Non-linear maps preserving solvability

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Abstract

Let $M_n$ be the algebra of all $n \times n$ complex matrices and let $\mathcal{L}$ be the general linear Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ or the special linear Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. A bijective (not necessarily linear) map $\phi : \mathcal{L} \to \mathcal{L}$ preserves solvability in both directions if both $\phi$ and $\phi^{-1}$ map every solvable Lie subalgebra of $\mathcal{L}$ into some solvable Lie subalgebra. If $n \geq 3$ then every such map is either a composition of a bijective lattice preserving map with a similarity transformation and a map $[a_{ij}] \mapsto [f(a_{ij})]$ induced by a field automorphism $f : \mathbb{C} \to \mathbb{C}$, or a map of this type composed with the transposition. We also describe the general form of such maps in the case when $n = 2$. Using Lie’s theorem we will reduce the proof of this statement to the problem of characterizing bijective maps on $M_n$ preserving triangularizability of matrix pairs in both directions. As a byproduct we will characterize bijective maps on $M_n$ that preserve inclusion for lattices of invariant subspaces in both directions. © 2004 Elsevier Inc. All rights reserved.

1. Introduction and statement of the results

Let $\mathcal{L}$ be a Lie algebra. All basic definitions and facts concerning Lie algebras needed in this note can be found in [19]. One of the fundamental concepts in this theory is that of a solvable Lie algebra. Recall that the derived Lie algebra $\mathcal{L}^{(1)}$ of $\mathcal{L}$ is the Lie ideal $[\mathcal{L}, \mathcal{L}]$
spanned by all \([X, Y]\), \(X, Y \in \mathcal{L}\). To each Lie algebra \(\mathcal{L}\) we associate the derived series: \(\mathcal{L} \supset \mathcal{L}^{(1)} \supset \mathcal{L}^{(2)} = (\mathcal{L}^{(1)})^{(1)} \supset \cdots\). The Lie algebra \(\mathcal{L}\) is solvable if there exists a positive integer \(r\) such that \(\mathcal{L}^{(r)} = [0]\).

In this note we will study bijective maps \(\phi: \mathcal{L} \to \mathcal{L}\) with the property that both \(\phi\) and its inverse map every solvable Lie subalgebra into some solvable Lie subalgebra. We will say that a bijective map \(\phi: \mathcal{L} \to \mathcal{L}\) preserves solvability in both directions if for every solvable Lie subalgebra \(M \subset \mathcal{L}\) there exist solvable Lie subalgebras \(\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{L}\) such that \(\phi(M) \subset \mathcal{L}_1\) and \(\phi^{-1}(M) \subset \mathcal{L}_2\). Note that we have not assumed that \(\phi\) is linear. The problem of characterizing such maps on an arbitrary Lie algebra seems to be difficult. We will solve this problem in the special case that \(\mathcal{L}\) is either the general linear Lie algebra or the special linear Lie algebra.

The set of all \(n \times n\) complex matrices will be denoted by \(M_n\) when considered as a set or a linear space or a ring or an algebra. As usual, \(n \times n\) matrices will be identified with linear operators acting on \(\mathbb{C}^n\). If the linear space \(M_n\) is equipped with the Lie product \([\cdot, \cdot]\), \([A, B] = AB - BA\), then it becomes a Lie algebra denoted also by \(gl(n, \mathbb{C})\). By \(sl(n, \mathbb{C}) \subset gl(n, \mathbb{C})\) we denote the special linear Lie algebra consisting of all trace zero matrices. The obvious examples of linear bijective maps on \(gl(n, \mathbb{C})\) preserving solvability in both directions are the transposition \(A \mapsto A^t\) and similarity transformations \(A \mapsto TAT^{-1}\). Here \(T\) is any invertible \(n \times n\) matrix. Let \(f: \mathbb{C} \to \mathbb{C}\) be any automorphism of the complex field. Recall that the identity and the complex conjugation are the only continuous automorphisms of \(\mathbb{C}\) [13]. The map \(A = [a_{ij}] \mapsto [f(a_{ij})]\) is a ring automorphism of \(M_n\). It follows easily that this is a bijective map on \(gl(n, \mathbb{C})\) preserving solvability in both directions. All the examples given so far were semilinear. To get nonadditive examples we have to recall Lie’s theorem [19, pp. 21–23] stating that every solvable Lie subalgebra of \(gl(n, \mathbb{C})\) is equivalent to a triangular one. In other words, a Lie subalgebra \(\mathcal{L} \subset gl(n, \mathbb{C})\) is solvable if and only if there exists a triangularizing chain of invariant subspaces for \(\mathcal{L}\). Here, of course, by an invariant subspace of \(\mathcal{L}\) we mean a subspace that is invariant under every member of \(\mathcal{L}\). Now, two matrices \(A\) and \(B\) are defined to be lattice-equal, denoted by \(A \sim B\), if they have exactly the same lattice of invariant subspaces. In many, but certainly not all, cases, this amounts to each of \(A\) and \(B\) being a polynomial function of the other. The complete description of this equivalence relation can be found in [2, Theorem 10.2.1] and [23]. Lie’s theorem yields that a bijective map \(\tau: gl(n, \mathbb{C}) \to gl(n, \mathbb{C})\) satisfying \(\tau(A) \sim A, A \in gl(n, \mathbb{C})\), preserves solvability in both directions. Such a map is just an arbitrary permutation on each of the equivalence classes with respect to \(\sim\). We will call every such a bijective map lattice preserving. Our aim is to show that every bijective map on \(gl(n, \mathbb{C})\) preserving solvability in both directions is a composition of the types of maps described in this paragraph.

