2 = \tfrac{1}{2}\|\nabla\hat{v}\|^2,$$

by (2.2) and the Plancherel theorem, where the gradient is in the ξ variables. Differentiating (2.4) with respect to t and taking the gradient, we have

$$\begin{aligned} \hat{v}(\xi, t) &= -\rho F(\xi) \sin \rho t + G(\xi) \cos \rho t, \\ (\nabla \hat{v})(\xi, t) &= -tF(\xi)(\cos \rho t)\, \xi - \rho(\sin \rho t)(\nabla F)(\xi) \\ &\quad - F(\xi)(\sin \rho t) \frac{\xi}{\rho} - tG(\xi)(\sin \rho t) \frac{\xi}{\rho} \\ &\quad + (\cos \rho t)(\nabla G)(\xi), \end{aligned}$$

so that

$$\begin{aligned} 2K^{(2)} &= \int_{\mathbb{R}^n} \cos^2 \rho t [t^2 \rho^2 |F|^2 - 2t \operatorname{Re}(F\xi) \cdot (\nabla G) + |\nabla G|^2]\, d\xi \\ &\quad + \int_{\mathbb{R}^n} \sin^2 \rho t [\rho^2 |\nabla F|^2 + 2 \operatorname{Re}(\nabla F) \cdot (F\xi) + 2t \operatorname{Re}(\nabla F) \cdot (G\xi) \\ &\quad + |F|^2 + 2t \operatorname{Re} F\bar{G} + t^2 |G|^2]\, d\xi \\ &\quad + \int_{\mathbb{R}^n} (\sin \rho t)(\cos \rho t)\, 2\operatorname{Re} \Big[t\rho (F\xi) \cdot (\nabla F) + t\rho |F|^2 \\ &\quad + t^2 \rho F \bar{G} - \rho (\nabla F) \cdot (\nabla G) - \left(F \frac{\xi}{\rho} \right) \cdot (\nabla G) \\ &\quad - t \left(G \frac{\xi}{\rho} \right) \cdot (\nabla G) \Big]\, d\xi \\ &\equiv 2 \int_{\mathbb{R}^n} (\cos^2 \rho t)\, X_1(\xi)\, d\xi + 2 \int_{\mathbb{R}^n} (\sin^2 \rho t)\, X_2(\xi)\, d\xi \\ &\quad + 2 \int_{\mathbb{R}^n} (\sin \rho t)(\cos \rho t)\, X_3(\xi)\, d\xi, \end{aligned} \tag{2.5}$$

where the dot product is conjugate linear in the second argument.

Using the trigonometric identities

$$\begin{aligned} \cos^2 \theta &= \frac{1 + \cos 2\theta}{2}, \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2}, \\ (\sin \theta)(\cos \theta) &= \frac{1}{2} \sin 2\theta, \end{aligned}$$

the above becomes

$$\begin{aligned}2K^{(2)} &= \tfrac{1}{2}t^2\,\|F\xi\|^2 - t\,\mathrm{Re}\langle F\xi, \nabla G\rangle + \tfrac{1}{2}\,\|\nabla G\|^2 \\ &\quad + \tfrac{1}{2}\langle \rho^2 \nabla F, \nabla F\rangle + \mathrm{Re}\langle \nabla F, F\xi\rangle \\ &\quad + t\,\mathrm{Re}\langle \nabla F, G\xi\rangle + \tfrac{1}{2}\,\|F\|^2 + t\,\mathrm{Re}\langle F, G\rangle \\ &\quad + \tfrac{1}{2}t^2\,\|G\|^2 + \int_{\mathbb{R}^n} (\cos 2\rho t)(X_1 - X_2)\,d\xi \\ &\quad + \int_{\mathbb{R}^n} (\sin 2\rho t)\,X_3\,d\xi. \end{aligned} \tag{2.6}$$

Claim 2.1. The two integral terms in (2.6) vanish for $t \geqslant b$.

Proof. We express the integrals in spherical coordinates ρ and Θ, where $\Theta = \xi/\rho$ is the coordinate on the unit sphere S^{n-1}. Denote by $d\Theta$ the measure on S^{n-1}; in this notation

$$\int_{\mathbb{R}^n} f(\xi)\,d\xi = \int_0^\infty \left(\rho^{n-1} \int_{S^{n-1}} f(\rho, \Theta)\,d\Theta\right) d\rho,$$

where $f(\rho, \Theta) = f(\xi) = f(\rho\Theta)$.

