Journal of Multivariate Analysis **76**, 63–89 (2001)
doi:10.1006/jmva.2000.1906, available online at http://www.idealibrary.com on IDEAL®

Stochastic Bounds and Dependence Properties of Survival Times in a Multicomponent Shock Model[1]

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Received September 25, 1998; published online November 7, 2000

Consider a system that consists of several components. Shocks arrive according to a counting process (which may be non-homogeneous and with correlated inter-arrival times) and each shock may simultaneously destroy a subset of the components. Shock models of this type arise naturally in reliability modeling in dynamic environments. Due to correlated shock arrivals, individual component lifetimes are statistically dependent, which makes the explicit evaluation of the joint distribution intractable. To facilitate the development of easily computable tight bounds and good approximations, an analytic analysis of the dependence structure of the system is needed. The purpose of this paper is to provide a general framework for studying the correlation structure of shock models in the setup of a multivariate, correlated counting process and to systematically develop upper and lower bounds for its joint component lifetime distribution and survival functions. The thrust of the approach is the interplay between a newly developed notion, *majorization with respect to weighted trees*, and various stochastic dependence orders, especially orthant dependence orders of random vectors and orthant dependence orders of stochastic processes. It is shown that the dependence nature of the joint lifetime is inherited from *spatial dependence* and *temporal dependence*; that is, dependence among various components due to simultaneous arrivals and dependence over different time instants introduced by the shock arrival process. The two types of dependency are investigated separately and their joint impact on the performance of the system is analyzed. The results are used to develop computable bounds for the statistics of the joint component lifetimes, which are tighter than the product-form bounds under certain conditions. The shock model with a non-homogeneous

[1] Supported in part by NSF Grant DMI 9812994.

0047-259X/00 $35.00

Poisson arrival process is studied as an illustrative example. The result is also applicable to the cumulative damage model with multivariate shock arrival processes.

AMS 1991 subject classifications: 60E15; 60K10.

Key words and phrases: shock models, correlated shock arrivals, Marshall–Olkin distributions, majorization with respect to weighted trees, positive and negative orthant dependence order, orthant dependence in time, non-homogeneous Poisson arrival processes, bounds.

1. INTRODUCTION

The purpose of this paper is to provide a general framework in which to study the correlation structure of shock models in the setup of a multivariate, correlated counting process and systematically develop upper and lower bounds for the joint component lifetime distribution and survival functions.

Consider the following shock model of a system consisting of s components with component index set $E=\{1, ..., s\}$. Assume that a counting process $\{N(t), t\geqslant 0\}$ governs the occurrence of shocks of multiple types that are fatal to the system components, where the counting process may be non-stationary (due to the impact of dynamic environments) and have dependent interarrival times. With probability P^K, where $K\subseteq E$ and $\sum_{K\subseteq E} P^K=1$, an arriving shock simultaneously destroys all the components $j\in K$ that are still alive but all of the other components $j\in E-K$ that are still alive survive. Such a shock is called a type-K shock. Note that with probability $P^{\varnothing}$, an arriving shock is a type-$\varnothing$ shock, which means that all the components in the system survive at the arrival epoch. Let $\{N^K(t), t\geqslant 0\}$ denote the type-K shock arrival process, determined by $\{N(t), t\geqslant 0\}$ and P^K, $K\subseteq E$. Let T_j denote the life length of component j for $j\in E$. Then

$$T_j=\inf\{t\geqslant 0 \mid N^K(t)\geqslant 1, j\in K\subseteq E\}, \qquad j\in E. \tag{1.1}$$

In general, $T=(T_1, ..., T_s)$ are statistically dependent, due to simultaneous component failures and the dependent nature of the shock arrival process.

A special case of the above model was first introduced in Marshall and Olkin (1967), where type-K shocks with rate λ^K arrive at the system according to independent Poisson processes, $\varnothing\neq K\subseteq E$. The joint distribution of T defined as in (1.1) is known as the Marshall–Olkin multivariate exponential distribution with a set of parameters $\{\lambda^K, \varnothing\neq K\subseteq E\}$. Equivalently, one may view the shock model as governed by a Poisson shock arrival process $N(t)=\sum_{\varnothing\neq K\subseteq E} N^K(t)$ with rate $\lambda=\sum_{\varnothing\neq K\subseteq E}\lambda^K$, and an arriving shock is of a type-K shock with probability $P^K=\lambda^K/\lambda$, $K\neq\varnothing$,

$K \subseteq E$. The Marshall–Olkin distribution, one of the most widely discussed multivariate lifetime distributions, plays a fundamental role in reliability modeling and survival analysis (Barlow and Proschan, 1981) and also is a prelude to recent research interest in modeling multivariate failure systems in dynamic environments (Singpurwalla, 1995). The s-dimensional Marshall–Olkin distribution can be computed explicitly and it is *positively associated* (see, for example, Barlow and Proschan, 1981). It is known that if a random vector is associated, then its joint distribution (survival) function is bounded below by the product of its marginal distribution (survival) functions (Tong, 1980).

However, if we move away from the Poissonian assumption, the statistics associated with T and other performance measures are no longer easily accessible. In studying such correlated systems operating in a random environment, we want to know how the dependence structure of the component lifetime vector T varies in response to the change of environmental input parameters (such as the probability laws that govern the shock arrival process and the pattern of simultaneous failures). For example, if both $\{P^K, K \subseteq E\}$ and $\{N(t), t \geqslant 0\}$ become more dependent in some sense (e.g., shocks are more synchronized and the interarrival times become more autocorrelated), then how and in what sense do these changes affect the dependence structure of the component lifetime vector T? Are the components of T still positively associated under a counting shock arrival process, regardless of the properties of $\{P^K, K \subseteq E\}$ (the answer is affirmative if $\{N(t), t \geqslant 0\}$ is a Poisson process)? If not, then under what conditions and in what sense are the components of T positively and negatively dependent? What are the advantages and disadvantages brought into the system by positive and negative dependence? To answer these questions, a structural dependence analysis that does not rely on the specific assumptions of the shock arrival process is desirable. Such an analysis will not only be helpful for the efficient and effective component/modular design and failure process control of reliability systems, but will also provide a vehicle for the development of easily computable, tight bounds for the statistics of the joint lifetimes and other system performance measures whose explicit computations are intractable.

The thrust of our approach is the interplay between a newly developed notion (Xu and Li, 1998; Li and Xu, 1999), *majorization with respect to weighted trees*, and various stochastic dependence orders, especially orthant dependence orders. Specifically, we consider the random vector (called the shock loading vector) characterized by the parameter set $\{P^K, K \subseteq E\}$. We view $\{P^K, K \subseteq E\}$ as a weighted tree and establish several partial orders between two weighted trees. By systematically varying the parameters of a tree, we can obtain the shock loading vector that is positively upper (respectively, negatively upper, positively lower, and negatively lower) orthant dependent (PUOD) (respectively, NUOD, PLOD, and NLOD). We also introduce several notions of *orthant dependence in time* for a real-valued stochastic process, which are weaker and broader than the notion of *association in time* (Esary and Proschan, 1970; Lindqvist, 1988). These notions of orthant dependence in time for the shock arrival process allows us to compare the structural difference of two shock arrival processes and to characterize their autocorrelation properties. Our method yields several structural results that are worth special attention.

1. We show that, unlike the Marshall–Olkin distribution, the joint component lifetime vector with a counting shock arrival process is not necessarily positively associated. Indeed, we illustrate that the component lifetime vector can even be negatively orthant dependent (see Remark 5.6). In the literature, association has been used extensively to study the positive dependence relation among components of a random vector and to develop the product-form lower bounds for its distribution and survival functions (see, for example, Baccelli and Makowski, 1989; and Szekli, 1995). However, we find that the notion of association is not only difficult to verify, but also fails to be valid in many applications. Instead, here we resort to the notions of positive and negative orthant dependence of a random vector. Using majorization of weighted trees as a tool, we are able to provide parametric characterizations of these notions and to study the dependence structure of the joint component lifetimes under weaker and broader conditions.

2. We show that the dependence nature of the joint lifetime vector T is induced by the superposition of two types of dependence; dependence among various components due to simultaneous arrivals and dependence over different time instants introduced by the shock arrival process. Such spatial dependence and temporal dependence can be studied separately and their combined effect characterizes the dependence nature of the performance of the system.

3. We derive the upper and lower bounds of the distribution and survival functions of the joint lifetime vector T. We identify the conditions under which the bounds outperform the product-form bounds and show that the improvement can be significant when the number of components is large or individual component failures rates are high.

4. We show that, of the shock loading vectors with identical marginal distributions, the shock loading vector that is both NUOD and PLOD is the most desirable among the four possible combinations PUOD and PLOD, PUOD and NLOD, PLOD and NUOD, and NUOD and NLOD, in the sense that it stochastically prolongs the times until the first and last components fail, whereas the shock loading vector that is both

PUOD and NLOD is the least desirable among the four combinations, in the sense that it stochastically shortens the times until the first and last components fail. Loosely speaking a NUOD (PUOD) shock loading vector is less (more) likely to destroy at least all the components in set K, for any $K \subseteq E$, than its independent counterpart does, and hence the lifetime until the last component fails with the NUOD (PUOD) shock loading vector is more likely to be larger (smaller) than that with the independent shock loading vector. Similarly, a PLOD (NLOD) shock loading vector is less (more) likely to destroy at most those components in set K, for any $K \subseteq E$, than its independent counterpart does, and hence the time until the first component fails with the PLOD (NLOD) shock loading vector is more likely to be larger (smaller) than that with the independent shock loading vector. It is a common perception that positive correlation can enhance the system performance. However, our result demonstrates that the system performance can benefit from positive as well as negative dependence.

Some previous study on shock models with simultaneous component failures can be found in Shaked and Shanthikumar (1986) and the references therein. Olkin and Tong (1994) and Shaked and Shanthikumar (1997) studied some classes of multivariate distributions arising from random vectors with common values. Shaked and Shanthikumar (1997) also obtained some dependence comparison results using the supermodular orders (also see Tchen, 1980). Li and Zhu (1994) discussed a similar fatal shock model where the shock arrival processes are independent renewal processes with NBU (or NBUE) interarrival times. They obtained some computable bounds for $(T_1, ..., T_s)$ using the increasing concave ordering method. In contrast, the bounding methodology used in this paper is the orthant dependence ordering with fixed one-dimensional marginals. This enables us to obtain the tighter bounds under more general assumptions.

