J. Math. Anal. Appl. 367 (2010) 260–272

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative ☆

Zhongli Wei [a,b,*], Qingdong Li [b], Junling Che [a]

[a] Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong, 250101, People's Republic of China
[b] School of Mathematics and System Sciences, Shandong University, Jinan, Shandong, 250100, People's Republic of China

ARTICLE INFO

Article history:
Received 20 September 2009
Available online 18 January 2010
Submitted by J. Shi

Keywords:
Initial value problem
Fractional differential equation
Riemann–Liouville sequential fractional derivatives
Upper solution and lower solution

ABSTRACT

In this paper, we shall discuss the properties of the well-known Mittag–Leffler function, and consider the existence and uniqueness of solution of the initial value problem for fractional differential equation involving Riemann–Liouville sequential fractional derivative by using monotone iterative method.

1. Introduction

Let $J = [a, b]$ be a compact interval on the real axis $\mathbb{R}$, and y be a measurable Lebesgue function, that is, $y \in L_1(a, b)$. Let $x \in J$ and $\alpha \in \mathbb{R}$ $(0 < \alpha \leqslant 1)$. The Riemann–Liouville fractional integrals I^{α}_{a+} and derivative D^{α}_{a+} are defined by (see, for example, [1–3])

$$\left(I^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}y(s)\,ds \quad \text{and} \quad \left(D^{\alpha}_{a+}y\right)(x) = \frac{d}{dx}\left(I^{1-\alpha}_{a+}y\right)(x). \tag{1.1}$$

We will work here following the definition of *sequential fractional derivative* presented by Miller and Ross in p. 209 of [4],

$$\begin{cases} \mathscr{D}^{\alpha}_{a+}y = D^{\alpha}_{a+}y, \\ \mathscr{D}^{k\alpha}_{a+}y = \mathscr{D}^{\alpha}_{a+}\mathscr{D}^{(k-1)\alpha}_{a+}y \quad (k = 2, 3, \ldots). \end{cases} \tag{1.2}$$

There is a close connection between the sequential fractional derivatives and the nonsequential Riemann–Liouville derivatives. For example, in the case $k = 2$, $0 < \alpha < 1/2$ and the Riemann–Liouville derivatives, the relationship between $\mathscr{D}^{k\alpha}_{a+}y$ and $D^{k\alpha}_{a+}y$ is given by

$$\left(\mathscr{D}^{2\alpha}_{a+}y\right)(x) = \left(D^{2\alpha}_{a+}\left[y(t) - \left(I^{1-\alpha}_{a+}y\right)(a+)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\right]\right)(x). \tag{1.3}$$

☆ Research supported by the NNSF-China (10971046) and the NSF of Shandong Province (ZR2009AM004).
* Corresponding author at: Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong, 250101, People's Republic of China.
E-mail address: jnwzl@yahoo.com.cn (Z. Wei).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.01.023

We shall consider the existence of minimal and maximal solutions and uniqueness of solution of the initial value problem for fractional differential equation involving Riemann–Liouville sequential fractional derivative, using the method of upper and lower solutions and its associated monotone iterative method.

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha} y)(x) = f(x, y, \mathscr{D}_{0+}^{\alpha} y), \quad x \in (0, T], \\ x^{1-\alpha} y(x)|_{x=0} = y_0, \quad x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} y)(x)|_{x=0} = y_1, \end{cases} \tag{1.4}$$

where $0 < T < +\infty$, and $f \in C([0, T] \times \mathbb{R} \times \mathbb{R})$.

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems etc. Recently, many researchers paid attention to existence result of solution of the initial value problem and boundary value problem for fractional differential equations, such as [5–16]. Some recent contributions to the theory of fractional differential equations can be seen in [17–21].

The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions of nonlinear differential equations, such as [22–27]. In [16], the existence and uniqueness of solution of the following initial value problem for fractional differential equation

$$\begin{cases} D^{\alpha} u(t) = f(t, u), \quad t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = u_0. \end{cases} \tag{1.5}$$

was discussed by using the method of upper and lower solutions and its associated monotone iterative method.

While for the existence of minimal and maximal solutions and uniqueness of solution of the initial value problem (1.4) for fractional differential equation involving Riemann–Liouville sequential fractional derivative has not been considered up to now, the research proceeds slowly and appears some new difficulties in obtaining comparison results.

Now, in this paper, we discuss the properties of the well-known Mittag–Leffler function, and consider the existence and uniqueness of solution of the initial value problem (1.4) for fractional differential equation involving Riemann–Liouville sequential fractional derivative by using monotone iterative method.

Definition 1.1. We call a function $y(x)$ a classical solution of problem (1.4), if:

(i) $y(x)$ is continuous on $(0, T]$; $x^{1-\alpha} y(x)$, $x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha} y)(x)$ are continuous on $[0, T]$, and its fractional integral $(I^{1-\alpha} y(t))(x)$, $(I^{1-\alpha} \mathscr{D}_{0+}^{\alpha} y(t))(x)$ are continuously differentiable for $(0, T]$;
(ii) $y(x)$ satisfies problem (1.4).

Let

$$C([0, T]) = \Big\{y\colon\ y(x) \text{ is continuous on } [0, T],\ \|y\|_C = \max_{t \in [0,1]} |y(t)|\Big\},$$

$$C_{1-\alpha}([0, T]) = \Big\{y \in C(0, T]\colon x^{1-\alpha} y(x) \in C([0, T]),\ \|y\|_{C_{1-\alpha}} = \|x^{1-\alpha} y\|_C\Big\},$$

$$C_{1-\alpha}^{\alpha}([0, T]) = \Big\{y \in C_{1-\alpha}([0, T])\colon\ x^{1-\alpha}(\mathscr{D}_{0+}^{\alpha} y)(x) \in C([0, T])\Big\}.$$

For problem (1.4), we have the following definitions of upper and lower solutions.

Definition 1.2. A function $p \in C_{1-\alpha}^{\alpha}([0, T])$ is called a lower solution of problem (1.4), if it satisfies

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha} p)(x) \leqslant f(x, p, \mathscr{D}_{0+}^{\alpha} p), \quad x \in (0, T], \\ x^{1-\alpha} p(x)|_{x=0} \leqslant y_0, \quad x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} p)(x)|_{x=0} \leqslant y_1. \end{cases} \tag{1.6}$$

Analogously, a function $q \in C_{1-\alpha}^{\alpha}([0, T])$ is called an upper solution of problem (1.4), if it satisfies

$$\begin{cases} (\mathscr{D}_{0+}^{2\alpha} q)(x) \geqslant f(x, q, \mathscr{D}_{0+}^{\alpha} q), \quad x \in (0, T], \\ x^{1-\alpha} q(x)|_{x=0} \geqslant y_0, \quad x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} q)(x)|_{x=0} \geqslant y_1. \end{cases} \tag{1.7}$$

In what follows, we assume that

$$\begin{cases} p(x) \leqslant q(x), \quad x \in (0, T]\colon x^{1-\alpha} p(x)|_{x=0} \leqslant x^{1-\alpha} q(x)|_{x=0}, \\ x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} p)(x)|_{x=0} \leqslant x^{1-\alpha} (\mathscr{D}_{0+}^{\alpha} q)(x)|_{x=0}. \end{cases} \tag{1.8}$$

and define that the ordered interval in space $C_{1-\alpha}^{\alpha}([0, T])$,

$$[p,q]=\{u\in C^{\alpha}_{1-\alpha}([0,T]):\ p(t)\leqslant u(t)\leqslant q(t),\ t\in(0,T],\ t^{1-\alpha}p(t)|_{t=0}\leqslant t^{1-\alpha}u(t)|_{t=0}\leqslant t^{1-\alpha}q(t)|_{t=0},$$
$$t^{1-\alpha}\big(\mathscr{D}^{\alpha}_{0+}p\big)(t)|_{t=0}\leqslant t^{1-\alpha}\big(\mathscr{D}^{\alpha}_{0+}u\big)(t)|_{t=0}\leqslant t^{1-\alpha}\big(\mathscr{D}^{\alpha}_{0+}q\big)(t)|_{t=0}\}. \tag{1.9}$$

The following is an existence result of the solution for the linear initial value problem for fractional differential equation and a property of Riemann–Liouville fractional calculus, which are important for us to obtain existence and uniqueness results of solutions for problem (1.4).