A main tool in the proof is Lie’s theorem. It reduces our problem to the problem of characterizing bijective maps on \(M_n\) preserving triangularizability of matrix pairs in both directions. We say that a bijective map \(\phi: M_n \to M_n\) preserves triangularizability of matrix pairs in both directions if for all \(A, B \in M_n\) the set \([A, B]\) is (simultaneously) triangularizable if and only if the set \((\phi(A), \phi(B))\) is. Triangularizability can be viewed, in various ways, as an approximation of commutativity [17]. This observation plays an important role (see Lemma 2.5) in the proof of our main result.
We have here two closely related problems. A bijective map \( \phi : M_n \to M_n \) preserves triangularizability in both directions if for every \( S \subset M_n \) the set \( \phi(S) \) is triangularizable if and only if \( S \) is triangularizable. We will show that a bijective map \( \phi : M_n \to M_n \) preserves triangularizability of matrix pairs in both directions if and only if it preserves triangularizability in both directions. It is less obvious that both of these two preserving assumptions are equivalent to the assumption of preserving the lattice inclusion in both directions. We say, of course, that a bijective map \( \phi \) on \( M_n \) preserves lattice inclusion in both directions if \( \text{Lat} A \subset \text{Lat} B \iff \text{Lat} \phi(A) \subset \text{Lat} \phi(B) \), \( A, B \in M_n \). Here, \( \text{Lat} A \) denotes the lattice of all invariant subspaces of \( A \).

We are now ready to state our main result.

**Theorem 1.1.** Let \( \phi : M_n \to M_n, n \geq 3 \), be a bijective map. The following are equivalent:

1. \( \phi \) preserves solvability in both directions,
2. \( \phi \) preserves triangularizability in both directions,
3. \( \phi \) preserves triangularizability of pairs of matrices in both directions,
4. \( \phi \) preserves lattice inclusion in both directions,
5. there exist an invertible \( T \in M_n \), a field automorphism \( f : \mathbb{C} \to \mathbb{C} \), and a bijective lattice preserving map \( \tau : M_n \to M_n \) such that either
   \[
   \phi([a_{ij}]) = T \tau([f(a_{ij})]) T^{-1} \text{ for every } [a_{ij}] \in M_n,
   \]
   or
   \[
   \phi([a_{ij}]) = T (\tau([f(a_{ij})]))^t T^{-1} \text{ for every } [a_{ij}] \in M_n.
   \]

The assumption that \( n \geq 3 \) is indispensable. The \( 2 \times 2 \) case will be treated in the last section.

The description of the equivalence relation \( \sim \) is quite complicated but well understood. It is somewhat surprising that one can understand our proof without knowing when two matrices are equivalent with respect to \( \sim \).

