Vertex pancyclic graphs

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Abstract

Let $G$ be a graph of order $n$. A graph $G$ is called pancyclic if it contains a cycle of length $k$ for every $3 \leq k \leq n$, and it is called vertex pancyclic if every vertex is contained in a cycle of length $k$ for every $3 \leq k \leq n$. In this paper, we shall present different sufficient conditions for graphs to be vertex pancyclic. © 2002 Elsevier Science B.V. All rights reserved.

1. Terminology

We consider finite, undirected, and simple graphs $G$ with the vertex set $V(G)$ and the edge set $E(G)$. The order of a graph $G$ is denoted by $n$ and the size by $m$, respectively. The complete graph of order $n$ is denoted by $K_n$, and the complete bipartite graph with the partite sets $A$ and $B$ with $|A| = p$ and $|B| = q$ is denoted by $K_{p,q}$. A graph $G$ is $k$-connected if $G - S$ is connected for every subset $S \subseteq V(G)$ with $|S| \leq k - 1$. A vertex $v$ of a connected graph $G$ is called a cut vertex if $G - v$ is disconnected. The distance $d_G(u,v) = d(u,v)$ of two vertices $u$ and $v$ in $G$ is the length of a shortest $uv$-path, which is a path connecting $u$ and $v$. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by $A$. The set of all neighbours of a vertex $x \in V(G)$ is denoted by $N(x) = N_G(x)$, and $d(x) = d_G(x) = |N(x)|$ is the degree of the vertex $x$. We write $N[x] = N_G[x]$ instead of $N(x) \cup \{x\}$. In addition, we define $N(X) = N_G(X) = \bigcup_{x \in X} N(x)$ and $N[X] = N_G[X] = N(X) \cup X$ for a subset $X$ of $V(G)$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of a graph $G$, respectively. Let $\sigma_2(G)$ denote the minimum degree sum of a pair of distinct nonadjacent vertices in $G$. A cycle $C_k$ of length $k$ is also called a $k$-cycle. A path of a graph $G$ is a Hamiltonian path if it contains all the vertices of $G$. A graph $G$ is said to be pancyclic if it has $k$-cycles for every $k$

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between 3 and \( n \). A vertex of a graph \( G \) is \( r \)-pancyclic if it is contained in a \( k \)-cycle for every \( k \) between \( r \) and \( n \), and \( G \) is vertex \( r \)-pancyclic if every vertex is \( r \)-pancyclic. A 3-pancyclic vertex and a vertex 3-pancyclic graph are also called pancyclic and vertex pancyclic, respectively. Analogously, \( G \) is called edge \( r \)-pancyclic for some \( 3 \leq r \leq n \), if every edge of \( G \) belongs to a cycle of length \( k \) for every \( r \leq k \leq n \).

A graph \( G \) is Hamiltonian-connected, if for every pair \( u, v \) of distinct vertices of \( G \), there exists a Hamiltonian path from \( u \) to \( v \). It is called \( s \)-Hamiltonian, if \( G - S \) is Hamiltonian for every vertex set \( S \) with \( |S| \leq s \).

Following Hendry [13], we call a cycle \( C \) in a graph \( G \) extendable if there is a cycle \( C' \) in \( G \) such that \( |V(C')| = |V(C)| + 1 \) and \( V(C) \subseteq V(C') \). If such a cycle \( C' \) exists, we will say that \( C \) can be extended to \( C' \) or that \( C' \) is an extension of \( C \). A graph \( G \) is cycle extendable if it has at least one cycle and every nonHamiltonian cycle in \( G \) is extendable. A graph \( G \) is fully cycle extendable if \( G \) is cycle extendable and every vertex of \( G \) lies on a cycle of length 3.

In [6], Bondy and Chvátal introduced the closure of a graph and the stability of a property. The \( k \)-closure \( Cl_k(G) \) is obtained from \( G \) by recursively joining pairs of nonadjacent vertices whose degree sum is at least \( k \), until no such pair remains. The \( k \)-closure is independent of the order of adjunction of the edges. Obviously, any graph of order \( n \) satisfies

\[
G = Cl_{2n-3}(G) \subseteq Cl_{2n-4}(G) \subseteq \cdots \subseteq Cl_1(G) \subseteq Cl_0(G) = K_n.
\]

A property \( P \) defined on all graphs of order \( n \) is said to be \( k \)-stable if for any graph \( G \) of order \( n \) that does not satisfy \( P \), the fact that \( uv \) is not an edge of \( G \) and that \( G + uv \) satisfies \( P \) implies \( d_G(u) + d_G(v) < k \). Vice versa, if \( uv \notin E(G) \), \( d_G(u) + d_G(v) \geq k \) and \( G + uv \) has property \( P \), then \( G \) itself has property \( P \). Every property is \((2n - 3)\)-stable and every \( k \)-stable property is \((k + 1)\)-stable.

## 2. Preliminaries and motivation

In 1971 Bondy [4] posed the following metaconjecture:

Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that it is pancyclic (except for maybe a simple family of exceptional graphs).

For several sufficient conditions Bondy’s metaconjecture has been verified. This is motivation to examine these sufficient conditions even for vertex pancyclicity, since vertex pancyclicity implies pancyclicity and pancyclicity implies Hamiltonicity.

The first sufficient condition for a graph to be Hamiltonian is due to Dirac [9] in 1952.

**Theorem 1** (Dirac [9]). If \( G \) is a graph of order \( n \geq 3 \) such that \( \delta(G) \geq n/2 \), then \( G \) is Hamiltonian.
An interesting improvement of Diracs condition was obtained by Ore [14].

**Theorem 2** (Ore [14]). Let $G$ be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then $G$ is Hamiltonian.

The proof of Theorem 2 is based on the following more general result of Ore [14].

**Lemma 3** (Ore [14]). Let $u$ and $v$ be distinct nonadjacent vertices of a graph $G$ of order $n \geq 3$ such that $d_G(u) + d_G(v) \geq n$. Then $G$ is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Lemma 3 motivated Bondy and Chvátal [6] to the definition of the closure. For recent overview on closure concepts the reader is referred to [8].

Diracs condition was also extended in the following direction.

**Theorem 4** (Erdős [10]). Let $G$ be a graph of size $m$ and order $n \geq 3$. If either $\sigma_0(G) = \sigma_2(G) \geq n + 1$ or

$$m > \max \left\{ \left( \frac{n - \delta}{2} \right) + \delta^2, \left( \frac{(n + 2)/2}{2} \right) + \left\lfloor \frac{n - 1}{2} \right\rfloor^2 \right\}$$

then $G$ is Hamiltonian.

**Theorem 5** (Ore [15]). Let $G$ be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n + 1$, then $G$ is Hamiltonian-connected.

### 3. Ore-type results

We start with a lemma that we prove by using a classical argument (cf. for example Williamson [17] or Faudree et al. [11]).

**Lemma 6.** Let $G$ be a graph of order $n \geq 3$ and $\sigma_2(G) \geq n + 1$. If the vertex $w$ is not pancyclic, then $d_G(w) \leq (n - 1)/2$.