Now let $u_0 = u(\cdot, 0)$ and $v_0 = v(\cdot, 0)$. $F = \hat{u}_0$ may be expressed as

$$F(\rho, \Theta) = (2\pi)^{-n/2} \int_{\{r \leqslant b\}} e^{i\rho x\cdot\Theta} u_0(x)\,dx. \tag{2.7}$$

For fixed Θ, (2.7) converges for all complex values of ρ, and in fact is an entire function of ρ of exponential type b:

$$|F(\rho)| \leqslant (2\pi)^{-n/2}\,\|u_0\|_{L^1(\mathbb{R}^n)}\,e^{b|\rho|}.$$

(The dependence of the entire extension of F on Θ is suppressed here.) Similarly $\bar{F}$, G, and $\bar{G}$ extend, for fixed Θ, to entire functions of ρ of exponential type b. Note that the extension of $\bar{F}$ is not the conjugate of the extension of F; in the integral representation corresponding to (2.7), the i is changed to $-i$ but the ρ is *not* conjugated. In addition, ∇F, $\nabla\bar{F}$, ∇G, $\nabla\bar{G}$, $F\xi$, $\bar{F}\xi$, $G\xi$, $\bar{G}\xi$, and $(\xi/\rho)\cdot(\nabla G) = \Theta \cdot G$ have entire extensions of the same exponential type, since, for example,

$$\frac{\partial F}{\partial \xi_j}(\rho, \Theta) = (2\pi)^{-n/2} \int_{\{r \leqslant b\}} e^{i\rho x\cdot\Theta}\, i x_j u_0(x)\,dx.$$

All these entire functions are Schwartz class in ξ and thus in real ρ for fixed Θ.

By Remark 1.2, ρF and $\rho\nabla F$ also have all these properties. Thus the functions X_1, X_2, X_3 of (2.5) are entire, of exponential type $2b$, and Schwartz class on the real axis. If

$$Y_m(\rho) = \rho^{n-1} \int_{S^{n-1}} X_m(\rho, \Theta)\, d\Theta, \qquad m = 1, 2, 3,$$

Y_m is entire, of exponential type $2b$, and Schwartz class (thus L^1) on the real axis. By (2.7), $F(-\rho, \Theta) = F(\rho, -\Theta)$, and similarly for $\bar{F}$, G, $\bar{G}$, ∇F, $\nabla \bar{F}$, ∇G, $\nabla \bar{G}$, $F\xi$, $\bar{F}\xi$, $G\xi$, and $\bar{G}\xi$; so that $X_m(-\rho, \Theta) = X_m(\rho, -\Theta)$ for $m = 1, 2$. On the other hand, $\Theta \cdot \nabla G$, ρF, and $\rho\nabla F$ behave in the opposite way: $X_3(-\rho, \Theta) = -X_3(\rho, -\Theta)$.

This means that Y_1 and Y_2 are even functions of ρ, and Y_3 is odd. For example, since the antipodal map is measure preserving and $n-1$ is even,

$$\begin{aligned} Y_1(-\rho) &= \rho^{n-1} \int_{S^{n-1}} X_1(-\rho, \Theta)\, d\Theta \\ &= \rho^{n-1} \int_{S^{n-1}} X_1(\rho, -\Theta)\, d(-\Theta) \\ &= Y_1(\rho). \end{aligned}$$

Thus the Fourier transform of $Y_1 - Y_2$ is a cosine transform, and that of Y_3 a sine transform. We have,

$$\tfrac{1}{2}(Y_1 - Y_2)^\wedge\,(2t) = \int_0^\infty (\cos 2\rho t)(Y_1 - Y_2)(\rho)\, d\rho,$$

$$\tfrac{1}{2}\hat{Y}_3(2t) = \int_0^\infty (\sin 2\rho t)\, Y_3(\rho)\, d\rho.$$

By Theorem 1.3 (Paley–Wiener), these vanish for $t > b$, and, by continuity, for $t = b$ also. This establishes the claim.

By the claim, the Plancherel theorem, (2.1), and (2.2), we may reduce (2.6) for $t \geqslant b$ to

$$\begin{aligned} 2K^{(2)} = {} & \tfrac{1}{2}t^2 \|\nabla u_0\|^2 - t\langle \nabla u_0, v_0 x\rangle + \tfrac{1}{2}\|v_0 x\|^2 \\ & - \tfrac{1}{2}\langle \Delta(u_0 x), u_0 x\rangle + \langle u_0 x, \nabla u_0\rangle \\ & + t\langle u_0 x, \nabla v_0\rangle + \tfrac{1}{2}\|u_0\|^2 + t\langle u_0, v_0\rangle \\ & + \tfrac{1}{2}t^2 \|v_0\|^2. \end{aligned} \tag{2.8}$$