The organization of this paper is as follows. Section 2 summarizes some preliminaries about orthant dependence comparisons and tree majorizations. Section 3 discusses the dependence structures of shock loading vectors and shock arrival processes. Section 4 considers the impact of the two types of dependence on the performance of the component lifetime vector. Section 5 derives several bounds for the statistics of the component lifetime vector. The shock model with non-homogeneous Poisson shock arrivals is also studied there as an illustrative example. Section 6 discusses several generalizations of our model.

Throughout this paper, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing", respectively, and the existence of expectations is assumed without explicit mention. The inequalities in two separate cases such as $A > a$ $(A \leqslant a)$ are always written in the compact form $A > (\leqslant)\, a$.

2. PRELIMINARIES

In this section we first review the definitions of various dependence orders and related properties. We then introduce the notion of *majorization of weighted trees* developed by Xu and Li (1998).

2.1. *Notions of Dependence*

Many different notions of dependence have been introduced and studied extensively in the literature (see, for example, Tong, 1980; Shaked and Shanthikumar, 1994; and Szekli 1995), but the following concepts of dependence are most relevant to this research.

DEFINITION 2.1. Let $\mathbf{X}=(X_1, ..., X_s)$ and $\mathbf{Y}=(Y_1, ..., Y_s)$ be two random vectors.

1. $\mathbf{X}$ is said to be positively upper (lower) orthant dependent (PUOD (PLOD)), if for any $\mathbf{x}=(x_1, ..., x_s)\in\mathscr{R}^s$, $P(\mathbf{X}>(\leqslant)\,\mathbf{x})\geqslant\prod_{j=1}^{s} P(X_j>(\leqslant)\,x_j)$. $\mathbf{X}$ is said to be negatively upper (lower) orthant dependent (NUOD, (NLOD)), if for any $\mathbf{x}\in\mathscr{R}^s$, $P(\mathbf{X}>(\leqslant)\,\mathbf{x})\leqslant\prod_{j=1}^{s} P(X_j>(\leqslant)\,x_j)$.
2. $\mathbf{X}$ is said to be larger (smaller) than $\mathbf{Y}$ in the upper (lower) orthant order, denoted $\mathbf{X}\geqslant_{\text{uo}}(\leqslant_{\text{lo}})\,\mathbf{Y}$, if $P(\mathbf{X}>(\leqslant)\,\mathbf{x})\geqslant P(\mathbf{Y}>(\leqslant)\,\mathbf{x})$, for all $\mathbf{x}\in\mathscr{R}^s$. If, in addition, X_j and Y_j have the same distribution for each $j\in E$ (denoted as $X_j=_{\text{st}}Y_j$ in the following), then $\mathbf{X}$ is said to be more positively upper (lower) orthant dependent than $\mathbf{Y}$.
3. $\mathbf{X}$ is said to be associated if $\operatorname{Cov}(f(\mathbf{X}), g(\mathbf{X}))\geqslant 0$ whenever f and g are increasing. $\mathbf{X}$ is said to be negatively associated if for every subset $K\subseteq\{1, ..., s\}$, $\operatorname{Cov}(f(X_i, i\in K), g(X_j, j\in K^c))\leqslant 0$ whenever f and g are increasing.
4. $\mathbf{X}$ is said to be larger than $\mathbf{Y}$ in the usual stochastic order, denoted as $\mathbf{X}\geqslant_{\text{st}}\mathbf{Y}$, if $P(\mathbf{X}\in U)\geqslant P(\mathbf{Y}\in U)$ for all upper sets $U\subseteq\mathscr{R}^s$ (U is said to be upper if $\mathbf{x}\in U$ and $\mathbf{x}\leqslant\mathbf{y}$ implies that $\mathbf{y}\in U$).

The following facts are easy to verify (see, for example, Tong, 1980, and Szekli, 1995):

$$\begin{aligned}\mathbf{X}\text{ is associated} &\Rightarrow \mathbf{X}\text{ is PUOD and PLOD}\\ &\Rightarrow \operatorname{Cov}(X_i, X_j)\geqslant \mathbf{0}, \qquad \text{for all } i, j, \qquad (2.1)\end{aligned}$$

$$\begin{aligned}\mathbf{X}\text{ is negatively associated} &\Rightarrow \mathbf{X}\text{ is NUOD and NLOD}\\ &\Rightarrow \operatorname{Cov}(X_i, X_j)\leqslant \mathbf{0}, \qquad \text{for all } i, j, \qquad (2.2)\end{aligned}$$

$$\mathbf{X}\geqslant_{\text{st}}\mathbf{Y}\Rightarrow\mathbf{X}\geqslant_{\text{uo}}\mathbf{Y} \qquad\text{and}\qquad \mathbf{X}\geqslant_{\text{lo}}\mathbf{Y}. \qquad (2.3)$$

The above notions, in various stochastic senses, express either the different degrees of dependence among the components of a random vector or between two random vectors. In general, we found that Definition 2.1, (1), (2), are the most useful notions in characterizing the dependence behavior of component lifetime vectors. This is because pairwise correlation is too weak to describe the dependence nature of the multivariate failure process and is unable to generate satisfactory bounds for the distribution and survival functions of the joint components lifetimes, yet the usual multivariate stochastic order and associations are too strong to be valid.

Remark 2.2. The PLOD (PUOD, NLOD, NUOD) property of a random vector means that its joint distribution or survival function can be bounded below or above by the products of its marginal distributions or survival functions. The ordering $\mathbf{X} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{Y}$, coupled with $X_j =_{\text{st}} Y_j$, $j = 1, ..., s$, emphasizes dependence comparisons of two random vectors by separating the marginals from consideration. For the literature studying negative dependence, see Joag-Dev and Proschan (1983) and Block *et al.* (1982), among others.

Some properties regarding orthant dependence comparisons are summarized below and will be used in Sections 3–5.

LEMMA 2.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two nonnegative n-dimensional random vectors.*

1. *If* $\mathbf{X} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{Y}$, *then* $\{X_j, j \in K\} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \{Y_j, j \in K\}$, *for any* $K \subseteq \{1, 2, ..., n\}$.
2. *If* $\mathbf{X} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{Y}$ *and if* f_j *is an increasing function,* $j = 1, ..., n$, *then*

$$(f_1(X_1), ..., f_n(X_n)) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) (f_1(Y_1), ..., f_n(Y_n)).$$

3. *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be another two n-dimensional nonnegative random vectors such that* $\mathbf{X} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{Y}$ *and* $\mathbf{U} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{V}$. *In addition,* $\mathbf{X}$ *and* $\mathbf{U}$ *are independent and* $\mathbf{Y}$ *and* $\mathbf{V}$ *are independent. Let* $f_j \colon \mathscr{R}_+^2 \to \mathscr{R}_+$ *be an increasing function,* $j = 1, 2, ..., n$. *Then*

$$(f_1(X_1, U_1), ..., f_n(X_n, U_n)) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) (f_1(Y_1, V_1), ..., f_n(Y_n, V_n)).$$

4. $\mathbf{X} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \mathbf{Y}$ *if and only if* $E[\prod_{j=1}^n f_j(X_j)] \geqslant E[\prod_{j=1}^n f_j(Y_j)]$, *for every collection* $\{f_1, ..., f_n\}$ *of univariate nonnegative increasing (decreasing) functions.*

Proof. Note that (1), (2), and (4) can be found in Shaked and Shanthikumar (1994), while (3) can be obtained from (2) and the standard conditioning arguments. ∎

2.2. *Majorization of Weighted Trees*

In order to compare the characteristics of two different yet similar systems, a common approach is to first establish an ordering between the two parameter sets of the two compared systems and then, according to the system dynamics, to show that the two system performance measures possess a certain stochastic ordering relation (Stoyan, 1983; Chang and Yao, 1994). Therefore, in order to compare dependence structures of two life length vectors given in (1.1), we need first to compare the distributions of the corresponding probability masses P^K, $K \subseteq E$, over a partially ordered index set $S(E) = \{K: K \subseteq E\}$ (define the partial order $L < K$ if $L \subseteq K$). To handle such situations, Xu and Li (1998) introduced the notion of majorization with respect to weighted trees. We briefly review their notion below.

Consider a set of parameter values $\Lambda = \{\lambda^K, K \subseteq E\}$ with a partially ordered index set $S(E)$, where $S(E)$ is the collection of all subsets of $E = \{1, ..., s\}$, including the empty set $\emptyset$. We treat each subset of E as a node. For two nodes J and K, if $J \subseteq K$, we say that J is a descendant of K and K is an ancestor of J. A node may have several immediate descendants and ancestors, and the root E (node $\emptyset$) is the ancestor (descendant) of everyone. For each node K, we assign a real number λ^K and call it the *weight* of K. As such, $(S(E), \Lambda)$, or simply Λ, can be thought of as an $(s+1)$-*generation weighted family tree*. (We count $\emptyset$ as a generation.) Note that for each node K, $K \subseteq E$, the set of all its ancestors (descendants) generates a subtree. For convenience, let a node be its own descendant and ancestor.

Definition 2.4 (Xu and Li, 1998). Let Λ and $\bar{\Lambda}$ be two $(s+1)$-generation weighted trees on the same index set $S(E)$, with the total weights $\lambda = \sum_{K \subseteq E} \lambda^K$ and $\bar{\lambda} = \sum_{K \subseteq E} \bar{\lambda}^K$, respectively. Λ is said to *majorize* $\bar{\Lambda}$ from *roots* (*leaves*), denoted by $\Lambda \geqslant_{\mathcal{T}_r} \bar{\Lambda}$ ($\Lambda \geqslant_{\mathcal{T}_l} \bar{\Lambda}$), if $\lambda = \bar{\lambda}$ and

$$\sum_{K \subseteq L} \lambda^L \geqslant \sum_{K \subseteq L} \bar{\lambda}^L, \qquad K \subseteq E. \tag{2.4}$$

$$\left(\sum_{L \subseteq K} \lambda^L \geqslant \sum_{L \subseteq K} \bar{\lambda}^L. \right) \tag{2.5}$$

In words, Definition 2.4 states that $\bar{\Lambda}$ is majorized by Λ from roots (leaves) if the total weights of the two trees, λ and $\bar{\lambda}$, are the same and, for each node K, $K \subseteq E$, the total weight of its ancestors (descendants) in $\bar{\Lambda}$ is less than its counterpart in Λ. Therefore, if Λ majorizes $\bar{\Lambda}$ from roots (leaves) then the value of Λ is more concentrated around the nodes in the earlier (later) generations than $\bar{\Lambda}$ is.

