A new bound on the feedback vertex sets in cubic graphs

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Abstract

In this paper a new upper bound for the feedback set of cubic graphs is obtained. This result answers a question posed by Speckenmeyer (1986, 1988) in the field of feedback vertex set and improves several former results due to Bondy et al. (1987). Also this new bound is sharp in some cases.

1. Introduction

Let $G = (V,E)$ be an undirected simple graph. A vertex set $F \subseteq V$ is called a feedback set (f.v.s) of $G$ if the graph $G - F$ is a forest. The cardinality of a minimum f.v.s of $G$ is denoted by $f(G)$. The girth $g = g(G)$ of a graph $G$ is the length of a shortest cycle in $G$.

For a connected cubic graph $G$ of order $n$ and girth $g$, Speckenmeyer [6] obtained that

$$f(G) \leq \frac{g + 1}{4g - 2}n + \frac{g - 1}{2g - 1}$$

and posed a question to improve this inequality.

Speckenmeyer also conjectured that for a biconnected cubic graph $G$ without triangles, $f(G) \leq \frac{1}{3}|V(G)|$. Bondy et al. [1] proved that for a connected cubic graph $G$ of girth at least 4, $f(G) \leq \lceil|V(G)|/3\rceil$. This problem was completely solved by Zheng

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and Lu [7], the answer is that for a connected cubic graph $G$ of girth at least 4, $f(G) \leq \frac{1}{3}|V(G)|$ if $|V(G)| \neq 8$, and the upper bound is sharp.

To answer Speckenmeyer's first problem, we have obtained a result in [4] by showing that

$$f(G) \leq \max \left\{ \frac{g+1}{4g-2}n + \frac{3-g}{2g-1}, \frac{g}{4(g-1)}, \frac{g-2}{2g-2}, \frac{2g+1}{8g-6}, \frac{4}{4g-3}, \frac{g+1}{4g-2}n + \frac{g-6}{2g-1} \right\}.$$

This paper is a continuing study of this problem. By combining the method used in [4] and a bipartite graph method used in [1], we can improve our result further. We will see the result in this paper includes the result in [7], improves the results in [1] and [6].

A few words about our notation and terminology. A vertex set $J \subseteq V$ is called a **nonseparating independent set** (n.s.i.s.) of $G$, if $J$ is an independent set of $G$, and $G - J$ is connected. The cardinality of a maximum n.s.i.s. of $G$ is denoted by $z(G)$. The set of vertices adjacent to a vertex $v$ is denoted by $N(v)$; $d(v) = |N(v)|$ is the degree of the vertex $v$. A graph $G$ is called a **cactus** if $G$ is connected and any two cycles of $G$ are disjoint. A vertex $v$ of a connected graph $G$ is called a cycle branching point (c.b.p.) of $G$ if $v$ has degree at least three and the graph $G - v$ is connected. A $q$-cycle of $G$ is a cycle of $G$ of length $q$. For the notation and terminology which do not appear here, we refer to [2].

### 2. Basic lemmas

In [6], Speckenmeyer gave an algorithm $B$ to find an n.s.i.s. $J$.

**Algorithm B:**

Input. Cubic connected graph $G$ and a minimum f.v.s. $F$ of $G$;
$J \leftarrow \emptyset$;
while there is a vertex $v \in F - J$, which is a c.b.p. of $G - J$, do $J \leftarrow J \cup \{v\}$;
Output. $J$;

He then proved that

**Theorem 1** (Speckenmeyer [6]). *For a cubic connected graph $G$ of order $n$ and girth $g$,*

$$f(G) \leq \frac{g+1}{4g-2}n + \frac{g-1}{2g-1}.$$

He also obtained the following two lemmas.
Lemma 2 (Speckenmeyer [6]). Let $F$ be a minimum f.v.s. and $J \subseteq F$ be an n.s.i.s. of a cubic graph $G$ obtained by algorithm $B$. Then $J$ is a maximal n.s.i.s., and $G - J$ is a cactus.

Lemma 3 (Speckenmeyer [6]). Let $G$ be a cubic connected graph with $n$ vertices. Then

$$f(G) = \frac{n}{2} - z(G) + 1.$$  

The following two lemmas were used in [4] and will be used in the proof of our main result. For completeness, we give proofs here.