In the proof we will repeatedly use the following fact. If \( \phi : M_n \to M_n \) is a map satisfying the hypothesis of the theorem and if after composing \( \phi \) with a map of any of the four basic types described above (similarities, ring automorphisms induced by automorphisms of \( \mathbb{C} \), the transposition, and bijective lattice preserving maps) we get a map of one of the two forms appearing in the conclusion of the theorem, then the map \( \phi \) has to be of one of these two forms as well. To check this we have to show that the inverse of any of the basic maps is a basic map of the same type and any composition of two basic maps can be written as a composition of two (possibly different) basic maps of the same type but in the reverse order. In particular, in the conclusion of the theorem the map \( \phi \) is described as a composition of a ring automorphism induced by an automorphism \( f \) of the complex field followed by a bijective lattice preserving map \( \tau \), possibly followed by the transposition and finally composed with a similarity transformation. But we could formulate the conclusion with these basic maps in any other prescribed order.
Our starting problem was to characterize solvability preserving maps. It turned out that it can be reduced to other preserver problems. We believe that these non-linear preserver problems are of independent interest. Namely, a lot of attention has been recently paid to linear preservers, that is, linear maps on $M_n$ that preserve a certain subset or a certain relation (see [14,16]). Besides linear preservers also additive and multiplicative preservers were considered in the literature. It is much more surprising that in some cases we can get nice structural results on preservers with no additional algebraic assumption. Already in the forties, Hua initiated the study of bijective maps (no linearity was assumed) on vector spaces of matrices that strongly preserve adjacent pairs of matrices [3–10]. Recall that two matrices $A$ and $B$ are adjacent if $\text{rank}(A - B) = 1$. In particular, he proved that up to a translation such maps are necessarily semilinear. For some recent improvements of his results we refer to [15,18,20,21]. The non-linear setting is much more interesting when we consider spectrum or commutativity preserving maps. Namely, there are many spectrum or commutativity preserving maps that are far from being semilinear or even additive. Just choose for every $A \in M_n$ an invertible matrix $T_A$ and define $\phi: M_n \rightarrow M_n$ by $\phi(A) = T_A A T_A^{-1}$, $A \in M_n$. Then clearly, $\phi$ preserves the spectrum, that is, $\sigma(\phi(A)) = \sigma(A)$, $A \in M_n$. Baribeau and Ransford proved the surprising result stating that every spectrum-preserving $C^1$-diffeomorphism of $M_n$ is of this form [1]. A locally polynomial map is an example of a commutativity preserving map that is far from being additive (a map $\phi: M_n \rightarrow M_n$ is a locally polynomial map if for every $A \in M_n$ there exists a polynomial $p$ depending on $A$ such that $\phi(A) = p(A)$). In [22] it was proved that every continuous bijective map on $M_n$ preserving commutativity in both directions is a composition of a semilinear commutativity preserving map and a locally polynomial map. The above result is another contribution to non-linear preserver problems. It is interesting to note that in contrast to the well-developed theory of linear preservers the characterizations of non-linear preservers discovered so far are all essentially mutually distinct.

It should be mentioned here that linear maps $\phi$ on $M_n$ satisfying $\text{Lat} \phi(A) \subset \text{Lat} A$ were treated in [12]. Another related linear preserver problem concerning maps preserving the isomorphism class of lattices of invariant subspaces was solved in [11].

It will not be difficult to deduce the following result from our main theorem.

**Corollary 1.2.** Let $\phi: sl(n, \mathbb{C}) \rightarrow sl(n, \mathbb{C})$, $n \geq 3$, be a bijective map preserving solvability in both directions. Then there exist an invertible $n \times n$ matrix $T$, a field automorphism $f: \mathbb{C} \rightarrow \mathbb{C}$, and a bijective lattice preserving map $\tau: sl(n, \mathbb{C}) \rightarrow sl(n, \mathbb{C})$ such that either

$$\phi([a_{ij}]) = T \tau([f(a_{ij})]) T^{-1} \quad \text{for every } [a_{ij}] \in sl(n, \mathbb{C}),$$

or

$$\phi([a_{ij}]) = T (\tau([f(a_{ij})]))^t T^{-1} \quad \text{for every } [a_{ij}] \in sl(n, \mathbb{C}).$$

2. Proofs

Throughout this section we will assume that $n \geq 3$. Let us start with some preliminary results.
Lemma 2.1. Let \( \phi : M_n \to M_n \) be a bijective map. The following are equivalent:

- \( \phi \) preserves solvability in both directions,
- \( \phi \) preserves triangularizability in both directions.