Lemma 1.1. *(See [1].) Suppose that $u\in C_{1-\alpha}([0,T])$, then the linear initial value problem*

$$\begin{cases}\mathscr{D}^{\alpha}_{0+}u(x)+Mu(x)=\sigma(x), & x\in(0,T],\\ x^{1-\alpha}u(x)|_{x=0}=u_0,\end{cases} \tag{1.10}$$

where $M\in\mathbb{R}$ is a constant and $\sigma\in C_{1-\alpha}[0,T]$, has the following integral representation of solution

$$u(x)=\Gamma(\alpha)u_0e_\alpha(-M,x)+\big[e_\alpha(-M,t)*\sigma(t)\big](x), \tag{1.11}$$

where

$$(g*f)(x)=\int_0^x g(x-t)f(t)\,\mathrm{d}t, \tag{1.12}$$

$$e_\alpha(\lambda,z)=z^{\alpha-1}E_{\alpha,\alpha}\big(\lambda z^\alpha\big)=z^{\alpha-1}\sum_{k=0}^{\infty}\lambda^k\frac{z^{\alpha k}}{\Gamma((k+1)\alpha)}, \tag{1.13}$$

$E_{\alpha,\alpha}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma((k+1)\alpha)}$ is Mittag–Leffler function (see [1,28]).

Remark 1.1. For $\alpha=1$, initial problem (1.10) is $u'(x)+Mu(x)=\sigma(x)$, $u(0)=u_0$ and the solution given by (1.11) is valid (it is the classical solution using the variation of constants formula).

Lemma 1.2. *Suppose that $u\in C^{\alpha}_{1-\alpha}([0,T])$, then the linear initial value problem*

$$\begin{cases}\big(\mathscr{D}^{2\alpha}_{0+}u\big)(x)+N\mathscr{D}^{\alpha}_{0+}u(x)+Mu(x)=\sigma(x), & x\in(0,T],\\ x^{1-\alpha}u(x)|_{x=0}=u_0, \quad x^{1-\alpha}\big(\mathscr{D}^{\alpha}_{0+}u\big)(x)|_{x=0}=u_1,\end{cases} \tag{1.14}$$

where $N,M\in\mathbb{R}$, $N^2>4M$ are constants and $\sigma\in C_{1-\alpha}[0,T]$, has the following representation of solution

$$u(x)=\Gamma(\alpha)u_0e_\alpha(\lambda_2,x)+\Gamma(\alpha)(u_1-\lambda_2u_0)\big[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)\big](x)+\big[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)*\sigma(t)\big](x), \tag{1.15}$$

where

$$\lambda_1=\frac{-N+\sqrt{N^2-4M}}{2},\qquad \lambda_2=\frac{-N-\sqrt{N^2-4M}}{2}<0. \tag{1.16}$$

Proof. Let

$$\big(\mathscr{D}^{\alpha}_{0+}-\lambda_2\big)u(x)=y(x),\quad x\in(0,T].$$

Then the problem in (1.14) is equivalent to

$$\begin{cases}\big(\mathscr{D}^{\alpha}_{0+}-\lambda_1\big)y(x)=\sigma(x), & x\in(0,T],\\ x^{1-\alpha}y(x)|_{x=0}=y_0=u_1-\lambda_2u_0,\end{cases} \tag{1.17}$$

and

$$\begin{cases}\big(\mathscr{D}^{\alpha}_{0+}-\lambda_2\big)u(x)=y(x), & x\in(0,T],\\ x^{1-\alpha}u(x)|_{x=0}=u_0.\end{cases} \tag{1.18}$$

By Lemma 1.1, we have that the linear initial value problems (1.16) and (1.17) have the following representation of solutions

$$y(x)=\Gamma(\alpha)y_0e_\alpha(\lambda_1,x)+\big[e_\alpha(\lambda_1,t)*\sigma(t)\big](x), \tag{1.19}$$
$$u(x)=\Gamma(\alpha)u_0e_\alpha(\lambda_2,x)+\big[e_\alpha(\lambda_2,t)*y(t)\big](x). \tag{1.20}$$

Substituting (1.19) into (1.20), we obtain (1.15). The proof of Lemma 1.2 is completed. □

By a direct computation

$$\begin{aligned}\big[e_\alpha(\lambda_1,t)*e_\alpha(\lambda_2,t)\big](x)&=\int_0^x\left(\sum_{i=0}^{\infty}\frac{\lambda_1^i(x-t)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}\right)\cdot\left(\sum_{j=0}^{\infty}\frac{\lambda_2^j t^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}\right)\mathrm{d}t\\&=\sum_{n=0}^{\infty}\sum_{i+j=n}\lambda_1^i\lambda_2^j\int_0^x\frac{(x-t)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}\cdot\frac{t^{(j+1)\alpha-1}}{\Gamma((j+1)\alpha)}\,\mathrm{d}t\\&=\sum_{n=0}^{\infty}\sum_{i+j=n}\lambda_1^i\lambda_2^j\frac{x^{(i+j+1+1)\alpha-1}}{\Gamma((i+j+1+1)\alpha)}\\&=\sum_{n=0}^{\infty}\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}\cdot\frac{x^{(n+1+1)\alpha-1}}{\Gamma((n+1+1)\alpha)}\\&=\frac{1}{\lambda_1-\lambda_2}\sum_{n=0}^{\infty}\frac{(\lambda_1^n-\lambda_2^n)x^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}\\&=\frac{1}{\lambda_1-\lambda_2}\big[e_\alpha(\lambda_1,t)-e_\alpha(\lambda_2,t)\big](x),\quad x\in\mathbb{R}.\end{aligned}$$

Hence, we obtain that

Lemma 1.3.

$$\big[e_\alpha(\lambda_2,t)*e_\alpha(\lambda_1,t)\big](x)=\big[e_\alpha(\lambda_1,t)*e_\alpha(\lambda_2,t)\big](x)=\frac{1}{\lambda_1-\lambda_2}\big[e_\alpha(\lambda_1,t)-e_\alpha(\lambda_2,t)\big](x),\quad x\in\mathbb{R}. \tag{1.21}$$

This paper is organized as follows. In Section 2 we give some preliminaries, including a property of Mittag–Leffler function which will be used in our main result, a comparison result. The main results are established in Section 3.

2. A property of Mittag–Leffler function and some lemmas

In the following, we shall use the definition and properties of the Γ function which listed as follows (see [29]):

$$\Gamma(\alpha)=\int_0^{+\infty}t^{\alpha-1}e^{-t}\,\mathrm{d}t, \tag{2.1}$$

$$\frac{1}{\Gamma(\alpha)}=\lim_{n\to\infty}\frac{1}{(n+1)^\alpha}\alpha(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right), \tag{2.2}$$

$$\frac{1}{\Gamma(1+\alpha)}=\lim_{n\to\infty}\frac{1}{(n+1)^\alpha}(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right). \tag{2.3}$$

Let

$$\omega_n(\alpha)=\alpha(1+\alpha)\left(1+\frac{\alpha}{2}\right)\cdots\left(1+\frac{\alpha}{n}\right). \tag{2.4}$$

Then

$$\frac{1}{\Gamma(\alpha)}=\lim_{n\to\infty}\frac{1}{(n+1)^\alpha}\omega_n(\alpha). \tag{2.5}$$

Lemma 2.1. *For $0<\alpha\leqslant 1$, there exist positive constants*

$$b_n^0>0,\quad b_n^1>0,\quad \ldots,\quad b_n^n>0,\quad \textit{such that}\quad \omega_n(k\alpha)=\sum_{i=0}^{n}b_n^iC_{k+i}^{i+1}. \tag{2.6}$$