Lemma 4. Let $F$ be a minimum f.v.s. and $J \subseteq F$ be an n.s.i.s. of a cubic graph $G$ obtained by algorithm $B$. Let $h(G - J)$ be the number of cycles of $G - J$. Then $f(G) = |J| + h(G - J)$ and no vertex of $J$ joins two cycles of $G - J$.

Proof. Since $G - J$ is a cactus, it is connected and any two cycles are disjoint. Let $C_1, C_2, \ldots, C_h$ be the cycles of $G - J$ and let $v_i \in C_i$. Suppose there is a vertex $v \in J$ which joins two cycles, say it joins $C_1, C_2$ at $v_1, v_2$, without loss of generality. Let $F' = (J \cup \{v_1, v_2, \ldots, v_h\}) - \{v\}$. It is easy to see that $F'$ is an f.v.s. of $G$. But $|F'| < |F|$, a contradiction. Therefore, no vertex of $J$ joins two cycles of $G - J$. Note that $F \cap C_i \neq \emptyset$, hence $f(G) = |F| = |J| + h$.  

In the following, we always assume that $F$ is a minimum f.v.s. of $G$ and $J$ is a maximum n.s.i.s. of $G$ obtained by the algorithm $B$. Then $G - J$ is a cactus. Let $h(= h(G - J))$ be the number of cycles of $G - J$. Let $U$ be the set of vertices of degree 1 in $G - J$, and $V$ be the set of vertices of degree 2 in $G - J$ which are not on any cycle of $G - J$, and $W$ be the set of vertices of degree 2 which are on cycles of $G - J$. Let $X$ be the vertices of degree 3 in $G - J$ which are not on any cycle and $Y$ be the vertices of degree 3 which are on cycles of $G - J$. Let $z = |J|, k_0 = |U|, k_1 = |V| \text{ and } k_2 = |X|$. Let $h_i$ be the number of $(g+i)$-cycles of $G - J$ for $i = 0, 1, \ldots, \frac{g}{2}$. Let $k = k(G - J) = k_0 + k_1 + k_2 + \sum ih_i$. Then $hg + z + k = n$.

Remark. We observe that both $h(G - J)$ and $k(G - J)$ are invariants of $G$ from Lemma 4.

Lemma 5. Let $G$ be a cubic graph of order $n$ and girth $g$ and $k$ be defined as above. Then

$$f(G) = \frac{g + 1}{4g - 2}n + \frac{g - k - 1}{2g - 1}.$$  

Proof. Let $F$ and $J$ be defined as above. By Lemma 4, we have that $F$ is obtained by adding one vertex from each cycle of $G - J$ to $J$ and no vertex of $J$ joins two cycles of $G - J$. This implies that $k > 0$.  

In any case, we have \( h = \frac{n - z - k}{g} \). Combining with \( f(G) = \frac{1}{2}n - z + 1 \) and \( f(G) = z + h \), we obtain

\[
f(G) = \frac{g + 1}{4g - 2}n + \frac{g - k - 1}{2g - 1}.
\]

The importance of this lemma is that it gives exactly \( f(G) \) in terms of \( g \) and \( k \). By using this result, we can obtain an upper bound for \( f(G) \) and we can see that in many cases, this upper bound is sharp.

Theorem 6 is Propositions 2 and 3 in [1].

**Theorem 6** (Bondy et al. [1]). Let \( G \) be a connected cubic graph with \( n \) vertices where \( n \gg 6 \). Then \( z(G) > \frac{(n + 6)}{8} \). Equality holds if and only if \( G \) is derived from a cubic tree by blowing up each degree three vertex to a triangle and attaching \( K_4 \) with one subdivided edge at each degree one vertex. (We denote this class of graphs by \( \mathcal{G} \).)

As a consequence of Theorem 6, Lemmas 3 and 5, we have

**Corollary 7.** Let \( G \) be a connected cubic graph of girth 3 with \( n \) vertices where \( n \gg 6 \). Then \( z(G) = k(G) \) if and only if \( G \in \mathcal{G} \).