Here and in the following, the scalar multiplication ($\gamma\Lambda$) and the summation ($\Lambda_1+\Lambda_2$) of trees defined on the same index set $S(E)$ are operated node-wise.

Note that a weighted tree as defined above is rather general. In particular, the weight of a node can be negative. In this paper, we shall mainly consider the *probability tree* where each of its components is between 0 and 1 and the total weight of the tree equals 1.

3. DEPENDENCE STRUCTURES OF SHOCK LOADING VECTORS AND SHOCK ARRIVAL PROCESSES

We first describe the model and introduce the notation used throughout the paper. Let $\{N(t), t\geqslant 0\}$ be a counting process. A shock arrives at the system (call it the system $\mathscr{S}$) according to $\{N(t), t\geqslant 0\}$ and destroys simultaneously, with probability P^K, all the components $j\in K\subseteq E$ that are still alive but all of the other components $j\in E-K$ that are still alive survive (type-K shock). For each $K\in S(E)$, let $N^K(t)$ be the number of type-K shocks received by time t. Clearly, $\{N^K(t), t\geqslant 0\}$ is a thinning of the counting process $\{N(t), t\geqslant 0\}$ with thinning probability P^K. Define the probability tree

$$\Lambda=\left\{\lambda^K=P^K, 0\leqslant P^K\leqslant 1, \sum_{K\subseteq E} P^K=1\right\}. \tag{3.1}$$

Let $P_j=\sum_{j\in L} P^L$ be the cumulative weight of all ancestors of node j. For $j\in E$, let $N_j(t)=\sum_{j\in K} N^K(t)$ be the number of shocks component j received by time t. Again, $\{N_j(t), t\geqslant 0\}$ is a thinning of $\{N(t), t\geqslant 0\}$, with thinning probability P_j, $j\in E$. In general, both multivariate processes $\{N^K(t), t\geqslant 0 \mid K\subseteq E\}$ and $\{N_j(t), t\geqslant 0 \mid j\in E\}$ are correlated.

Let $\delta_{n,j}$ be 1 if the nth shock destroys the component j and zero otherwise, then the vector $\delta_n=\{\delta_{n,j} \mid j\in E\}$ is the set of components in system $\mathscr{S}$ destroyed by the nth shock if they have not been destroyed already by previous shocks. We shall call δ_n the shock loading vector. We have

$$P(\delta_n=\mathbf{e}^K)=P^K, \qquad K\subseteq E, \tag{3.2}$$

where $\mathbf{e}^K$ is the s-dimensional vector with its jth element, $j\in K$, being 1 and the others zero, $K\subseteq E$. Observe that Λ may be regarded as the joint probability mass function of the indicator random vector δ_n. As such, $\delta_{n,j}$ is the indicator (Bernoulli) variable of a type-j shock, with probability P_j, $j\in E$.

Clearly, δ_n, $n=1, 2, ...,$ are independent and identically distributed (i.i.d.) random vectors. δ_n's are also independent of $\{N(t), t \geqslant 0\}$.

Let $\{\bar{N}(t), t \geqslant 0\}$ be another counting process governing shock arrival streams and let $\bar{\delta}_n$ be the shock loading vector of the nth shock defined by the probability tree $\bar{\Lambda} = \{\bar{\lambda}^K = \bar{P}^K, 0 \leqslant \bar{P}^K \leqslant 1, \sum_{K \subseteq E} \bar{P}^K = 1\}$, where

$$P(\bar{\delta}_n = \mathbf{e}^K) = \bar{P}^K, \qquad K \subseteq E. \tag{3.3}$$

We call the system with shock arrival process $\{\bar{N}(t), t \geqslant 0\}$ and probability tree $\bar{\Lambda}$ the system $\bar{\mathscr{S}}$. We use δ $(\bar{\delta})$ to represent the generic version of δ_n $(\bar{\delta}_n)$.

The remainder of this section studies the dependence structures of δ and the shock arrival process $\{N(t), t \geqslant 0\}$. Let $\delta = \{\delta_1, ..., \delta_s\}$ be a vector of Bernoulli random variables with marginal weights P_j, $j \in E$. Let $\delta^I = \{\delta_1^I, ..., \delta_s^I\}$ be a vector of independent random variables that have the same one-dimensional marginal distribution as that of δ. Let Λ^I be the probability tree corresponding to δ^I. Then

$$P(\delta^I = \mathbf{e}^K) = \prod_{j \in K} P^j \prod_{j \in E-K} (1 - P_j), \qquad K \subseteq E. \tag{3.4}$$

Next we present a lemma. It states that the root and leaf majorization orders between two probability trees are the parametric characterizations of the upper and lower orthant orders between the shock-loading vectors.

Lemma 3.1. *Let δ and $\bar{\delta}$ be the shock loading vectors corresponding to probability trees Λ and $\bar{\Lambda}$, respectively. Then*

1. $\Lambda \geqslant_{\mathscr{T}_r (\mathscr{T}_l)} \bar{\Lambda}$ *if and only if* $\delta \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \bar{\delta}$.

2. $\Lambda \geqslant_{\mathscr{T}_r (\mathscr{T}_l)} \bar{\Lambda}$ *and* $P_j = \bar{P}_j$, $j \in E$, *if and only if* δ *is more positively upper (lower) orthant dependent than* $\bar{\delta}$.

3. δ *is PUOD (PLOD, NUOD, NLOD) if and only if* $\Lambda \geqslant_{\mathscr{T}_r} (\geqslant_{\mathscr{T}_l}, \leqslant_{\mathscr{T}_r}, \leqslant_{\mathscr{T}_l}) \Lambda^I$, *where* Λ^I *is the probability tree of* δ^I *given in* (3.4).

Proof. We observe that

$$P(\delta \geqslant \mathbf{e}^K) - P(\bar{\delta} \geqslant \mathbf{e}^K) = \sum_{K \subseteq L} P^L - \sum_{K \subseteq L} \bar{P}^L, \tag{3.5}$$

$$P(\delta \leqslant \mathbf{e}^K) - P(\bar{\delta} \leqslant \mathbf{e}^K) = \sum_{L \subseteq K} P^L - \sum_{L \subseteq K} \bar{P}^L. \tag{3.6}$$

Then (1) is the immediate consequence of Definition 2.4. To establish (2), we note, from (3.2) that

$$P_j - \bar{P}_j = P(\delta_j = 1) - P(\bar{\delta}_j = 1) = \sum_{j \in L} P^L - \sum_{j \in L} \bar{P}^L, \qquad j \in E.$$

Since δ and $\bar{\delta}$ are binary random vectors, this implies that they have the same one-dimensional distribution if and only if $P_j = \bar{P}_j$, $j \in E$. This, together with (1), implies (2). Finally, (3) follows from Lemma 3.1(2) and Definition 2.1(1). ∎

Remark 3.2. If bivariate random vectors **X** and **Y** have identical marginal distributions, $X_j =_{\mathrm{st}} Y_j$, $j = 1, 2$, then $\mathbf{X} \geqslant_{\mathrm{uo}} \mathbf{Y}$ implies $\mathbf{X} \leqslant_{\mathrm{lo}} \mathbf{Y}$ and vice versa (see, for example, Marshall and Olkin, 1979). It is easy to see that for the $s+1=3$ generation trees, if $P_j = \bar{P}_j$, $j = 1, 2$,

$$\Lambda \geqslant_{\mathcal{T}_r} \bar{\Lambda} \Leftrightarrow \Lambda \geqslant_{\mathcal{T}_l} \bar{\Lambda} \qquad \text{and}$$
$$(\delta_1, \delta_2) \geqslant_{\mathrm{uo}} (\bar{\delta}_1, \bar{\delta}_2) \Leftrightarrow (\delta_1, \delta_2) \leqslant_{\mathrm{lo}} (\bar{\delta}_1, \bar{\delta}_2).$$

Example 3.3. Suppose the distribution of δ is given by

$$\Lambda = m\beta\varepsilon^E + \sum_{j=1}^{s} \beta\varepsilon^j + \gamma\varepsilon^{\varnothing},$$

where ε^K, $K \subseteq E$, denotes the $(s+1)$-generation weighted tree with the weight on node K being one and zero otherwise. Lemma 3.1(3) states that δ is PUOD if and only if $\Lambda \geqslant_{\mathcal{T}_r} \Lambda^I$. Clearly, to ensure $\Lambda \geqslant_{\mathcal{T}_r} \Lambda^I$, we need that for $2 \leqslant k \leqslant s$, $m\beta \geqslant [(m+1)\beta]^k$, which holds if $m\beta \geqslant [(m+1)\beta]^2$, for $s \geqslant 2$. Since this condition is equivalent to $m + 2 + 1/m \leqslant 1/\beta$, together with $s\beta + m\beta + \gamma = 1$, we have that δ is PUOD if $m \geqslant 1/(s-2)$.

Observe that when m is large (e.g., $m \geqslant 1/(s-2)$), Λ has a heavy weight on the root node E; this subsequently ensures that Λ majorizes Λ^I from roots. If we "rotate" Λ by 180°, we obtain (we still use Λ to denote the rotated tree)

$$\Lambda = \gamma\varepsilon^E + \sum_{j=1}^{s} \beta\varepsilon^{E-j} + m\beta\varepsilon^{\varnothing}.$$

Using an argument similar to that made before, we have that δ is PLOD if $m \geqslant 1/(s-2)$.