**Proof.** In view of Theorem 2, $G$ is Hamiltonian, and $G - w$ also contains a Hamiltonian cycle $C(w)$. Let $d_G(w) = s$ and $x_1, x_2, \ldots, x_s$ be the neighbours of $w$. If $G$ contains no $k$-cycle $C$ for some $k$ with $3 \leq k \leq n - 1$ with $w \in V(C)$, then no vertex $y_i$, $1 \leq i \leq s$, is adjacent to $w$, where $y_i$ is the vertex at distance $k - 2$ from $x_i$ on the arbitrarily oriented cycle $C(w)$. Thus $2d_G(w) \leq n - 1$. □

**Corollary 7** (Hendry [13]). Let $G$ be a graph of order $n \geq 3$ with $\delta(G) \geq (n + 1)/2$. Then $G$ is vertex pancyclic.

Hendry has even shown in [13] that a graph $G$ with $\delta(G) \geq (n + 1)/2$ is fully cycle extendable.
We define a very important family $H_{n/2}$ of graph classes which will be useful in the following to provide bounds for vertex pancyclicity and fully extendability.

**Remark 8.** A graph $G$ of order $n$ and with minimum degree $\delta$ is a member of the class $H_{n/2}$ if there exists a vertex $w$ of minimum degree $\delta$ s.t. $N_G(w)$ induces an independent set and every vertex $u \in N_G(w)$ is adjacent to every vertex $x \in V(G) - N_G[w]$. The unique member $G \in H_{n/2}$ with the additional property that $V(G) - N_G[w]$ induces a complete graph is denoted by $H_{n/2}$.

The class $H_{n/2}$ with $n$ even shows that Corollary 7 is best possible, since every member $G$ of $H_{n/2}$ satisfies $\delta(G) \geq n/2$ and it is easy to see that the vertex $w$ of minimum degree is not contained in a 3-cycle.

**Theorem 9.** Let $G$ be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq [4n/3] - 1$. Then every vertex of $G$ is contained in a 3-cycle.

**Proof.** Suppose to the contrary that there exists a vertex $w$ that is not contained in a 3-cycle. Let $u$ and $v$ be the two neighbours of $w$ on the Hamiltonian cycle $C$ which exists by Theorem 2. If $x_1, x_2, \ldots, x_s$ are the neighbours of $u$ different from $w$ in $G$, then these vertices are not adjacent to $w$. Since $u$ and $v$ are not adjacent, we conclude $d_G(w) + d_G(u) \leq n$ and analogously $d_G(w) + d_G(v) \leq n$.

If $d_G(w) < n/3 + 1$ and $y$ is a nonadjacent vertex of $w$, then, we obtain the following contradiction to the hypothesis

$$\sigma_2(G) \leq d_G(w) + d_G(y) < n/3 + 1 + n - 2 = 4n/3 - 1 \leq [4n/3] - 1.$$ 

In the remaining case $d_G(w) \geq n/3 + 1$, we obtain the contradiction

$$\sigma_2(G) \leq d_G(u) + d_G(v) \leq 2n - 2d_G(w) \leq 4n/3 - 2 < [4n/3] - 1. \quad \square$$

Theorem 9 is best possible. Consider for instance the graph $H_{(n-1)/3+1,n}$ defined in Remark 8. Note that the vertex $w \in H_{(n-1)/3+1,n}$ of minimum degree is not contained in a 3-cycle but for all nonadjacent vertices $u$ and $v$ of $H_{(n-1)/3+1,n}$ we conclude $d(u) + d(v) = [4n/3] - 2$.

The proofs of the following results in this section strongly depend on the following theorem of Hendry [13]. It will also be useful in Section 4.

**Theorem 10** (Hendry [13]). Let $G$ be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq n$ and let $C$ be a nonextendable cycle of length $k$ in $G$ with $3 \leq k \leq n - 1$. Then either

(i) we have $k \leq n/2$, the vertices of $C$ induce a complete graph, every vertex $x$ of $C$ is adjacent to at least one vertex $x' \in V(G) - V(C)$ and each vertex $y \in V(G) - V(C)$ is adjacent to at most one vertex of the cycle $C$, or
(ii) the integers $n$ and $k$ are even, the vertices of $C$ induce a balanced complete bipartite graph with the partite sets $U_i, U_2$, $V(G) = V(C) \cup V_1 \cup V_2$ with $|V_1| = |V_2|$ and the vertices $V_1 \cup V_2$ induce a graph having minimum degree at least $(n-k)/2$.

In addition, if $x \in U_i$ and $y \in V_j$ for $i, j \in \{1, 2\}$, then $xy \in E(G)$ if and only if $i = j$.

**Corollary 11.** Let $G$ be a graph of order $n \geq 3$ such that $\sigma_2(G) \geq \lceil 4n/3 \rceil - 1$. Then $G$ is fully cycle extendable.

**Proof.** In view of Theorem 9, every vertex of $G$ is contained in a 3-cycle. If we realize that for all graphs $G$ described in (i) and (ii) of Theorem 10 $\sigma_2(G) \leq \lceil 4n/3 \rceil - 2$ holds, we obtain the desired result. \(\Box\)

For the next result, we consider the graphs $G$ described in Theorem 10 with respect to vertex pancyclicity. It turns out that the Ore condition already ensures that $G$ is vertex 4-pancyclic unless $n$ is even and $G \cong K_{n/2, n/2}$. Here we give a short proof based on Theorem 10.

**Corollary 12.** Let $G$ be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$. Then $G$ is vertex 4-pancyclic unless $n$ is even and $G \cong K_{n/2, n/2}$.

**Proof.** Let $G$ be a graph of order $n \geq 4$ such that $\sigma_2(G) \geq n$, and let $x$ be an arbitrary vertex of $G$. If there exists a vertex $y$ nonadjacent to $x$ we deduce by $\sigma_2(G) \geq n$ that both vertices share at least two common neighbours. If otherwise $x$ is adjacent to all other vertices of $G$, it is easy to check that $G - x$ contains a path of length two. In both cases, $x$ belongs to a 4-cycle.

Suppose now that $G$ is not vertex 4-pancyclic and let $k$ be the minimal value such that there exists a vertex $x_1$ of $G$ lying on a $k$-cycle but not on a $(k + 1)$-cycle. Obviously, $k \geq 4$ and $C$ is not extendable. In view of Theorem 10, we then consider two cases.

