Differential Geometry and its Applications 28 (2010) 301–312

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Spaces admitting homogeneous G_2-structures

Frank Reidegeld *

Department Mathematik, Bereich Analysis und Differentialgeometrie, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

ARTICLE INFO

Article history:
Received 3 February 2009
Received in revised form 25 August 2009
Available online 4 November 2009
Communicated by Th. Friedrich

MSC:
53C10
53C15
53C30

Keywords:
Cosymplectic G_2-structures
Homogeneous spaces

ABSTRACT

We classify all seven-dimensional manifolds which admit a homogeneous cosymplectic G_2-structure. The motivation for this classification is that each of these spaces is a possible principal orbit of a parallel Spin(7)-manifold of cohomogeneity one.

1. Introduction

The aim of this article is to classify all manifolds which admit a homogeneous cosymplectic G_2-structure. More precisely, we want to solve the following classification problem: Find all triples (M, G, φ) of a seven-dimensional compact manifold M, a Lie group G, and a group action $\varphi : G \times M \to M$ such that φ is transitive and M admits a G-invariant cosymplectic G_2-structure.

We will identify two triples (M, G, φ) and (M', G', φ') with each other if there exists a Lie group isomorphism $\phi : G \to G'$ and a G-equivariant diffeomorphism $f : M \to M'$, i.e. $f(\varphi(g, p)) = \varphi'(\phi(g), f(p))$ for all $g \in G$ and $p \in M$. If M admits a G-invariant cosymplectic G_2-structure, M' also admits such a structure. For reasons of simplicity, we will often identify (M, G, φ) and (M', G', φ'), too, if there exists a third triple $(\widetilde{M}, \widetilde{G}, \psi)$ such that $\widetilde{G}$ covers G and G' and there are $\widetilde{G}$-equivariant covering maps $\pi : \widetilde{M} \to M$ and $\pi' : \widetilde{M} \to M'$. In this situation, we will say that (M, G, φ) and (M', G', φ') are *the same up to a covering*.

The above identifications allow us to assume that M is a coset space G/H. For each G/H we calculate the dimension n_{G_2} of the space of all G-invariant G_2-structures. By way of comparison, we also calculate the dimension $n_{O(7)}$ of the space of all G-invariant metrics on G/H.

In the literature, many homogeneous cosymplectic G_2-structures are known. Friedrich, Kath, Moroianu, and Semmelmann [14] classify all simply connected, compact manifolds which admit a homogeneous nearly parallel G_2-structure. The product of a manifold with a homogeneous $SU(3)$-structure and a circle carries a canonical homogeneous G_2-structure. The examples from the article of Cleyton and Swann [7] which admit a homogeneous $SU(3)$-structure should therefore be mentioned in this context, too.

* Tel.: +49 40 42838 5183.
E-mail address: frank.reidegeld@math.uni-hamburg.de.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2009.10.013

Table 1

G	H	G/H	$n_{O(7)}$	n_{G_2}
$U(1)^7$	$\{e\}$	T^7	28	35
$SU(2) \times U(1)^4$	$\{e\}$	$S^3 \times T^4$	28	35
$SU(2)^2 \times U(1)$	$\{e\}$	$S^3 \times S^3 \times S^1$	28	35
$SU(2)^2 \times U(1)^2$	$U(1)$	$S^3 \times S^3 \times S^1$	10	13
$SO(4) \times U(1)^2$	$SO(2)$	$V^{4,2} \times T^2$	10	13
$SU(2)^3 \times U(1)$	$SU(2)$	$S^3 \times S^3 \times S^1$	4	5
$SU(3) \times U(1)^2$	$SU(2)$	$S^5 \times T^2$	7	10
$SU(3) \times U(1)$	$U(1)^2$	$SU(3)/U(1)^2 \times S^1$	4	5
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	$\mathbb{CP}^3 \times S^1$	3	4
$G_2 \times U(1)$	$SU(3)$	$S^6 \times S^1$	2	3

Table 2

G	H	G/H		$n_{O(7)}$	n_{G_2}
$SU(3)$	$U(1)$	$N^{1,1}$		10	13
$SU(3)$	$U(1)$	$N^{1,0}$		6	7
$SU(3)$	$U(1)$	$N^{k,l}$	with $k, l \in \mathbb{Z}$, $k \geqslant l \geqslant 0$, $k \cdot l > 1$	4	5
$SO(5)$	$SO(3)$	$V^{5,2}$		4	5
$Sp(2)$	$Sp(1)$	S^7		7	10
$SO(5)$	$SO(3)$	B^7		1	1
$SU(2)^3$	$U(1)^2$	$Q^{1,1,1}$		4	5
$SU(3) \times U(1)$	$U(1)^2$	$N^{k,l}$	with $k, l \in \mathbb{Z}$ arbitrary	4	5
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$M^{1,1,0}$		3	4
$SU(3) \times SU(2)$	$SU(2) \times U(1)$	$N^{1,1}$		2	2
$Sp(2) \times U(1)$	$Sp(1) \times U(1)$	S^7		3	4
$Sp(2) \times Sp(1)$	$Sp(1) \times Sp(1)$	S^7		2	2
$SU(4)$	$SU(3)$	S^7		2	3
Spin(7)	G_2	S^7		1	1

One reason for our interest in this kind of manifolds is that any principal orbit of a parallel Spin(7)-manifold of cohomogeneity one carries a homogeneous cosymplectic G_2-structure. Conversely, any homogeneous cosymplectic G_2-structure can be extended to a parallel Spin(7)-manifold of cohomogeneity one. A discussion of these facts can be found in Hitchin [17]. The following theorem sums up the results of this article:

Theorem 1.

1. *Let G/H be a seven-dimensional, compact, connected, homogeneous space which admits a G-invariant G_2-structure. We assume that G/H is a product of a circle and another homogeneous space and that G acts almost effectively on G/H. Furthermore, we assume that G and H are both connected. In this situation, G, H, and G/H are up to a covering one of the spaces from Table 1 and the dimensions n_{G_2} ($n_{O(7)}$) of the space of all G-invariant G_2-structures (metrics) on G/H are shown in Table 1.*
2. *Let G, H, and G/H satisfy the same conditions as before with the single exception that G/H is not a product of a circle and another homogeneous space. In this situation, G, H, and G/H are up to a covering one of the spaces from Table 2 and the dimensions n_{G_2} ($n_{O(7)}$) of the space of all G-invariant G_2-structures (metrics) on G/H are shown in Table 2.*
3. *Any of the G/H from Table 1 or Table 2 admits a G-invariant cosymplectic G_2-structure. If G/H is from Table 2, it even admits a G-invariant nearly parallel G_2-structure.*

In Tables 1 and 2, $N^{k,l}$ denotes an Aloff–Wallach space, $V^{4,2}$ ($V^{5,2}$) denotes the Stiefel manifold of all orthonormal pairs in $\mathbb{R}^4$ ($\mathbb{R}^5$), and B^7 is the seven-dimensional Berger space. Why it suffices to consider Aloff–Wallach spaces $N^{k,l}$ with $k \geqslant l \geqslant 0$ will be explained in Section 6. In the fourth, fifth, and sixth row of Table 1 and in the first, second, seventh and ninth row of Table 2, the embedding of H into G has to be special in order to make G/H a homogeneous space which admits a G-invariant G_2-structure. The details of those embeddings are described in Sections 5 and 6. In the other cases, the information in Tables 1 and 2 is sufficient to determine the embedding of H into G.

From the theorem it follows that either G/H is a product of a circle and a manifold which admits a homogeneous $SU(3)$-structure or that it cannot be decomposed into factors of lower dimension. We remark that we not only prove the existence of a homogeneous cosymplectic G_2-structure on each of the manifolds but also the existence of cosymplectic G_2-structures which are invariant under any of the transitive group actions. The homogeneous space $V^{4,2} \times T^2$ admits a homogeneous G_2-structure but seems not to be mentioned in the literature before.

This article is organized as follows: After two introductory sections, we review the results of Dynkin [10,11] on the connected Lie subgroups of G_2. This is necessary, since in the situation of the theorem H can be embedded into G_2.

In Sections 5 and 6, we classify all G/H which admit a G-invariant, but not necessarily cosymplectic G_2-structure. The question how many G-invariant G_2-structures and metrics exist on G/H is investigated in Section 7. In order to finish the proof of Theorem 1, we have to prove the existence of a G-invariant cosymplectic G_2-structure on all of the manifolds G/H. This will be done in Section 8.

2. The group G_2

Before we classify the connected subgroups of G_2, we collect some facts on this group. For a more comprehensive introduction into this issue, see Baez [2] or Bryant [3].

The group G_2 can be defined with help of the octonions: We recall that a *normed division algebra* is a pair $(A, \langle\cdot,\cdot\rangle)$ of a real, not necessarily associative algebra with a unit element and a scalar product which satisfies $\langle x\cdot y, x\cdot y\rangle = \langle x,x\rangle\langle y,y\rangle$ for all $x, y \in A$. There exists up to isomorphisms exactly one eight-dimensional normed division algebra, namely the *octonions* $\mathbb{O}$.