**Proof.** Since \( g(G) = 3 \), by Lemmas 3 and 5, it follows that

\[
\frac{n}{2} - z + 1 = \frac{4}{10}n + \frac{2 - k}{5}.
\]

(a) If \( z = k \), then \( z(G) = k(G) = (n + 6)/8 \) from (\( \ast \)). By Theorem 6, \( G \in \mathcal{G} \).

(b) If \( G \in \mathcal{G} \), then \( z(G) = (n + 6)/8 \) by Theorem 6. By using (\( \ast \)), we also have that \( k(G) = (n + 6)/8 \). This proves Corollary 7.

**Examples.** By \( G_1, G_2 \) we denote the cubic graphs shown in Fig. 1.
3. Main result

Theorem 8. Let G be a cubic graph of order n and girth g where \( n \geq 4 \). Then

\[
f(G) \leq \frac{g}{4(g-1)}n + \frac{g-3}{2g-2},
\]

except for \( G \in \{K_4, G_1, G_2\} \cup \mathcal{G}\).

If \( g = 3 \) and \( G \in \mathcal{G} \),

\[
f(G) = \frac{3}{8}n + \frac{1}{4}.
\]

Proof. Let \( F, J, U, k_0, V, k_1, W, k_2, h, z, k \) be defined as above. By Lemma 5, we have

\[
f(G) = \frac{g+1}{4g-2}n + \frac{g-k-1}{2g-1}.
\]

The proof of this theorem is organized as follows. First in part A, we shall prove that \( k \geq z \) for any \( G - J \) where \( J \) is a maximum n.s.i.s. of \( G \). It then follows that

\[
f(G) \leq \frac{g}{4(g-1)}n + \frac{g-2}{2g-2}.
\]

Next in part B, we shall determine the extremal case when \( k = z \) and we shall see that except for \( G \in \{K_4, G_1, G_2\} \cup \mathcal{G} \),

\[ k > z, \]

it then follows that

\[
f(G) \leq \frac{g}{4(g-1)n} + \frac{g-3}{2g-2}.
\]

Part A: Now we need to estimate \( k \). The following facts are useful.

Facts. If a cycle \( C \) of \( G - J \) has length at most \( 3g - 7 \), then no vertex of \( J \) joins \( C \) by three edges. If a cycle \( C \) of \( G - J \) has length at most \( 2g - 5 \), then no vertex of \( J \) joins \( C \) by two edges.

These facts are true because otherwise we will have a cycle in \( G \) with length less than \( g \).

Let \( B_J = (U \cup V \cup W, J) \) be an induced bipartite subgraph of \( G \), that is, the vertices \( U \cup V \cup W \cup J \) together with the edges between \( U \cup V \cup W \) and \( J \). Let \( B_1, B_2, \ldots, B_m \) be the components of \( B_J \). For each \( B_i \), and \( A = U, V, W, J \), let \( A_i = A \cap V(B_i) \). Then by counting the edges of \( B_i \) in two ways, we have

\[ 2|U_i| + |V_i| + |W_i| = 3|J_i|. \]
Claim 1. If $W_i$ is not empty, $W_i$ must be contained in one cycle of $G - J$.

Suppose, to the contrary, that vertices $x, y$ of $W_i$ belong to distinct cycles of $G - J$. Let $P$ be an $(x, y)$-path in $B_i$, and for $A = J, U$, set $A \cap V(P) = A_P$. Then

$$(J - J_P) \cup U_P \cup \{x, y\}$$

is an n.s.i.s. of $G$ of cardinality $|J| + 1$, a contradiction. Note that if $|W_i| = 1$ then it is trivial that $W_i$ is contained in one cycle of $G - J$.