Hence, we have

$$(k-1)\omega_n(k\alpha)=\sum_{i=0}^{n}(i+2)b_n^iC_{k+i}^{i+2}, \tag{2.7}$$

$$(1+k\alpha)\left(1+\frac{k\alpha}{2}\right)\cdots\left(1+\frac{k\alpha}{n}\right)=\frac{1}{\alpha}\sum_{i=0}^{n}\frac{1}{i+1}b_n^i C_{k+i}^i. \tag{2.8}$$

Proof. When $n=0$, we can choose $b_0^0=\alpha$. Now, let

$$\omega_{n-1}(k\alpha)=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}.$$

Then

$$\begin{aligned}
\omega_n(k\alpha)&=\omega_{n-1}(k\alpha)\left(1+\frac{k\alpha}{n}\right)\\
&=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(1+\frac{k\alpha}{n}\right)\\
&=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(\frac{n-(i+1)\alpha}{n}+\frac{(k+i+1)\alpha}{n}\right)\\
&=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(\frac{n-(i+1)\alpha}{n}\right)+\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(\frac{(k+i+1)\alpha}{n}\right)\\
&=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(\frac{n-(i+1)\alpha}{n}\right)+\sum_{i=0}^{n-1}\frac{\alpha}{n}b_{n-1}^i C_{k+i+1}^{i+2}(i+2)\\
&=\sum_{i=0}^{n-1}b_{n-1}^i C_{k+i}^{i+1}\left(\frac{n-(i+1)\alpha}{n}\right)+\sum_{i=1}^{n}\frac{(i+1)\alpha}{n}b_{n-1}^{i-1} C_{k+i}^{i+1}\\
&=\sum_{i=0}^{n}b_n^i C_{k+i}^{i+1},
\end{aligned}$$

where

$$\begin{aligned}
&b_n^0=b_{n-1}^0\cdot\frac{n-\alpha}{n},\qquad b_n^n=b_{n-1}^{n-1}\cdot\frac{\alpha(n+1)}{n},\\
&b_n^i=b_{n-1}^i\cdot\frac{n-\alpha(i+1)}{n}+b_{n-1}^{i-1}\cdot\frac{\alpha(i+1)}{n}\quad(0<i<n).
\end{aligned}$$

By means of $0\leqslant i\leqslant n-1$ and $0<\alpha\leqslant 1$, we have

$$0<(i+1)\alpha<i+1<n\quad\Rightarrow\quad n-(i+1)\alpha>0.$$

Then $b_{n-1}^0>0$, $b_{n-1}^1>0$, $\ldots$, $b_{n-1}^{n-1}>0$ imply that $b_n^0>0$, $b_n^1>0$, $\ldots$, $b_n^n>0$. Hence, (2.6) holds. The proof of Lemma 2.1 is completed. □

Note

$$\begin{cases}F(x)=E_{\alpha,\alpha}(x)=\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{\Gamma((k+1)\alpha)},\quad g(x)=\sum_{k=1}^{\infty}\frac{kx^{k-1}}{\Gamma((k+1)\alpha)},\\ h(x)=\displaystyle\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+1)}.\end{cases} \tag{2.9}$$

Lemma 2.2. *For $0<\alpha\leqslant 1$, we have*

$$F(x)>0,\qquad g(x)>0,\qquad h(x)>0,\quad \forall x\in\mathbb{R}=(-\infty,+\infty). \tag{2.10}$$

Proof. It is easy to see that the convergence domains of $F(x)$, $g(x)$ and $h(x)$ are $\mathbb{R}$, and

$$F(x)>0,\qquad g(x)>0,\qquad h(x)>0,\quad \forall x>0.$$

Let

$$F(x)F(-x)=\sum_{k=0}^{\infty}a_kx^{2k},\qquad g(x)g(-x)=\sum_{k=0}^{\infty}b_kx^{2k},\qquad h(x)h(-x)=\sum_{k=0}^{\infty}c_kx^{2k},\quad \forall x\in\mathbb{R},$$

where

$$a_0=\left(\frac{1}{\Gamma(\alpha)}\right)^2>0,\qquad b_0=\left(\frac{1}{\Gamma(2\alpha)}\right)^2>0,\qquad c_0=1>0,$$

and

$$\begin{aligned}
a_k&=\sum_{m=1}^{2k+1}\frac{(-1)^{m-1}}{\Gamma(m\alpha)\Gamma((2k+2-m)\alpha)}\\
&=\lim_{n\to\infty}\sum_{m=1}^{2k+1}(-1)^{m-1}\omega_n(m\alpha)\omega_n\big((2k+2-m)\alpha\big)(n+1)^{-(2k+2)\alpha},\quad k=1,2,\ldots,\\
b_k&=\sum_{m=2}^{2k+2}\frac{(-1)^m(m-1)(2k+3-m)}{\Gamma(m\alpha)\Gamma((2k+4-m)\alpha)}\\
&=\lim_{n\to\infty}\left[\sum_{m=2}^{2k+2}(-1)^m(m-1)(2k+3-m)\omega_n(m\alpha)\omega_n\big((2k+4-m)\alpha\big)(n+1)^{-(2k+4)\alpha}\right],\quad k=1,2,\ldots,\\
c_k&=\sum_{m=0}^{2k}\frac{(-1)^m}{\Gamma(m\alpha+1)\Gamma((2k-m)\alpha+1)}\\
&=\lim_{n\to\infty}\frac{1}{(n+1)^{2k\alpha}}\sum_{m=0}^{2k}(-1)^m(1+m\alpha)\left(1+\frac{m\alpha}{2}\right)\cdots\left(1+\frac{m\alpha}{n}\right)\\
&\quad\cdot\big(1+(2k-m)\alpha\big)\left(1+\frac{(2k-m)\alpha}{2}\right)\cdots\left(1+\frac{(2k-m)\alpha}{n}\right),\quad k=1,2,\ldots,
\end{aligned}$$

where $\omega_n(x)$ is given by (2.4). By the proof of Lemma 2.1, there exist positive constants $b_n^0>0$, $b_n^1>0,\ldots,b_n^n>0$, such that (2.6), (2.7) and (2.8) hold.

Therefore,

$$\begin{aligned}
&\forall x\in\mathbb{R},\quad F(x)>0,\qquad g(x)>0,\qquad h(x)>0\\
\iff\ &\forall k\in\mathbb{N}_+,\ a_k\geqslant 0,\ b_k\geqslant 0,\ c_k\geqslant 0\\
\iff\ &\forall k,n\in\mathbb{N}_+,\quad \begin{cases}\displaystyle\sum_{m=1}^{2k+1}(-1)^{m-1}\omega_n(m\alpha)\omega_n\big((2k+2-m)\alpha\big)\geqslant 0,\\ \displaystyle\sum_{m=2}^{2k+2}(-1)^m(m-1)(2k+3-m)\omega_n(m\alpha)\omega_n\big((2k+4-m)\alpha\big)\geqslant 0,\\ \displaystyle\sum_{m=0}^{2k}(-1)^m(1+m\alpha)\left(1+\frac{m\alpha}{2}\right)\cdots\left(1+\frac{m\alpha}{n}\right)\\ \displaystyle\quad\cdot\big(1+(2k-m)\alpha\big)\left(1+\frac{(2k-m)\alpha}{2}\right)\cdots\left(1+\frac{(2k-m)\alpha}{n}\right)\geqslant 0\end{cases}\\
\iff\ &\forall k,n\in\mathbb{N}_+,\quad \begin{cases}\displaystyle\sum_{m=1}^{2k+1}(-1)^{m-1}\left(\sum_{i=0}^{n}b_n^iC_{m+i}^{i+1}\right)\left(\sum_{i=0}^{n}b_n^iC_{2k+2-m+i}^{i+1}\right)\geqslant 0,\\ \displaystyle\sum_{m=2}^{2k+2}(-1)^m\left(\sum_{i=0}^{n}(i+2)b_n^iC_{m+i}^{i+2}\right)\left(\sum_{i=0}^{n}(i+2)b_n^iC_{2k+4-m+i}^{i+2}\right)\geqslant 0,\\ \displaystyle\sum_{m=0}^{2k}(-1)^m\frac{1}{\alpha^2}\left(\sum_{i=0}^{n}\frac{1}{i+1}b_n^iC_{m+i}^{i}\right)\left(\sum_{i=0}^{n}\frac{1}{i+1}b_n^iC_{2k-m+i}^{i}\right)\geqslant 0\end{cases}
\end{aligned}$$