The quaternions $\mathbb{H}$ are a subalgebra of $\mathbb{O}$. We fix an octonion ϵ in the orthogonal complement of $\mathbb{H}$ such that $\|\epsilon\| = 1$. We call $(x_0,\ldots,x_7) := (1, i, j, k, \epsilon, i\epsilon, j\epsilon, k\epsilon)$ the *standard basis of* $\mathbb{O}$. Let $\mathrm{Im}(\mathbb{O}) := \mathrm{span}(1)^\perp$ be the *imaginary space of* $\mathbb{O}$. The map

$$\begin{aligned} &\omega : \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \times \mathrm{Im}(\mathbb{O}) \to \mathbb{R}, \\ &\omega(x, y, z) := \langle x\cdot y, z\rangle \end{aligned} \tag{1}$$

is a three-form. From now on, we denote $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ shortly by $dx^{i_1\ldots i_k}$. With this notation, we have:

$$\omega = dx^{123} + dx^{145} - dx^{167} + dx^{246} + dx^{257} + dx^{347} - dx^{356}. \tag{2}$$

Remark 2.1. The multiplication table of $\mathbb{O}$ is uniquely determined by the coefficients of ω. Let ϵ' be an octonion with the same properties as ϵ. Since there exists an automorphism of $\mathbb{O}$ which is the identity on $\mathbb{H}$ and maps ϵ to ϵ', ω is independent of the choice of ϵ.

We are now able to define the Lie group G_2:

Definition and Lemma 2.2.

1. *Any automorphism φ of $\mathbb{O}$ satisfies $\varphi(\mathrm{Im}(\mathbb{O})) \subseteq \mathrm{Im}(\mathbb{O})$ and thus can be identified with a map from $\mathrm{Im}(\mathbb{O})$ onto itself. G_2 is defined as the stabilizer group of ω or equivalently as the automorphism group of $\mathbb{O}$.*
2. *The Lie algebra of G_2 we denote by $\mathfrak{g}_2$.*
3. *The seven-dimensional representation which is induced by the action of G_2 on $\mathrm{Im}(\mathbb{O})$ by automorphisms we call the* standard representation *of G_2.*

A proof of the fact that the stabilizer of ω is the same as the automorphism group of $\mathbb{O}$ can be found in Bryant [3]. The Hodge dual $*\omega \in \bigwedge^4 \mathrm{Im}(\mathbb{O})^*$ of ω with respect to $\langle\cdot,\cdot\rangle$ and the orientation which makes $(x_1,\ldots,x_7)$ positive can be written as:

$$*\omega = -dx^{1247} + dx^{1256} + dx^{1346} + dx^{1357} - dx^{2345} + dx^{2367} + dx^{4567}. \tag{3}$$

Finally, we fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}_2$, which we will need for our explicit calculations. With respect to the standard basis of $\mathrm{Im}(\mathbb{O})$, let $\mathfrak{t}$ be the following set of matrices:

$$\mathfrak{t} := \left\{ \left(\begin{array}{ccccccc} 0 & & & & & & \\ & 0 & \lambda_1 & & & & \\ & -\lambda_1 & 0 & & & & \\ & & & 0 & \lambda_2 & & \\ & & & -\lambda_2 & 0 & & \\ & & & & & 0 & \lambda_1+\lambda_2 \\ & & & & & -\lambda_1-\lambda_2 & 0 \end{array} \right) \middle| \lambda_1, \lambda_2 \in \mathbb{R} \right\}. \tag{4}$$

3. Some remarks on G_2-structures

In this section, we define the notion of a G_2-structure. We refer the reader to Bryant [3] or the books of Joyce [18] and Salamon [21] for further reading. A G_2-structure can be defined as a three-form which is at each point stabilized by G_2:

Definition 3.1. Let M be a seven-dimensional manifold and ω be a three-form on M with the following property: For any $p \in M$ there exists a neighborhood U of p and vector fields $X_1, \ldots, X_7$ on U such that

$$\omega_q(X_i, X_j, X_k) = \omega(x_i, x_j, x_k) \quad \forall q \in U,\ i, j, k \in \{1, \ldots, 7\}. \tag{5}$$

The ω on the right-hand side of (5) is the three-form (2) and x_i, x_j, and x_k are elements of the standard basis of $\mathbb{O}$. In this situation, ω is called a *G_2-structure on M* and the pair (M, ω) is called a *G_2-manifold*.

On any G_2-manifold (M, ω) there exist a metric g and a volume form vol which are defined by:

$$g(X, Y)\,\mathrm{vol} := -\frac{1}{6}(X \lfloor \omega) \wedge (Y \lfloor \omega) \wedge \omega. \tag{6}$$

We call g the *associated metric* and vol the *associated volume form*. The Hodge dual $*\omega$ with respect to g and vol is a four-form $*\omega$, which is invariant under the stabilizer G_2 of ω. On the flat G_2-manifold $(\mathbb{R}^7, \omega)$ this four-form coincides with (3). In this article, we consider the following types of G_2-structures:

Definition 3.2. A G_2-manifold (M, ω) is called

1. *parallel* if $d\omega = 0$ and $d * \omega = 0$,
2. *nearly parallel* if there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ such that $d\omega = \lambda * \omega$ and thus $d * \omega = 0$,
3. *cosymplectic* if $d * \omega = 0$.

Further information on the different types of G_2-structures can be found in the article by Fernández and Gray [12]. We will deal first of all with homogeneous G_2-manifolds:

Definition 3.3. A G_2-manifold (M, ω) is called *homogeneous* if there exists a transitive smooth action by a Lie group G which leaves ω invariant.

In the above situation, M is G-equivariantly diffeomorphic to a quotient G/H. The group H acts on the tangent space of G/H by its isotropy representation. Since G_2 acts on the tangent space as the stabilizer of ω and ω is G-invariant, we have proven the following lemma:

Lemma 3.4. *Let G/H be a seven-dimensional homogeneous space which admits a G-invariant G_2-structure. We assume that G acts effectively on G/H. In this situation, there exists a vector space isomorphism $\varphi : T_pG/H \to \mathbb{R}^7$ such that $\varphi H \varphi^{-1} \subseteq G_2$, where H is identified with its isotropy representation and G_2 with its seven-dimensional irreducible representation.*

The converse of the above lemma is also true:

Lemma 3.5. *Let G/H be a seven-dimensional homogeneous space such that G acts effectively and there exists a vector space isomorphism $\varphi : T_pG/H \to \mathbb{R}^7$ with $\varphi H \varphi^{-1} \subseteq G_2$. In this situation, there exists a G-invariant G_2-structure on G/H.*

Proof. The action of G on the tangent bundle determines a G-invariant H-structure on G/H. Its extension to a principal bundle with structure group G_2 is a G-invariant G_2-structure. □

4. Subgroups of G_2

In this section, we describe all connected subgroups of G_2. The semisimple subalgebras of all semisimple Lie algebras including $\mathfrak{g}_2$ have been classified by Dynkin [11]. Moreover, Dynkin [10] has proven that G_2 contains exactly three maximal subgroups which are isomorphic to $SO(3)$, $SO(4)$, and $SU(3)$. These results yield the complete list of all connected subgroups of G_2. An explicit description of all nonabelian subalgebras of $\mathfrak{g}_2$ can be found in Friedrich [15]. In Cacciatori et al. [5], there is a nice description of the maximal subgroup $SO(4)$. The results of the above papers can be summed up as follows:

Theorem 4.1. *Let H be a connected Lie subgroup of G_2. We denote the Lie algebra of H by $\mathfrak{h}$. The standard representation of G_2 induces an action of H on* $\mathrm{Im}(\mathbb{O})$*. In this situation, $\mathfrak{h}$, H, and the splitting of* $\mathrm{Im}(\mathbb{O})$ *with respect to H are contained in Table 3. Moreover, any two connected Lie subgroups of G_2 whose action on* $\mathrm{Im}(\mathbb{O})$ *is equivalent are conjugate not only by an element of $GL(7)$ but even by an element of G_2.*

The subscripts of the modules in Table 3 denote the weights of the H-action and the superscript indicates if the module is complex or real.