To evaluate $k$, let $C^1, \ldots, C^t$ (generally, they are not all the cycles of $G - J$) be all the cycles of $G - J$ such that $l_j$, the number of vertices on $C^j$ having degree 3 in $G - J$, satisfies $1 \leq l_j < |C^j|$ for $j = 1, \ldots, t$. Then

$$k = |U| + |V| + |X| + \sum_{j=1}^{t} r_j \geq |U| + |V| + |X| + \sum_{j=1}^{t} (|C^j| - g)$$

$$= |U| + |V| + |X| + \sum_{j=1}^{t} \frac{|C^j| - g}{|C^j| - l_j} (|C^j| - l_j). \quad (1)$$

We have that $W_i \cap W_j = \emptyset$ if $i \neq j$, and $\bigcup W_i = (\bigcup C^j) - Y$. By Claim 1, we can assume that $W_i \subset C^h$. Then $|W_i| \leq |C^h| - l_h$. If, say $W_{r_1}, W_{r_2}, \ldots, W_{r_p} \subset C^r$, then $|W_{r_1}| + |W_{r_2}| + \cdots + |W_{r_p}| \leq |C^r| - l_r$. Hence

$$|W_{r_1}| \frac{|C^{h_1} - g}{|C^{h_1} - l_{h_1}} + \cdots + |W_{r_p}| \frac{|C^{h_p} - g}{|C^{h_p} - l_{h_p}} \leq (|C^r| - l_r) \frac{|C^r| - g}{|C^r| - l_r}$$

(where $C^{h_1} = \cdots = C^{h_p} = C^r$ and $l_{h_1} = \cdots = l_{h_p} = l_r$). Note also, if $W_i = \emptyset$, then $|W_i| = 0$. For convenience, we will choose any $C^{h}$ and consider $W_i$ as a subset of $C^h$.

Therefore, by (1) we have

$$k \geq |X| + \sum_{i=1}^{m} \left( |U_i| + |V_i| + |W_i| \frac{|C^{h} - g}{|C^{h} - l_{h}} \right). \quad (2)$$

Note that the number of $W_i$'s may not equal the number of cycles in $G - J$. Our objective now is to prove that

$$\sum_{i=1}^{m} \left( |U_i| + |V_i| + |W_i| \frac{|C^{h} - g}{|C^{h} - l_{h}} \right) \geq \sum_{i=1}^{m} |J_i| = |J| = z.$$

Since $B_i = (U_i \cup V_i \cup W_i, J_i)$ is connected and the vertices in $V_i \cup W_i$ of $B_i$ have degree 1 in $B_i$, the subgraph $(U_i, J_i)$ of $B_i$ is connected. It has $|U_i| + |J_i|$ vertices and $2|U_i|$ edges. Therefore,

$$2|U_i| \geq |U_i| + |J_i| - 1.$$

Hence,

$$|U_i| \geq |J_i| - 1.$$
Two cases arise.

Case 1: \( |U_i| \geq |J_i| \).

Subcase 1.1: \( |U_i| \geq |J_i| + 1 \) or \( |U_i| = |J_i| \) and \( |V_i| \geq 1 \). We have that
\[
|U_i| + |V_i| + |W_i| \left| \frac{C_{ij}}{C_i^h} - \frac{g}{l_j} \right| \geq |J_i| + 1.
\]

Subcase 1.2: \( |U_i| = |J_i| \) and \( |V_i| = 0 \). In this subcase, \( |W_i| \neq 0 \) by (**). We have that
\[
|U_i| + |V_i| + |W_i| \left| \frac{C_{ij}}{C_i^h} - \frac{g}{l_j} \right| \geq |J_i|.
\]

Equality holds only if \( |C_i^h| = g \).

Case 2: \( |U_i| = |J_i| - 1 \). We have \( |V_i| + |W_i| = |J_i| + 2 \) by (**). If \( |V_i| \geq 2 \), then
\[
|U_i| + |V_i| > |J_i|.
\]
If \( |V_i| \leq 1 \), then \( |W_i| \geq |J_i| + 1 \), and hence \( W_i \neq \emptyset \). Looking at \( (U_i, J_i) \) again, we see that there are two vertices of \( J_i \) having degree 1 in the subgraph \( (U_i, J_i) \).

This implies that one of the two vertices is adjacent to two vertices of the cycle \( C_i^h \) of \( G - J \). Hence, \( C_i^h \) has length at least 2\( g - 4 \), (if \( g = 3 \), \( C_i^h \) has length at least 3).