$$\iff \quad \forall 0 \leqslant i, j \leqslant n,\ i, j \in \mathbb{N}_+, \quad \begin{cases} \lambda_{i,j}(k)_F = \sum_{m=1}^{2k+1} (-1)^{m-1} \big(C_{m+i}^{i+1} C_{2k+2-m+j}^{j+1} + C_{m+j}^{j+1} C_{2k+2-m+i}^{i+1} \big) \geqslant 0, \\ \lambda_{i,j}(k)_g = \sum_{m=2}^{2k+2} (-1)^{m} \big(C_{m+i}^{i+2} C_{2k+4-m+j}^{j+2} + C_{m+j}^{j+2} C_{2k+4-m+i}^{i+2} \big) \geqslant 0, \\ \lambda_{i,j}(k)_h = \sum_{m=0}^{2k} (-1)^{m} \big(C_{m+i}^{i} C_{2k-m+j}^{j} + C_{m+j}^{j} C_{2k-m+i}^{i} \big) \geqslant 0. \end{cases}$$

For $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$, we have

$$\frac{1}{(1-x)^{i+2}} = \sum_{m=1}^{\infty} C_{m+i}^{i+1} x^{m-1}, \qquad \frac{1}{(1+x)^{i+2}} = \sum_{m=1}^{\infty} (-1)^{m-1} C_{m+i}^{i+1} x^{m-1} \quad \big(|x| < 1\big).$$

Hence, we know that

$$\begin{cases} \frac{1}{(1-x)^{i+2}} \frac{1}{(1+x)^{j+2}} + \frac{1}{(1-x)^{j+2}} \frac{1}{(1+x)^{i+2}} = \sum_{k=0}^{\infty} \lambda_{i,j}(k)_F x^{2k} \quad \big(|x| < 1\big), \\ \frac{1}{(1-x)^{i+3}} \frac{1}{(1+x)^{j+3}} + \frac{1}{(1-x)^{j+3}} \frac{1}{(1+x)^{i+3}} = \sum_{k=0}^{\infty} \lambda_{i,j}(k)_g x^{2k} \quad \big(|x| < 1\big), \\ \frac{1}{(1-x)^{i+1}} \frac{1}{(1+x)^{j+1}} + \frac{1}{(1-x)^{j+1}} \frac{1}{(1+x)^{i+1}} = \sum_{k=0}^{\infty} \lambda_{i,j}(k)_h x^{2k} \quad \big(|x| < 1\big), \end{cases}$$

$$\frac{1}{(1-x)^{i+2}} \frac{1}{(1+x)^{j+2}} + \frac{1}{(1-x)^{j+2}} \frac{1}{(1+x)^{i+2}} = \frac{(1-x)^{|i-j|} + (1+x)^{|i-j|}}{(1-x^2)^{\max\{i,j\}+2}}, \quad \forall x \in \mathbb{R}.$$

Since every coefficient for the expansion of $(1-x)^{|i-j|} + (1+x)^{|i-j|}$ and $\frac{1}{(1-x^2)^{\max\{i,j\}+2}}$ is nonnegative. Hence, $\lambda_{i,j}(k)_F \geqslant 0$, $i, j = 1, 2, \ldots, n$, $k = 1, 2, \ldots$. Similarly, we can prove that

$$\lambda_{i,j}(k)_g \geqslant 0, \qquad \lambda_{i,j}(k)_h \geqslant 0, \quad k = 1, 2, \ldots.$$

Therefore,

$$\forall x \in \mathbb{R}, \quad F(x) > 0, \qquad g(x) > 0, \qquad h(x) > 0.$$

This completes the proof. □

The following result will play a very important role in this paper.

Lemma 2.3 *(A comparison result). If $w \in C_{1-\alpha}([0,T])$ and satisfies the relations*

$$\begin{cases} D^{\alpha} w(t) + M w(t) \geqslant 0, \quad t \in (0, T], \\ t^{1-\alpha} w(t)|_{t=0} \geqslant 0, \end{cases} \tag{2.11}$$

where $M \in \mathbb{R}$ is a constant. Then $w(t) \geqslant 0$, $t \in (0, T]$.

Proof. By Lemma 2.2, we know that $E_{\alpha,\alpha}(-Mt^{\alpha}) > 0$, $t \in (0, T]$. Hence $e_{\alpha}(-M, t) > 0$, $t \in (0, T]$. Let $t^{1-\alpha} w(t)|_{t=0} = w_0$, $D^{\alpha} w(t) + M w(t) = \sigma(t)$, $t \in (0, T]$. Then $w_0 \geqslant 0$, $\sigma(t) \geqslant 0$, $t \in (0, T]$. By the formula (1.11) of Lemma 1.1, we obtain that $w(t) \geqslant 0$, $t \in (0, T]$. □

Remark. In this result, we delete the condition $M > -\frac{\Gamma(1+\alpha)}{T^{\alpha}}$ in Lemma 2.1 of paper [16], so this result is an essential improvement of the paper [16].

Lemma 2.4 *(A comparison result). If $w \in C_{1-\alpha}^{\alpha}([0,T])$ and satisfies the relations*

$$\begin{cases} \big(\mathscr{D}_{0+}^{2\alpha} w\big)(x) + N \mathscr{D}_{0+}^{\alpha} w(x) + M w(x) = \sigma(x) \geqslant 0, \quad x \in (0, T], \\ x^{1-\alpha} w(x)|_{x=0} = w_0 \geqslant 0, \quad x^{1-\alpha} \big(\mathscr{D}_{0+}^{\alpha} w\big)(x)|_{x=0} = w_1 \geqslant 0, \end{cases} \tag{2.12}$$

where $N, M \in \mathbb{R}$, $N^2 > 4M$ *are constants such that*

$$\lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2} \geqslant 0 > \lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2}.$$

Then $w(t) \geqslant 0$, $t \in (0, T]$.

Proof. By $\lambda_2 < 0$, we have $w_1 - \lambda_2 w_0 \geqslant 0$. By means of Lemma 2.2, we know that $e_\alpha(\lambda_1, x) > 0$, $e_\alpha(\lambda_2, x) > 0$, $x \in (0, T]$. Therefore,

$$\left[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t)\right](x) \geqslant 0, \qquad \left[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t) * \sigma(t)\right](x) \geqslant 0, \quad x \in (0, T].$$

By the formula (1.15) of Lemma 1.2, we obtain that $w(t) \geqslant 0$, $t \in (0, T]$. The proof of Lemma 2.4 is completed. □

3. Main results

On the basis of Lemmas 1.2 and 2.3, using the monotone iterative method, we shall show the existence theorem of extremal solutions for IVP (1.4). In the following, we shall assume that f satisfies the following condition:

(H1): there exist constants $N, M \in \mathbb{R}$, $N^2 > 4M$ such that

$$f\left(t, q, \mathscr{D}_{0+}^\alpha q\right) - f\left(t, p, \mathscr{D}_{0+}^\alpha p\right) \geqslant -N\left(\mathscr{D}_{0+}^\alpha q - \mathscr{D}_{0+}^\alpha p\right) - M(q - p), \tag{3.1}$$

$p, q \in C_{1-\alpha}^\alpha([0, T])$ are lower and upper solutions of problem (1.4);

(H2): there exist constants $N, M \in \mathbb{R}$, $N^2 > 4M$ such that (H1) holds, and for $x \in (0, T]$, $p(x) \leqslant y_2 \leqslant y_1 \leqslant q(x)$, $D_1(x) \leqslant z_i \leqslant D_2(x)$, $i = 1, 2$ such that

$$f(x, y_1, z_1) - f(t, y_2, z_2) \geqslant -N(z_1 - z_2) - M(y_1 - y_2), \tag{3.2}$$

where

$$\begin{cases} D_1(x) = \left(\mathscr{D}_{0+}^\alpha p\right)(x) + \lambda_2\left(q(x) - p(x)\right), \\ D_2(x) = \left(\mathscr{D}_{0+}^\alpha q\right)(x) - \lambda_2\left(q(x) - p(x)\right), \quad x \in (0, T], \\ \lambda_1 = \dfrac{-N + \sqrt{N^2 - 4M}}{2} \geqslant 0 > \lambda_2 = \dfrac{-N - \sqrt{N^2 - 4M}}{2}; \end{cases} \tag{3.3}$$

(H3): there exist constants $N, M \in \mathbb{R}$, $N^2 > 4M$ such that (3.3) holds, and for $x \in (0, T]$, $p(x) \leqslant y_2 \leqslant y_1 \leqslant q(x)$, $D_1(x) \leqslant z_i \leqslant D_2(x)$, $i = 1, 2$ such that

$$f(x, y_1, z_1) - f(t, y_2, z_2) \leqslant N(z_1 - z_2) + M(y_1 - y_2). \tag{3.4}$$

In view of (3.2), the function

$$f(t, u, v) + Mu + Nv$$

is monotone nondecreasing in u, v for $u, v \in C_{1-\alpha}([0, T])$.