Assume first that (i) of Theorem 10 holds. Let $x_2, x_3, x_4$ be three different vertices of $V(C - x_1)$ and let $x_i' \in (V(G) - V(C)) \cap N_G(x_i)$ for $i = 1, 2, \ldots, 4$. Note that $|\{x_i' \mid i = 1, 2, \ldots, 4\}| = 4$. If $x_i' x_2' \in E(G)$, then, since the vertices of $C$ induce a complete graph, there exists a $(k + 1)$-cycle containing the vertices $V(C - x_4) \cup \{x_1', x_2'\}$ in $G$, a contradiction. If $x_1' x_2' \not\in E(G)$, then because of $\sigma_2(G) \geq n$, there exists a common neighbour $z$ of the vertices $x_1'$ and $x_2'$ (contained in $V(G) - V(C)$). But then there exists also a $(k + 1)$-cycle containing the vertices $V(C - \{x_3, x_4\}) \cup \{x_1', x_2', z\}$ in $G$, a contradiction. Suppose now that (ii) of Theorem 10 holds. Say $x_1, x_2$ are contained in $U_1$ and $x_3, x_4$ in $U_2$. If there exist vertices $z_1, z_2$ of $V_1$ with $z_1 z_2 \in E(G)$ or $z_3, z_4$ of $V_2$ with $z_3 z_4 \in E(G)$, then there is a $(k + 1)$-cycle in $G$ containing the vertices $V(C - \{x_4\}) \cup \{z_1, z_2\}$ or $V(C - \{x_2\}) \cup \{z_3, z_4\}$, respectively, a contradiction. Therefore, $V_1$ and $V_2$ are independent sets in $G$, and with the additional minimum degree...
condition we deduce that the graph induced by \( V_1 \cup V_2 \) is a balanced complete bipartite graph. But then \( G \cong K_{n/2,n/2} \) and we are done. \( \square \)

The next theorem, an extension of Corollary 12, deals with graphs \( G \) satisfying the local Ore-type condition

\[
d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|
\]

for any path \( uuvv \) with \( uv \notin E(G) \). This condition is obviously satisfied by graphs \( G \) with \( \sigma_2(G) \geq n \), even if this only holds for all vertices of distance two in a graph. The above local Ore-type condition was first successfully studied by Asratian (cf. [1–3]).

**Theorem 13** (Asratian and Sarkisian [3]). Let \( G \) be a connected graph of order \( n \geq 4 \) such that \( N(u) \cup N(v) \cup N(w) \) for any path \( uvw \) with \( uv \notin E(G) \). Then \( G \) is vertex 4-pancyclic unless \( n \) is even and \( G \cong K_{n/2,n/2} \).

**Corollary 14** (Song and Zhang [16]). Let \( G \) be a graph of order \( n \geq 4 \) such that \( d(u) + d(v) \geq n \) for any path \( uuvv \) with \( uv \notin E(G) \). Then \( G \) is vertex 4-pancyclic unless \( n \) is even and \( G \cong K_{n/2,n/2} \).

If \( G \) is a graph of order \( n \geq 3 \), then the condition \( \sigma_2(G) \geq n \) obviously ensures the existence of a triangle in \( G \) unless \( n \) is even and \( G \cong K_{n/2,n/2} \). In view of Corollary 12, we deduce a result on pancyclicity which is due to Bondy [5] in 1971.

**Corollary 15** (Bondy [5]). Let \( G \) be a graph of order \( n \geq 3 \) such that \( \delta(G) \geq n \). Then \( G \) is pancyclic unless \( n \) is even and \( G \cong K_{n/2,n/2} \).

To close this section, we will use the preceding results to consider the question of vertex pancyclicity in graphs satisfying a Dirac-type condition.

**Lemma 16.** Let \( G \) be a graph of even order \( n = 2p \geq 4 \) and \( \delta(G) \geq p \). If \( G \) is not an element of the family \( H_{p,2p} \), defined in Remark 8, then every vertex of \( G \) is contained in a 3-cycle.

**Proof.** Assume that the vertex \( v \) is not contained in a 3-cycle, and let \( N_G(v) = \{u_1, u_2, \ldots, u_s\} \). Then \( s \geq p \) and \( N_G(v) \) an independent set. Since \( d_G(u_i) \geq p \), we conclude that \( s = p \) and that \( u_i \) is adjacent to every vertex of \( V(G) - N_G[v] \) for \( i = 1, 2, \ldots, s \). Now the edges between vertices in \( V(G) - N_G[v] \) are arbitrary, and thus \( G \in H_{p,2p} \), a contradiction. \( \square \)

Corollaries 7, 12 and Lemma 16 yield the following extension of Dirac’s Theorem 1.

**Corollary 17.** Let \( G \) be a graph of order \( n \geq 3 \) with \( \delta(G) \geq n/2 \). Then \( G \) is vertex pancyclic or \( n \) is even and \( G \) is an element of the family \( H_{n/2,n} \).
4. Closure results

We first present some results which will be used in the proofs of this section.

**Theorem 18** (Bondy and Chvátal [6]). The property “$G$ contains a cycle $C_k$” is $(2n - k)$-stable for $5 \leq k \leq n$.

**Sketch of the proof.** Suppose $G + uv$ contains a cycle $C_k$ but $G$ not. Then there is a cycle $C_k$ with vertices labelled $v_1, v_2, \ldots, v_k$ and $u = v_1, v = v_k$. By the degree condition $d(u) + d(v) \geq 2n - k$ there exists an integer $i$, where $2 \leq i \leq k - 2$, such that $v_{i-1}v_i, v_iy_k \in E(G)$. But then $v_1v_2\ldots v_{i-1}v_iy_kv_{i+1}\ldots v_{k-1}v_1$ is a cycle $C_k$, a contradiction. □

It is easy to see that Theorem 18 remains valid, if we prescribe an arbitrary vertex $w$ to be contained in the $k$-cycle.

**Theorem 19.** The property “$G$ contains a cycle $C_k$ with vertex $w$” is $(2n - k)$-stable for $5 \leq k \leq n$.

Next observe that there exist two integers $i$ and $j$ with $2 \leq i < j \leq k - 2$, such that $v_1v_{i+1}, v_iy_k, v_1v_{j+1}, v_jy_k \in E(G)$, if we increase the degree requirement by one, i.e. $d(u) + d(v) \geq 2n - k + 1$. Hence, we can prescribe an arbitrary edge $e$ to be contained in the $k$-cycle.

**Theorem 20.** The property “$G$ contains a cycle $C_k$ with edge $e$” is $(2n - k + 1)$-stable for $5 \leq k \leq n$.

Moreover, Bondy and Chvátal considered the stability of $s$-Hamiltonicity.

**Theorem 21** (Bondy and Chvátal [6]). The property “$G$ is $s$-Hamiltonian” is $(n + s)$-stable.

**Corollary 22.** If $\text{Cl}_{n+p}(G)$ is complete, then $G$ is $s$-Hamiltonian for $1 \leq s \leq p$.

**Lemma 23.** If $\text{Cl}_{n+p}(G)$ is complete, then the graph $G$ is $(p + 2)$-connected.

**Proof.** Let $S$ be a minimum cutset of $G$. The first time that two vertices $u$ and $v$ in different components of $G - S$ are joined, then

$$n + p \leq d(u) + d(v) \leq (n - |S| - 2) + 2|S| = n + |S| - 2.$$

Hence, $|S| \geq p + 2$. □

**Corollary 24.** If $\text{Cl}_{n+p}(G)$ of a graph $G$ is complete, then $\delta(G) \geq p + 2$. 

Now we extend Lemma 6 of the previous section in the following way.

**Lemma 25.** Let \( C \) be a cycle of length \( k \) in a graph \( G \) and \( u, v, w \) be three (not necessarily distinct) vertices in \( V(G) - V(C) \) such that there is a uv-path on \( q \) vertices containing \( w \). Suppose that there is no \( l \)-cycle containing \( w \) for some \( l \) with \( q + 1 \leq l \leq q + k \). Then

(i) \( d_C(w) \leq k/2 \), if \( q = 1 \), and

(ii) \( d_C(u) + d_C(v) \leq k \), if \( q \geq 2 \).