Table 3

$\mathfrak{h}$	H	Splitting of $\mathrm{Im}(\mathbb{O})$ into irreducible summands
$\{0\}$	$\{e\}$	
$\mathfrak{u}(1)$	$U(1)$	$\mathbb{V}^{\mathbb{C}}_a \oplus \mathbb{V}^{\mathbb{C}}_b \oplus \mathbb{V}^{\mathbb{C}}_{-a-b} \oplus \mathbb{V}^{\mathbb{R}}_0$
$2\mathfrak{u}(1)$	$U(1)^2$	$\mathbb{V}^{\mathbb{C}}_{1,0} \oplus \mathbb{V}^{\mathbb{C}}_{0,1} \oplus \mathbb{V}^{\mathbb{C}}_{1,1} \oplus \mathbb{V}^{\mathbb{R}}_{0,0}$
$\mathfrak{su}(2)$	$SU(2)$	$\mathbb{V}^{\mathbb{C}}_1 \oplus 3\mathbb{V}^{\mathbb{R}}_0$
$\mathfrak{su}(2)$	$SU(2)$	$\mathbb{V}^{\mathbb{R}}_2 \oplus \mathbb{V}^{\mathbb{C}}_1$
$\mathfrak{su}(2)$	$SO(3)$	$2\mathbb{V}^{\mathbb{R}}_2 \oplus \mathbb{V}^{\mathbb{R}}_0$
$\mathfrak{su}(2)$	$SO(3)$	$\mathbb{V}^{\mathbb{R}}_6$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\mathbb{V}^{\mathbb{C}}_1 \oplus 3\mathbb{V}^{\mathbb{R}}_0$ w.r.t. $\mathfrak{su}(2)$
$\mathfrak{su}(2) \oplus \mathfrak{u}(1)$	$U(2)$	$\mathbb{V}^{\mathbb{R}}_2 \oplus \mathbb{V}^{\mathbb{C}}_1$ w.r.t $\mathfrak{su}(2)$
$2\mathfrak{su}(2)$	$SO(4)$	$\mathbb{V}^{\mathbb{R}}_{2,0} \oplus \mathbb{V}^{\mathbb{C}}_{1,1}$
$\mathfrak{su}(3)$	$SU(3)$	$\mathbb{V}^{\mathbb{C}}_{1,0} \oplus \mathbb{V}^{\mathbb{R}}_{0,0}$
$\mathfrak{g}_2$	G_2	$\mathbb{V}^{\mathbb{R}}_{1,0}$

We describe some of the above subgroups more explicitly: In Cacciatori et al. [5], the authors introduce the following Lie group homomorphism:

$$\begin{aligned} &\varphi : Sp(1) \times Sp(1) \to G_2, \\ &\varphi(h,k)(x + y\epsilon) := hxh^{-1} + \big(kyh^{-1}\big)\epsilon, \end{aligned} \tag{7}$$

where $x, y \in \mathbb{H}$ and $Sp(1)$, which is isomorphic to $SU(2)$, is identified with the unit quaternions. The kernel of φ is $\{(1,1), (-1,-1)\}$ and its image thus is isomorphic to $SO(4)$. The first factor of $Sp(1) \times Sp(1)$ acts irreducibly on $\mathrm{Im}(\mathbb{H})$ and $\mathbb{H}\epsilon$ and the second factor acts irreducibly on $\mathbb{H}\epsilon$ and trivially on its orthogonal complement. $\mathrm{Im}(\mathbb{O})$ therefore splits in the same way into irreducible $2\mathfrak{su}(2)$-modules as we have stated in the theorem. By a straightforward calculation, it follows that the group $Sp(1)$ which is diagonally embedded into $Sp(1) \times Sp(1)$ acts irreducibly on $\mathrm{Im}(\mathbb{H})$ and $\mathrm{Im}(\mathbb{H})\epsilon$ and trivially on $\mathrm{span}(\epsilon)$. The two ideals and the diagonal subalgebra of $2\mathfrak{su}(2)$ therefore describe three out of four subalgebras of type $\mathfrak{su}(2)$. According to the nonzero weights of their action on $\mathrm{Im}(\mathbb{O})$, we denote the four subalgebras by $\mathfrak{su}(2)_1$, $\mathfrak{su}(2)_{1,2}$, $\mathfrak{su}(2)_{2,2}$, and $\mathfrak{su}(2)_6$.

The two subalgebras of type $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ are the direct sum of an ideal of $2\mathfrak{su}(2)$ and a one-dimensional subalgebra of the other ideal. We finally remark that the subgroup $SU(3)$ is the stabilizer of $i \in \mathbb{O}$.

5. The reducible case

We divide the manifolds which admit a homogeneous G_2-structure into two classes:

Definition 5.1. Let G be a compact connected Lie group and H be a closed connected subgroup of G. We call G/H *S^1-reducible* if there exists a Lie group G' and a covering map $\pi : G' \times U(1) \to G$ such that $H \subseteq \pi(G')$. Otherwise, G/H is called *S^1-irreducible*.

In this section, we classify all S^1-reducible spaces which admit a homogeneous G_2-structure, and in the next section, we classify the S^1-irreducible ones. We will see that none of the S^1-irreducible homogeneous spaces is covered by a product of lower-dimensional homogeneous spaces. The S^1-irreducible spaces which we will find are thus irreducible in the classical sense, too.

Throughout this article we denote the Lie algebra of G by $\mathfrak{g}$ and the Lie algebra of H by $\mathfrak{h}$. In order to simplify our considerations, we assume that G/H is compact and that G is connected and acts almost effectively on G/H, i.e. the subgroup of G which acts as the identity map is finite. Moreover, we classify the G/H and G only up to coverings. Before we start our classification, we collect some helpful facts:

Lemma 5.2. *Let G/H be a compact homogeneous space which admits a G-invariant G_2-structure. Moreover, let G act almost effectively on G/H. In this situation, the following statements are true*:

1. $\dim \mathfrak{g} = \dim \mathfrak{h} + 7$.
2. *G is compact and $\mathfrak{g}$ is the direct sum of a semisimple and an abelian Lie algebra.*
3. $\operatorname{rank} \mathfrak{h} \in \{0, 1, 2\}$ *and* $\operatorname{rank} \mathfrak{g} \not\equiv \operatorname{rank} \mathfrak{h} \pmod 2$. *If* $\operatorname{rank} \mathfrak{h} = 1$, *the dimension of the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is less or equal* 3. *If* $\operatorname{rank} \mathfrak{h} = 2$, $\dim \mathfrak{z}(\mathfrak{g}) \leqslant 1$.
4. *Let $G = G' \times U(1)$ and $H = H' \times U(1)$. If the second factor of H is transversely embedded into the product $G' \times U(1)$, G/H is G'-equivariantly covered by G'/H'.*

5. *Let* $\mathfrak{m}$ *be the orthogonal complement of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *with respect to an* Ad_H*-invariant metric on* $\mathfrak{g}$*. The restriction of the adjoint action* Ad_G *to a map* $H \to \mathfrak{gl}(\mathfrak{m})$ *is equivalent to the isotropy action of* H *on the tangent space.*

Proof. Most of the lemma consists of well-known facts on Lie groups and homogeneous spaces. We therefore only prove the nonobvious statements: Any compact Riemannian manifold has a compact isometry group. Since G/H is compact and G leaves the metric on G/H invariant, G is compact, too.

$\mathfrak{h}$ can be considered as a subalgebra of $\mathfrak{g}_2$ and thus is trivial or of rank 1 or 2. Since the roots of a semisimple Lie algebra are paired, we have $\dim \mathfrak{k} \equiv \operatorname{rank} \mathfrak{k} \pmod 2$ for any Lie algebra $\mathfrak{k}$ of a compact Lie group. It follows from $\dim \mathfrak{g} = \dim \mathfrak{h} + 7$ that $\operatorname{rank} \mathfrak{g} \not\equiv \operatorname{rank} \mathfrak{h} \pmod 2$.

The Cartan subalgebra of $\mathfrak{h}$ has to act on the tangent space in the same way as a one-dimensional subalgebra of $\mathfrak{t}$ on $\mathrm{Im}(\mathbb{O})$. The maximal trivial $\mathfrak{h}$-submodule of the tangent space therefore is at most three-dimensional. It follows that the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is at most three-dimensional, too.

If $\operatorname{rank} \mathfrak{h} = 2$, its Cartan subalgebra has to act as $\mathfrak{t}$ on $\mathrm{Im}(\mathbb{O})$. The maximal trivial $\mathfrak{h}$-submodule therefore is at most one-dimensional and we have $\dim \mathfrak{z}(\mathfrak{g}) \leqslant 1$.

The covering map in Lemma 5.2(4) is given by $\pi : G'/H' \to G/H$ with $\pi(gH') := gH$. The preimage $\pi^{-1}(eH)$ is the discrete group $G' \cap U(1)$, where $U(1)$ is the second factor of $H' \times U(1)$. □

If G/H is S^1-reducible, it is covered by $G'/H \times S^1$, where G' is a suitable Lie group. Conversely, any product $G'/H \times S^1$ admits a transitive action by $G' \times U(1)$ which makes it S^1-reducible. In this section, we therefore assume that $G = G' \times U(1)$ and $G/H = G'/H \times S^1$. G'/H admits a G'-invariant $SU(3)$-structure. We can prove by similar arguments as in Lemmas 3.4 and 3.5 that our task reduces to classifying all six-dimensional G'-homogeneous spaces G'/H with $H \subseteq SU(3)$. The possibilities for $\mathfrak{h}$ are thus fewer than in the general situation.

We prove our classification result, by considering each $\mathfrak{h} \subseteq \mathfrak{su}(3)$ separately. Lemma 5.2 reduces the number of $\mathfrak{g}$ which we have to consider. For reasons of brevity, we mostly mention only those $\mathfrak{g}$ which cannot be excluded by the techniques of the lemma.

$\underline{\mathfrak{h} = \{0\}}$: In this case, G/H simply is a seven-dimensional, compact, connected Lie group. Up to coverings, the only groups of this kind are $U(1)^7$, $SU(2) \times U(1)^4$, and $SU(2)^2 \times U(1)$.