Subcase 2.1: (a) If \( |J_i| = 1 \), \( |V_i| = 1 \), then \( |W_i| = 2 \), and \( C_i^h \) has length at least 2\( g - 4 \) (if \( g = 3 \), \( C_i^h \) has length at least 3). Thus
\[
|U_i| + |V_i| + |W_i| \left| \frac{C_{ij}}{C_i^h} - \frac{g}{l_j} \right| \geq |U_i| + |V_i| + 2\frac{2g - 6}{3g - 7} > |J_i|.
\]

(b) If \( |J_i| = 1 \), \( |V_i| = 0 \), then \( |W_i| = 3 \) and \( C_i^h \) has length at least 3\( g - 6 \) (if \( g = 3 \), \( C_i^h \) has length at least 4). Thus
\[
|U_i| + |V_i| + |W_i| \left| \frac{C_{ij}}{C_i^h} - \frac{g}{l_j} \right| \geq |U_i| + |V_i| + 3\frac{2g - 6}{3g - 7} > |J_i|.
\]

Subcase 2.2: \( g = 3 \) and \( |J_i| \geq 2 \).

(a) If \( |V_i| = 1 \), then \( |W_i| \geq 3 \) and \( C_i^h \) has length at least 4 if \( G \neq K_4 \). Thus,
\[
|U_i| + |V_i| + |W_i| \left| \frac{C_{ij}}{C_i^h} - \frac{3}{4 - 1} \right| \geq |J_i| + \frac{4 - 3}{4 - 1} > |J_i|.
\]
(b) $|V_i| = 0$, then $|W_i| \geq 4$ and $C^h$ has length at least 5. Thus

$$\frac{|C^h| - 3}{|C^h| - I_i} \geq \frac{4(5 - 3)}{5 - 1} > 1.$$ 

Therefore

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - 3}{|C^h| - I_i} > |J_i|.$$ 

Subcase 2.3: $g = 4$ and $|J_i| \geq 2$.

(a) $|V_i| = 0$. If $|J_i| \geq 3$, then

$$|W_i| \frac{|C^h| - 4}{|C^h| - I_i} \geq \frac{5 - 4}{5 - 1} > 1.$$ 

Therefore

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - 4}{|C^h| - I_i} \geq |U_i| + |V_i| + \frac{5 - 4}{5 - 1} > |J_i|.$$ 

If $|J_i| = 2$, then $|W_i| = 4$, and

$$|W_i| \frac{|C^h| - 4}{|C^h| - I_i} \geq \frac{5 - 4}{5 - 1} = 1.$$ 

Equality holds only if $|C^h| = 5$.

(b) $|V_i| = 1$. If $|C^h| \geq 5$, then

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_i} \geq |U_i| + |V_i| + \frac{5 - 4}{5 - 1} > |J_i|.$$ 

If $|C^h| = 4$, then $|U_i| = |V_i| = 1$, $|J_i| = 2$, $|W_i| = 3$, and $C^h$ has only one vertex belonging to $Y$. Therefore

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_i} \geq |U_i| + |V_i| + \frac{4 - 4}{4 - 1} = |J_i|.$$ 

Equality holds only if $|C^h| = 4$.

Subcase 2.4: $g = 5$ and $|J_i| \geq 2$.

(a) $|V_i| = 1$. Then

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_i} \geq |U_i| + |V_i| + \frac{g - 4}{2g - 5} > |J_i|.$$ 

(b) $|V_i| = 0$. If $|J_i| \geq 4$, then

$$|U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_i} \geq |U_i| + |V_i| + \frac{g - 4}{2g - 5} > |J_i|.$$
If \( |J_i| = 3 \), then \( |W_i| = 5 \), and then \( C^J_i \) has length at least 6. Therefore

\[
|U_i| + |V_i| + |W_i| \geq \left| \frac{C^J_i - g}{C^J_i - I_j} \right| |U_i| + 5 \frac{6 - 5}{6 - 1} \geq |J_i|.
\]

Equality holds only if \( C^J_i \) has length 6.

Now suppose that \( |J_i| = 2 \), then \( |W_i| = 4 \).