**Proof.** Assume first that \( \phi \) preserves solvability in both directions and let \( S \subset M_n \) be a triangularizable subset. Then there exists an invertible matrix \( T \) such that \( S \subset T T_n T^{-1} \), where \( T_n \) denotes the full upper triangular algebra. As \( T T_n T^{-1} \) is a solvable Lie subalgebra of \( gl(n, \mathbb{C}) \), its \( \phi \)-image has to be contained in some solvable Lie subalgebra which, by Lie’s theorem must be triangularizable. Hence, \( \phi(S) \) is triangularizable. Similarly, \( \phi^{-1} \) preserves triangularizability.

Assume next that \( \phi \) preserves triangularizability in both directions and let \( M \subset M_n \) be a solvable subalgebra of \( gl(n, \mathbb{C}) \). By Lie’s theorem, \( M \) is triangularizable. By hypothesis, \( \phi(M) \) and \( \phi^{-1}(M) \) are triangularizable. So, there exist invertible matrices \( T, S \in M_n \) such that \( \phi(M) \subset T T_n T^{-1} \) and \( \phi^{-1}(M) \subset S T_n S^{-1} \). Therefore \( \phi \) preserves solvability in both directions. \( \square \)

Lemma 2.2. For a matrix \( A \in M_n \) the following two statements are equivalent:

- \( A = \lambda I + N \) for some scalar \( \lambda \) and some nilpotent \( N \) with \( N^{n-1} \neq 0 \),
- if for any \( B, C \in M_n \) the pairs \( \{ A, B \} \) and \( \{ A, C \} \) are both triangularizable, then so is the pair \( \{ B, C \} \).

**Proof.** Since the triangularizing chain of \( A \) is unique, so is that of any triangularizable pair \( \{ A, T \} \). Thus, the first statement implies the second one.

For the converse, assume that \( A \) is not of the form described. Then, it has either at least two eigenvalues, or one eigenvalue with the geometric multiplicity at least two. In both cases we may assume, after applying a similarity, that

\[
A = \begin{bmatrix}
\lambda_1 & 0 & * & * & \ldots & * \\
0 & \lambda_2 & * & * & \ldots & * \\
0 & 0 & * & * & \ldots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & * & \ldots & * \\
0 & 0 & 0 & 0 & \ldots & * 
\end{bmatrix}
\]

Take

\[
B = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 
\end{bmatrix}
\quad \text{and} \quad
C = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 
\end{bmatrix}
\]
and observe that both pairs $\{A, B\}$ and $\{A, C\}$ are triangularizable. Indeed, all we have to verify is that the upper left $2 \times 2$ corners of $A$ and $C$ form a triangularizable pair. As the triangularizing chain of $B$ is unique, the pair $\{B, C\}$ is not triangularizable, as desired.

Corollary 2.3. Let $\phi : M_n \to M_n$ be a bijective map. The following are equivalent:

- $\phi$ preserves triangularizability in both directions,
- $\phi$ preserves triangularizability of pairs of matrices in both directions.

Proof. Clearly, the assumption of preserving triangularizability in both directions yields that $\phi : M_n \to M_n$ is a bijective map preserving triangularizability of pairs of matrices in both directions. To prove the reverse implication assume that $\phi : M_n \to M_n$ is a bijective map preserving triangularizability of pairs of matrices in both directions. Because $\phi$ and $\phi^{-1}$ have the same properties it is enough to show that the triangularizability of $S$ yields that of $\phi(S)$. So, assume that $S$ is triangularizable and adjoin to $S$ a nilpotent $N$ of maximal nilindex, if necessary, such that the enlarged set is still triangularizable. The $\phi$-image of this enlarged set has the property that every pair in it is triangularizable and it contains $\phi(N)$, which, by our preservation assumption and the previous lemma, has a unique triangularizing chain. Thus, $\phi(S)$ is triangularizable.

Proposition 2.4. Let $\phi : M_n \to M_n$ be a bijective map preserving triangularizability of pairs of matrices in both directions. Let $D$ denote the subset of $M_n$ consisting of all diagonalizable matrices. Then $\phi(D) = D$ and two diagonalizable matrices $A$ and $B$ commute if and only if $\phi(A)$ and $\phi(B)$ do.