$\underline{\mathfrak{h} = \mathfrak{u}(1)}$: Since $\dim \mathfrak{g} = 8$ and coset spaces of type $SU(3)/U(1)$ are irreducible, the only remaining possibilities for G are $SU(2) \times U(1)^5$ and $SU(2)^2 \times U(1)^2$. The first case can be excluded, since the center of G is too large. If $G = SU(2)^2 \times U(1)^2$, H is embedded into G by a map of type:

$$e^{i\varphi} \mapsto \left(\begin{pmatrix} e^{ik_1\varphi} & 0 \\ 0 & e^{-ik_1\varphi} \end{pmatrix}, \begin{pmatrix} e^{ik_2\varphi} & 0 \\ 0 & e^{-ik_2\varphi} \end{pmatrix}, e^{ik_3\varphi}, e^{ik_4\varphi} \right), \tag{8}$$

where $k_1, \ldots, k_4 \in \mathbb{Z}$. We repeat the argument from Lemma 5.2(4) twice and see that G/H is covered by $S^3 \times S^3 \times S^1$ or that $H \subseteq SU(2)^2$. Depending on k_1 and k_2, the action of H on the tangent space has at most two nonzero weights. We compare the weights of that action with the weights with which the one-dimensional subgroups of G_2 act on $\mathrm{Im}(\mathbb{O})$. After that, we see that we can assume $|k_1| = |k_2| = 1$. Since we obtain the same manifold for different choices of the signs of k_1 and k_2, we can even assume that $k_1 = k_2 = 1$. If $(k_3, k_4) = (1, 0)$, G/H is diffeomorphic to $S^3 \times S^3 \times S^1$, and if $(k_3, k_4) = (0, 0)$, we obtain the only manifold which is not covered by $S^3 \times S^3 \times S^1$. In that situation, G/H is of type $SU(2)^2/U(1) \times T^2$. The five-dimensional manifold $SU(2)^2/U(1)$ coincides up to the double covering of $SO(4)$ with the Stiefel manifold $V^{4,2} = SO(4)/SO(2)$. The reason for this is that both manifolds are simply connected and the isotropy group $U(1)$ acts with the same weights on $2\mathfrak{su}(2)$ or $\mathfrak{so}(4)$ respectively.

$\underline{\mathfrak{h} = \mathfrak{su}(2)}$: In this situation, G has to be a ten-dimensional compact Lie group. On the one hand, $\dim \mathfrak{z}(\mathfrak{g})$ has to be positive, since G/H is S^1-reducible. On the other hand, we have $\dim \mathfrak{z}(\mathfrak{g}) \leqslant 3$. The only remaining possibilities for G therefore are $SU(2)^3 \times U(1)$ and $SU(3) \times U(1)^2$.

In the first case, we can embed H diagonally, i.e. by the map $g \mapsto (g, g, g, 1)$. The action of H on the tangent space is the same as of $\mathfrak{su}(2)_{2,2}$ on $\mathrm{Im}(\mathbb{O})$ and G/H is diffeomorphic to $S^3 \times S^3 \times S^1$. If we had embedded H differently, it would act as the identity on a four-dimensional subspace, which is impossible.

In the second case, there are two possible embeddings of H into $SU(3)$: The first embedding is induced by he standard representation of $SO(3)$ on $\mathbb{R}^3 \subseteq \mathbb{C}^3$. The only elements of $SU(3)$ which commute with all of $SO(3)$ are the multiples of the identity. Since those elements are a discrete set, the action of H splits the tangent space into a trivial and a five-dimensional irreducible submodule. There is no connected subgroup of G_2 which acts in this way on $\mathrm{Im}(\mathbb{O})$ and we thus can exclude this case. The second embedding is given by the following map from $SU(2)$ to $SU(3)$:

$$A \mapsto \begin{pmatrix} \boxed{A} & \\ & \boxed{1} \end{pmatrix}. \tag{9}$$

In this situation, H acts as $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$ on the tangent space. Since $\mathfrak{su}(2)_1$ acts in the same way, we have to put $SU(3)/SU(2) \times U(1)^2 = S^5 \times T^2$ on our list.

All connected subgroups of $SU(3)$ are known from Theorem 4.1, since $SU(3) \subseteq G_2$. There are only two subgroups of type $SU(2)$ or $SO(3)$ which act trivially on a one-dimensional subspace of $\mathrm{Im}(\mathbb{O})$. Therefore, there are no further embeddings of H into $SU(3)$.

$\mathfrak{h} = 2\mathfrak{u}(1)$: Since $\operatorname{rank}\mathfrak{h} = 2$ and G/H is S^1-reducible, we have $\dim \mathfrak{z}(\mathfrak{g}) = 1$. The group G has to be nine-dimensional. Therefore, we can assume that $G = SU(3) \times U(1)$. Since $\mathfrak{su}(3) \subseteq \mathfrak{g}_2$ and $\operatorname{rank}\mathfrak{su}(3) = \operatorname{rank}\mathfrak{g}_2$, any Cartan subalgebra of $\mathfrak{su}(3)$ acts on $\mathbb{C}^3$ in the same way as $\mathfrak{t}$ on $\operatorname{span}(j, k, \dots, k\epsilon)$. We thus have to put $G/H = SU(3)/U(1)^2 \times U(1)$ on our list.

$\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$: As above, we have $\dim \mathfrak{z}(\mathfrak{g}) = 1$. With help of the classification of the semisimple Lie algebras, we see that $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$. We describe a possible G/H in detail. $Sp(2)$ has a subgroup of type $Sp(1) \times U(1)$ which is given by:

$$H = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} \middle| h_1 \in \mathbb{H},\ h_2 \in \mathbb{C},\ |h_1| = |h_2| = 1 \right\}. \tag{10}$$

G/H is diffeomorphic to $\mathbb{CP}^3 \times S^1$. The Lie algebra of H acts in the same way on the tangent space of $\mathbb{CP}^3 \times S^1$, as $\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$ on $\mathrm{Im}(\mathbb{O})$. The kernel of the isotropy representation of H is isomorphic to $\mathbb{Z}_2$. Therefore, we have an effective action by $(Sp(1) \times U(1))/\mathbb{Z}_2$ on the tangent space. Since that group is isomorphic to $U(2)$, our example does not contradict the fact that G_2 contains no subgroup of type $Sp(1) \times U(1)$.

We exclude the existence of further manifolds of the above kind. If $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ and $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{u}(1)$, either G/H is covered by the sphere $Sp(2)/Sp(1)$, which is not reducible, or $\mathfrak{h} \subseteq \mathfrak{sp}(2)$. There are three embeddings of $\mathfrak{sp}(1)$ into $\mathfrak{sp}(2)$, which is isomorphic to $\mathfrak{so}(5)$. In the first case, $\mathfrak{sp}(1)$ acts as $\mathfrak{so}(3)$ on $\mathbb{R}^3 \subseteq \mathbb{R}^5$, in the second case, it acts as $\mathfrak{su}(2)$ on $\mathbb{C}^2 \cong \mathbb{R}^4 \subseteq \mathbb{R}^5$, and in the last case, it acts irreducibly on $\mathbb{R}^5$. The second embedding yields the homogeneous space $\mathbb{CP}^3 \times S^1$, which we have described above. If the semisimple part of $\mathfrak{h}$ was embedded by the first map, it would act as $\mathfrak{su}(2)_{2,2}$ on the tangent space. Since $\mathfrak{g}_2$ has no subalgebra of type $\mathfrak{su}(2)_{2,2} \oplus \mathfrak{u}(1)$, this is not possible. It follows from Schur's lemma that there is no nonzero element of $\mathfrak{so}(5)$ which commutes with the third embedding of the semisimple part. The third case can therefore be excluded, too.

$\mathfrak{h} = \mathfrak{su}(3)$: As in the previous cases, G has to be a product of a 14-dimensional semisimple Lie group G' and $U(1)$. With help of the classification of the semisimple Lie algebras, we conclude that G' is $SU(3) \times SU(2)^2$ or G_2. In the first case, $SU(3)$ acts trivially on G/H and in the second case we obtain $G_2/SU(3) \times U(1)$, which is diffeomorphic to $S^6 \times S^1$. We can verify that H acts in the same way as the subgroup $SU(3)$ of G_2 on $\mathrm{Im}(\mathbb{O})$. Therefore, we have to put this manifold on our list and have finally proven the first part of Theorem 1.

Remark 5.3. There is a one-to-one correspondence between the manifolds from Theorem 1(1) and the six-dimensional manifolds which admit a homogeneous $SU(3)$-structure. These manifolds are considered by Cleyton and Swann [7], too. They obtain a list of homogeneous spaces which coincides with our list with the single exception of $V^{4,2} \times T^2$, which seems to be missing in [7].

6. The irreducible case

In this section, we classify the S^1-irreducible manifolds which admit a homogeneous G_2-structure. Analogously to the previous section, we consider each subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2$ separately.

$\mathfrak{h} = \{0\}$: Since any seven-dimensional compact Lie group is covered by a product of a semisimple Lie group and a torus of positive dimension, G/H cannot be S^1-irreducible.

$\mathfrak{h} = \mathfrak{u}(1)$: In the previous section, we have already proven that if $\mathfrak{h} = \mathfrak{u}(1)$ and G/H is S^1-irreducible, we necessarily have $G = SU(3)$. Since any closed one-dimensional subgroup of $SU(3)$ is conjugate to a

$$U(1)_{k,l} := \left\{ \begin{pmatrix} e^{ikt} & 0 & 0 \\ 0 & e^{ilt} & 0 \\ 0 & 0 & e^{-i(k+l)t} \end{pmatrix} \middle| t \in \mathbb{R} \right\} \tag{11}$$

with $k, l \in \mathbb{Z}$, G/H has to be an Aloff–Wallach space $N^{k,l} := SU(3)/U(1)_{k,l}$. The manifolds $N^{k,l}$ and $N^{-k,-l}$ obviously coincide. Moreover, any permutation σ of $(k, l, -(k+l))$ induces an $SU(3)$-equivariant diffeomorphism of $N^{k,l}$ and $N^{\sigma(k),\sigma(l)}$. For this reason, we can assume that $k \geqslant l \geqslant 0$.