If \( C^J_i \) is a 6-cycle and there is only one vertex which is in \( Y \), then we have a \( W_i \) which is a singleton vertex set and is contained in \( C^J_i \) (\( C^J_i = C^{J_i} \)). In this case, we have that either \( |U_i| > |J_i| \) or \( |U_i| = |J_i| \) and \( |V_i| \neq 0 \) or \( |U_i| = |J_i| - 1 \) and \( |V_i| \geq 2 \). This implies that \( |U_i| + |V_i| \geq |J_i| + 1 \). Therefore,

\[
|U_i| + |V_i| + |W_i| \geq \left| \frac{C^J_i - g}{C^J_i - I_j} \right| |U_i| + |V_i| + |W_i| \geq |J_i| + |J_i|.
\]

If \( C^J_i \) is a 6-cycle and there are two vertices which are in \( Y \), then \( I_{j_i} = 2 \). We have

\[
|U_i| + |V_i| + |W_i| \geq |U_i| + |V_i| + 4 \frac{1}{6 - 2} \geq |J_i|.
\]

Equality holds only if \( |U_i| = 1, |V_i| = 0, |J_i| = 2, |W_i| = 4, \) \( C^J_i \) is a 6-cycle and there are two vertices of \( C^J_i \) which are in \( Y \).

If \( C^J_i \) is a 7-cycle, then

\[
|U_i| + |V_i| + |W_i| \geq |U_i| - 1 + 4 \frac{7 - 5}{7 - 1} > |J_i|.
\]

Subcase 2.5: (a) If \( |J_i| \geq 2, |V_i| = 1 \) and \( g \geq 6 \), then

\[
|U_i| + |V_i| + |W_i| \geq |U_i| + |V_i| + 3 \frac{g - 4}{2g - 5} > |J_i|.
\]

(b) If \( |J_i| \geq 2, |V_i| = 0 \) and \( g \geq 6 \), then

\[
|U_i| + |V_i| + |W_i| \geq |U_i| + |V_i| + 4 \frac{g - 4}{2g - 5} > |J_i|.
\]

By above arguments, we always have for any \( i \), either

\[
|U_i| + |V_i| + |W_i| \geq |J_i|
\]

or

\[
|U_i| + |V_i| + |W_i| \geq |U_i| + |V_i| + |W_i| \geq |J_i| + |J_i|
\]

for some \( i' \neq i \). Therefore, if \( g \geq 3 \), then

\[
k \geq \sum_{i=1}^m \left( |U_i| + |V_i| + |W_i| \right) \geq \sum_{i=1}^m |J_i| = |J| = z.
\]

Note that (3) is true for any \( G - J \) where \( J \) is a maximum n.s.i.s of \( G \).
Combining with
\[ f(G) = \frac{g + 1}{4g - 2}n + \frac{g - k - 1}{2g - 1} \quad \text{and} \quad f(G) = \frac{n}{2} - z + 1, \]
we obtain
\[ f(G) \leq \frac{g}{4(g - 1)}n + \frac{g - 2}{2g - 2}. \]

**Part B:** Next we shall investigate the extremal case when \( k = z \). And we shall obtain that if \( G \not\in \{K_4, G_1, G_2\} \cup \mathcal{G} \), then
\[ k \geq \sum_i \left( |U_i| + |V_i| + |W_i| \right) \geq \sum_i |J_i| = |J| = z, \]
and hence \( k \geq z + 1 \).

In the proof of Subcase 1.2 of case 1, we met that
\[ |U_i| + |V_i| + |W_i| \geq |J_i|. \]
Equality holds only if \( |C^i| = g \). We prove that this case cannot occur except \( g = 4 \) and \( n = 8 \).

Observe that the bipartite graph induced by \((U_i, J_i)\) is 2-regular and connected in \( G - J \), and hence it is a cycle of \( G - J \). \( J_i \) and \( U_i \) have the same property as a subset of an n.s.i.s of \( G \), that is, if we let \( J' = (J - J_i) \cup U_i \), then \( J' \) is a maximal n.s.i.s and \( G - J' \) has the same system cycles as that of \( G - J \). From this observation, we can see that every vertex in \( U_i \) is adjacent to a common cycle by Claim 1. Let \( |J_i| = l \geq 2 \).