Lemma 3.1. *Let* (H1) *be satisfied. Then*

$$\mathscr{D}_{0+}^\alpha (q - p)(x) - \lambda_2 (q - p)(x) \geqslant 0, \quad x \in (0, T]. \tag{3.5}$$

Hence,

$$\mathscr{D}_{0+}^\alpha (q)(x) - \lambda_2 (q - p)(x) \geqslant \mathscr{D}_{0+}^\alpha (p)(x) \geqslant \mathscr{D}_{0+}^\alpha (p)(x) + \lambda_2 (q - p)(x), \quad x \in (0, T],$$

where $\lambda_2 < 0$ *is given by* (3.3).

Proof. Let $z(x) = \mathscr{D}_{0+}^\alpha (q - p)(x) - \lambda_2 (q - p)(x)$, $x \in (0, T]$. Then

$$\begin{cases} \mathscr{D}_{0+}^\alpha z(x) - \lambda_1 z(x) = \mathscr{D}_{0+}^{2\alpha} (q - p)(x) - (\lambda_1 + \lambda_2)\mathscr{D}_{0+}^\alpha (q - p)(x) + \lambda_1\lambda_2 (q - p)(x) \\ \qquad \geqslant f(x, q, \mathscr{D}_{0+}^\alpha q) - f(x, p, \mathscr{D}_{0+}^\alpha p) + N\mathscr{D}_{0+}^\alpha (q - p)(x) + M(q - p)(x) \geqslant 0, \quad x \in (0, T], \\ x^{1-\alpha} z(x)|_{x=0} \geqslant 0. \end{cases}$$

By Lemma 2.2, we have that $z(x) \geqslant 0$, $x \in (0, T]$. This complete the proof of Lemma 3.1. □

Lemma 3.2. *Let* (H1) *be satisfied. Then*

$$\Omega = \{\eta \in [p, q]:\ D_1(x) \leqslant (\mathscr{D}^{\alpha}_{0+}\eta)(x) \leqslant D_2(x),\ x \in (0, T]\} \tag{3.6}$$

is a convex closed set.

Theorem 3.1. *Assume that $p, q \in C^{\alpha}_{1-\alpha}([0, T])$ are lower and upper solutions of problem* (1.4)*, such that* (1.8) *holds, $f \in C([0, T] \times \mathbb{R} \times \mathbb{R})$, and satisfies* (H1) *and* (H2)*. Then there exist sequences $\{p_n(t)\}, \{q_n(t)\} \subset C^{\alpha}_{1-\alpha}([0, T])$ with $p_0 = p$, $q_0 = q$ such that $\lim_{n\to\infty} p_n(t) = \rho(t)$, $\lim_{n\to\infty} q_n(t) = \gamma(t)$ on $(0, T]$ and ρ, γ are minimal and maximal solutions on the ordered interval $[p, q]$ for* IVP (1.4)*, respectively, that is ρ, γ are two solutions of* IVP (1.4)*, and for any solution u of* IVP (1.4) *such that $u \in \Omega$, we have*

$$p \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots \leqslant \rho \leqslant u \leqslant \gamma \leqslant \cdots \leqslant q_n \leqslant \cdots \leqslant q_2 \leqslant q_1 \leqslant q. \tag{3.7}$$

Also, if condition (3.4) *holds, then problem* (1.4) *has one unique solution in the ordered interval $[p, q]$.*

Proof of Theorem 3.1. Let

$$\sigma(\eta)(x) = f\big(x, \eta(x), \mathscr{D}^{\alpha}_{0+}\eta(x)\big) + N\mathscr{D}^{\alpha}_{0+}\eta(x) + M\eta(x), \quad x \in (0, T]. \tag{3.8}$$

For any $\eta \in \Omega$, consider the linear IVP

$$\begin{cases} (\mathscr{D}^{2\alpha}_{0+}u)(x) + N\mathscr{D}^{\alpha}_{0+}u(x) + Mu(x) = \sigma(\eta)(x), \quad x \in (0, T], \\ x^{1-\alpha}u(x)|_{x=0} = y_0, \quad x^{1-\alpha}(\mathscr{D}^{\alpha}_{0+}u)(x)|_{x=0} = y_1. \end{cases} \tag{3.9}$$

By Lemmas 1.2 and (1.3), (3.9) has exactly one solution $u \in C^{\alpha}_{1-\alpha}([0, T])$ given by

$$\begin{aligned} u(x) &= (A\eta)(x) \\ &= \Gamma(\alpha)y_0 e_\alpha(\lambda_2, x) + \Gamma(\alpha)(y_1 - \lambda_2 y_0)\big[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t)\big](x) + \big[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t) * \sigma(\eta)(t)\big](x), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \big(\mathscr{D}^{\alpha}_{0+}A\eta\big)(x) &= \Gamma(\alpha)y_0\lambda_2 e_\alpha(\lambda_2, x) + \Gamma(\alpha)(y_1 - \lambda_2 y_0)\frac{1}{\lambda_1 - \lambda_2}\big[\lambda_1 e_\alpha(\lambda_1, t) - \lambda_2 e_\alpha(\lambda_2, t)\big](x) \\ &\quad + \frac{1}{\lambda_1 - \lambda_2}\big[\lambda_1 e_\alpha(\lambda_1, t) * \sigma(\eta)(t) - \lambda_2 e_\alpha(\lambda_2, t) * \sigma(\eta)(t)\big](x). \end{aligned} \tag{3.11}$$

And then A is an operator from Ω into $C^{\alpha}_{1-\alpha}([0, T])$ and η is a solution of IVP (1.4) if and only if $\eta = A\eta$. By $\lambda_1 \geqslant 0 > \lambda_2$ in (3.3), we have

$$\frac{1}{\lambda_1 - \lambda_2}\big[\lambda_1 e_\alpha(\lambda_1, t) - \lambda_2 e_\alpha(\lambda_2, t)\big](x) \geqslant 0, \quad x \in (0, T].$$

By means of Lemmas 2.3 and 2.4, (1.6), (1.7), (3.3), (3.10) and (3.11) we can obtain

$$p \leqslant Ap \leqslant A\eta \leqslant Aq \leqslant q, \quad \forall \eta \in \Omega, \tag{3.12}$$

and

$$\begin{cases} \text{if } p \leqslant \eta_1 \leqslant \eta_2 \leqslant q, \quad \eta_i \in \Omega,\ i = 1, 2, \quad \text{then} \\ \sigma(\eta_1) \leqslant \sigma(\eta_2), \quad A\eta_1 \leqslant A\eta_2, \quad \text{and} \quad \mathscr{D}^{\alpha}_{0+}A\eta_1 \leqslant \mathscr{D}^{\alpha}_{0+}A\eta_2. \end{cases} \tag{3.13}$$

By the proof of Lemma 3.1, we know that

$$z_1(x) = \mathscr{D}^{\alpha}_{0+}(A\eta - p)(x) - \lambda_2(A\eta - p)(x) \geqslant 0, \quad x \in (0, T],\ \forall \eta \in \Omega.$$

Hence,

$$\begin{aligned} \mathscr{D}^{\alpha}_{0+}(A\eta)(x) &\geqslant \mathscr{D}^{\alpha}_{0+}(p)(x) + \lambda_2(A\eta - p)(x) \\ &\geqslant \mathscr{D}^{\alpha}_{0+}(p)(x) + \lambda_2(q - p)(x) = D_1(x), x \in (0, T], \quad \forall \eta \in \Omega. \end{aligned}$$

Similarly, we can obtain that $\mathscr{D}^{\alpha}_{0+}(A\eta)(x) \leqslant D_2(x)$, $x \in (0, T]$, $\forall \eta \in \Omega$. Therefore, $A(\Omega) \subset \Omega$.