**Proof.** Let \( l \) with \( q + 1 \leq l \leq q + k \) such that there is no \( l \)-cycle containing \( w \). Assume first that \( q \geq 2 \). Then for every vertex \( x \in N_C(u) \) (respectively, \( x \in N_C(v) \)) there is no vertex \( y \in N_C(v) \) (\( y \in N_C(u) \)) at distance \( l - q - 1 \) on the cycle. Hence,

\[
|N_C(u)| \leq |V(C) - N_C(v)| \quad \text{and} \quad |N_C(v)| \leq |V(C) - N_C(u)|
\]

implying \( d_C(u) + d_C(v) \leq k \). If \( q = 1 \), then \( u = v = w \) and we are done analogously. \( \square \)

**Theorem 26** (Faudree et al. [11]). If \( Cl_{n+1}(G) \) is complete, then the graph \( G \) is pancyclic.

This theorem can be extended as follows.

**Theorem 27.** Let \( G \) be a graph of order \( n \geq 3 \) and define \( T = \{v \in V(G) \mid d(v) \geq n/2 \} \). If \( Cl_{n+1}(G) \) is complete, then every vertex \( v \in T \) is pancyclic.

**Proof.** If \( G \cong K_n \), then we are done. Hence, we may assume that \( G \not\cong K_n \) implying \( \Delta(G) \geq (n + 1)/2 \). Therefore, \( T \neq \emptyset \). Since \( Cl_{n+1}(G) \) is complete, \( G \) is 1-Hamiltonian by Corollary 22. Suppose there is a vertex \( w \in T \) which is not pancyclic and let \( C \) be a Hamiltonian cycle in \( G - w \). Then

\[
d(w) = d_C(w) \leq \frac{n - 1}{2}
\]

by Lemma 25, a contradiction. \( \square \)

**Theorem 28.** If \( Cl_{[(4n-3)/3]}(G) \) is complete, then \( G \) is vertex pancyclic.

**Proof.** By Theorem 27, every vertex \( w \in V(G) \) with \( d(w) \geq n/2 \) is pancyclic. Suppose there is a vertex \( w \) with \( d(w) \leq (n - 1)/2 \) which is not pancyclic. By Theorem 19, we know that \( w \) is contained in a \( k \)-cycle for \( n - [(n - 3)/3] \leq k \leq n \). Now choose a vertex set \( S \) containing \( w \) and \( p - 1 \) vertices not adjacent to \( w \), where \( p = [(n - 3)/3] \). Then \( G - S \) has a Hamiltonian cycle \( C \) by Corollary 22. According to the assumption, \( w \) is not contained in an \( l \)-cycle for some \( l \) with \( 3 \leq l \leq n - p - 1 \). Then by
Corollary 24 and Lemma 25, we obtain

\[2(p + 2) \leq 2\delta(G) \leq 2d(w) = 2d_C(w) \leq n - p\]

implying \(p \leq (n - 4)/3\), a contradiction. \(\square\)

The class of graphs showing that Theorem 9 is best possible also illustrates that the value \(\lceil (4n - 3)/3 \rceil\) cannot be decreased unless we do not ask the vertices to be contained in small cycles.

**Theorem 29.** Let \(c > 0\) and \(cn \geq 6\lceil 1/c \rceil - 2\). If \(C_{\lfloor c(1+c)n\rfloor}(G)\) is complete, then every vertex is contained in a \(k\)-cycle for \(6\lceil 1/c \rceil - 1 \leq k \leq n\).

**Proof.** Put \(f_{\text{SO}} = f_{\text{SO}}(G)\) and let \(w \in V(G)\) be an arbitrary vertex. Furthermore, let \(u\) and \(v\) be distinct nonadjacent vertices with \(d(u) + d(v) \geq (1 + c)n\) and let \(q\) denote the number of vertices of a shortest \(uv\)-path \(P\) containing \(w\). Note that such a path exists, since \(G\) is 2-connected by Lemma 23.

Let the vertices of \(P\) be denoted \(u = v_1, \ldots, v_q = v\) with \(w = v_k\) for \(1 \leq k \leq q\). W.l.o.g. \(|\{v_1, \ldots, v_{k-1}\}| \geq |\{v_{k+1}, \ldots, v_q\}|\). Denote by \(P'\) the subpath of \(P\) from \(v_{k+1}\) to \(v_q\). Let \(S = \{v_1, v_4, \ldots, v_{3s-2}, v_k\}\) be a (independent!) set s.t. \(k \geq 3s + 1\) and \(S\) is maximal. Then \(q \leq 3(2s + 1) + 2 = 6s + 5\).

Note that there exists no pair of edges \(v_i v_j, v_i v_j\) in \(G\) s.t. \(i_1, i_2 \in S\) with \(i_1 < i_2\) and \(j_1, j_2 \in V(P')\) with \(j_1 < j_2\), since \(P\) is a shortest \(uv\)-path containing \(w\). Hence

\[\sum_{x \in S} |N(x) \cap V(P')| \leq q - k + (s + 1) - 1.\]

Next observe that \(x_1, x_2 \in S\) cannot have a common neighbour outside the path \(P\). Therefore

\[n \geq \left| \left( \bigcup_{x \in S} N[x] \right) \cup V(P') \right| \geq \left( \sum_{x \in S} |N[x]| \right) - (q - k + s) + |V(P')| \geq (s + 1)(\delta + 1) - (q - k + s) + (q - k) = (s + 1)\delta + 1.\]

Now \(\delta \geq cn + 2\) implies \(s \leq n/\delta - 1 < 1/c - 1\). Thus

\[q \leq 6s + 5 < 6/c - 1 \leq 6 \left\lfloor \frac{1}{c} \right\rfloor - 1\]

and therefore

\[q \leq 6 \left\lfloor \frac{1}{c} \right\rfloor - 2.\]
Thus we conclude that
\[ d(u) + d(v) \geq (1 + c)n \geq n + 6 \left\lfloor \frac{1}{c} \right\rfloor - 2 \geq n + q = (n - q + 4) + (2q - 4). \]

Since \((1 + c)n \geq n + q\), \(G\) is \(q\)-Hamiltonian by Corollary 22. Choose a Hamiltonian cycle \(C_1 = G[V(G) - V(P)]\) and suppose to the contrary that \(w\) is not contained in an \(l\)-cycle for some \(q + 1 \leq l \leq n\). Then \(d_C(u) + d_C(v) \leq n - q\) by Lemma 25 and we derive the contradiction
\[ d(u) + d(v) \leq (n - q) + (2q - 4) = n + q - 4 < n + q. \]

In the following we present classes \(G_{l,r}\) of graphs such that for every member \(G \in G_{l,r}\) its corresponding closure \(Cl_l(G)\) is complete, but there exists at least one vertex not contained on a cycle \(C\) of length \(l\).

A graph \(G\) of order \(n\) is a member of \(G_{2l+1,(1+1/(3,2^{l-1}-1))(n-1)-1}\) with \(l \geq 2\), if the following holds: there exists a vertex \(x\) of \(G\) such that with \(N_l = \{v \in V(G) \mid d(x,v) = i\}\) for \(i = 0,\ldots, l\), we have \(V(G) = \bigcup_{i=0}^{l+1} N_i\), \(|N_1| = (n-1)/(3 \cdot 2^{l-1} - 1)\), \(|N_i| = 2|N_{i-1}|\) for \(i = 2,\ldots, l\), and \(|N_{l+1}| = |N_l| = (2^{l-1}(n-1))/((3 \cdot 2^{l-1} - 1))\). (Then \(\sum_{i=1}^{l+1} |N_i| = (n - 1)/(3 \cdot 2^{l-1} - 1)\) to arbitrary even cycles. A graph

\[ G_{2l+1,(1+1/(3,2^{l-1}-1))(n-1)-1}(G) \] is complete and \(x\) does not belong to an odd cycle of length \(2l + 1\).