Proof. We have already proved that $\phi$ preserves the collection of triangularizable sets in both directions. In particular, it induces a bijective correspondence on the collection of all maximal triangularizable sets, i.e., the sets that are similar to $T_n$. For any triangularizable set $E$ let $C_E$ denote the set of all triangularizing chains for $E$.

First observe that for any triangularizable $E$, $C_E$ and $C_{\phi(E)}$ have the same cardinality. This follows from the fact that the cardinality of $C_E$ coincides with that of the collection of those maximal triangularizable sets that contain $E$.

We next verify that the matrix $A$ has $n$ distinct eigenvalues if and only if $C_{\{A\}}$ has exactly $n!$ members. Since every invariant subspace of a diagonalizable $A$ with distinct eigenvalues is a direct sum of some of its eigenspaces, such an operator has exactly $n!$ triangularizing chains. To show the converse we first observe that for any matrix $A$ having an eigenspace of dimension at least two, $C_{\{A\}}$ has infinite cardinality. Thus, we can assume that the Jordan canonical form of $A$ has precisely one cell corresponding to each eigenvalue. Let $A$ have the Jordan form $A_1 \oplus \cdots \oplus A_k$, where each $A_j$ is a Jordan cell $\lambda_j I + N_j$ acting on a subspace $V_j$. It is clear that every invariant subspace of $A$ is of the form $W_1 \oplus \cdots \oplus W_k$, where each $W_j \subset V_j$ is a kernel of some power of the nilpotent $(A_j - \lambda_j I)$. Hence, the cardinality of $C_{\{A\}}$ is less than $n!$ unless all Jordan cells of $A$ are $1 \times 1$.

Clearly, two diagonalizable matrices commute if and only if they are simultaneously diagonalizable. Since any composition of $\phi$ with a similarity transformation satisfies the assumptions of our theorem, it is sufficient to show that the set $\Delta_n$ of all diagonal matrices
Lemma 2.5. Let $\phi(D_0)$ has $n$ distinct eigenvalues and we may assume with no loss of generality that it is diagonal. To each of the $n!$ triangularizing chains for $D_0$ there corresponds the full triangular algebra with that chain of invariant subspaces, and their intersection is precisely $\Delta_n$. The $\phi$-images of these triangular algebras are again $n!$ full triangular algebras containing $\phi(D_0)$. Since $\phi(D_0)$ has distinct eigenvalues, this intersection has to coincide with $\Delta_n$. $\square$

We denote by $\mathcal{E}_1 \subset \mathcal{D}$ the subset of all simple matrices, that is, the matrices of the form $\mu P + \lambda I$, where $\mu$ and $\lambda$ are scalars, $\mu \neq 0$, and $P$ is an idempotent of rank one. Clearly, if $A = \mu P + \lambda I$ is a simple matrix, then $\mu$, $P$, and $\lambda$ are uniquely determined.

Lemma 2.5. Let $\psi: \mathcal{D} \to \mathcal{D}$ be a bijective map such that $AB = BA$ if and only if $\psi(A)\psi(B) = \psi(B)\psi(A)$, $A, B \in \mathcal{D}$. Then there exist an invertible matrix $T \in M_n$, an automorphism $f$ of the complex field, and a bijective map $\tau: \mathcal{D} \to \mathcal{D}$ with $\tau(A) \sim A$, $A \in \mathcal{D}$, such that either $\psi([aij]) = T\tau((f(aij)))T^{-1}$ for all $A \in \mathcal{D}$, or $\psi([aij]) = T(\tau((f(aij))))T^{-1}$ for all $A \in \mathcal{D}$.

Proof. The proof, essentially reproduced here for the sake of completeness, comes from [22]. Denote by $\mathcal{D}_k$, $k = 1, \ldots, n$, the set of all diagonalizable matrices with exactly $k$ distinct eigenvalues. For any subset $\mathcal{M} \subset \mathcal{D}$ we define $\mathcal{M}' = \{B \in \mathcal{D}: AB = BA \text{ for every } A \in \mathcal{M}\}$. We call this set the first commutant of $\mathcal{M}$ in $\mathcal{D}$. The second commutant $\mathcal{M}''$ is defined by $\mathcal{M}'' = (\mathcal{M}')'$. If $\mathcal{M} = \{A\}$, then we write shortly $\{A\}' = A'$. Clearly, $\psi(\mathcal{M}') = (\psi(\mathcal{M}))'$, $\mathcal{M} \subset \mathcal{D}$. Obviously, a diagonalizable matrix $A$ belongs to $\mathcal{D}_1$, that is, the set of all scalar matrices, if and only if $A' = \mathcal{D}$. So, $\mathcal{D}_1$ is mapped by $\psi$ onto itself. Observe that for $A \in \mathcal{D}$ the following two statements are equivalent:

- $A \in \mathcal{D}_2$,
- $A \notin \mathcal{D}_1$ and every matrix $B \in \mathcal{D}$ satisfying $A' \subset B'$ and $A' \neq B'$ belongs to $\mathcal{D}_1$.