Example 3.4. Consider the *uniform tree*, where the unit weight of the tree is evenly distributed among its 2^s nodes: $P^K = 1/2^s$, $K \subseteq E$. It follows from (3.4) that the uniform tree corresponds to the i.i.d. Bernoulli random variables with parameter 1/2. Now consider the following $s+1=4$ generation-weighted trees:

$$\Lambda=\tfrac{1}{4}(\varepsilon^{123}+\varepsilon^{1}+\varepsilon^{2}+\varepsilon^{3}),\qquad \Lambda^{I}=\tfrac{1}{8}\sum_{K\subseteq\{1,2,3\}}\varepsilon^{K},\qquad\text{and}$$

$$\bar{\Lambda}=\tfrac{1}{4}(\varepsilon^{12}+\varepsilon^{13}+\varepsilon^{23}+\varepsilon^{\varnothing}).$$

We have $P_j=P_j^I=\bar{P}_j=1/2$. Clearly, $\Lambda\geqslant_{\mathcal{T}_r}\Lambda^I\geqslant_{\mathcal{T}_r}\bar{\Lambda}$, but $\Lambda\leqslant_{\mathcal{T}_l}\Lambda^I\leqslant_{\mathcal{T}_l}\bar{\Lambda}$. Therefore, by Lemma 3.1(3),

$$P(\delta\geqslant\mathbf{e}^K)\geqslant\prod_{j\in K}P_j\geqslant P(\bar{\delta}\geqslant\mathbf{e}^K),\qquad\text{and}$$

$$P(\delta\leqslant\mathbf{e}^K)\leqslant\prod_{j\in E-K}(1-P_j)\leqslant P(\bar{\delta}\leqslant\mathbf{e}^K).$$

That is, δ is PUOD and NLOD, and $\bar{\delta}$ is NUOD and PLOD.

We close this section by discussing briefly the autocorrelation structure of shock arrival processes. The following notions of dependence over time can be thought of as generalizations of Definition 2.1(1, 2) to stochastic processes.

DEFINITION 3.5. Let $\{X(t), t\geqslant 0\}$ and $\{Y(t), t\geqslant 0\}$ be two real-valued stochastic processes.

1. $\{X(t), t\geqslant 0\}$ is said to be PUOD (PLOD, NUOD, NLOD) *in time* if for any set $\{t_1, t_2, ..., t_n\}$, the random vector $\{X(t_1), X(t_2), ..., X(t_n)\}$ is PUOD (PLOD, NUOD, NLOD).

2. $\{X(t), t\geqslant 0\}$ is said to be larger (smaller) than $\{Y(t), t\geqslant 0\}$ in the upper (lower) orthant order, denoted as $\{X(t), t\geqslant 0\}\geqslant_{\text{uo}}(\leqslant_{\text{lo}})$ $\{Y(t), t\geqslant 0\}$, if for any choice of $\{t_1, t_2, ..., t_n\}$, $(X(t_1), ..., X(t_n))\geqslant_{\text{uo}}(\leqslant_{\text{lo}})$ $(Y(t_1), ..., Y(t_n))$.

Esary and Proschan (1970) introduced the notion of *time association* for stochastic processes (also see Lindqvist, 1988) in order to obtain bounds for the reliability of certain systems with dependent components. A real-valued stochastic process $\{X(t), t\geqslant 0\}$ is said to be associated in time, if for any set $\{t_1, t_2, ..., t_n\}$, $(X(t_1), X(t_2), ..., X(t_n))$ is associated. Obviously (see (2.1)), association in time implies both PUOD in time and PLOD in time. However, the reverse is not true, in general.

It is worth mentioning that if $\{N(t), t\geqslant 0\}\geqslant_{\text{st}}\{\bar{N}(t), t\geqslant 0\}$, that is, $E\phi(\{N(t), t\geqslant 0\})\geqslant E\phi(\{\bar{N}(t), t\geqslant 0\})$ for all real increasing functions ϕ, then $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}}(\geqslant_{\text{lo}})\{\bar{N}(t), t \geqslant 0\}$ (see, for example, Shaked and Shanthikumar, 1994). The reader is referred to Shaked and Szekli (1995) for other notions of stochastic comparisons of processes and their applications.

4. DEPENDENCE COMPARISONS OF JOINT LIFETIMES OF COMPONENTS

We now turn our attention to the dependence structure of the joint component lifetime vector T. We show that the spatial dependence and temporal dependence introduced by $\{P^K, K \subseteq E\}$ and $\{N(t), t \geqslant 0\}$, respectively, determine the dependence nature of T.

Let $\mathscr{S}$ ($\bar{\mathscr{S}}$) be the shock model with shock arrival process $\{N(t), t \geqslant 0\}$ ($\{\bar{N}(t), t \geqslant 0\}$) and the shock loading vector δ ($\bar{\delta}$). Let Λ ($\bar{\Lambda}$) be the probability tree associated with δ ($\bar{\delta}$). The following theorem states that if $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}}(\leqslant_{\text{lo}})\{\bar{N}(t), t \geqslant 0\}$ and $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$, then the numbers of shocks received by components up to times $\mathbf{t} = (t_1, ..., t_s)$ in the two systems preserve the orthant dependence order.

THEOREM 4.1. *If* $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ *and* $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}}(\leqslant_{\text{lo}})\{\bar{N}(t), t \geqslant 0\}$, *then for any* $(t_1, ..., t_s)$,

$$(N_1(t_1), ..., N_s(t_s)) \geqslant_{\text{uo}}(\leqslant_{\text{lo}})(\bar{N}_1(t_1), ..., \bar{N}_s(t_s)).$$

If, in addition, $N(t) =_{\text{st}} \bar{N}(t)$, *for each fixed* $t \geqslant 0$, *and* $P_j = \bar{P}_j$, $j \in E$, *then* $(N_1(t_1), ..., N_s(t_s))$ *is more positively upper (lower) orthant dependent than* $(\bar{N}_1(t_1), ..., \bar{N}_s(t_s))$.

Proof. Recall that $N_j(t)$ ($\bar{N}_j(t)$) is the number of shocks received by the component j in the system $\mathscr{S}$ ($\bar{\mathscr{S}}$) before time t. Without loss of generality, we now assume that $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$. Since $N(t)$ is increasing almost surely, we have, for $j = 1, ..., s$,

$$N_j(t_j) = \sum_{n=1}^{N(t_j)} \delta_{n,j} = \sum_{k=1}^{j} \sum_{n=N(t_{k-1})+1}^{N(t_k)} \delta_{n,j}, \tag{4.1}$$

where $N(t_0) = 0$ and $\sum_{n=l}^{k} \delta_n$ is understood as 0 if $k < l$. Therefore,

$$\begin{aligned}(N_1(t_1), N_2(t_2), ..., N_s(t_s)) \\ = \sum_{n=1}^{N(t_1)} \delta_n + v^1 \sum_{n=N(t_1)+1}^{N(t_2)} \delta_n + \cdots + v^{s-1} \sum_{n=N(t_{s-1})+1}^{N(t_s)} \delta_n,\end{aligned} \tag{4.2}$$

where v^j is the s-dimensional vector with the first j elements being zeros and the others 1. Here and in the following, a product of two vectors is understood as the element-wise product. Similarly, for system $\bar{\mathscr{S}}$, we have

$$(\bar{N}_1(t_1), \bar{N}_2(t_2), ..., \bar{N}_s(t_s)) = \sum_{n=1}^{\bar{N}(t_1)} \bar{\delta}_n + v^1 \sum_{n=\bar{N}(t_1)+1}^{\bar{N}(t_2)} \bar{\delta}_n + \cdots + v^{s-1} \sum_{n=\bar{N}(t_{s-1})+1}^{\bar{N}(t_s)} \bar{\delta}_n. \tag{4.3}$$

From Lemma 3.1(1), $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ implies that $\delta_n \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \bar{\delta}_n$. Thus, by Lemma 2.3,

$$\sum_{n=k}^{l} \delta_n \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \sum_{n=k}^{l} \bar{\delta}_n, \qquad k \leqslant l.$$

Because δ_n ($\bar{\delta}_n$) are i.i.d. random vectors, from Lemma 2.3 we obtain, for any $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$, that

$$\sum_{n=1}^{k_1} \delta_n + v^1 \sum_{n=k_1+1}^{k_2} \delta_n + \cdots + v^{s-1} \sum_{n=k_{s-1}+1}^{k_s} \delta_n \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \sum_{n=1}^{k_1} \bar{\delta}_n + v^1 \sum_{n=k_1+1}^{k_2} \bar{\delta}_n + \cdots + v^{s-1} \sum_{n=k_{s-1}+1}^{k_s} \bar{\delta}_n. \tag{4.4}$$

From (4.3), (4.4), and the monotone sample path property of $\bar{N}(t)$, we obtain that $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ implies that, for any $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$,

$$\left(\sum_{n=1}^{\bar{N}(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{\bar{N}(t_s)} \delta_{n,s} \right) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) (\bar{N}_1(t_1), \bar{N}_2(t_2), ..., \bar{N}_s(t_s)). \tag{4.5}$$

Since $(N(t_1), N(t_2), ..., N(t_s)) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) (\bar{N}(t_1), \bar{N}(t_2), ..., \bar{N}(t_s))$, by conditioning on δ_n, we have

$$\left(\sum_{n=1}^{N(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{N(t_s)} \delta_{n,s} \right) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \left(\sum_{n=1}^{\bar{N}(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{\bar{N}(t_s)} \delta_{n,s} \right). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain that

$$(N_1(t_1), N_2(t_2), ..., N_s(t_s)) \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) (\bar{N}_1(t_1), \bar{N}_2(t_2), ..., \bar{N}_s(t_s)).$$

The last claim of the theorem follows from Lemma 3.1 and the fact that if $\delta_{n,j} =_{\text{st}} \bar{\delta}_{n,j}$ and $N(t_j) =_{\text{st}} \bar{N}(t_j)$ then $N_j(t_j) =_{\text{st}} \bar{N}_j(t_j)$. ∎

Our main result of this section is a direct consequence of Theorem 4.1. Loosely speaking, the theorem states that if each arrival shock is more likely to destroy a large (small) set of components simultaneously, and the numbers of shocks received over time instants $\{t_1, t_2, ..., t_n\}$ are more dependent in some sense, then its components are more likely to fail (survive) jointly. Let $\{N(t), t \geqslant 0\}$ and $\{\bar{N}(t), t \geqslant 0\}$ be two counting processes governing the shock arrival processes of $\mathscr{S}$ and $\bar{\mathscr{S}}$, respectively. Let $T = (T_1, ..., T_s)$ and $\bar{T} = (\bar{T}_1, ..., \bar{T}_s)$ be the corresponding joint lifetimes of the components.