A graph of order \(n\) is a member of \(G_{4,n-2+p}\) with \(p = O(\sqrt{n})\) (exact value \(p = (\sqrt{8n} - 7 - 1)/4\)), if the following holds: There exists a vertex \(x\) of \(G\) such that with \(N_l = \{v \in V(G) \mid d(x,v) = i\}\), we have \(V(G) = \bigcup_{i=0}^{2} N_i\), \(|N_1| = (v_1,\ldots,v_p)\), and \(N_2 = \{w_1^1,\ldots,w_1^{p},w_1^2,\ldots,w_1^{p}\}\) such that every vertex \(v_i\) of \(N_1\) is only adjacent to \(x\) and to all vertices \(w_j^i\) of \(N_2\). Furthermore \(N_2\) should induce a complete subgraph. Clearly, \(Cl_{n-2+p}(G)\) is complete but \(x\) is not contained in a cycle of length 4. Note that it is not very difficult to enhance these examples, but they will remain to be a member of a class \(G_{4,n-2+O(\sqrt{n})}\).

Now we extend the “4-cycle-class” \(G_{4,n-2+O(\sqrt{n})}\) to arbitrary even cycles. A graph \(G\) of order \(n\) is a member of \(G_{2l,n-2+p}\) with \(p = O(\sqrt{n})\) and \(l \geq 2\), if the following holds: there exists a vertex \(x\) of \(G\) such that with \(N_l = \{v \in V(G) \mid d(x,v) = i\}\) for \(i = 0,1,\ldots, l\), we have \(V(G) = \bigcup_{i=0}^{l+1} N_i\), \(|N_1| = (v_1,\ldots,v_p)\), \(|N_i| = \{w_1^1,\ldots,w_1^{p},w_2^1,\ldots,w_2^{p}\}\) such that every vertex \(v_i\) of \(N_1\) is only adjacent to \(x\) and to all vertices \(w_j^i\) of \(N_2\), \(N_3 = \{w_1^1,\ldots,w_1^{p/(2p^{2/2})},w_2^1,\ldots,w_2^{p/(2p^{2/2})},w_3^1,\ldots,w_3^{p/(2p^{2/2})}\}\) such that every vertex \(w_j^i\) of \(N_2\) is furthermore only adjacent to all vertices \(w_j^i\) of \(N_3\), \(N_4 = \{w_1^1,\ldots,w_1^{p/(2p^{2/2})},w_2^1,\ldots,w_2^{p/(2p^{2/2})},w_3^1,\ldots,w_3^{p/(2p^{2/2})}\}\)...

Furthermore, \(N_l\) should induce a complete subgraph. Analogously to the case \(l = 2\), it is not very difficult to enhance these examples but that they will be a member of the class \(G_{2l,n-2+O(\sqrt{n})}\).
Theorem 30. If $\text{Cl}_{[(3n-3)/2]}(G)$ is complete, then $G$ is edge pancyclic.

Proof. By Theorem 20, we know that every edge $uv$ is contained in a $k$-cycle for every $2n - [(3n - 3)/2] + 1 \leq k \leq n$. Moreover, Corollary 24 yields $\delta = \delta(G) \geq (n + 1)/2$ implying $\sigma_2(G) \geq n + 1$. Now choose a vertex set $S$ of size at least $n - p$, where $p = [(n - 3)/2]$, containing all neighbours of $u$ and $v$. By Corollary 22, $S$ has a Hamiltonian cycle $C$. Suppose the edge $uv$ does not belong to an $l$-cycle for some $l$ with $3 \leq l \leq n - p + 1$. Then by Corollary 24 and Lemma 25, we get

$$2(p + 1) \leq 2(\delta - 1) \leq (d(u) - 1) + (d(v) - 1) = d_C(u) + d_C(v) \leq n - 2$$

implying $p \leq (n - 4)/2$, a contradiction. \hfill \Box

The value $[(3n - 3)/2]$ in Theorem 30 is sharp. Consider the graph described in Case (ii) of Theorem 10, but for $k = 2$. Then $G[U_1 \cup U_2]$ consists of an edge, say $u_1u_2$, which does not belong to a triangle. If $G[V_1 \cup V_2]$ is complete, then $\text{Cl}_{[(3n-4)/2]}(G)$ is complete.

Corollary 31. If $\text{Cl}_{[(3n-3)/2]}(G)$ is complete, then $G$ is panconnected, i.e., any pair of vertices $u, v \in V(G)$ is connected by a path of length $k$ for every $2 \leq k \leq n - 1$.

Corollary 32 (Faudree and Schelp [12], Williamson [17]). If $\sigma_2(G) \geq (3n-2)/2$, then $G$ is panconnected.

Theorem 33. If $\text{Cl}_{3n/2-2}(G)$ is complete, then $G$ is fully cycle extendable.

Proof. By Corollary 24, we obtain $\delta(G) \geq n/2$ implying $\sigma_2(G) \geq n$. Suppose there is a nonextendable cycle $C$ of length $k$ with $3 \leq k \leq n - 1$. Again, we apply Theorem 10.

In Case (i), let the vertices of $K_k$ be labelled $u_1, u_2, \ldots, u_k$ such that $u_i$ is the $i$th vertex in the construction of the closure which is joined to a vertex $v$ of $R = V(G) - V(C)$. Since $d(u_i) + d(v) \geq \frac{3}{2}n - 2$, we conclude

$$d_R(u_i) \geq \frac{3}{2}n - 2 - (k - 1) - (n - k - 1) - (1 + (i - 1)) = \frac{n}{2} - i$$

for $1 \leq i \leq k$. Thus we derive the contradiction

$$n - 3 \geq n - k = |R| \geq \sum_{i=1}^{k} d_R(u_i) \geq d_R(u_1) + d_R(u_2) + d_R(u_3)$$

$$\geq \left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} - 2\right) + 1 = n - 2.$$

If Case (ii) holds, then $d(x) = n/2$ for all vertices $x \in V(C)$ and $d(y) = n - (k/2) - 1 \leq n - 3$ for all other vertices. Hence, $\text{Cl}_{3n/2-2}(G)$ cannot be complete, a contradiction.

Therefore, every cycle of $G$ is extendable. Clearly, $\frac{3}{2}n - 2 \geq \frac{4}{3}n - 1$ for $n \geq 6$ and thus every vertex $w \in V(G)$ is contained in a 3-cycle by Theorem 9 which gives the desired result. \hfill \Box
The graph constructed in Case (ii) of Theorem 10 shows that the value $\frac{3}{2}n - 2$ is sharp for both “cycle extendable” and “fully cycle extendable”.