By an explicit calculation, we see that there exists a one-dimensional Lie subalgebra of $\mathfrak{t}$ which acts in the same way on $\mathrm{Im}(\mathbb{O})$ as the Lie algebra of $U(1)_{k,l}$ on the tangent space. Any $N^{k,l}$ therefore satisfies the conditions of Theorem 1.

$\mathfrak{h} = \mathfrak{su}(2)$: Since G/H is S^1-irreducible, $\mathfrak{z}(\mathfrak{g})$ has to be trivial. The only remaining possibility for $\mathfrak{g}$ therefore is $\mathfrak{so}(5)$. As we have mentioned before, there are three embeddings of $\mathfrak{su}(2)$ into $\mathfrak{so}(5)$, which are distinguished by the splitting of $\mathbb{R}^5$ with respect to $\mathfrak{su}(2)$:

1. $\mathbb{R}^5 = \mathbb{V}_2^{\mathbb{R}} \oplus 2\mathbb{V}_0^{\mathbb{R}}$: In this situation, G/H is the Stiefel manifold $V^{5,2} = SO(5)/SO(3)$ of all orthonormal pairs in $\mathbb{R}^5$. The action of $\mathfrak{su}(2)$ splits the tangent space into $2\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$. Since $\mathfrak{su}(2)_{2,2}$ splits $\mathrm{Im}(\mathbb{O})$ in the same way, $V^{5,2}$ admits an $SO(5)$-invariant G_2-structure.

2. $\mathbb{R}^5 = \mathbb{V}_1^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}$: If this is the case, G/H is covered by the seven-sphere $Sp(2)/Sp(1)$. The action of $Sp(1)$ splits the tangent space into $\mathbb{V}_1^{\mathbb{C}} \oplus 3\mathbb{V}_0^{\mathbb{R}}$. $\mathfrak{su}(2)_1$ acts in the same way on $\mathrm{Im}(\mathbb{O})$ and S^7 thus admits an $Sp(2)$-invariant G_2-structure.
3. $\mathbb{R}^5 = \mathbb{V}_4^{\mathbb{R}}$: If $\mathfrak{su}(2)$ acts irreducibly on $\mathbb{R}^5$, it also acts irreducibly on the tangent space of G/H. Since the action of $\mathfrak{su}(2)_6$ on $\mathrm{Im}(\mathbb{O})$ is irreducible, too, we have found another manifold which admits a homogeneous G_2-structure, namely the seven-dimensional Berger space B^7.

$\mathfrak{h} = 2\mathfrak{u}(1)$: Since $\mathfrak{h}$ is of rank 2, $\dim \mathfrak{z}(\mathfrak{g})$ is either 0 or 1. If the center is one-dimensional, we have $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{u}(1)$ and $\mathfrak{h}$ is transversely embedded into that direct sum. In this situation, G/H is covered by an Aloff–Wallach space $N^{k,l}$, on which $SU(3) \times U(1)$ acts transitively. The group $SU(3)$ acts as usual by left multiplication on Nk, l. Moreover, a certain one-dimensional subgroup of the normalizer $\mathrm{Norm}_{SU(3)}\, U(1)_{k,l}$ acts on $N^{k,l}$ by right multiplication. This subgroup can be identified with the second factor of $SU(3) \times U(1)$.

If $\mathfrak{g}$ is semisimple, we can assume that $\mathfrak{g} = 3\mathfrak{su}(2)$. We describe the possible embeddings of $2\mathfrak{u}(1)$ into $3\mathfrak{su}(2)$. A Cartan subalgebra of $3\mathfrak{su}(2)$ is given by:

$$\left\{ \left(\begin{array}{cccccc} ix & 0 & & & & \\ 0 & -ix & & & & \\ & & iy & 0 & & \\ & & 0 & -iy & & \\ & & & & iz & 0 \\ & & & & 0 & -iz \end{array} \right) \middle|\, x, y, z \in \mathbb{R} \right\}. \tag{12}$$

We fix the biinvariant metric $q(X, Y) := -\operatorname{tr}(XY)$ on $3\mathfrak{su}(2)$. Let $\mathfrak{k}_{k,l,m}$, where $k, l, m \in \mathbb{Z}$, be the one-dimensional subalgebra of $3\mathfrak{su}(2)$ which is generated by the matrix with $x = k$, $y = l$, and $z = m$. Furthermore, let $2\mathfrak{u}(1)_{k,l,m}$ be the q-orthogonal complement of $\mathfrak{k}_{k,l,m}$ in the Cartan subalgebra (12). Any connected two-dimensional Lie subgroup of $SU(2)^3$ is conjugate to a connected subgroup with a Lie algebra of type $2\mathfrak{u}(1)_{k,l,m}$. As Castellani [6], we denote the quotient of $SU(2)^3$ by that subgroup by $Q^{k,l,m}$.

By the action of the group $(\mathbb{Z}_2)^3 \rtimes S_3$ of outer automorphisms of $3\mathfrak{su}(2)$, we can change the signs and the order of (k, l, m) arbitrarily. We may therefore assume without loss of generality that $k \geqslant l \geqslant m \geqslant 0$. The isotropy representation of $2\mathfrak{u}(1)_{k,l,m}$ on the tangent space of $Q^{k,l,m}$ is with respect to a suitable basis given by:

$$\left\{ \left(\begin{array}{ccccccc} 0 & x & & & & & \\ -x & 0 & & & & & \\ & & 0 & y & & & \\ & & -y & 0 & & & \\ & & & & 0 & z & \\ & & & & -z & 0 & \\ & & & & & & 0 \end{array} \right) \middle|\, xk + yl + zm = 0 \right\}. \tag{13}$$

By comparing (13) with the Cartan subalgebra (4) of $\mathfrak{g}_2$, we see that only $Q^{1,1,1}$ admits an $SU(2)^3$-invariant G_2-structure.

$\mathfrak{h} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$: Since $\operatorname{rank} \mathfrak{su}(2) \oplus \mathfrak{u}(1) = 2$, the center of $\mathfrak{g}$ is at most one-dimensional. $\mathfrak{g}$ has to be an eleven-dimensional Lie algebra and therefore is either $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ or $\mathfrak{so}(5) \oplus \mathfrak{u}(1)$.

We start with the first of the two cases. The semisimple part of $\mathfrak{h}$ we denote by $\mathfrak{su}(2)'$. In order to classify the homogeneous spaces which we can obtain in this situation, we have to describe the possible embeddings of $\mathfrak{su}(2)'$ into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. $\mathfrak{su}(2)' \cap \mathfrak{su}(3)$ has to be nontrivial. Otherwise, G would not act almost effectively on G/H. The projection of $\mathfrak{su}(2)'$ onto $\mathfrak{su}(3)$ therefore has to be one of the two maps which we have described on page 306. If $\mathfrak{su}(2)'$ acted irreducibly on $\mathbb{C}^3$, the tangent space of G/H would contain a five-dimensional $\mathfrak{su}(2)'$-submodule. This follows by the same arguments as on page 306. Since no subalgebra of $\mathfrak{g}_2$ acts in this way on $\mathrm{Im}(\mathbb{O})$, $\mathfrak{su}(2)'$ has to split $\mathbb{C}^3$ into $\mathbb{V}_1^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{C}}$. We consider the projection of $\mathfrak{su}(2)'$ onto the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$. We first assume that $\mathfrak{su}(2)' \subseteq \mathfrak{su}(3)$. In this situation, the center of $\mathfrak{h}$ is without loss of generality generated by a matrix of type

$$\left(\begin{array}{ccccc} ki & 0 & 0 & & \\ 0 & ki & 0 & & \\ 0 & 0 & -2ki & & \\ & & & li & 0 \\ & & & 0 & -li \end{array} \right) \tag{14}$$

where k and l are integers. $\mathfrak{su}(2)'$ acts as $\mathfrak{su}(2)_1$ on the tangent space of G/H. There exists up to conjugation a unique one-dimensional subalgebra of $\mathfrak{g}_2$ which commutes with $\mathfrak{su}(2)_1$. Therefore, the weights with which the center of $\mathfrak{h}$ acts on the tangent space are uniquely determined. By computing the action of the above matrix on the tangent space, we see that we necessarily have $l = \pm 3k$. The quotient G/H is in both cases up to an $SU(3) \times SU(2)$-equivariant diffeomorphism the same and admits an $SU(3) \times SU(2)$-invariant G_2-structure. We again use the same notation as Castellani [6] and call our manifold $M^{1,1,0}$.