THEOREM 4.2. *If* $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ *and* $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \{\bar{N}(t), t \geqslant 0\}$, *then*

$$T \leqslant_{\text{lo}} (\geqslant_{\text{uo}})\, \bar{T}.$$

If, in addition, $N(t) =_{\text{st}} \bar{N}(t)$, *for each fixed* $t \geqslant 0$ *and* $P_j = \bar{P}_j$, $j \in E$, *then* T *is more positively lower (upper) orthant dependent than* $\bar{T}$.

Proof. First we consider the lower orthant case. Clearly,

$$\begin{aligned} T_j &= \inf\{t \geqslant 0 \mid N_j(t) \geqslant 1\}, \\ \bar{T}_j &= \inf\{t \geqslant 0 \mid \bar{N}_j(t) \geqslant 1\}, \qquad j = 1, ..., s. \end{aligned} \tag{4.7}$$

Since $N_j(t)$ is increasing almost surely, we have $T_j \leqslant t_j$ if and only if $N_j(t_j) \geqslant 1$, for $j = 1, ..., s$. From this, we obtain that for any $t_1 \geqslant 0, ..., t_s \geqslant 0$,

$$P(T_1 \leqslant t_1, ..., T_s \leqslant t_s) = P(N_1(t_1) \geqslant 1, ..., N_s(t_s) \geqslant 1).$$

Similarly,

$$P(\bar{T}_1 \leqslant t_1, ..., \bar{T}_s \leqslant t_s) = P(\bar{N}_1(t_1) \geqslant 1, ..., \bar{N}_s(t_s) \geqslant 1).$$

Thus, from Theorem 4.1, $\Lambda \geqslant_{\mathscr{T}_r} \bar{\Lambda}$ and $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}} \{\bar{N}(t), t \geqslant 0\}$ imply that $T \leqslant_{\text{lo}} \bar{T}$.

Next we consider the upper orthant case. We observe that

$$P(T_1 > t_1, ..., T_s > t_s) = P(N_1(t_1) = 0, ..., N_s(t_s) = 0),$$

and

$$P(\bar{T}_1 > t_1, ..., \bar{T}_s > t_s) = P(\bar{N}_1(t_1) = 0, ..., \bar{N}_s(t_s) = 0).$$

Again, from Theorem 4.1, $\Lambda \geqslant_{\mathscr{T}_l} \bar{\Lambda}$ and $\{N(t), t \geqslant 0\} \leqslant_{\text{lo}} \{\bar{N}(t), t \geqslant 0\}$ imply that $T \geqslant_{\text{uo}} \bar{T}$.

The second claim of the theorem is a direct consequence of the result on dependence comparison obtained in Theorem 4.1. ∎

Remark 4.3. If $N(t)$ is a thinning of $\bar{N}(t)$ (or vice versa) with thinning probability α, then we can normalize the two arrival processes as follows. Let Λ and $\bar{\Lambda}$ be the corresponding probability trees. Let both systems have the common arrival process $\{\bar{N}(t), t \geqslant 0\}$. Let us modify the probability tree Λ to, say $\tilde{\Lambda}$, where

$$\tilde{P}^K = \alpha P^K, \qquad K \neq \varnothing, \qquad \text{and} \qquad \tilde{P}^\varnothing = (1-\alpha) + \alpha P^\varnothing.$$

Clearly, the system governed by $\{N(t), t \geqslant 0\}$ and Λ is stochastically equivalent to the system governed by $\{\bar{N}(t), t \geqslant 0\}$ and $\tilde{\Lambda}$. The examples that one process can be regarded as a thinning of another include Poisson processes and non-homogeneous Poisson processes.

EXAMPLE 4.4. Let systems $\mathscr{S}$ and $\bar{\mathscr{S}}$ have the same shock arrival process $\{N(t), t \geqslant 0\} =_{\text{st}} \{\bar{N}(t), t \geqslant 0\}$. Let $\Lambda = (P^{12}, P^1, P^2, P^\varnothing) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $\bar{\Lambda} = (\bar{P}^{12}, \bar{P}^1, \bar{P}^2, \bar{P}^\varnothing) = (0, \frac{1}{2}, \frac{1}{2}, 0)$. Thus, $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ and $P_j = \bar{P}_j = \frac{1}{2}$. Therefore, $T \geqslant_{\text{uo}} \bar{T} \geqslant_{\text{lo}} T$. For example, if $\{N(t), t \geqslant 0\}$ is a Poisson process with rate 1, then the joint survival function of T is characterized by the bivariate distribution of the Marshall–Olkin type,

$$P(T_1 > t_1, T_2 > t_2) = e^{-(t_1 + t_2 + \max\{t_1, t_2\})/4}. \tag{4.8}$$

On the other hand, because shocks arrive individually under $\bar{\Lambda}$, $\bar{T}_1$ and $\bar{T}_2$ are independent. Hence

$$P(\bar{T}_1 > t_1, \bar{T}_2 > t_2) = P(\bar{T}_1 > t_1)\, P(\bar{T}_2 > t_2) = e^{-(t_1 + t_2)/2}. \tag{4.9}$$

Evidently, (4.9) bounds (4.8) from below.

The following corollary follows immediately from Theorem 4.2.

COROLLARY 4.5. $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \bar{\Lambda}$ *and* $\{N(t), t \geqslant 0\} \geqslant_{\text{uo}} (\leqslant_{\text{lo}}) \{\bar{N}(t), t \geqslant 0\}$ *imply that*

$$\max\{a_1 T_1, ..., a_s T_s\} \leqslant_{\text{st}} \max\{a_1 \bar{T}_1, ..., a_s \bar{T}_s\}$$

$$(\min\{a_1 T_1, ..., a_s T_s\} \geqslant_{\text{st}} \min\{a_1 \bar{T}_1, ..., a_s \bar{T}_s\}),$$

where $a_j \geqslant 0$, $j = 1, 2, ..., s$.

Remark 4.6. The above corollary has some interesting implications. Let us assume that both of the systems $\mathscr{S}$ and $\bar{\mathscr{S}}$ have identical shock arrival processes. Let the shock loading vectors to $\mathscr{S}$ and $\bar{\mathscr{S}}$ be δ and δ^{I}, respectively, where δ^{I} is the independent counterpart of δ. Let $T_{(1)} = \min\{T_1, ..., T_s\}$ and $T_{(s)} = \max\{T_1, ..., T_s\}$. Then the corollary states that:

$$\delta \text{ is PUOD and PLOD} \Rightarrow T_1 \geqslant_{\mathrm{st}} \bar{T}_{(1)} \text{ and } T_s \leqslant_{\mathrm{st}} \bar{T}_{(s)};$$

$$\delta \text{ is NUOD and NLOD} \Rightarrow T_{(1)} \leqslant_{\mathrm{st}} \bar{T}_{(1)} \text{ and } T_{(s)} \geqslant_{\mathrm{st}} \bar{T}_{(s)};$$

$$\delta \text{ is PUOD and NLOD} \Rightarrow T_{(1)} \leqslant_{\mathrm{st}} \bar{T}_{(1)} \text{ and } T_{(s)} \leqslant_{\mathrm{st}} \bar{T}_{(s)};$$

$$\delta \text{ is NUOD and PLOD} \Rightarrow T_{(1)} \geqslant_{\mathrm{st}} \bar{T}_{(1)} \text{ and } T_{(s)} \geqslant_{\mathrm{st}} \bar{T}_{(s)}.$$

In particular, note that if δ is PUOD and NLOD, then system $\mathscr{S}$ underperforms system $\bar{\mathscr{S}}$ in the sense that it stochastically decreases the first and last component failure times. Note also that if δ is NUOD and PLOD, then system $\mathscr{S}$ outperforms $\bar{\mathscr{S}}$ in the sense that it stochastically increases the first and last component failure times. For example, let Λ, Λ^{I}, and $\bar{\Lambda}$ be defined as in Example 3.4. Let δ, δ^{I}, and $\bar{\delta}$ be their corresponding shock loading vectors. We have shown that δ is PUOD and NLOD and $\bar{\delta}$ is NUOD and PLOD. By Corollary 4.5,

$$T_{(1)} \leqslant_{\mathrm{st}} T^I_{(1)} \leqslant_{\mathrm{st}} \bar{T}_{(1)} \qquad \text{and} \qquad T_{(3)} \leqslant_{\mathrm{st}} T^I_{(3)} \leqslant_{\mathrm{st}} \bar{T}_{(3)}.$$

Thus, with other conditions being equal, system $\mathscr{S}$ underperforms the system with independent loading vector, which subsequently underperforms system $\bar{\mathscr{S}}$, as measured by the lengths of the first and last component failure times. The lesson learned from this example may aid in the decision for the component design and failure process control of reliability systems. ∎

For more examples concerning the class of PUOD and NLOD (NUOD and PLOD) random vectors, the reader is referred to Xu and Li (1998) and Li and Xu (1999), where the authors use the so-called *inclusion–exclusion* transform to systematically generate a sequence of PUOD and NLOD (NUOD and PLOD) random vectors.

5. BOUNDS FOR THE JOINT LIFETIME DISTRIBUTION AND SURVIVAL FUNCTIONS

Using Lemma 3.1 and Theorem 4.2, in this section we derive bounds for the shock model with multiple types of correlated shocks. First, we need the following.

THEOREM 5.1. *Let $\{N(t), t \geqslant 0\}$ be a counting process governing the shock arrivals and let δ be the shock loading vector. Let Λ^{I} be the probability tree corresponding to δ^{I}, the independent counterpart of δ.*

1. *If $\{N(t), t \geqslant 0\}$ is positively upper (lower) orthant dependent in time and $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \Lambda^{\mathrm{I}}$, then $(T_1, ..., T_s)$ is PLOD (PUOD).*

2. *If $\{N(t), t \geqslant 0\}$ is negatively upper (lower) orthant dependent in time and $\Lambda \leqslant_{\mathscr{T}_r(\mathscr{T}_l)} \Lambda^{\mathrm{I}}$, then $(T_1, ..., T_S)$ is NLOD (NUOD).*

Proof. First, we observe that if $(N_1(t_1), N_2(t_2), ..., N_s(t_s))$ is PUOD (PLOD, NUOD, NLOD) for any given $(t_1, ..., t_s)$, then $(T_1, ..., T_s)$ is PLOD (PUOD, NLOD, NUOD). We only need to prove that $(N_1(t_1), N_2(t_2), ..., N_s(t_s))$ is PUOD (PLOD, NUOD, NLOD) under the given conditions. We prove Statement 1 only. Statement 2 can be established similarly.