Now, let $p_0 = p$, $q_0 = q$, $p_n = Ap_{n-1}$, $q_n = Aq_{n-1}$ $(n = 1, 2, \ldots)$. By (3.12) and (3.13), we have

$$p \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n \leqslant \cdots \leqslant q_n \leqslant \cdots \leqslant q_2 \leqslant q_1 \leqslant q, \tag{3.14}$$

$$D_1(x) \leqslant \mathscr{D}^{\alpha}_{0+}p_1 \leqslant \mathscr{D}^{\alpha}_{0+}p_2 \leqslant \cdots \leqslant \mathscr{D}^{\alpha}_{0+}p_n \leqslant \cdots \leqslant \mathscr{D}^{\alpha}_{0+}q_n \leqslant \cdots \leqslant \mathscr{D}^{\alpha}_{0+}q_2 \leqslant \mathscr{D}^{\alpha}_{0+}q_1 \leqslant D_2(x). \tag{3.15}$$

By (3.12) and (3.13), we see that the upper sequence $\{q_k\}$ is monotone nonincreasing and is bounded from below and that the lower sequence $\{p_k\}$ is monotone nondecreasing and is bounded from above. Moreover, $\mathscr{D}_{0+}^{\alpha}(p_k)(x), \mathscr{D}_{0+}^{\alpha}(q_k)(x) \in [D_1(x), D_2(x)]$. Let $B = \{p_n\colon n = 1, 2, 3, \ldots\}$. In the following, we shall show that B is a relatively compact set in $C_{1-\alpha}^{\alpha}[0, T]$. For any $\eta \in \Omega$, by (1.6), (1.7) and (3.2), we have

$$\begin{cases} \big(\mathscr{D}_{0+}^{2\alpha}p\big)(x) + N\big(\mathscr{D}_{0+}^{\alpha}p\big)(x) + Mp(x) \leqslant f\big(x, p, \mathscr{D}_{0+}^{\alpha}p\big) + N\big(\mathscr{D}_{0+}^{\alpha}p\big)(x) + Mp(x) \\ \quad \leqslant f\big(x, \eta, \mathscr{D}_{0+}^{\alpha}\eta\big) + N\big(\mathscr{D}_{0+}^{\alpha}\eta\big)(x) + M\eta(x) \leqslant f\big(x, q, \mathscr{D}_{0+}^{\alpha}q\big) + N\big(\mathscr{D}_{0+}^{\alpha}q\big)(x) + Mq(x) \\ \quad \leqslant \big(\mathscr{D}_{0+}^{2\alpha}q\big)(x) + N\big(\mathscr{D}_{0+}^{\alpha}q\big)(x) + Mq(x), \quad x \in (0, T]. \end{cases}$$

Since $B, \Omega \subset C_{1-\alpha}^{\alpha}[0, T]$ are bounded sets, therefore, $\{\sigma(\eta)(t) = f(x, \eta, \mathscr{D}_{0+}^{\alpha}\eta) + N(\mathscr{D}_{0+}^{\alpha}\eta)(x) + M\eta(x) \mid \eta \in \Omega\}$ is a bounded set also. Hence, there exists a constant $L > 0$ such that

$$\begin{cases} \big\|\sigma(p_k)\big\| = \max\limits_{0\leqslant t\leqslant T}\big|t^{1-\alpha}\sigma(p_k)(t)\big| \leqslant L, \quad \forall k = 1, 2, 3, \ldots, \\ \iff \quad \big|\sigma(p_k)(t)\big| \leqslant Lt^{\alpha-1}, \quad \forall t \in (0, T],\ k = 1, 2, 3, \ldots, \end{cases} \tag{3.16}$$

On the other hand, from (1.21), $\{p_k(t) \mid k \in \mathbb{N}\}$ satisfies

$$\begin{aligned} p_k(x) = {} & \Gamma(\alpha)y_0 e_\alpha(\lambda_2, x) + \Gamma(\alpha)(y_1 - \lambda_2 y_0)\big[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t)\big](x) \\ & + \big[e_\alpha(\lambda_2, t) * e_\alpha(\lambda_1, t) * \sigma(p_{k-1})(t)\big](x), \end{aligned} \tag{3.17}$$

$$\begin{aligned} \big(\mathscr{D}_{0+}^{\alpha}p_k\big)(x) = {} & \Gamma(\alpha)y_0\lambda_2 e_\alpha(\lambda_2, x) + \Gamma(\alpha)(y_1 - \lambda_2 y_0)\frac{1}{\lambda_1 - \lambda_2}\big[\lambda_1 e_\alpha(\lambda_1, t) - \lambda_2 e_\alpha(\lambda_2, t)\big](x) \\ & + \frac{1}{\lambda_1 - \lambda_2}\big[\lambda_1 e_\alpha(\lambda_1, t) * \sigma(p_{k-1})(t) - \lambda_2 e_\alpha(\lambda_2, t) * \sigma(p_{k-1})(t)\big](x). \end{aligned} \tag{3.18}$$

Let

$$G(\lambda_i, t) = t^{1-\alpha}\big[e_\alpha(\lambda_i, t) * \sigma(p_{k-1})(t)\big], \quad t \in [0, T],\ i = 1, 2. \tag{3.19}$$

(Without loss of generality, we assume $0 \leqslant t_1 < t_2 \leqslant T$.) From $\lambda_2 < 0 \leqslant \lambda_1$, we have

$$\big|G(\lambda_2, t_1) - G(\lambda_2, t_2)\big| \leqslant \frac{L\Gamma(\alpha)}{|\lambda_1|}\big|E_{\alpha,\alpha}\big(\lambda_2 t_1^\alpha\big) - E_{\alpha,\alpha}\big(\lambda_2 t_2^\alpha\big)\big| + \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)}(t_2 - t_1)^\alpha, \tag{3.20}$$

and

$$\big|G(\lambda_1, t_1) - G(\lambda_1, t_2)\big| \leqslant \left(\frac{L\Gamma(\alpha)}{|\lambda_1|} + \frac{LT^\alpha}{\alpha}\right)\big|E_{\alpha,\alpha}\big(\lambda_1 t_1^\alpha\big) - E_{\alpha,\alpha}\big(\lambda_1 t_2^\alpha\big)\big| + \frac{2L\Gamma^2(\alpha)}{\Gamma(2\alpha)}E_{\alpha,\alpha}\big(\lambda_1 T^\alpha\big)(t_2 - t_1)^\alpha. \tag{3.21}$$