If the projection of $\mathfrak{su}(2)'$ onto the second summand of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ is bijective, there exists up to conjugation a unique one-dimensional subalgebra of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ which commutes with $\mathfrak{su}(2)'$. In this situation, G/H is diffeomorphic to the exceptional Aloff–Wallach space $N^{1,1}$. $SU(3)$ acts on a $gU(1)_{1,1}$ by matrix multiplication from the left. Since $U(1)_{1,1}$ commutes with $S(U(2) \times U(1))$ which is isomorphic to $SU(2)$, $gU(1)_{1,1} \mapsto gh^{-1}U(1)_{1,1}$ defines a left action by $SU(2)$ on $N^{1,1}$ which commutes with the action of $SU(3)$. The isotropy group of the $SU(3) \times SU(2)$-action is $SU(2) \times U(1)$. The embedding of its Lie algebra into $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ is the same as we have described above. We thus have found another group action on $N^{1,1}$ which we have to include in our list.

In both of the above two cases, there exists an element of G which is of order two and acts trivially on G/H. For the same reasons as on page 307, the fact that G_2 contains no subgroup of type $SU(2) \times U(1)$ therefore does not contradict the statement of our theorem.

Next, we assume that $\mathfrak{g} = \mathfrak{so}(5) \oplus \mathfrak{u}(1)$. The embedding of $\mathfrak{su}(2)'$ into $\mathfrak{so}(5)$ has to be one of the three subalgebras which we have described on page 307. Furthermore, the projection of $\mathfrak{z}(\mathfrak{h})$ onto $\mathfrak{so}(5)$ should not be trivial. If $\mathfrak{su}(2)'$ was embedded by its five-dimensional representation into $\mathfrak{so}(5)$, there would be no element of $\mathfrak{so}(5)$ left which commutes with $\mathfrak{su}(2)'$. Since this contradicts our statement on $\mathfrak{z}(\mathfrak{h})$, we can exclude this case. If $\mathfrak{su}(2)'$ was embedded by its three-dimensional representation, it would decompose its complement in $\mathfrak{so}(5)$ into $2\mathbb{V}_2^{\mathbb{R}} \oplus \mathbb{V}_0^{\mathbb{R}}$. $\mathfrak{g}_2$ has no subalgebra of type $\mathfrak{su}(2)_{2,2} \oplus \mathfrak{u}(1)$ and we thus can exclude this case, too. The only remaining case is where $\mathfrak{su}(2)'$ is embedded by its two-dimensional complex representation. Since $\mathfrak{z}(\mathfrak{h})$ has to commute with $\mathfrak{su}(2)'$, its projection onto $\mathfrak{so}(5)$ has to be an element of the second summand of the Lie subalgebra $\mathfrak{so}(4)$, which splits into $\mathfrak{su}(2)'$ and another $\mathfrak{su}(2)$. If $\mathfrak{h} \subseteq \mathfrak{so}(5)$, we obtain $\mathbb{CP}^3 \times S^1$, which we already have considered in the previous section. If this is not the case, G/H is covered by S^7, which is equipped with an action of $Sp(2) \times U(1)$.

<u>$\mathfrak{h} = 2\mathfrak{su}(2)$</u>: Since $\dim \mathfrak{h} = 6$, the dimension of $\mathfrak{g}$ has to be 13. There is no nonzero element of $\mathrm{Im}(\mathbb{O})$ on which the subalgebra $2\mathfrak{su}(2)$ of $\mathfrak{g}_2$ acts trivially. Therefore, $\mathfrak{z}(\mathfrak{g})$ has to be trivial. The only remaining possibility for $\mathfrak{g}$ is $\mathfrak{so}(5) \oplus \mathfrak{su}(2)$.

It follows from Lemma 3.4 and Theorem 4.1 that $\mathfrak{h}$ has to decompose the tangent space into $\mathbb{V}_{2,0}^{\mathbb{R}} \oplus \mathbb{V}_{1,1}^{\mathbb{C}}$. Let $\iota : 2\mathfrak{su}(2) \to \mathfrak{so}(5) \oplus \mathfrak{su}(2)$ be the embedding of $\mathfrak{h}$ into $\mathfrak{g}$, $\pi_1 : \mathfrak{so}(5) \oplus \mathfrak{su}(2) \to \mathfrak{so}(5)$ be the projection on the first summand, and $\pi_2 : \mathfrak{so}(5) \oplus \mathfrak{su}(2) \to \mathfrak{su}(2)$ be the projection on the second one. The tangent space contains a submodule of type $\mathbb{V}_{1,1}^{\mathbb{C}}$ only if $(\pi_1 \circ \iota)(2\mathfrak{su}(2))$ is the standard embedding of $\mathfrak{so}(4)$ into $\mathfrak{so}(5)$. The first summand of $2\mathfrak{su}(2)$ has to act irreducibly on a three-dimensional submodule of the tangent space and we therefore can assume that

$$(\pi_2 \circ \iota)(x, y) = x \quad \forall x, y \in \mathfrak{su}(2). \tag{15}$$

We are now able to describe G/H explicitly. Let $S^7 \subseteq \mathbb{H}^2$ be the seven-sphere. $Sp(2)$ acts on S^7 from the left by matrix multiplication. We identify $Sp(1)$ with the group of all unit quaternions. Since the scalar multiplication on a quaternionic vector space acts from the right, scalar multiplication with h^{-1} where $h \in Sp(1)$ defines a left action of $Sp(1)$ on S^7. We thus have constructed a transitive $Sp(2) \times Sp(1)$-action on S^7. The isotropy group of this action is $Sp(1) \times Sp(1)$ and the isotropy action has the properties which we have demanded above. Analogously to the case where $H = SU(2) \times U(1)$, the kernel of the isotropy representation of $Sp(1) \times Sp(1)$ is $\mathbb{Z}_2$ and the group which acts effectively on the tangent space is in fact $(Sp(1) \times Sp(1))/\mathbb{Z}_2$, which is isomorphic to $SO(4)$.

<u>$\mathfrak{h} = \mathfrak{su}(3)$</u>: G has to be a Lie group of dimension 15 which contains $SU(3)$. With help of the classification of the compact Lie groups, we see that G is covered either by a product of $SU(3)$ and a seven-dimensional Lie group or by $SU(4)$. In the first case, G would not act almost effectively on G/H. In the second case, G/H is covered by S^7.

<u>$\mathfrak{h} = \mathfrak{g}_2$</u>: For similar reasons as above, we have $\mathfrak{g} = \mathfrak{so}(7)$. Therefore, G/H is covered by the seven-dimensional sphere $\mathrm{Spin}(7)/G_2$ and we have completed the proof of Theorem 1(2).

Remark 6.1. Friedrich, Kath, Moroianu, and Semmelmann [14] have classified all manifolds which admit a homogeneous nearly parallel G_2-structure. In particular, the authors prove that all manifolds from Theorem 1(2) admit such a G_2-structure.

7. The space of invariant G_2-structures

Until now, we have only proven that on each of the G/H there exists at least one G-invariant G_2-structure. It is natural to ask how many of such structures exist on G/H. This question can be answered with help of the following lemma:

Lemma 7.1. *Let G be a compact Lie group and H be a closed subgroup of G such that G/H admits a G-invariant G_2-structure. As usual, let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an Ad_H-invariant scalar product on $\mathfrak{g}$. We denote the set of all H-invariant elements of an H-module V by V^H. The set of all G-invariant G_2-structures on G/H can be bijectively identified with a subset of $(\bigwedge^3 \mathfrak{m}^*)^H$. Moreover, this subset is open.*

Proof. The first statement of the lemma follows from the fact that the tangent space of G/H can be identified with $\mathfrak{m}$. The $GL(7, \mathbb{R})$-orbit of $\omega \in \bigwedge^3 \mathrm{Im}(\mathbb{O})^*$ is an open subset of $\bigwedge^3 \mathrm{Im}(\mathbb{O})^*$. Therefore, the second statement follows, too. □

Table 4

$\mathfrak{h}$	$n_{O(7)}$	n_{G_2}
$\{0\}$	28	35
$\mathfrak{u}(1)$	4, 6, 10	5, 7, 13
$2\mathfrak{u}(1)$	4	5
$\mathfrak{su}(2)_1$	7	10
$\mathfrak{su}(2)_{1,2}$	2	2
$\mathfrak{su}(2)_{2,2}$	4	5
$\mathfrak{su}(2)_6$	1	1
$\mathfrak{su}(2)_1 \oplus \mathfrak{u}(1)$	3	4
$\mathfrak{su}(2)_{1,2} \oplus \mathfrak{u}(1)$	2	2
$2\mathfrak{su}(2)$	2	2
$\mathfrak{su}(3)$	2	3
$\mathfrak{g}_2$	1	1

Because of the above lemma, it makes sense to speak of the dimension of the space of all G-invariant G_2-structures on G/H. We denote this dimension by n_{G_2}. For the same reasons as above, we can identify the set of all G-invariant metrics on G/H with an open subset of $(S^2\mathfrak{m}^*)^H$. We denote its dimension by $n_{O(7)}$, since a metric on a seven-dimensional manifold is an $O(7)$-structure. The values of $n_{O(7)}$ we calculate, too, and compare them with n_{G_2}. Any element of $(S^2\mathfrak{m}^*)^H$ can be identified with help of a background metric q with an H-equivariant q-symmetric endomorphism of $\mathfrak{m}$. We can therefore use Schur's lemma to calculate $n_{O(7)}$. n_{G_2} can be calculated by means of representation theory, too. $n_{O(7)}$ and n_{G_2} depend only on the action of H on $\mathfrak{m}$. Moreover, that action is equivalent to the action of a subgroup of G_2 on $\mathrm{Im}(\mathbb{O})$. Our next step is therefore to calculate the dimension of $(S^2\,\mathrm{Im}(\mathbb{O})^*)^H$ and $(\bigwedge^3 \mathrm{Im}(\mathbb{O})^*)^H$ for each $H \subseteq G_2$:

Theorem 7.2. *Let H be a connected Lie subgroup of G_2 and $\mathfrak{h}$ be its Lie algebra. The action of $\mathfrak{g}_2$ on $\mathrm{Im}(\mathbb{O})$ makes $S^2\,\mathrm{Im}(\mathbb{O})^*$ ($\bigwedge^3 \mathrm{Im}(\mathbb{O})^*$) an $\mathfrak{h}$-module. The dimension of $(S^2\,\mathrm{Im}(\mathbb{O})^*)^H$ ($(\bigwedge^3 \mathrm{Im}(\mathbb{O})^*)^H$), which we also denote by $n_{O(7)}$ (n_{G_2}), can be found in Table 4. The meaning of the numbers in the second row is as follows: There are infinitely many nonconjugate embeddings of $U(1)$ into G_2. More precisely, for any $a, b \in \mathbb{Z}$ there exists an embedding such that $\mathfrak{u}(1)$ splits $\mathrm{Im}(\mathbb{O})$ into $\mathbb{V}_a^{\mathbb{C}} \oplus \mathbb{V}_b^{\mathbb{C}} \oplus \mathbb{V}_{-a-b}^{\mathbb{C}} \oplus \mathbb{V}_0^{\mathbb{R}}$. If one of the numbers a, b, or $-a-b$ is zero, we have $n_{O(7)} = 10$ and $n_{G_2} = 13$. If they are all nonzero but two of the numbers coincide up to a sign, we have $n_{O(7)} = 6$ and $n_{G_2} = 7$. In all other cases, we have $n_{O(7)} = 4$ and $n_{G_2} = 5$.*

We omit the proof of the above theorem, since it merely consists of the decomposition of various H-modules. Since we have described the isotropy action of $\mathfrak{h}$ on the tangent space of the manifolds G/H in Sections 5 and 6, we can use Theorem 7.2 to verify the numbers from Tables 1 and 2 described in Theorem 1. In Theorems 1 and 7.2 we always have $n_{O(7)} \leqslant n_{G_2}$. There is in fact a deeper reason behind this:

Lemma 7.3. *In the situation of Lemma 7.1, the map*

$$\phi : \left(\bigwedge^3 \mathfrak{m}^*\right)^H \to \left(S^2\mathfrak{m}^*\right)^H \tag{16}$$

which maps a G_2-structure to its associated metric is surjective.

Proof. Let ω be a fixed G-invariant G_2-structure on G/H and let g be its associated metric. Furthermore, let $(e_1, \ldots, e_7)$ be a basis of $\mathfrak{m}$ such that the coefficients of ω with respect to that basis are the same as of (2). Finally, let h be an arbitrary G-invariant metric on G/H and $A : \mathfrak{m} \to \mathfrak{m}$ be defined by $g(Ax, y) := h(x, y)$. Since A is self-adjoint and positive definite $A^{\frac{1}{2}}$ exists. Moreover, A is invertible and H-equivariant. $(A^{-\frac{1}{2}}e_1, \ldots, A^{-\frac{1}{2}}e_7)$ is orthonormal with respect to h. The three-form ω' which is defined by

$$\omega'(u, v, w) := \omega\big(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v, A^{\frac{1}{2}}w\big) \tag{17}$$

is an H-invariant G_2-structure and satisfies $\phi(\omega') = h$. $\square$

The statement of the above lemma can be understood as follows: For any G-invariant metric g on G/H there exists a G-invariant G_2-structure such that its associated metric is g. Moreover, the space of all such G_2-structures is of dimension $n_{G_2} - n_{O(7)}$.

Many of the manifolds from Theorem 1 are diffeomorphic to each other. For example, some of the Aloff–Wallach spaces are diffeomorphic or homeomorphic to each other. This phenomenon is discussed by Kreck and Stolz [19] in detail. Since none of the $N^{k,l}$ with $\gcd x(k, l) = 1$ and $k \geqslant l \geqslant 0$ are $SU(3)$-equivariantly diffeomorphic to each other, they should be treated as different in our context.

Some of the manifolds appear twice as G/H and G'/H' with $G \subsetneq G'$ in our list. We want to know if there exists a G-invariant G_2-structure on G/H which is not G'-invariant. The answer to this question can be found with help of Theorem 1:

Corollary 7.4. *Let G/H satisfy the conditions of Theorem 1. Moreover, let $G' \supsetneq G$ act transitively on G/H. We denote the isotropy group of the G'-action by H'. For reasons of clarity, we assign to n_{G_2} the homogeneous space it belongs to. In this situation, we have*

$$n_{G_2}(G/H) > n_{G_2}(G'/H'), \tag{18}$$

except if $G = SU(3)$, $H = U(1)_{k,l}$, $N^{k,l}$ is generic, and $G' = SU(3) \times U(1)$.

As a consequence, all triples $(G, H, G/H)$ from Theorem 1 should be considered as different geometric objects, except for the case from the above corollary.

8. Existence of the cosymplectic G_2-structures

In Sections 5 and 6, we have classified all manifolds which admit a homogeneous G_2-structure. The aim of this section is to prove that a transitive group action which leaves at least one G_2-structure invariant also leaves a cosymplectic G_2-structure invariant. In the irreducible case, we even establish the existence of an invariant nearly parallel G_2-structure. We prove these facts by a case-by-case analysis. Although most of this work has already been done by other authors, there are still some cases left open.

The article of Friedrich et al. [14] answers our question for many subcases of the irreducible case. More precisely, we only have to consider those irreducible spaces on which we have more than one transitive group action. It is known (cf. Bär [1]) that the sphere $\mathrm{Spin}(7)/G_2$ admits a $\mathrm{Spin}(7)$-invariant nearly parallel G_2-structure ω. The associated metric on S^7 has constant sectional curvature. Since we have $Sp(2) \subseteq SU(4) \subseteq \mathrm{Spin}(7)$, ω is invariant with respect to the action of the three groups. In [14], the authors describe a homogeneous nearly parallel G_2-structure on S^7. The associated metric on S^7 is the squashed one and its isometry group is $Sp(2) \times Sp(1)$. Since the G_2-structure is homogeneous, it has to be at least $Sp(2)$-invariant. We assume that the second factor of $Sp(2) \times Sp(1)$ does not preserve the G_2-structure. In that situation, there exists a one-dimensional subgroup of $Sp(1)$ which generates a continuous family of nearly parallel G_2-structures but preserves the associated metric. Any nearly parallel G_2-structure induces a Killing spinor and the dimension of the space of all Killing spinors thus is at least two. Since it is known (cf. [14]) that this dimension is in fact one, the G_2-structure is $Sp(2) \times Sp(1)$- and in particular $Sp(2) \times U(1)$-invariant. All in all, we have found for each transitive action on S^7 an invariant nearly parallel G_2-structure.

Next, we consider the Aloff–Wallach spaces. Page and Pope [20] have proven that any Aloff–Wallach space admits two $SU(3)$-invariant nearly parallel G_2-structures, which coincide on $N^{1,0}$. It is known (cf. [14]) that the isometry group of the associated metric is $SU(3) \times U(1)$ if $(k,l) \neq (1,1)$. Since in this situation the space of all Killing spinors is one-dimensional (cf. [14,20]), we can conclude by the same arguments as above that both G_2-structures are not only $SU(3)$- but also $SU(3) \times U(1)$-invariant. The nearly parallel G_2-structure on $N^{1,1}$ which is considered in [14] is preserved by $SU(3) \times SU(2)$. Since $SU(3) \subseteq SU(3) \times U(1) \subseteq SU(3) \times SU(2)$, that G_2-structure is invariant with respect to all of the three group actions from Theorem 1(2) and we have proven our statement for the irreducible case.

We proceed to the reducible case. Butruille [4] has proven that the only six-dimensional manifolds which admit a homogeneous nearly Kähler structure are S^6, $\mathbb{CP}^3$, $SU(3)/U(1)^2$, and $S^3 \times S^3$. These four manifolds have also been considered by Grunewald [16] and Bär [1], since they carry a real Killing spinor. The groups which preserve the nearly Kähler structure on the first three homogeneous spaces are G_2, $Sp(2)$, and $SU(3)$. Bär [1] has also proven that the nearly Kähler structure on $S^3 \times S^3$ (or equivalently on $SU(2) \times SU(2)$) is not only left-invariant but is preserved by an $SU(2)^3$-action. The isotropy group of this action is $SU(2)$, which is embedded as the diagonal subgroup by

$$g \mapsto (g, g, g). \tag{19}$$

We denote the metric, the real two-form, and the complex $(3,0)$-form which determine the $SU(3)$-structure by g, α, and θ. Furthermore, we denote the real (imaginary) part of θ by θ^{Re} (θ^{Im}). We have $d\alpha = 3\lambda\theta^{\mathrm{Re}}$ and $d\theta^{\mathrm{Im}} = -2\lambda\alpha \wedge \alpha$ for a $\lambda \in \mathbb{R} \setminus \{0\}$, since the four manifolds are nearly Kähler. These equations are discussed in more detail by Hitchin [17]. On a product of a circle and a nearly Kähler manifold of real dimension six, we can define a G_2-structure by $\omega := \alpha \wedge dt + \theta^{Im}$. Here, "$t$" denotes the coordinate of the circle. By a straightforward calculation, it follows that $d * \omega = 0$. All in all, we have proven our statement for the last three manifolds from Theorem 1(1) and for all three actions on $S^3 \times S^3 \times S^1$.