Then \((U_i, J_i)\) is a cycle of length \( 2l \), and hence \( g \leq 2l \).

There are two cases in this part.

(a) \( U_i \) joins to \( C^i \). Then \( g = |C^i| \geq |J_m| + |U'_m| \geq 2l \). But if \( G - J \) has at least two cycles, then \( C^i \) has at least one vertex which is adjacent to a vertex in another cycle of \( G - J \) and hence \( g = |C^i| \geq 2l + 1 \). This is a contradiction. So \( G - J \) has only one cycle of girth \( g \). Then \( n = 2g \) and \( g = 2l \). If \( g = 4 \), we obtain \( n = 8 \), and this graph does exist. Let \( g \geq 6 \). Then \( G \) is a graph obtained by joining the two \( 2l \)-cycles by an \( l \)-matching. Note that \( G \) has girth \( g = 2l \). To obtain \( G \), one can draw a cycle of length \( 2l \) and then draw an edge at each vertex of this cycle. So we have \( 2l \) vertices of degree 1 now. To make it cubic and no more new vertex added, this is possible only if \( 2l = 4 \). If \( 2l = 6 \), the smallest order of such a graph is 14 and this graph is unique. If \( 2l > 6 \), then all the degree 1 vertices are independent, otherwise, the girth of \( G \) is less than \( 2l \) (easy calculation). This means we must add new vertices to make it cubic. So this kind of graph does not exist since a cubic graph of girth \( g \) must have order greater than \( 2g \) if \( g \geq 6 \).

(b) \( U_i \) does not join to \( C^i \).

Let \( U_i \) join to \( C^i \). Then there is a path \( uabv \) in \( G \) such that \( u \in C^i, a \in J_i, b \in U_i \) and \( v \in C^i \). Let \( J' = (J - \{a\}) \cup \{u, v\} \). Then it is easy to see that \( J' \) is an n.s.i.s. This contradicts the maximality of \( J \).
We see that in this case, $k = z$ only when $g = 4$, $n = 8$ and this graph is the graph $G_1$ described in [7]. So in this case, we always have

$$ |U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_{j_i}} \geq |J_i| + |W_i| \frac{1}{g-2}. $$

From the proof above, we have that in the extremal case when $k = z$ we must have that $g \leq 5$.

**Case B1**: $g = 5$. In the proof of subcase 2.4, we met equality only in the following cases.

(a) $|J_i| = 3$, $|W_i| = 5$ and $C^h$ has length at least 6 or, (b) $|U_i| = 1$, $|V_i| = 0$, $|J_i| = 2$, $|W_i| = 4$, $C^h$ is a 6-cycle and there are two vertices of $C^h$ which are in $Y$.

We claim that either there is an $i'$ such that

$$ |U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_{j_i}} + |U_{i'}| + |V_{i'}| + |W_{i'}| \frac{|C^{i'}| - g}{|C^{i'}| - I_{j_{i'}}} > |J_i| + |J_{i'}|, \tag{4} $$

or $|X| \neq 0$, then

$$ |X| + |U_i| + |V_i| + |W_i| \frac{|C^h| - g}{|C^h| - I_{j_i}} > |J_i|. \tag{5} $$

Now we show that the above claim is correct. We have $|U_i| \geq 1$. If there is a vertex of $U_i$ joining a cycle of $G - J$, say $u$ in $U_i$ which is adjacent to $x$. There is a path $xuvy$ such that $v \in J_i$ and $y \in W_i$. Let $J' = (J - \{v\}) \cup \{x, y\}$. Then $J'$ is an n.s.i.s. which contains more elements than $J$. This is a contradiction. Suppose that all vertices in $U_i$ are not adjacent to any cycle of $G - J$, then we have that either there is an $i'$ such that $|V_{i'}| \neq 0$,

$$ |U_{i'}| + |V_{i'}| + |W_{i'}| \frac{|C^{i'}| - g}{|C^{i'}| - I_{j_{i'}}} > |J_{i'}|, $$

and hence (4) holds, or $|X| \neq 0$, and hence (5) holds.