From Lemma 3.1, δ is PUOD (PLOD) if and only if $\Lambda \geqslant_{\mathscr{T}_r(\mathscr{T}_l)} \Lambda^{\mathrm{I}}$. Using (4.2) and (4.3), and an argument similar to that in Theorem 4.1, we obtain that if $\delta \geqslant_{\mathrm{uo}} (\leqslant_{\mathrm{lo}}) \delta^I$, then

$$(N_1(t_1), N_2(t_2), ..., N_s(t_s)) = \left(\sum_{n=1}^{N(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{N(t_s)} \delta_{n,s} \right)$$
$$\geqslant_{\mathrm{uo}} (\leqslant_{\mathrm{lo}}) \left(\sum_{n=1}^{N(t_1)} \delta^{\mathrm{I}}_{n,1}, ..., \sum_{n=1}^{N(t_s)} \delta^{\mathrm{I}}_{n,s} \right). \quad (5.1)$$

Since for each $j = 1, ..., s$, $\sum_{n=1}^{k_j} \delta^{\mathrm{I}}_{n,j}$ is stochastic increasing (in the usual sense) with respect to k_j, then $P(\sum_{n=1}^{k_j} \delta^{\mathrm{I}}_{n,j} > (\leqslant) x)$ is increasing (decreasing) in k_j. Since $\{N(t), t \geqslant 0\}$ is positively upper (lower) orthant dependent in time, we have that $\{N(t_1), ..., N(t_s)\}$ is PUOD (PLOD) for any $(t_1, ..., t_s)$. Because $\sum_{n=1}^{k_1} \delta^{\mathrm{I}}_{n,1}, ..., \sum_{n=1}^{k_s} \delta^{\mathrm{I}}_{n,s}$ are independent, from Lemma 2.3 we have

$$P\left(\sum_{n=1}^{N(t_1)} \delta^{\mathrm{I}}_{n,1} > (\leqslant) x_1, ..., \sum_{n=1}^{N(t_s)} \delta^{\mathrm{I}}_{n,s} > (\leqslant) x_s \right)$$
$$= \sum_{k_1, ..., k_s} \prod_{j=1}^{s} P\left(\sum_{n=1}^{k_j} \delta^{\mathrm{I}}_{n,j} > (\leqslant) x_j \right) P(N(t_1) = k_1, ..., N(t_s) = k_s)$$
$$\geqslant \prod_{j=1}^{s} \sum_{k_j} P\left(\sum_{n=1}^{k_j} \delta^{\mathrm{I}}_{n,j} > (\leqslant) x_j \right) P(N(t_j) = k_j)$$
$$= \prod_{j=1}^{s} P\left(\sum_{n=1}^{N(t_j)} \delta^{\mathrm{I}}_{n,j} > (\leqslant) x_j \right).$$

We thus obtain that $(\sum_{n=1}^{N(t_1)} \delta^{\mathrm{I}}_{n,1}, ..., \sum_{n=1}^{N(t_s)} \delta^{\mathrm{I}}_{n,s})$ is PUOD (PLOD). Therefore, from (5.1), we have, for any given $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$, that $(N_1(t_1), N_2(t_2), ..., N_s(t_s))$ is PUOD (PLOD). ∎

The proof of the above lemma actually illustrates the following idea: The combination of dependence among the components and dependence over time determines the dependence nature of $(N_1(t_1), N_2(t_2), ..., N_s(t_s))$ for any $(t_1, t_2, ..., t_s)$, which characterizes the dependence structure of the multivariate arrival process.

For any sequence of s-dimensional binary random vectors δ_n and any non-negative integer-valued random vector $\mathbf{N} = \{N_1, ..., N_s\}$, we let $M(\delta, \mathbf{N}) = (M_1(\delta_1, N_1), ..., M_s(\delta_s, N_s))$ where

$$M_j(\delta_j, N_j) = \sum_{n=1}^{N_j} \delta_{n,j}, \qquad j = 1, 2, ..., s. \tag{5.2}$$

For any $\mathbf{t} = (t_1, ..., t_s)$, denote $\mathbf{N}(\mathbf{t}) = (N(t_1), ..., N(t_s))$, where $\{N(t), t \geqslant 0\}$ is a counting process. Let $\mathbf{N}^{\mathbf{I}}(\mathbf{t}) = (N^I(t_1), N^I(t_2), ..., N^I(t_s))$ be an independent version of $\mathbf{N}(\mathbf{t})$, that is, $N^I(t_i) =_{\mathrm{st}} N(t_i)$ for $i = 1, ..., s$ and $N^{\mathrm{I}}(t_1), N^{\mathrm{I}}(t_2), ..., N^{\mathrm{I}}(t_s)$ are independent. Note that $M_j(\delta_j, N(t_j)) = N_j(t_j)$ for $j = 1, ..., s$. Also, denote the lifetimes of the components of the system with shock loading vector δ^{I} and shock arrival process $\{N(t), t \geqslant 0\}$ as

$$T_j(\delta^{\mathrm{I}}, \mathbf{N}) = \inf\{t \geqslant 0 \mid M_j(\delta^{\mathrm{I}}_j, N(t)) \geqslant 1\}, \qquad j = 1, 2, ..., s. \tag{5.3}$$

The following result states that if the system only has spatial dependence, then the joint distribution (or survival) function of T is bounded by that of $T(\delta^{\mathrm{I}}, \mathbf{N}) = (T_1(\delta^{\mathrm{I}}, \mathbf{N}), ..., T_s(\delta^{\mathrm{I}}, \mathbf{N}))$. If the system only has temporal dependence, then the joint distribution (or survival) function of T is bounded by the joint survival (or distribution) function of $M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t}))$.

THEOREM 5.2. 1. *If $\Lambda \geqslant_{\mathcal{T}_l(\mathcal{T}_r)} \Lambda^I$ (i.e., δ is PLOD (PUOD)),*

$$P(T \geqslant \mathbf{t}) \geqslant P(T(\delta^{\mathrm{I}}, \mathbf{N}) \geqslant \mathbf{t}) \qquad (P(T \leqslant \mathbf{t}) \geqslant P(T(\delta^{\mathrm{I}}, \mathbf{N}) \leqslant \mathbf{t})).$$

Also, if $\Lambda \leqslant_{\mathcal{T}_l(\mathcal{T}_r)} \Lambda^{\mathrm{I}}$ (i.e., δ is NLOD (NUOD)),

$$P(T(\delta^{\mathrm{I}}, \mathbf{N}) \geqslant \mathbf{t}) \geqslant P(T \geqslant \mathbf{t}) \qquad (P(T(\delta^{\mathrm{I}}, \mathbf{N}) \leqslant \mathbf{t}) \geqslant P(T \leqslant \mathbf{t})).$$

2. *If the process* $\{N(t), t \geqslant 0\}$ *is PLOD (PUOD) in time,*

$$P(T > \mathbf{t}) \geqslant P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) = \mathbf{0}) \qquad (P(T \leqslant \mathbf{t}) \geqslant P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \geqslant \mathbf{1})).$$

Also, if $\{N(t), t \geqslant 0\}$ *is NLOD (NUOD) in time,*

$$P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) = \mathbf{0}) \geqslant P(T > \mathbf{t}) \qquad (P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \geqslant \mathbf{1}) \geqslant P(T \leqslant \mathbf{t})).$$

Here $\mathbf{0} = (0, ..., 0)$, $\mathbf{1} = (1, ..., 1)$ *with appropriate dimension.*

Proof. Part 1 follows from Theorem 4.2 and we now prove Part 2. If the process $\{N(t), t \geqslant 0\}$ is PLOD (PUOD) in time, then for any $(t_1, ..., t_s)$ we have

$$(N(t_1), ..., N(t_s)) \leqslant_{\mathrm{lo}} (\geqslant_{\mathrm{uo}}) (N^I(t_1), ..., N^I(t_s)).$$

From Lemma 2.3(2), we obtain that

$$\left(\sum_{n=1}^{N(t_1)} i_{n,1}, ..., \sum_{n=1}^{N(t_s)} i_{n,s} \right) \leqslant_{\mathrm{lo}} (\geqslant_{\mathrm{uo}}) \left(\sum_{n=1}^{N^I(t_1)} i_{n,1}, ..., \sum_{n=1}^{N^I(t_s)} i_{n,s} \right),$$

for any non-negative integers $(i_{n,1}, ..., i_{n,s})$. Thus

$$\left(\sum_{n=1}^{N(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{N(t_s)} \delta_{n,s} \right) \leqslant_{\mathrm{lo}} (\geqslant_{\mathrm{uo}}) \left(\sum_{n=1}^{N^I(t_1)} \delta_{n,1}, ..., \sum_{n=1}^{N^I(t_s)} \delta_{n,s} \right). \tag{5.4}$$

The inequalities in (2) now follow from (5.4). ∎

Using Theorems 5.1 and 5.2, we are able to derive the bounds for the distribution and survival functions of $(T_1, ..., T_s)$ via either analyzing one type of dependence individually or analyzing them jointly. To compare bounds obtained in Theorems 5.1 and 5.2, we list the following corollary, whose proof is straightforward.

COROLLARY 5.3. 1. *If* $\Lambda \geqslant_{\mathcal{T}_l(\mathcal{T}_r)} \Lambda^{\mathrm{I}}$ *(i.e.,* δ *is PLOD (PUOD)) and the process* $\{N(t), t \geqslant 0\}$ *is PLOD (PUOD) in time, then*

$$P(T > (\leqslant)\, \mathbf{t}) \geqslant P(T(\delta^{\mathrm{I}}, \mathbf{N}) > (\leqslant)\, \mathbf{t}) \geqslant P(M(\delta^{\mathrm{I}}, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0})$$

$$= \prod_{j \in E} P(T_j > (\leqslant)\, t_j),$$

$$P(T > (\leqslant)\, \mathbf{t}) \geqslant P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0}) \geqslant P(M(\delta^{\mathrm{I}}, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0})$$

$$= \sum_{j \in E} P(T_j > (\leqslant)\, t_j).$$

2. *If* $\Lambda \leqslant_{\mathcal{T}_l(\mathcal{T}_r)} \Lambda^I$ *(i.e.,* δ *is NLOD (NUOD)) and the process* $\{N(t), t \geqslant 0\}$ *is NLOD (NUOD) in time, then*

$$\prod_{j \in E} P(T_j > (\leqslant)\, t_j) = P(M(\delta^{\mathbf{I}}, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0})$$

$$\geqslant P(T(\delta^{\mathbf{I}}, \mathbf{N}) > (\leqslant)\, \mathbf{t}) \geqslant P(T > (\leqslant)\, \mathbf{t}),$$

$$\prod_{j \in E} P(T_j > (\leqslant)\, t_j) = P(M(\delta^{I}, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0})$$

$$\geqslant P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) \leqslant (>)\, \mathbf{0}) \geqslant P(T > (\leqslant)\, \mathbf{t}).$$

The above corollary suggests that it is a worthwhile effort to evaluate the distribution and survival functions of $T(\delta^I, \mathbf{N})$ or $M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t}))$ because they can improve the product-form lower and upper bounds under the conditions specified in Corollary 5.3. To illustrate, we derive the expression of $P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) = \mathbf{0})$.