On the torus T^7, we have the flat G_2-structure, which is of course cosymplectic. On $\mathbb{C}^2 \times T^4$ ($\mathbb{C}^3 \times T^2$), there exists a flat $\mathrm{Spin}(7)$-structure Ω. It is preserved by the action of $SU(2) \times U(1)^4$ ($SU(3) \times U(1)^2$), where the first factor acts on $\mathbb{C}^2$ ($\mathbb{C}^3$) and the second one by translations on the torus. The principal orbits of this action, which is of cohomogeneity one, are $S^3 \times T^4$ ($S^5 \times T^2$). Ω induces an $SU(2) \times U(1)^4$ ($SU(3) \times U(1)^2$)-invariant G_2-structure on any principal orbit. This G_2-structure is cosymplectic, since $d\Omega = 0$ and the principal orbits are hypersurfaces.

The only remaining manifold is $V^{4,2} \times T^2$. The issue of homogeneous G_2-structures on this manifold is not yet discussed in the literature. In the following, we construct an explicit $SO(4) \times U(1)^2$-invariant cosymplectic G_2-structure on $V^{4,2} \times T^2$. On $V^{4,2}$, there exists an $SO(4)$-invariant Einstein–Sasaki structure (cf. [13]). An Einstein–Sasaki structure on a five-dimensional manifold M can be defined as an $SU(2)$-structure with a special intrinsic torsion. As a G_2- or $\mathrm{Spin}(7)$-structure, an $SU(2)$-structure is determined by certain differential forms. More precisely, there exists a one-form α and two-forms ω_1, ω_2, and ω_3 such that for each T_p^*M with $p \in M$ there exists a basis $(e^1, \ldots, e^5)$ with

$$\alpha = e^5, \qquad \omega_1 = e^{12} + e^{34},$$
$$\omega_2 = e^{13} - e^{24}, \qquad \omega_3 = e^{14} + e^{23}. \tag{20}$$

Analogously to Section 2, $e^{i_1 \dots i_k}$ is an abbreviation of $e^{i_1} \wedge \dots \wedge e^{i_k}$. The Einstein–Sasaki condition is equivalent to

$$d\alpha = -2\omega_1, \qquad d\omega_2 = 3\alpha \wedge \omega_3, \qquad d\omega_3 = -3\alpha \wedge \omega_2. \tag{21}$$

We remark that $(\alpha, \omega_1, \omega_2, \omega_3)$ determines all other geometric structures, e.g. the metric, on M by which one usually defines an Einstein–Sasaki structure. The above facts on Einstein–Sasaki structures are proven in the articles of Conti and Salamon [8,9]. Since G_2 contains a subgroup of type $SU(2)$ which acts irreducibly on a four-dimensional subspace of $\mathrm{Im}(\mathbb{O})$ and trivially on its orthogonal complement, the Einstein–Sasaki structure $(\alpha, \omega_1, \omega_2, \omega_3)$ on $V^{4,2}$ can be extended to a G_2-structure on $V^{4,2} \times T^2$. Let (e^6, e^7) be a basis of the cotangent space of T^2. We define the following four-form:

$$*\omega := -\frac{1}{2}\omega_1 \wedge \omega_1 + e^6 \wedge \alpha \wedge \omega_3 - e^7 \wedge \alpha \wedge \omega_2 + e^{67} \wedge \omega_1. \tag{22}$$

Let $(e_i)_{1 \leqslant i \leqslant 7}$ be defined by $e^j(e_i) := \delta_i^j$. If we identify the basis $(e_6, e_7, e_5, e_1, e_3, e_4, -e_2)$ by a linear map with the standard basis of $\mathrm{Im}(\mathbb{O})$, $*\omega$ turns into the four-form (3). $*\omega$ therefore is a four-form which is induced by a G_2-structure. In fact, $*\omega$ determines a G_2-structure up to the sign of the three-form. It follows with help of (21) that we have $d * \omega = 0$. This calculation finishes the proof of Theorem 1.

Remark 8.1.

1. In general, the space of all G-invariant nearly parallel G_2-structures on G/H is much smaller than the space of all G-invariant G_2-structures. As an illustration of this fact, we consider the generic Aloff–Wallach spaces: It follows from the results of Page and Pope [20] that there are up to homotheties only two $SU(3)$-invariant nearly parallel G_2-structures on $N^{k,l}$. The space of all $SU(3)$-invariant G_2-structures on $N^{k,l}$ is five-dimensional. If we identify homothetic metrics with each other, it is still four-dimensional and thus much larger than the discrete set of nearly parallel G_2-structures.
2. Our proof of Theorem 1(3) is done by a case-by-case analysis. The author suspects that it is possible to prove these facts more directly.

Acknowledgement

This work was supported by the Collaborative Research Centre 676 of the Deutsche Forschungsgemeinschaft (DFG).

References

[1] Christian Bär, Real Killing spinors and holonomy, Commun. Math. Phys. 154 (3) (1993) 509–521.
[2] John C. Baez, The octonions, Bull. Amer. Math. Soc. 39 (2002) 145–205, Bull. Amer. Math. Soc. 42 (2005) 213 (Errata).
[3] Robert Bryant, Metrics with exceptional holonomy, Ann. of Math. 126 (1987) 525–576.
[4] Jean-Baptiste Butruille, Classification des variété approximativement Kähleriennes homogénes, Ann. Global Anal. Geom. 27 (3) (2005) 201–225. Online available at arXiv:math/0401152v2 [math.DG], or as "Homogeneous nearly Kähler manifolds", arXiv:math/0612655v1 [math.DG].
[5] Sergio L. Cacciatori, Bianca L. Cerchiai, Alberto Della Vedova, Giovanni Ortenzi, Antonio Scotti, Euler angles for G_2, Lawrence Berkeley National Laboratory. Paper LBNL-57265, March 10, 2005. Online available at http://repositories.cdlib.org/lbnl/LBNL-57265 or arXiv:hep-th/0503106v2.
[6] L. Castellani, L.J. Romans, N.P. Warner, A classification of compactifying solutions for $d = 11$ supergravity, Nuclear Physics B 241 (1984) 429–462.
[7] Richard Cleyton, Andrew Swann, Cohomogeneity-one G_2-structures, J. Geom. Phys. 44 (2–3) (2002) 202–220. Online available at arXiv:math/0111056v1 [math.DG].
[8] Diego Conti, Simon Salamon, Generalized Killing spinors in dimension 5, Trans. Am. Math. Soc. 359 (11) (2007) 5319–5343. Online available at arXiv:math/0508375v2 [math.DG].
[9] Diego Conti, Cohomogeneity one Einstein–Sasaki 5-manifolds, Commun. Math. Phys. 274 (3) (2007) 751–774. Online available at arXiv:math/0606323v3 [math.DG].
[10] E.B. Dynkin, The maximal subgroups of classical groups, Tr. Mosk. Mat. Obshch. 1 (1952) 39–166.
[11] E.B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, in: American Mathematical Society Translations Series 2, vol. 6, 1957, pp. 111–244.
[12] M. Fernández, A. Gray, Riemannian manifolds with structure group G_2, Annali di Matematica Pura ed Applicata 132 (1982) 19–45.
[13] Th. Friedrich, I. Kath, Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator, J. Differ. Geom. 29 (1989) 263–279.
[14] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, On nearly parallel G_2-structures, J. Geom. Phys. 23 (3–4) (1997) 259–286.
[15] Th. Friedrich, G_2-manifolds with parallel characteristic torsion, Differ. Geom. Appl. 25 (6) (2007) 632–648.
[16] Ralf Grunewald, Six-dimensional Riemannian manifolds with a real Killing spinor, Ann. Global Anal. Geom. 8 (1) (1990) 43–59.
[17] Nigel Hitchin, Stable forms and special metrics, in: Marisa Fernández (Ed.), Global Differential Geometry: The Mathematical Legacy of Alfred Gray, Proceedings of the International Congress on Differential Geometry Held in Memory of Professor Alfred Gray, Bilbao, Spain, September 18–23, 2000, in: AMS Contemporary Mathematical Series, vol. 288, 2001, pp. 70–89. Online available at arXiv:math/0107101v1 [math.DG].
[18] Dominic D. Joyce, Compact Manifolds with Special Holonomy, Oxfor University Press, New York, 2000.
[19] Matthias Kreck, Stefan Stolz, Some nondiffeomorphic homeomorphic homogeneous 7-manifolds with positive sectional curvature, J. Differ. Geom. 33 (2) (1991) 465–486, J. Differ. Geom. 49 (1998) 203–204 (Errata).
[20] D.N. Page, C.N. Pope, New squashed solutions of $d = 11$ supergravity, Phys. Lett. B 147 (1984) 55–60.
[21] Simon Salamon, Riemannian Geometry and Holonomy Groups, Essex, New York, 1989.