**Case B2**: $g = 4$. In the proof of subcase 2.3, we met two cases that might produce the equality.

One is that $|J_i| = 2$, $|U_i| = 1$, $|V_i| = 0$, $|W_i| = 4$ and $C^h$ has length 5. This can occur only if $G - J$ has only one 5-cycle and $G$ is the graph $G_2$ which is a graph of order 8 as described in Lu and Zheng's paper [7]. To prove this, let $J_i = \{a, b\}$, and $W_i = \{w_1, w_2, w_3, w_4\}$, and $U_i = \{u\}$, and $C^h = w_1w_2w_3w_4w_5w_1$. Now let $aw_1 \in E$ and $bw_1 \notin E$. Let $J' = (J - a) \cup \{w_1\}$. Then $J'$ must be a maximum n.s.i.s. of $G$. If $uw_5 \in E$, then $G = G_2$. Note that $\{b, w_1\}$ belongs to the same component in $B_{j_i}$.

If $uw_5 \notin E$, the component consisting of $J_i' = \{b, w_1\}$ in $B_{j_i'}$ does not produce the equality in $G - J'$.

The other is that $|C^h| = 4$, $|U_i| = |V_i| = 1$, $|J_i| = 2$, $|W_i| = 3$, and $C^h$ has only one vertex belonging to $Y$. In this case, let $J_i = \{a, b\}$, and $U_i = \{u\}$, and $V_i = \{v\}$, and $C^h = w_1w_2w_3w_4w_1$, and let $aw_2 \in E$, and $bw_1, bw_3 \in E$. Let $J' = (J - \{a, b\}) \cup \{u, w_1\}$, then $J'$ must be a maximum n.s.i.s. of $G$. Note that $\{u, w_1\}$ belongs to the same
component in \( B_{J'} \). It follows that the component consisting of \( J' = \{u, w_1\} \) in \( B_{J'} \) does not produce equality in \( G - J' \).

**Case B3:** \( g = 3 \). By Corollary 7, we see that if \( k = z \) then \( G \in \mathcal{G} \). Combining with

\[
f(G) = \frac{g + 1}{4g - 2} n + \frac{g - k - 1}{2g - 1}
\]

and

\[
f(G) = \frac{n}{2} - z + 1
\]

again, we obtain

\[
f(G) \leq \frac{g}{4(g - 1)} n + \frac{g - 3}{2g - 2}.
\]

This completes the proof. \( \square \)

**Corollary 9.** If \( g \geq 3 \), then

\[
f(G) \leq \frac{3}{8} n
\]

except for \( G \in \{K_4\} \cup \mathcal{G} \).

This improves a result of Bondy et al. [1].

**Corollary 10 (Zheng and Lu [7]).** If \( G \) is a connected cubic graph of order \( n \) and girth \( g \) with \( g \geq 4 \), then

\[
f(G) \leq \frac{n}{3},
\]

except for \( G = G_1 \), or \( G_2 \).

**Proof.** Let

\[
B(g) = \frac{g}{4(g - 1)} n + \frac{g - 3}{2g - 2}.
\]

It is easy to check \( B(g) \) is a decreasing function. Therefore, \( f(G) \leq B(4) = \frac{5}{3} + \frac{1}{6} \).

We have that \( n \) is an even integer. If \( n = 6m \), then \( f(G) \leq 2m + \frac{1}{6} \). Thus \( f(G) \leq 2m = n/3 \)

If \( n = 6m + 2 \), then \( f(G) \leq 2m + \frac{2}{3} + \frac{1}{6} = 2m + \frac{5}{6} \) and hence \( f(G) \leq 2m < n/3 \).

If \( n = 6m + 4 \), then \( f(G) \leq 2m + 1 + \frac{1}{3} + \frac{1}{6} = 2m + 1 + \frac{1}{2} \) and hence \( f(G) \leq 2m + 1 < n/3 \).

This shows that \( f(G) \leq \frac{5}{6} \). \( \square \)

Note that we improve the result in [6] by

\[
\frac{g - 2}{4(g - 1)(2g - 1)} n + \frac{3g - 1}{2(g - 1)(2g - 1)}.
\]
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References