It follows easily that $\psi(\mathcal{D}_2) = \mathcal{D}_2$. If we replace in the above two statements $\mathcal{D}_2$ by $\mathcal{D}_3$ and $\mathcal{D}_1$ by $\mathcal{D}_3 \cup \mathcal{D}_2$ we obtain a new pair of equivalent statements characterizing $\mathcal{D}_3$. This characterization implies that $\psi(\mathcal{D}_3) = \mathcal{D}_3$. Repeating this procedure we get $\psi(\mathcal{D}_k) = \mathcal{D}_k$, $k = 1, \ldots, n$.

In our next step we will prove that $\psi$ maps the set $\mathcal{E}_1$ onto itself. In the case $n = 3$ we have $\mathcal{E}_1 = \mathcal{D}_2$ and so, there is nothing to prove. Therefore we will assume in this paragraph that $n \geq 4$. We will verify that for $A \in \mathcal{D}_2$ the following two statements are equivalent:

- $A \in \mathcal{E}_1$,
- for every $B \in A' \cap \mathcal{D}_2$ we have $\{A, B\}'' \subset \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$.

Assume for a moment that we have already proved this. Then, because $\psi$ preserves the first commutants in $\mathcal{D}$, it has to preserve also the second commutants in $\mathcal{D}$, and since it preserves $\mathcal{D}_k$, $k = 1, 2, 3$, we have necessarily $\psi(\mathcal{E}_1) = \mathcal{E}_1$, as desired. So, assume that
A = μP + λI ∈ \mathcal{E}_1 and B ∈ A' \cap D_2. A matrix C commutes with A if and only if it commutes with P. So, there is no loss of generality in assuming that A is already an idempotent of rank one, and after applying a similarity, if necessary, we may assume that A = E_{11}. Moreover, two diagonalizable matrices commute if and only if they are simultaneously diagonalizable, and therefore, there is no loss of generality in assuming that

\[ B = η(E_{11} + \cdots + E_{kk}) + δ(E_{k+1,k+1} + \cdots + E_{nn}), \quad 1 ≤ k ≤ n - 1, \ η ≠ δ. \]

If k = 1, then \{A, B\}′′ = span\{E_{11}, I - E_{11}\} ⊂ D_1 ∪ D_2, and if 2 ≤ k ≤ n - 1, then

\[ \{A, B\}′′ = \text{span}\{E_{11}, E_{22} + \cdots + E_{kk}, I - (E_{11} + \cdots + E_{kk})\} \subset D_1 ∪ D_2 ∪ D_3. \]

To prove the other direction assume that A ∈ D_2 \ \mathcal{E}_1. As before there is no loss of generality in assuming that A = E_{11} + \cdots + E_{kk} for some k, 2 ≤ k ≤ n - 2. Take B = E_{11} + E_{k+1,k+1} and observe that then \{A, B\}′′ = span\{E_{11}, E_{22} + \cdots + E_{kk}, E_{k+1,k+1}, I - (E_{11} + \cdots + E_{k+1,k+1})\} contains matrices with four different eigenvalues.

To each A ∈ \mathcal{E}_1 we associate the unique idempotent P of rank one satisfying A = μP + λI, λ, μ ∈ \mathbb{C}. If A, B ∈ \mathcal{E}_1 and P and Q are the corresponding idempotents of rank one, then P = Q if and only if A′ = B′. Thus, ψ induces a bijective map ξ : I_n → I_n. Here, I_n ⊂ M_n stands for the set of all rank-one idempotents. We say that P, Q ∈ I_n are orthogonal, P ⊥ Q, if PQ = QP = 0. It is easy to see that P ⊥ Q if and only if A and B commute and A′ ≠ B′. Thus, the map ξ preserves the orthogonality in both directions. By [22, Theorem 2.3], there exists a nonsingular matrix T ∈ M_n and an automorphism f : \mathbb{C} → \mathbb{C} such that either ξ([p_{ij}]) = T[f(p_{ij})]T^{-1}, [p_{ij}] ∈ I_n, or ξ([p_{ij}]) = T[f(p_{ij})]^Tf^{-1}, [p_{ij}] ∈ I_n.