From $E_{\alpha,\alpha}(t) \in C[0, T]$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$, when $|t_1 - t_2| < \delta$ (without loss of generality, we assume $0 \leqslant t_1 < t_2 \leqslant T$), we have

$$\big|E_{\alpha,\alpha}\big(\lambda_1 t_1^\alpha\big) - E_{\alpha,\alpha}\big(\lambda_1 t_2^\alpha\big)\big| < \frac{\varepsilon}{8L_1}, \tag{3.22}$$

$$\big|E_{\alpha,\alpha}\big(\lambda_2 t_1^\alpha\big) - E_{\alpha,\alpha}\big(\lambda_2 t_2^\alpha\big)\big| < \frac{\varepsilon}{8L_2}, \tag{3.23}$$

$$(t_2 - t_1)^\alpha < \frac{\varepsilon}{8L_3}, \tag{3.24}$$

where

$$\begin{aligned} L_1 &= \max\left\{\frac{|\Gamma(\alpha)(y_1 - \lambda_2 y_0)\lambda_1|}{|\lambda_1 - \lambda_2|}, \frac{L}{|\lambda_1 - \lambda_2|}\left(\Gamma(\alpha) + \frac{|\lambda_1|T^\alpha}{\alpha}\right)\right\}, \\ L_2 &= \max\left\{\big|\Gamma(\alpha)y_0\lambda_2\big|, \frac{|\Gamma(\alpha)(y_1 - \lambda_2 y_0)\lambda_1|}{|\lambda_1 - \lambda_2|}, \frac{L\Gamma(\alpha)}{|\lambda_1 - \lambda_2|}\right\}, \\ L_3 &= \frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1 - \lambda_2|}\big(|\lambda_2| + |\lambda_1|\Gamma(\alpha)E_{\alpha,\alpha}\big(|\lambda_1|T^\alpha\big)\big). \end{aligned}$$

From (3.18), (3.18)–(3.24) and by a direct computation, we obtain that

$$\begin{aligned}
&\left|t_1^{1-\alpha}\left(\mathscr{D}_{0+}^{\alpha}p_k\right)(t_1)-t_2^{1-\alpha}\left(\mathscr{D}_{0+}^{\alpha}p_k\right)(t_2)\right|\\
&\quad\leqslant\left|\Gamma(\alpha)y_0\lambda_2\right|\left|E_{\alpha,\alpha}\left(\lambda_2t_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda_2t_2^{\alpha}\right)\right|\\
&\qquad+\frac{|\Gamma(\alpha)(y_1-\lambda_2y_0)|}{|\lambda_1-\lambda_2|}\left[|\lambda_1|\left|E_{\alpha,\alpha}\left(\lambda_1t_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda_1t_2^{\alpha}\right)\right|+|\lambda_2|\left|E_{\alpha,\alpha}\left(\lambda_2t_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda_2t_2^{\alpha}\right)\right|\right]\\
&\qquad+\frac{L}{|\lambda_1-\lambda_2|}\left[\left(\Gamma(\alpha)+\frac{|\lambda_1|T^{\alpha}}{\alpha}\right)\left|E_{\alpha,\alpha}\left(\lambda_1t_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda_1t_2^{\alpha}\right)\right|+\Gamma(\alpha)\left|E_{\alpha,\alpha}\left(\lambda_2t_1^{\alpha}\right)-E_{\alpha,\alpha}\left(\lambda_2t_2^{\alpha}\right)\right|\right]\\
&\qquad+\frac{2L\Gamma(\alpha)}{\Gamma(2\alpha)|\lambda_1-\lambda_2|}\left(|\lambda_2|+|\lambda_1|\Gamma(\alpha)E_{\alpha,\alpha}\left(|\lambda_1|T^{\alpha}\right)\right)(t_2-t_1)^{\alpha}<\varepsilon.
\end{aligned}$$

This means B is equi-continuity in $C_{1-\alpha}^{\alpha}[0,T]$, by means of the Arzela–Ascoli theorem, we have that B is a relatively compact set of $C_{1-\alpha}^{\alpha}[0,T]$. Similarly, we can also show that $\{q_k(t)\}$ is a relatively compact set of $C_{1-\alpha}^{\alpha}[0,T]$. Therefore, $\{p_k(t)\}$ and $\{q_k(t)\}$ converges to $\rho(t)$ and $\gamma(t)$ uniformly on $[0,T]$ respectively, that is

$$\begin{cases}\lim\limits_{k\to\infty}p_k(x)=\rho(x),\quad \lim\limits_{k\to\infty}q_k(x)=\gamma(x), & x\in(0,T],\\ \lim\limits_{k\to\infty}\mathscr{D}_{0+}^{\alpha}p_k(x)=\mathscr{D}_{0+}^{\alpha}\rho(x),\quad \lim\limits_{k\to\infty}\mathscr{D}_{0+}^{\alpha}q_k(x)=\mathscr{D}_{0+}^{\alpha}\gamma(x), & x\in(0,T].\end{cases}\tag{3.25}$$

Moreover, by (3.14) and (3.15), the limits ρ,γ satisfy

$$p\leqslant p_1\leqslant p_2\leqslant\cdots\leqslant p_n\leqslant\cdots\leqslant\rho(x)\leqslant\gamma(x)\leqslant\cdots\leqslant q_n\leqslant\cdots\leqslant q_2\leqslant q_1\leqslant q,\quad x\in(0,T],\tag{3.26}$$

and

$$\begin{aligned}D_1(x)&\leqslant\mathscr{D}_{0+}^{\alpha}p_1\leqslant\mathscr{D}_{0+}^{\alpha}p_2\leqslant\cdots\leqslant\mathscr{D}_{0+}^{\alpha}p_n\leqslant\cdots\leqslant\mathscr{D}_{0+}^{\alpha}\rho(x)\leqslant\mathscr{D}_{0+}^{\alpha}\gamma(x)\leqslant\cdots\leqslant\mathscr{D}_{0+}^{\alpha}q_n\leqslant\cdots\\&\leqslant\mathscr{D}_{0+}^{\alpha}q_2\leqslant\mathscr{D}_{0+}^{\alpha}q_1\leqslant D_2(x),\quad x\in(0,T].\end{aligned}\tag{3.27}$$

In the following, to prove that ρ and γ are solutions of initial value problem (1.4). By the assumption of function f, the function σ is continuous and is monotone nondecreasing in p, the convergence of p_k to ρ (see (3.25)) implies that $\sigma(p_k)(x)$ converges to $\sigma(\rho)(x)$, $x\in(0,T]$. Let $k\to\infty$ in (3.17) and apply the dominated convergence theorem, ρ satisfies the integral equation

$$\begin{aligned}\rho(x)&=(A\rho)(x)\\&=\Gamma(\alpha)y_0e_{\alpha}(\lambda_2,x)+\Gamma(\alpha)(y_1-\lambda_2y_0)\left[e_{\alpha}(\lambda_2,t)*e_{\alpha}(\lambda_1,t)\right](x)+\left[e_{\alpha}(\lambda_2,t)*e_{\alpha}(\lambda_1,t)*\sigma(\rho)(t)\right](x),\\&\quad x\in(0,T].\end{aligned}\tag{3.28}$$

That is, $\rho(x)$ is an integral representation of the solution to problem (3.9), that is, $\rho(t)$ is an integral representation of the solution to problem (1.4). By assumptions of functions f and Lemma 1.2, ρ is a classical solution of initial value problem (1.4). This proves that the lower sequence p_k converges to a solution ρ of problem (1.4). Similarly, we also can prove that the upper sequence q_k converges to a solution γ of problem (1.4), and satisfies relation $\rho(x)\leqslant\gamma(x)$, $\mathscr{D}_{0+}^{\alpha}\rho(x)\leqslant\mathscr{D}_{0+}^{\alpha}\gamma(x)$, $x\in(0,T]$. It follows by using standard arguments that (3.7) holds and $\rho(x)$ and $\gamma(x)$ are minimal and maximal solutions of IVP (1.4) on the ordered interval $[p,q]$ respectively.