Now if ϕ and ψ are C_0^∞ functions on $\mathbb{R}^n$, integration by parts gives

$$\langle \nabla\phi, \psi x\rangle = -\langle \phi, \nabla\cdot(\psi x)\rangle = -\langle \phi x, \nabla\psi\rangle - n\langle \phi, \psi\rangle,$$

and in particular $\langle \nabla\phi, \phi x\rangle = -(n/2)\,\|\phi\|^2$. In addition, if e_j is the jth standard basis vector in $\mathbb{R}^n$,

$$\begin{aligned}
\langle \Delta(u_0 x), u_0 x\rangle &= \sum \langle \Delta(u_0 x_j), u_0 x_j\rangle \\
&= -\sum \langle \nabla(u_0 x_j), \nabla(u_0 x_j)\rangle \\
&= -\sum \langle x_j \nabla u_0 + u_0 e_j, x_j \nabla u_0 + u_0 e_j\rangle \\
&= -\|r\nabla u_0\|^2 - 2\langle \nabla u_0, u_0 x\rangle - n\,\|u_0\|^2 \\
&= -\|r\nabla u_0\|^2.
\end{aligned}$$

Equation (2.8) thus simplifies, for $t \geqslant b$, to

$$\begin{aligned}
2K^{(2)} = {} & \frac{1}{2}\|rv_0\|^2 + \frac{1}{2}\|r\nabla u_0\|^2 + \frac{1}{2}t^2\,\|v_0\|^2 + \frac{1}{2}t^2\,\|\nabla u_0\|^2 \\
& - 2t\langle \nabla u_0, v_0 x\rangle - \frac{n-1}{2}\|u_0\|^2 - (n-1)\,t\langle u_0, v_0\rangle.
\end{aligned}$$

It is remarkable that this expression for $2K^{(2)} = \|rv(\cdot, t)\|^2$ duplicates the formal appearance of the conserved quantity I_0, except for the sign of the terms mixing u_0 and v_0.

Evaluating the constants H and I_0 at time 0, we find

$$H = \langle \nabla u_0, v_0 x\rangle + \frac{n-1}{2}\langle u_0, v_0\rangle,$$

$$I_0 = \frac{1}{2}\|rv_0\|^2 + \frac{1}{2}\|r\nabla u_0\|^2 - \frac{n-1}{2}\|u_0\|^2.$$

This means

$$2K^{(2)} = 2K^{(2)}(t) = I_0 - 2tH + t^2 E, \qquad t \geqslant b.$$

This, along with the discussion in Section 1, establishes the theorem.

3. Remarks

(1) A good check on (1.1) can be made by translating the time scale: let $t = s + \delta$. The constants H and I_0 are different in the s scale:

$$H^t = H^s + \delta E, \tag{3.1}$$

$$I_0^t = I_0^s + 2s\delta E + \delta^2 E + 2\delta\langle \nabla u, vx\rangle + (n-1)\,\delta\langle u, v\rangle. \tag{3.2}$$

The eventual value of $\|rv\|^2$ should be equal to both $Es^2 - 2H^s s + I_0^s$ and $Et^2 - 2H^t + I_0^t$. Equations (3.1) and (3.2) imply that these expressions are equal.

(2) When $n = 1$, the proof of Theorem 1.6 shows that in

$$I_0 = \tfrac{1}{2}\|xv\|^2 + \tfrac{1}{2}\|xu'\|^2 + \tfrac{1}{2}t^2\|v\|^2 + \tfrac{1}{2}t^2\|xu'\|^2 + 2t\langle xv, u'\rangle,$$

where $u' = \partial u/\partial x$, each of the first two terms eventually becomes $\frac{1}{2}Et^2 - Ht + \frac{1}{2}I_0$, and the rest obey (1.3)–(1.5). Thus we have an eventual partition theorem for $n = 1$ also.

Though the eventual behavior of $\|u\|^2$ is not an issue here, Zachmanoglou's proof of Theorem 1.5 shows that $\|u\|^2$ is a linear function of t for $t \geqslant b$.

(3) Application of Duffin's method to the conserved quantity $B_j = \int_{\mathbb{R}^n} b_j$ shows that eventually, under the assumptions of Theorem 1.4, $\langle x_j v, v\rangle = \langle x_j \nabla u, \nabla u\rangle = B_j - P_j t$, where $P_j = \int_{\mathbb{R}^n} p_j$. This partitions B_j into three linear functions of t.

(4) A similar result holds for the Maxwell equations, for which Dassios [1] has proved the energy equipartition theorem. These calculations will appear separately.

(5) Of course, an asymptotic partition theorem corresponding to Theorem 1.6 holds if the compact support assumption is replaced by suitable spatial decay assumptions.

References

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