For any $\mathbf{k} = (k_1, ..., k_s)$, let k_{i_j} be the jth smallest number in $\mathbf{k}$, $j = 1, ..., s$ (arrange tied numbers in a randomly selected order). Then $(i_1, ..., i_s)$ is a permutation of $(1, 2, ..., s)$. Now let $v_{\mathbf{k}}^{\ell}$ be the s-dimensional vector with its elements i_j, $j = 1, 2, ..., \ell$, being zero and others 1. Also denote $v_{\mathbf{k}}^0 = (1, ..., 1)$. For example, suppose $\mathbf{k} = (k_1, k_2, k_3) = (5, 3, 7)$, then $(i_1, i_2, i_3) = (2, 1, 3)$, $v_{\mathbf{k}}^1 = (1, 0, 1)$, and $v_{\mathbf{k}}^2 = (0, 0, 1)$. Then

$$\begin{aligned} &P(M(\delta, \mathbf{N}^{\mathbf{I}}(\mathbf{t})) = \mathbf{0}) \\ &\quad = P\left(\sum_{n=1}^{N^I(t_1)} \delta_{n,1} = 0, ..., \sum_{n=1}^{N^I(t_s)} \delta_{n,s} = 0\right) \\ &\quad = \sum_{\text{all } \mathbf{k}} \prod_{j=1}^{s} \left[P\left(v_{\mathbf{k}}^{j-1} \sum_{n=k_{i_{j-1}}+1}^{k_{i_j}} \delta_n = \mathbf{0}\right) P(N^I(t_{i_j}) = k_{i_j}) \right] \\ &\quad = \sum_{\text{all } \mathbf{k}} \prod_{j=1}^{s} \left[\left(\sum_{L \subseteq \{\varnothing, i_1, ..., i_{j-1}\}} p^L \right)^{k_{i_j} - k_{i_{j-1}}} P(N(t_{i_j}) = k_{i_j}) \right], \end{aligned}$$

where $k_{i_0} = 0$. The evaluation of the above expression is relatively straightforward since it only involves parameter tree Λ and the marginal distribution of $\{N(t), t \geqslant 0\}$.

In general, the evaluation of the joint distribution and survival distributions of $T(\delta^I, \mathbf{N})$ is not an easy task, because one must deal with the joint distribution function of $(N(t_1), N(t_2), ..., N(t_s))$. However, when $\{N(t), t \geqslant 0\}$ has independent increments, the closed form solutions of $P(T(\delta^I, \mathbf{N}) \leqslant \mathbf{t})$ and $P(T(\delta^I, \mathbf{N}) \geqslant \mathbf{t})$ can be derived relatively easily. We illustrate this with the following example.

EXAMPLE 5.4. Let $\{N(t), t \geqslant 0\}$ be a non-homogeneous Poisson process with intensity function $\lambda(t)$ and mean value function $\int_0^t \lambda(x)\,dx = m(t)$, $t \geqslant 0$. Because $\{N(t), t \geqslant 0\}$ has independent increments, then for $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$,

$$P(N(t_1)=k_1, ..., N(t_s)=k_s) = \prod_{j=1}^{s} P(N(t_j) - N(t_{j-1}) = \tilde{k}_j),$$

where $N(t_0)=0$, $k_0=0$, and $\tilde{k}_j = k_j - k_{j-1}$. Clearly, $\{N(t), t \geqslant 0\}$ is associated in time, and so $\{N(t), t \geqslant 0\}$ is positively upper and lower orthant dependent in time. Let us derive the survival function of $T(\delta^I, \mathbf{N})$, for $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s$,

$$\begin{aligned}
&P(T(\delta^{\mathrm{I}}, \mathbf{N}) > \mathbf{t})\\
&\quad = P(N_1^{\mathrm{I}}(t_1)=0, ..., N_s^{\mathrm{I}}(t_s)=0)\\
&\quad = \sum_{k_1 \leqslant \cdots \leqslant k_s} \prod_{j=1}^{s} P\left(\sum_{n=1}^{k_j} \delta^{\mathrm{I}}_{n,j}=0\right) P(N(t_j) - N(t_{j-1}) = k_j - k_{j-1})\\
&\quad = \sum_{\tilde{k}_1=0}^{\infty} \cdots \sum_{\tilde{k}_s=0}^{\infty} \prod_{j=1}^{s} (1-P_j)^{\sum_{i=1}^{j} \tilde{k}_i} e^{-[m(t_j)-m(t_{j-1})]}\\
&\qquad \times \frac{[m(t_j)-m(t_{j-1})]^{\tilde{k}_j}}{\tilde{k}_j!}\\
&\quad = \prod_{j=1}^{s} e^{-[m(t_j)-m(t_{j-1})]} \sum_{\tilde{k}_j=0}^{\infty} \frac{\left[\prod_{i=j}^{s}(1-P_i)(m(t_j)-m(t_{j-1}))\right]^{\tilde{k}_j}}{\tilde{k}_j!}\\
&\quad = e^{-\sum_{j=1}^{s}(1-\prod_{i=j}^{s}(1-P_i))[m(t_j)-m(t_{j-1})]}\\
&\quad = e^{-\sum_{j=1}^{s} P_j \prod_{i=j+1}^{s}(1-P_i)\, m(t_j)},
\end{aligned} \tag{5.5}$$

where $\prod_{i=l}^{k} a_i$ is defined as 1 whenever $l > k$. Let us compare this expression with the product form solution,

$$\prod_{j=1}^{s} P(T_j > t_j) = \prod_{j=1}^{s} P(N_j(t_j)=0) = e^{-\sum_{j=1}^{s} P_j m(t_j)}. \tag{5.6}$$

If $\delta \leqslant_{\mathrm{lo}} \delta^{\mathrm{I}}$, then both (5.5) and (5.6) serve as the lower bounds for the survival function of T. However, since $\prod_{i=j+1}^{s}(1-P_i) \leqslant 1$, the lower bound in (5.5) is always larger than the bound in (5.6). Especially, for large P_j's or large s, the former can be significantly sharper than the latter. Finally, to obtain the joint distribution function of $T(\delta^I, \mathbf{N})$, we have

$$
\begin{aligned}
P(T(\delta^{\mathrm{I}}, \mathbf{N}) \leqslant \mathbf{t})
&= P(N_1^{\mathrm{I}}(t_1) > 0, ..., N_s^{\mathrm{I}}(t_s) > 0) \\
&= 1 - \sum_{j=1}^{s} P(N_j^{\mathrm{I}}(t_j) = 0) + \sum_{i<j} P(N_i^{\mathrm{I}}(t_i) = 0, N_j^{\mathrm{I}}(t_j) = 0) \\
&\quad - \cdots + (-1)^s P(N_1^{\mathrm{I}}(t_1) = 0, ..., N_s^{\mathrm{I}}(t_s) = 0) \\
&= 1 - \sum_{j=1}^{s} e^{-P_j m(t_j)} + \sum_{i<j} e^{-P_i (1-P_j) m(t_i) - P_j m(t_j)} \\
&\quad - \cdots + (-1)^s e^{-\sum_{j=1}^{s} P_j \prod_{i=j+1}^{s} (1-P_i) m(t_j)}.
\end{aligned} \tag{5.7}
$$

In Remark 4.3, we show that if $\{N(t), t \geqslant 0\}$ is a thinning of $\{\bar{N}(t), t \geqslant 0\}$ or vice versa, then one can normalize the two arrival processes by modifying the arrival process and probability tree Λ. The following example shows that for certain shock arrival processes we can use the thinning as a tool to construct bounds of the lifetime vector. Consider an arbitrary renewal process $\{\tilde{N}(t), t \geqslant 0\}$. Since for any t, $\tilde{N}(t)$ is a decreasing functional of the sequence of independent interarrival times, $(\tilde{N}(t_1), ..., \tilde{N}(t_n))$ is associated for any $t_1, ..., t_n$ (see Lindqvist, 1988). Thus any renewal process is associated in time and therefore positively upper and lower orthant dependent in time.

EXAMPLE 5.5. Let system $\mathscr{S}$ be governed by a renewal process $\{N(t), t \geqslant 0\}$ with the generic interarrival time X. Assume $X = \sum_{i=1}^{N} Y_i$, where Y_i's are i.i.d. non-negative random variables, and that N has geometric distribution with parameter α. Let Λ be the probability tree of $\mathscr{S}$, with marginal weight $P_j = \theta$, $j \in E$. Let T be the corresponding lifetime vector of components. We aim at deriving the lower bounds for the distribution and survival functions of T. To this end, we introduce a renewal process $\{\tilde{N}(t), t \geqslant 0\}$ with interarrival times Y_i. We treat $\{N(t), t \geqslant 0\}$ as a thinning of $\{\tilde{N}(t), t \geqslant 0\}$, with thinning probability α. Now introduce the probability tree $\tilde{\Lambda}$, where

$$\tilde{P}^K = \alpha P^K, \qquad \varnothing \neq K \subseteq E, \qquad \text{and} \qquad \tilde{P}^{\varnothing} = 1 - \alpha + \alpha P^{\varnothing}.$$

Note that the marginal weights of $\tilde{\Lambda}$ are $\tilde{P}_j = \alpha\theta$, $j \in E$. Let $\tilde{T}$ be the lifetime vector of components corresponding to $\{\tilde{N}(t), t \geqslant 0\}$ and $\tilde{\Lambda}$. From

Remark 4.3, $T =_{\mathrm{st}} \tilde{T}$. Therefore, it is sufficient to derive stochastic bounds for $\tilde{T}$.