Finally, if condition (H3) holds, then $\rho=\gamma$ is a unique solution of problem (1.4). It is sufficient to prove $\rho(t)\geqslant\gamma(t)$, $\mathscr{D}_{0+}^{\alpha}\rho(t)\geqslant\mathscr{D}_{0+}^{\alpha}\gamma(t)$, $t\in(0,T]$ by $\rho(t)\leqslant\gamma(t)$, $\mathscr{D}_{0+}^{\alpha}\rho(t)\leqslant\mathscr{D}_{0+}^{\alpha}\gamma(t)$, $t\in(0,T]$ obtained in former. In fact, by (1.4) and (H3), the function $w=\rho-\gamma$ satisfies the relations

$$\begin{cases}\left(\mathscr{D}_{0+}^{2\alpha}w\right)(x)+N\mathscr{D}_{0+}^{\alpha}w(x)+Mw(x)\\ \quad=-\left(f\left(t,\gamma,\mathscr{D}_{0+}^{\alpha}\gamma\right)-f\left(t,\rho,\mathscr{D}_{0+}^{\alpha}\rho\right)\right)-N\mathscr{D}_{0+}^{\alpha}(\gamma-\rho)(x)-M(\gamma-\rho)(x)\\ \quad=\phi(w)(x)\geqslant0,\quad x\in(0,T],\\ x^{1-\alpha}w(x)|_{x=0}=0,\quad x^{1-\alpha}\left(\mathscr{D}_{0+}^{\alpha}w\right)(x)|_{x=0}=0,\end{cases}\tag{3.29}$$

$$\left(\mathscr{D}_{0+}^{\alpha}w\right)(x)=\frac{1}{\lambda_1-\lambda_2}\left[\left(\lambda_1e_{\alpha}(\lambda_1,t)-\lambda_2e_{\alpha}(\lambda_2,t)\right)*\phi(w)(t)\right](x)\geqslant0,\tag{3.30}$$

then, Lemma 2.4 implies that $w(x)\geqslant0$, $x\in(0,T]$. This and (3.30) prove $\rho(x)\geqslant\gamma(x)$, $\mathscr{D}_{0+}^{\alpha}\rho(x)\geqslant\mathscr{D}_{0+}^{\alpha}\gamma(x)$, $x\in(0,T]$, and therefore we obtain that $\rho=\gamma$ is a unique solution of problem (1.4). Thus, we complete this proof. □

Corollary 3.1. *Assume that $p, q \in C_{1-\alpha}^{\alpha}([0,T])$ are lower and upper solutions of problem* (1.4)*, such that* (1.8) *holds, $f \in C([0,T] \times \mathbb{R} \times \mathbb{R})$, and satisfies Lipschitz condition*

$$\left|f(t,u_1,v_1) - f(t,u_2,v_2)\right| \leqslant M|u_1 - u_2| + N|v_1 - v_2|, \quad u_1, u_2 \in [p,q],\ v_1, v_2 \in [D_1, D_2], \tag{3.31}$$

where $M > 0$, $N > 0$, $N^2 > 4M$ are Lipschitz constant such that (3.3)*, then problem* (1.4) *has one unique solution in the ordered interval $[p,q]$.*

Proof. From (3.31), we obtain that

$$\begin{aligned} -M(u_1 - u_2) - N(v_1 - v_2) &\leqslant f(t,u_1,v_1) - f(t,u_2,v_2) \\ &\leqslant M(u_1 - u_2) + N(v_1 - v_2), \quad p \leqslant u_2 \leqslant u_1 \leqslant q,\ v_1, v_2 \in [D_1, D_2], \end{aligned}$$

that is, (3.2) and (3.4) hold, then Theorem 3.1 implies that problem (1.4) has one unique solution in the ordered interval $[p,q]$. □

Example. Consider the following IVP

$$\begin{cases} \left(\mathscr{D}_{0+}^{2\alpha} u\right)(x) = q(x)\dfrac{(1-2\alpha)}{2\Gamma(2-2\alpha)} x^{\alpha}(1-x)^{1/3} + u^{\delta_1}\left(\mathscr{D}_{0+}^{\alpha} u\right)^{\delta_2}, \quad x \in (0,1], \\ x^{1-\alpha} u(x)|_{x=0} = 0, \quad x^{1-\alpha}\left(\mathscr{D}_{0+}^{\alpha} u\right)(x)|_{x=0} = 0, \end{cases} \tag{3.32}$$

where $0 < \alpha < 1/2$, $0 < \delta_1$, $0 < \delta_2 < 2$. Then IVP (3.32) has a solution u such that $0 \leqslant u(x) \leqslant q(x)$, $x \in (0,1]$, where

$$p(x) = 0, \qquad q(x) = \left[\frac{\Gamma(1-\alpha)\cdot(1-2\alpha)}{2\Gamma(2-2\alpha)}\right]^{1/(\delta_1+\delta_2-1)},$$

p is a lower solution and q an upper solution. The proof is omitted.

Acknowledgments

The authors would like to thank the referee for some valuable suggestions and comments.

References

[1] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[2] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.
[3] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.
[4] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley and Sons, New York, 1993.
[5] M.A. Al-Bassam, Some existence theorems on differential equations of generalized order, J. Reine Angew. Math. 218 (1) (1965) 70–78.
[6] M. Rivero, L. Rodriguez-Germa, J.J. Trujillo, Linear fractional differential equations with variable coefficients, Appl. Math. Lett. 21 (2008) 892–897.
[7] W. Ibrahim Rabha, Maslina Darus, Subordination and superordination for univalent solutions for fractional differential equations, J. Math. Anal. Appl. 345 (2008) 871–879.
[8] Zhang Shuqin, Positive solution for some class of nonlinear fractional differential equation, J. Math. Anal. Appl. 278 (1) (2003) 136–148.
[9] J.V. Devi, V. Lakshmikantham, Nonsmooth analysis and fractional differential equations, Nonlinear Anal. 70 (12) (2009) 4151–4157.
[10] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal. 70 (7) (2009) 2521–2529.
[11] V. Lakshmikantham, S. Leela, A Krasnoselskii–Krein-type uniqueness result for fractional differential equations, Nonlinear Anal. 71 (2009) 3421–3424.
[12] Mohammed Belmekki, Juan J. Nieto, Rosana Rodríguez-López, Existence of periodic solution for a nonlinear fractional differential equation, Bound. Value Probl. 2009 (2009), Art. ID 324561.
[13] A. Arara, M. Benchohra, N. Hamidi, J.J. Nieto, Fractional order differential equations on an unbounded domain, Nonlinear Anal. 72 (2010) 580–586.
[14] Chang Yong-Kui, Juan J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, Math. Comput. Modelling 49 (2009) 605–609.
[15] Nickolai Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal. 70 (7) (2009) 2521–2529.
[16] Zhang Shuqin, Monotone iterative method for initial value problem involving Riemann–Liouville fractional derivatives, Nonlinear Anal. 71 (2009) 2087–2093.
[17] Ravi P. Agarwal, V. Lakshmikantham, Juan J. Nieto, On the concept of solution for fractional differential equations with uncertainty, Nonlinear Anal. 72 (6) (2010) 2859–2862.
[18] Ma Qing-Hua, Josip Pečarić, Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations, J. Math. Anal. Appl. 341 (2) (2008) 894–905.
[19] Bashir Ahmad, Juan J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, Comput. Math. Appl. 58 (9) (2009) 1838–1843.
[20] Shi Kehua, Wang Yongjin, On a stochastic fractional partial differential equation driven by a Lévy space–time white noise, J. Math. Anal. Appl. 364 (1) (2010) 119–129.
[21] Junyi Liu, Mingyu Xu, Some exact solutions to Stefan problems with fractional differential equations, J. Math. Anal. Appl. 351 (2) (2009) 536–542.
[22] G.S. Ladde, V. Lakshmikantham, A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman Pub. Co., Boston, 1985.

[23] B. Ahmad, S. Sivasundaram, Existence results and monotone iterative technique for impulsive hybrid functional differential systems with anticipation and retardation, Appl. Math. Comput. 197 (2008) 515–524.
[24] J.J. Nieto, R. Rodriguez-Lopez, Monotone method for first-order functional differential equations, Comput. Math. Appl. 52 (2006) 471–484.
[25] D. Jiang, J.J. Nieto, W. Zuo, On monotone method for first and second order periodic boundary value problems and periodic solutions of functional differential equations, J. Math. Anal. Appl. 289 (2004) 691–699.
[26] J.J. Nieto, R. Rodriguez-Lopez, Boundary value problems for a class of impulsive functional equations, Comput. Math. Appl. 55 (2008) 2715–2731.
[27] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett. 21 (2008) 828–834.
[28] B. Bonilla, M. Rivero, L. Rodríguez-Germá, J.J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Appl. Math. Comput. 187 (2007) 79–88.
[29] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.