5. Extremal results

Broersma [7] has shown that the minimum number $m_n$ of edges of a vertex pancyclic graph on $n$ vertices satisfies

$$3n/2 < m_n \leq \left\lceil \frac{5n}{3} \right\rceil \quad \text{for } n \geq 7.$$ 

The edge maximal graph $H_{f,SO;n}$ of the class $H_{f,SO;n}$ (Remark 8) reveals that a graph of order $n$, minimum degree $f_{SO}$ and size $m = \binom{n}{2} - \binom{f_{SO}}{2} - n + f_{SO} + 1$ need not be vertex pancyclic. In the following we will show that all graphs of order $n$ and minimum degree $f_{SO}$ containing at least

$$g(n, f_{SO}) := \binom{n}{2} - \binom{f_{SO}}{2} - n + f_{SO} + 2$$

edges are vertex pancyclic. We first prove that all these graphs are Hamiltonian by Theorem 4 of Erdős.

**Lemma 34.** Let $G$ be a graph of size $m$ and order $n \geq 5$. If $2 \leq \delta \leq (n-1)/2$ and $m \geq g(n, \delta)$, then $G$ is Hamiltonian.

**Proof.** Let $2 \leq \delta \leq (n-1)/2$. It is straightforward to verify the estimation

$$\binom{n}{2} - \binom{\delta}{2} - n + \delta + 2 > \binom{n-\delta}{2} + \delta^2$$

for $n \geq 5$. In addition, it is a simple matter to obtain

$$\binom{n}{2} - \binom{\delta}{2} - n + \delta + 2 > \left\lceil \frac{(n+2)/2}{2} \right\rceil + \left\lfloor \frac{n-1}{2} \right\rfloor^2$$

for $n \geq 5$. In all these cases, the graph $G$ is Hamiltonian by Theorem 4 of Erdős.

**Theorem 35.** Let $G$ be a graph of size $m$ and order $n \geq 3$. If either $\delta = \delta(G) \geq (n+1)/2$ or

$$2 \leq \delta \leq n/2 \text{ and } m \geq g(n, \delta) \quad \text{with } g(n, \delta) = \binom{n}{2} - \binom{\delta}{2} - n + \delta + 2,$$

then $G$ is vertex pancyclic.

**Proof.** Let $G$ be a graph of order $n \geq 3$, size $m$ and minimum degree $\delta = \delta(G)$. It is easy to check the claim for $n = 3, 4$. Hence, let $n \geq 5$ in the following. If $\delta \geq (n + 1)/2$, then we are done by Corollary 7. Therefore, let $2 \leq \delta \leq n/2$ and $m \geq g(n, \delta)$. If $\delta = 2$, then $m \geq g(n, 2) = \binom{n}{2} - n + 3$. Applying the handshaking lemma, we deduce
we deduce that not imply that $G$ is vertex pancyclic. Hence this example shows that Theorem 35 is best possible. (Remark 8) reveals that a graph of order $m = \binom{n}{2} - n + 3$. This identity implies $G - x = K_{a-1}$, when $x$ is a vertex of minimum degree. Obviously, $G$ is vertex pancyclic in this case. Now let $G$ be a counterexample of minimal order $n$, i.e. $G$ is not vertex pancyclic. Then $\delta \geq 3$. By Theorem 1 and Lemma 34, $G$ is Hamiltonian. Note that

$$g(n, \delta) \geq g(n - 1, \delta - 1) + \delta \quad \text{and} \quad g(n, \delta) \geq g(n - 1, \delta') + n - 2 \quad \text{if} \quad \delta' \geq \delta.$$

(1)

Let $x$ be a vertex of minimum degree. Because of (1) and $\delta(G - x) \geq \delta - 1$ we have $m(G - x) \geq g(n - 1, \delta(G - x))$. Because of the minimality of $G$ we deduce that $G - x$ is vertex pancyclic. Since $G$ is Hamiltonian, every vertex $y \in V(G - x)$ is vertex pancyclic in $G$. If $\delta(G - x) = \delta - 1$ there exists a vertex $y \in N_G(x)$ with $d_G(y) = \delta$. Analogously to the former case we obtain that $G - y$ is vertex pancyclic, especially $x$ is vertex pancyclic in $G - y$ and hence we are done. So let $d_G(y) \geq \delta + 1$ for all $y \in N_G(x)$. If there exists $y \in V(G) - N_G[x]$ with $d_G(y) = \delta$ then we conclude analogously to the former cases that $G$ is vertex pancyclic. Otherwise, $d_G(y) \geq \delta + 1$ for all $y \in V(G - x)$. But then $\delta(G - y) = \delta$ for all $y \in V(G) - N_G[x]$. Let $y \in V(G) - N_G[x]$. Because of (1) and $d_G(y) \leq n - 2$ we deduce that $m(G - y) \geq g(n - 1, \delta)$. Because of the minimality of $G$ we deduce that $G - y$ is vertex pancyclic, which completes the proof of Theorem 35.

As already mentioned at the beginning of this section the graph $H_{\delta,n}^k$ of the class $H_{\delta,n}$ (Remark 8) reveals that a graph of order $n$, minimum degree $\delta$ and size $g(n, \delta) - 1$ need not be vertex pancyclic. Hence this example shows that Theorem 35 is best possible. If $\delta \geq (n + 1)/2$, the sharpness of the result is already considered in Corollary 7.

The following examples will show that the conditions of Theorem 35 in general do not imply that $G$ is cycle extendable. Let $p \geq 3$ be an integer, $a_1, a_2, \ldots, a_p$ be the vertices of a complete graph $K_p$, and $x_1, x_2, \ldots, x_s$ be the vertices of a complete graph $K_s$ with $s = 2p - 1$. The graph $H$ consists of the disjoint union of $K_p$ and $K_s$ together with the edges $a_i x_j$ such that $1 \leq i \leq p$ and $2(i - 1)p^2 + 1 \leq j \leq 2ip^2$. First, we observe that the cycle $a_1 a_2 \ldots a_p a_1$ is not extendable. Second, it is straightforward to verify that $2 \leq \delta(H) \leq n(H)/2$ and $m(H) \geq g(n(H), \delta(H))$.

In [13], Hendry characterized the fully extendable graphs $G$ with

$$m(G) \geq \binom{n}{2} - (n - 2).$$

Next we will give different extensions of this result.

**Lemma 36.** Let $G$ be a graph of order $n \geq 4$ and $3 \leq k \leq n - 1$ with $\delta \geq 2$ if $k = n - 1$. Every $k$-cycle is extendable, if

$$m = m(G) \geq \binom{n}{2} - \frac{k + 1}{2} (n + 2 - k) + 5 := f(n,k) \quad \text{for} \quad k \text{ odd},$$

$$m = m(G) \geq \binom{n}{2} - \frac{k}{2} (n + 2 - k) + 4 := h(n,k) \quad \text{for} \quad k \text{ even}.$$
Fourth, let \( H = v_1v_2 \ldots v_kv_1 \) be a nonextendable cycle in \( G \), where \( 3 \leq k \leq n-1 \). Then each vertex of \( V(G) - V(C) \) has at most \( \lfloor k/2 \rfloor \) neighbours in \( C \). In addition, if one vertex of \( V(G) - V(C) \) has at least two neighbours in \( C \), then \( G[V(C)] \) is not Hamilton-connected, and therefore it follows from Theorem 5

\[
m(G[V(C)]) \leq \binom{k}{2} - k + 3.
\]