To construct the bounds of the distribution and survival functions of $\tilde{T}$ for $s > 2$, we introduce an "extremal" tree,

$$\hat{\Lambda} = \frac{1}{s-2}\,\beta\varepsilon^{E} + \sum_{i=1}^{s} \beta\varepsilon^{i} + \gamma\varepsilon^{\varnothing},$$

where ε^K is defined as in Example 3.3. Let $\hat{T}$ be the joint lifetime vector of components for the system with arrival process $\{\tilde{N}(t), t \geqslant 0\}$ and tree $\hat{\Lambda}$. From Theorem 5.1 and Example 3.3, $\hat{T}$ is PLOD.

If $\theta \leqslant 1/\alpha(s-1)$ and $P^E \geqslant (1/(s-1))\,\theta$, then set $\beta = ((s-2)/(s-1))\,\alpha\theta$ and $\gamma = 1-(s-1)\,\alpha\theta \geqslant 0$. Thus $\alpha P^E \geqslant \beta/(s-2)$, and hence it is easy to show that $\tilde{\Lambda} \geqslant_{\mathscr{T}_r} \hat{\Lambda}$ and that the marginal weights of $\tilde{\Lambda}$ and $\hat{\Lambda}$ are the same. From Theorem 4.2, $\tilde{T}$ is more lower orthant dependent than $\hat{T}$ is, and so $\tilde{T}$ (or T) is PLOD.

Similarly, we can also obtain the conditions under which $\tilde{T}$ (or T) is PUOD.

It is worth emphasizing that our approach in this example is the following: (a) there exists a "super" arrival process (in our example, $\{\tilde{N}(t), t \geqslant 0\}$) such that the arrival process is a thinning of this super arrival process; (b) corresponding to the super arrival process, we can construct an extremal tree that is majorized from roots and leaves by another tree of the identical marginals; and (c) the joint distribution and survival functions of the lifetime vector corresponding to the super arrival process and the extremal tree has a product form bound.

Remark 5.6. To illustrate that simultaneous arrivals may also introduce negative dependence, suppose that the shock arrival process $\{N(t), t \geqslant 0\}$ has a deterministic sample path (admittedly unrealistic). It is easy to see that in such a case $\{N(t), t \geqslant 0\} = \{N^{\mathrm{I}}(t), t \geqslant 0\}$. In other words, the counting process with constant interarrival times is simultaneously PUOD, PLOD, NUOD and NLOD in time. Then, examining the proof of Theorem 5.1, one sees that $T(\delta^{\mathrm{I}}, \mathbf{N})$ are independent random variables. Hence, if $\Lambda \leqslant_{\mathscr{T}_l} \Lambda^{\mathrm{I}}$, then T is NUOD; if $\Lambda \leqslant_{\mathscr{T}_r} \Lambda^{\mathrm{I}}$, then T is NLOD.

6. GENERALIZATIONS

Our results derived in Sections 3 and 4 still hold under the following generalizations.

1. Suppose that the nth shock loading vector is governed by $\Lambda_n = \{P^K(n) \mid K \subseteq E\}$, where $P^K(n)$ is the probability that the nth shock is a type-K shock, $n = 1, 2, \ldots$. Let δ_n be the random variable defined by Λ_n, $n = 1, 2, \ldots$, and assume that they are independent. Then if we replace the condition "$\Lambda \geqslant_{\mathcal{T}_r(\mathcal{T}_l)} \bar{\Lambda}$" by the condition "$\Lambda_n \geqslant_{\mathcal{T}_r(\mathcal{T}_l)} \bar{\Lambda}_n$ for all n," in Sections 3 and 4, then all the major results carry.

2. Suppose that component j has n_j spare parts, $j \in E$, which can be used to replace component j when it fails. The fatal failure of component j occurs when it receives the $(n_j + 1)$ st shock, when no spare of its own type is available to replace the failed component, $j \in E$. As before, let $\{N_1(t_1), \ldots, N_s(t_s)\}$ be the multivariate counting process. Let $\{T_1, \ldots, T_s\}$ be the component lifetime vector. Then,

$$T_j = \inf\{t \geqslant 0 \mid N_j(t) \geqslant n_j + 1\}, \qquad j \in E.$$

Hence,

$$P(T_1 \leqslant (>)\, t_1, \ldots, T_s \leqslant (>)\, t_s) = P(N_1(t_1) \geqslant (<)\, n_1 + 1, \ldots, N_s(t_s) \geqslant (<)\, n_s + 1).$$

Note that only Theorem 4.2 is affected by this new assumption. But it is trivial to observe that Theorem 4.2 remains valid with the modified definition of T.

3. Consider a system with s components that are subjected to multiple types of shocks, whose occurrences are governed by a counting process. With probability P^K, $K \subseteq E$, a shock simultaneously inflicts random damage on the components in set K. Suppose that for each component $j \in E$, the damage of the nth shock to component j is $D_{n,j}$, if the nth shock inflicts component j. Suppose that $(D_{n,1}, \ldots, D_{n,s})$, $n = 1, 2, \ldots$, are i.i.d. random vectors and are also independent of the shock arrival process $\{N(t), t \geqslant 0\}$ as well as of K. Suppose that the damages accumulate additively. Let T_j denote the life length of component j, for $j \in E$, that is, the first time that the total damage accumulated at component j exceeds its design threshold, say d_j. Marshall and Shaked (1979) and Savits and Shaked (1981) studied a similar multivariate cumulative damage model, assuming that the shock arrival process is Poisson. This model also includes, as special cases, several cumulative damage shock models that appeared in the literature (e.g., Li and Zhu, 1994, and Kijima *et al.*, 1998). The lifetime of component j can be expressed by

$$T_j = \inf\left\{t \geqslant 0 \;\middle|\; \sum_{n=1}^{N_j(t)} \delta_{n,j} D_{n,j} > d_j\right\}, \qquad j \in E,$$

where the nth shock loading vector $\delta_n = \{\delta_{n,1}, ..., \delta_{n,s}\}$ is independent of the nth shock damage vector $D_n = \{D_{n,1}, ..., D_{n,s}\}$. Then

$$P(T_1 \leqslant (>)\, t_1, ..., T_s \leqslant (>)\, t_s)$$
$$= P\left(\sum_{n=1}^{N_1(t_1)} \delta_{n,1} D_{n,1} > (\leqslant)\, d_1, ..., \sum_{n=1}^{N_s(t_s)} \delta_{n,s} D_{n,s} > (\leqslant)\, d_s \right).$$

Again, only Theorem 4.2 is affected by this new assumption. Using the independence of δ_n and D_n and Lemma 2.3, it is straightforward to show that Theorem 4.2 is still true.

ACKNOWLEDGMENTS

We thank the two anonymous referees for their very careful reviews of the paper and insightful comments.

REFERENCES

1. F. Baccelli and A. M. Makowski, Multidimensional stochastic ordering and associated random variables, *Oper. Res.* **37** (1989), 478–487.
2. R. E. Barlow and F. Proschan, "Statistical Theory of Reliability and Life Testing," To Begin With, Silver Spring, MD, 1981.
3. H. W. Block, T. H. Savits, and M. Shaked, Some concepts of negative dependence, *Ann. Probab.* **10** (1982), 765–772.
4. C. S. Chang and D. Yao, Rearrangement, majorization and stochastic scheduling, *Math. Oper. Res.* (1994).
5. J. D. Esary and F. Proschan, A reliability bound for systems of maintained, independent components, *J. Amer. Statist. Assoc.* **65** (1970), 329–338.
6. K. Joag-Dev and F. Proschan, Negative association of random variables with applications, *Ann. Statist.* **11** (1983), 286–295.
7. M. Kijima, H. Li, and M. Shaked, Stochastic processes in reliability, preprint, 1998.
8. H. Li and H. Zhu, Stochastic equivalence of ordered random variables with applications in reliability theory, *Statist. Probab. Lett.* **20** (1994), 383–393.
9. H. Li and S. H. Xu, On the dependent structure and bounds of correlated parallel queues and their applications to synchronized stochastic systems, preprint, 1999.
10. H. Lindqvist, Association of probability measures on partially ordered spaces, *J. Multivariate Anal.* **26** (1988), 111–131.
11. A. W. Marshall and I. Olkin, A multivariate exponential distribution, *J. Amer. Statist. Assoc.* **2** (1967), 84–98.
12. A. W. Marshall and I. Olkin, "Inequalities: Theory of Majorization and Its Applications," Academic Press, New York, 1979.
13. A. W. Marshall and M. Shaked, Multivariate shock models for distributions with increasing hazard rate average, *Ann. Probab.* **7** (1979), 343–358.
14. I. Olkin and Y. L. Tong, Positive dependence of a class of multivariate exponential distributions, *SIAM J. Control. Optim.* **32** (1994), 965–974.

15. T. H. Savits and M. Shaked, Shock models and the MIFRA property, *Stochast. Process. Appl.* **11** (1981), 273–283.
16. M. Shaked and J. G. Shanthikumar, Multivariate imperfect repair, *Oper. Res.* **34** (1986), 435–448.
17. M. Shaked and J. G. Shanthikumar, "Stochastic Orders and Their Applications," Academic Press, New York, 1994.
18. M. Shaked and J. G. Shanthikumar, Supermodular stochastic order and positive dependence of random vectors, *J. Multivariate Anal.* **61** (1997), 86–101.
19. M. Shaked and R. Szekli, Comparison of replacement policies via point processes, *Adv. Appl. Probab.* **27** (1995), 1079–1103.
20. N. Singpurwalla, Survival in dynamic environments, *Statist. Sci.* **10** (1995), 86–103.
21. D. Stoyan, "Comparison Methods for Queues and Other Stochastic Models," Wiley, New York, 1983.
22. R. Szekli, "Stochastic Ordering and Dependence in Applied Probability," Springer-Verlag, New York, 1995.
23. A. H. Tchen, Inequalities for distributions with given marginals, *Ann. Prob.* **8** (1980), 814–827.
24. Y. L. Tong, "Probability Inequalities in Multivariate Distributions," Academic Press, New York, 1980.
25. S. H. Xu and H. Li, Majorization of weighted trees: A new tool to study correlated stochastic systems, preprint, 1998.