Replacing ψ by [a_{ij}] ↦ T^{-1}ψ([f^{-1}(a_{ij})])T, and composing the obtained map with the transposition, if necessary, we may assume without loss of generality that for every idempotent P of rank one the set of all matrices of the form μP + λI, λ ≠ 0, is mapped bijectively onto itself. In other words, ψ(A) ~ A for every A ∈ \mathcal{E}_1 ∪ \mathcal{C}_I. Note that for T, S ∈ D we have T ~ S if and only if T′ = S′. Moreover, T′ = S′ if and only if T′ ∩ \mathcal{E}_1 = S′ ∩ \mathcal{E}_1. It follows that ψ(A) ~ A for every A ∈ D, as desired. □

We are now ready to complete the proof of our main result. It remains to prove that the third condition implies the fifth one, the fifth one the fourth one, and finally, that the fourth condition yields the third one.

So, assume that ϕ : M_n → M_n is a bijective map preserving triangularizability of pairs of matrices in both directions. By Proposition 2.4, it maps the set of diagonalizable matrices onto itself and the restriction of ϕ to D preserves commutativity in both directions. So, we can apply Lemma 2.5. After composing ϕ with a similarity transformation, the transposition, if necessary, an appropriate bijective lattice preserving map (which acts like the identity on all nondiagonalizable matrices), and a ring automorphism of M_n induced by an appropriate field automorphism of \mathbb{C}, we may assume that ϕ(A) = A for every A ∈ D.

In order to complete the proof of the implication (3) ⇒ (5) we have to show that ϕ(A) ~ A for every A ∈ M_n. The inverse of ϕ has the same properties as ϕ. Hence, it is enough to show that for every A ∈ M_n and every subspace V ⊂ \mathbb{C}^n that is invariant under A we have ϕ(A)V ⊂ V. If V is any such subspace, then there exists a triangularizing chain C of A containing V. Let S be the set of all matrices leaving the chain C invariant. Then, S, and hence also ϕ(S) is a maximal triangularizable set. We know that ϕ(D) = D for every
D ∈ D ∩ S. Thus, D ∩ S ⊂ φ(S), which further implies φ(S) = S. As φ(A) ∈ φ(S) we have φ(A)V ⊂ V, as desired.

It is not difficult to see that the last statement in of our main result implies the fourth condition of this theorem. So, it remains to prove that the third condition follows from the fourth one. The fourth assumption yields that every matrix with a minimal lattice of invariant subspaces is mapped into a matrix of the same type. Equivalently, the set of all matrices of the form \( λI + N \), where \( λ \in \mathbb{C} \) and \( N \) is a nilpotent of maximal nilindex, is mapped by \( φ \) onto itself. Note also that a pair of matrices \( A, B \) is simultaneously triangularizable if and only if there exists a nilpotent \( N \) of maximal nilindex such that \( \text{Lat} N \subset \text{Lat} A \) and \( \text{Lat} N \subset \text{Lat} B \). Thus, \( φ \) as well as \( φ^{-1} \) preserve the triangularizability of matrix pairs.

**Proof of Corollary 1.2.** Let \( φ : sl(n, \mathbb{C}) \rightarrow sl(n, \mathbb{C}) \) be a bijective map preserving solvability in both directions. Extend \( φ \) to a bijective map \( θ : gl(n, \mathbb{C}) \rightarrow gl(n, \mathbb{C}) \) defined by

\[
θ(A) = \frac{\text{tr}A}{n} I + φ(A - \frac{\text{tr}A}{n} I), \quad A \in gl(n, \mathbb{C}).
\]