Altogether we obtain the estimations

\[
f(n,k), h(n,k) \leq m \leq \binom{k}{2} + \binom{n-k}{2} + \max\{(n-k)\lfloor k/2 \rfloor - k + 3, n-k\}.
\]

For \( 3 \leq k \leq n-2 \) it is a simple matter to verify that \( (n-k)\lfloor k/2 \rfloor - k + 3 \geq n-k \), and since \( \delta \geq 2 \) in the case \( k=n-1 \), it follows that

\[
f(n,k), h(n,k) \leq m \leq \binom{k}{2} + \binom{n-k}{2} + (n-k)\lfloor k/2 \rfloor - k + 3.
\]

In both cases \( k \) odd and \( k \) even it is straightforward to obtain a contradiction, and hence the lemma is proved. \( \Box \)

The graph \( K_1 + (K_1 \cup K_{n-2}) \) shows that \( \delta \geq 2 \) is necessary for \( k=n-1 \) in Lemma 36. In addition, Lemma 36 is best possible for \( k=3,4,5 \) and \( k=n-2 \) odd.

First, let \( G \) consist of \( K_3 \cup K_{n-3} \) such that every vertex \( x \) of \( K_{n-3} \) is adjacent to exactly one vertex of \( K_3 \). Clearly, the cycle \( K_3 \) is not extendable and we have

\[
m(G) = \binom{n}{2} - 2n + 6 < \binom{n}{2} - 2n + 7 = f(n,3).
\]

Second, let \( H \) be a cycle \( x_1x_2x_3x_4x_1 \) together with the chord \( x_1x_3 \). If \( K_{n-4} \) consists of the vertices \( u_1, u_2, \ldots, u_{n-4} \), then let \( G \) be the union of \( H \) and \( K_{n-4} \) together with the edges \( x_1u_i \) and \( x_3u_i \) for \( 1 \leq i \leq n-4 \). Then the cycle \( x_1x_2x_3x_4x_1 \) is not extendable and

\[
m(G) = \binom{n}{2} - 2n + 7 < \binom{n}{2} - 2n + 8 = h(n,4).
\]

Third, let \( H \) be a cycle \( x_1x_2x_3x_4x_5x_1 \) together with the chords \( x_1x_3 \), \( x_1x_4 \), and \( x_3x_5 \). If \( K_{n-5} \) consists of the vertices \( u_1, u_2, \ldots, u_{n-5} \), then let \( G \) be the union of \( H \) and \( K_{n-5} \) together with the edges \( x_1u_i \) and \( x_3u_i \) for \( 1 \leq i \leq n-5 \). Then the cycle \( x_1x_2x_3x_4x_5x_1 \) is not extendable and

\[
m(G) = \binom{n}{2} - 3n + 13 < \binom{n}{2} - 3n + 14 = f(n,5).
\]

Fourth, let \( x, y \) be the two vertices of the complete graph \( K_{n-2} \) and \( u, v \) two further vertices. In addition, we add the edges \( uv, xu, \) and \( yv \). In the resulting graph \( H \) there are obviously \( (n-2) \)-cycles that are not extendable, and for \( n-2 \) odd we see that

\[
m(H) = \binom{n}{2} - 2n + 6 < \binom{n}{2} - 2n + 7 = f(n,n-2).
\]
Corollary 37. Let $G$ be a graph of order $n \geq 5$ with $\delta(G) \geq 2$. Then $G$ is cycle extendable, if

$$m(G) \geq \left( \frac{n}{2} \right) - \frac{3n}{2} + 5 \text{ for } n \text{ even},$$

or

$$m(G) \geq \left( \frac{n}{2} \right) - \frac{3(n-1)}{2} + 4 \text{ for } n \text{ odd}.$$

Proof. First, let us observe that $G$ has at least one cycle. Second, $f(n,n-1)$ or $h(n,n-1)$ is the maximum in Lemma 36 when $n$ is even or odd, respectively, and hence the corollary is proved. \qed

For small $\delta$, Corollary 37 yields immediately the following improvement of Lemma 34.

Corollary 38. Let $G$ be a graph of order $n \geq 8$ with $2 \leq \delta(G) \leq \sqrt{n}$. Then $G$ is Hamiltonian, if

$$m(G) \geq \left( \frac{n}{2} \right) - 3[n/2] + 5.$$

The next result is a completion of Theorem 35.

Theorem 39. Let $G$ be a graph of order $n \geq 8$. If $2 \leq \delta = \delta(G) \leq \sqrt{2n}$ and

$$m(G) \geq \left( \frac{n}{2} \right) - \left( \frac{\delta}{2} \right) - n + \delta + 2$$

then $G$ is fully cycle extendable.

Proof. According to Theorem 35, every vertex is contained in a 3-cycle and $G$ is Hamiltonian. Therefore, every $(n-1)$-cycle is extendable. For $3 \leq k \leq n-2$ and $2 \leq \delta \leq \sqrt{2n}$ it is easy to check that

$$m(G) \geq \left( \frac{n}{2} \right) - \left( \frac{\delta}{2} \right) - n + \delta + 2 \geq f(n,k), h(n,k)$$

and hence the desired result follows from Lemma 36. \qed

Theorem 40. Let $G$ be a graph of order $n \geq 8$ and $\delta = \delta(G) \geq 4$. Then $G$ is fully cycle extendable, if

$$m = m(G) \geq \left( \frac{n}{2} \right) - n.$$

Proof. According to Theorem 35, every vertex is contained in a 3-cycle and $G$ is Hamiltonian. Therefore, every $(n-1)$-cycle is extendable. For $3 \leq k \leq n-2$ and $n \geq 8$
it is easy to check that

\[ m(G) \geq \binom{n}{2} - n \geq f(n,k), h(n,k) \]

and hence Theorem 40 follows from Lemma 36. □

6. Edge pancyclic graphs

As our investigations of vertex pancyclicity in graphs have underlined (cf. Theorem 9, Corollary 17), the most difficult part is the consideration of small cycle lengths. For edge pancyclic graphs, the situation is similar which will be illustrated in the following.

Faudree and Schelp [12] proved that any graph \( G \) on \( n \) vertices with \( f_{ESC}(G) \geq n+1 \) is 4-panconnected, i.e., any pair of vertices \( u, v \in V(G) \) is connected by a path of length \( k \) for every \( 3 \leq k \leq n - 1 \). Hence, the following holds.

**Theorem 41** (Faudree and Schelp [12]). Let \( G \) be a graph of order \( n \) such that \( f_{ESC}(G) \geq n+1 \). Then \( G \) is edge 5-pancyclic.

This result is sharp in the sense that there might be edges \( xy \) not belonging to a cycle of length 3 or 4. Firstly, note that \( \sigma_2 \geq n+1 \) yields that every vertex not in \( N(x) \cup N(y) \) is adjacent to a vertex in \( (N(x) \cup N(y)) - \{x, y\} \). If \( xy \) is not contained in a 3-cycle, then clearly \( N(x) \cap N(y) = \emptyset \). If \( xy \) does not belong to a 4-cycle, then it follows that \( N(x) - \{y\} = N(y) - \{x\} \) and this is an independent vertex set. Indeed, assume that \( xu \in E(G) \) and \( yu \notin E(G) \) for some \( u \in V(G) - \{x, y\} \). Then \( d(u) + d(y) \geq n + 1 \) and hence, there exists \( v \in (N(u) \cap N(y)) - \{x\} \) which leads to the 4-cycle \( xyvuux \). It is easy to finish the construction of the corresponding graphs such that \( \sigma_2 \geq n+1 \) and we omit the details.