Let \( \mathcal{L} \subset gl(n, \mathbb{C}) \) be a solvable Lie subalgebra. By Lie’s theorem there exists an invertible matrix \( S \) such that \( \mathcal{L} \subset S \mathcal{T} S^{-1} = \mathcal{C}I \oplus \mathcal{M} \), where \( \mathcal{M} = S(\mathcal{T} \cap sl(n, \mathbb{C})) S^{-1} \). By the solvability preserving assumption, \( φ(\mathcal{M}) \) is contained in some solvable Lie subalgebra \( \mathcal{K} \subset sl(n, \mathbb{C}) \). Thus, \( θ(\mathcal{L}) \) is contained in a solvable Lie subalgebra \( \mathcal{C}I \oplus \mathcal{K} \). Similarly, \( θ^{-1} \) defined by

\[
θ^{-1}(A) = \frac{\text{tr}A}{n} I + φ^{-1}(A - \frac{\text{tr}A}{n} I), \quad A \in gl(n, \mathbb{C})
\]

preserves solvability. Thus, we can apply our main result to conclude that there exist an invertible \( n \times n \) matrix \( T \), a field automorphism \( f : \mathbb{C} \rightarrow \mathbb{C} \), and a bijective lattice preserving map \( τ : gl(n, \mathbb{C}) \rightarrow gl(n, \mathbb{C}) \) such that either \( θ([a_{ij}]) = T τ([f(a_{ij})]) T^{-1}, [a_{ij}] ∈ gl(n, \mathbb{C}), \) or \( θ([a_{ij}]) = T τ([f(a_{ij})]) T^{-1}, [a_{ij}] ∈ gl(n, \mathbb{C}) \). The Lie subalgebra \( sl(n, \mathbb{C}) \) is invariant under \( θ \), the similarity \( A \mapsto T A T^{-1} \), the transposition, and the ring automorphism \([a_{ij}] \mapsto [f(a_{ij})] \). It follows that \( τ(sl(n, \mathbb{C})) = sl(n, \mathbb{C}) \). This completes the proof. \( \square \)

3. The distinct case of \( n = 2 \)

In this section we will treat bijective maps \( φ : M_2 \rightarrow M_2 \) preserving triangularizability of pairs of matrices in both directions. The algebra \( M_2 \) is a disjoint union of \( \mathbb{C}I, \mathbb{N} \), and \( D \), where \( \mathbb{N} \) is the set of all matrices of the form \( λI + N \) with \( N \neq 0 \) and \( N^2 = 0 \), and \( D \) is the set of all nonscalar diagonalizable operators. Two matrices \( A, B \in M_2 \) form a triangularizable pair if and only if they have a common one-dimensional invariant subspace. Every one-dimensional subspace of \( \mathbb{C}^2 \) is invariant under every member of \( \mathbb{C}I \). If \( λI + N \in \mathbb{N} \) and the range space of \( N \) is \([x] \in \mathbb{P}^2 \), then \([x] \) is the unique one-dimensional invariant subspace of \( λI + N \). Each operator \( D \in D \) has two eigenspaces, say \([x], [y] \in \mathbb{P}^2 \), and
these are the only two one-dimensional invariant subspaces of $D$. Now, an argument similar to that used in the higher dimensional case shows that each of the sets $CI$, $N$, and $D$ is invariant under $\phi$.

The set $N$ is a disjoint union of the sets $N_{[x]}$, $[x] \in \mathbb{CP}^2$, where $N_{[x]}$ is the set of all operators $\lambda I + N$ with $N$ being a nilpotent whose range space is $[x]$. Two members $\lambda I + M$ and $\mu I + N$ of $N$ form a triangularizable pair if and only if $M$ and $N$ have the same range space. Thus, $\phi$ induces a bijective map $\varphi$ on the projective space $\mathbb{CP}^2$. Each subset $N_{[x]}$, $[x] \in \mathbb{CP}^2$, is mapped by $\phi$ bijectively onto $N_{\varphi([x])}$. Clearly, $\phi$ maps the set of scalar operators bijectively onto itself. Every $D \in D$ with eigenspaces $[x]$ and $[y]$ is simultaneously triangularizable with a nilpotent $N$ with the range space $[x]$ and with a nilpotent $M$ with the range space $[y]$. It follows that $D$ is mapped into an operator from $D$ with eigenspaces $\varphi([x])$ and $\varphi([y])$. Subject to this and bijectivity, the behavior of $\phi$ on $D$ is arbitrary. Every map of this form is bijective and preserves the triangularizability of matrix pairs in both directions.

It is now clear that the assumption $n \geq 3$ is indispensable in the results presented in this paper.

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References

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