Note that each edge is contained either in cycle of length 3 or 4. Moreover, it is not difficult to check that there is at most one edge of a graph in consideration not belonging to a 4-cycle. In contrast to this, the example \( H_{\delta,n}^c \) with \( 3 \leq \delta \leq (n-1)/2 \) shows that there might be many edges which are not contained in a cycle of length 3.

For more information on edges in these graphs not belonging to small cycles, we refer the reader to [19].

Analogously to Section 3, we now switch from the Ore-type condition to a condition on the minimum degree. We investigate the edge pancyclicity of graphs by considering the vertex pancyclicity of a related digraph. In a first step, a degree condition for vertex pancyclic digraphs is developed.

**Theorem 42** (Woodall [18]). Let \( D \) be a digraph of order \( n \). If \( d^+(x) + d^-(y) \geq n \) for every pair of distinct vertices \( x \) and \( y \) such that there is no arc from \( x \) to \( y \), then \( D \) is Hamiltonian.
Lemma 43. Let $D$ be a digraph of order $n \geq 3$ such that $d^+(x) + d^-(y) \geq n + 1$ for every pair of distinct vertices $x, y \in V(D)$ where $x$ does not dominate $y$. If $w \in V(D)$ is not pancyclic then $\min\{d^+(w), d^-(w)\} \leq (n - 1)/2$.

Proof. Let $H = D - w$. By Theorem 42, $H$ contains a Hamiltonian cycle $C$. Let $d^+(w) = s$ and $x_1, x_2, \ldots, x_s$ be the positive neighbours of $w$. If $w$ is not contained in a cycle of length $k$ for some $3 \leq k \leq n$ it is easy to see that $w$ is not dominated by $y_i$ for every $1 \leq i \leq s$ where $y_i$ is the vertex at distance $k - 2$ from $x_i$ when following the orientation of $C$. Therefore, $d^+(w) + d^-(w) \leq n - 1$ and we are done. \qed

Corollary 44. Let $D$ be a digraph of order $n \geq 3$ with $\min\{\delta^+, \delta^-\} \geq (n+1)/2$. Then $D$ is vertex pancyclic.

Theorem 45. Let $G$ be graph of order $n$ such that $\delta \geq (n + 2)/2$. Then $G$ is edge pancyclic.

Proof. Let $ab \in E(G)$ be an arbitrary edge of $G$. Since $d_G(a), d_G(b) \geq (n + 2)/2$, the vertices $a$ and $b$ have a common neighbour in $G - \{a, b\}$ and hence $ab$ is contained in a cycle of length 3.

Now consider the digraph $D_G$ where $V(D_G) = \{z\} \cup \{V(G) - \{a, b\}\}$ and for $x, y \in V(D_G) - \{z\}$ let

$z \rightarrow x$ if and only if $x \in N_G(a)$,
$x \rightarrow z$ if and only if $x \in N_G(b)$,
$x \rightarrow y \rightarrow x$ if and only if $x, y$ are adjacent in $G$.

It is easy to see that $d_{D_G}^+(z) = d_G(a)$ and $d_{D_G}^-(z) = d_G(b)$. For $x \in V(D_G) - \{z\}$ we have four possible situation. Clearly, $d_{D_G}^+(x) = d_{D_G}^-(x) = d_G(x)$, if $x$ is neither adjacent to $a$ nor to $b$ in $G$. If $x$ is adjacent to $a$ but not to $b$ in $G$ then $d_{D_G}^+(x) = d_G(x)$ and $d_{D_G}^-(x) = d_G(x) - 1$. Analogously, $d_{D_G}^+(x) = d_G(x) - 1$ and $d_{D_G}^-(x) = d_G(x)$, if $x$ is adjacent to $b$ but not to $a$ in $G$ and $d_{D_G}^+(x) = d_G(x) - 1$ and $d_{D_G}^-(x) = d_G(x) - 1$, if $x$ is adjacent to $a$ and to $b$ in $G$. Altogether we see that $d_{D_G}^+(v), d_{D_G}^-(v) \geq n/2 = (n(G) + 1)/2$ for every $v \in V(D_G)$.

By Corollary 44, $D_G$ is vertex pancyclic and hence $z$ is contained in an oriented cycle of length $k$ for every $3 \leq k \leq n(D_G) = n - 1$. Let $zu_1u_2\ldots u_{k-1}z$ be such a cycle. By the definition of $D_G$, $u_1$ is adjacent to $a$ and $u_{k-1}$ is adjacent to $b$ in $G$. Therefore, we obtain the $(k+1)$-cycle $au_1u_2\ldots u_{k-1}ba$ in $G$ which contains the edge $ab$. Hence $ab$ is 4-pancyclic and we are done. \qed
and let $x_jx_k \in E(D)$ for some $3 \leq j,k \leq p + 1$ such that $d(x_i,A - \{x_1,x_2\}) \geq 1$ for every $3 \leq i \leq p + 1$.

The following observations show that Theorem 45 is best possible.

**Remark 46.** Let $G$ be a graph of order $n = 2p \geq 4$ such that $\delta \geq p$. Then every edge of $G$ is contained in a cycle of length 3 if and only if $G \not\in D_{2p}$.

**Proof.** If $G \in D_{2p}$, then the edge $x_1y_1$ is not contained in a cycle of length 3. Conversely, let $G$ be a graph with $\delta \geq p$ containing an edge $uv$ that does not belong to any cycle of length 3 in $G$. Obviously, $N(u) \cap N(v) = \emptyset$ and since $d(u),d(v) \geq p$, we see that $G \in D_{2p}$ where $A = N(v)$ and $B = N(u)$.

**Theorem 47.** Let $G$ be a graph of order $n = 2p + 1 \geq 7$ such that $\delta \geq p + 1$. Then every edge of $G$ is contained in a cycle of length 4 if and only if $G \not\in D_{2p+1}$.

**Proof.** For $G \in D_{2p+1}$, it is easy to see that the edge $x_1x_2$ does not belong to any cycle of length 4 in $G$. Conversely, let $G$ be a graph with $\delta \geq p + 1$ where the edge $uv$ is not contained in a cycle of length 4. Clearly, $N(u) \cap N(v) \neq \emptyset$, so $x \in N(u) \cap N(v)$. Define $X = (N(u) \cup N(v)) - \{u,v\}$. By the assumption, $x$ is not adjacent to any vertex in $X$ and therefore $d(x) \leq n - |X|$. Since $d(x) \geq (n + 1)/2$, we have $|X| \leq (n - 1)/2$ and obviously equality holds. Hence $X$ is an independent set of vertices and every vertex of $X$ is adjacent to $u$ and $v$ and to every vertex not in $N(u) \cup N(v)$. To satisfy the condition on the vertex degree every vertex in $V(G) - (N(u) \cup N(v))$ is adjacent to at least one vertex of this set and it finally follows that $G \in D_{2p+1}$, where $B = X$.

**References**