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The semigroup generated by the idempotents of a partition monoid

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ABSTRACT

We study the idempotent generated subsemigroup of the partition monoid. In the finite case this subsemigroup consists of the identity and all the singular partitions. In the infinite case, the subsemigroup is described in terms of certain parameters that measure how far a partition is from being a permutation. As one of several corollaries, we deduce Howie's description from 1966 of the semigroup generated by the idempotents of a full transformation semigroup.

1. Introduction

The motivation for studying partition monoids comes from several directions. Partition algebras were introduced in the context of statistical mechanics [23], but are of great interest to algebraists because of their connection to the symmetric groups and Schur–Weyl duality [12]. Because of their geometric definition, in terms of multiplicative properties of diagrammatic basis elements, they contain a number of important diagram algebras as subalgebras, including the well-known Brauer algebras [2], Temperley–Lieb algebras [11], and Jones algebras [19]. Partition algebras are twisted

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[1] This research was completed while the first author was a postdoctoral research fellow at the University of Sydney.

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semigroup algebras of partition monoids [30]. The partition monoids $\mathcal{P}_X$ occupy a place of fundamental importance in semigroup theory because they contain (isomorphic copies of) such monoids as the symmetric group $\mathcal{S}_X$, the full transformation semigroup $\mathcal{T}_X$, and the symmetric and dual symmetric inverse monoids $\mathcal{I}_X$ and $\mathcal{J}_X$[2]; see [7,10,17,21,22]. It is natural to ask which properties of these monoids also hold in $\mathcal{P}_X$. It is already known [7] that for X finite, $\mathcal{P}_X$ is generated by its idempotents and units, as is the case for $\mathcal{T}_X$ [1] and $\mathcal{I}_X$ [28]. The idempotents of finite $\mathcal{P}_X$ generate $\{1\} \cup (\mathcal{P}_X \setminus \mathcal{S}_X)$; see [8]. This is exactly analogous to Howie's description [16] of finite $\mathcal{T}_X \setminus \mathcal{S}_X$. In [16], infinite $\mathcal{T}_X$ was also considered; in this case, $\mathcal{T}_X \setminus \mathcal{S}_X$ is not closed under multiplication, but the subgroup generated by the idempotents was described. Analogous results have been obtained for semigroups of integer matrices and, more generally, semigroups of endomorphisms of various kinds of (finite and infinite dimensional) algebras; see for example [29] as references therein. It is the main purpose of the current work to prove corresponding results about the idempotent generated submonoid of (finite and infinite) $\mathcal{P}_X$. The key idea in [16] was a description of a uniformity property satisfied by the idempotents of $\mathcal{T}_X$, in terms of parameters called the collapse, defect and shift, and the observation that this property extends to products of idempotents. An explicit description of the idempotents of $\mathcal{T}_X$ was crucial to establish this. By contrast, the idempotents of $\mathcal{P}_X$ have not been systematically investigated, and a combinatorial description is not currently known; to the authors' knowledge, a formula for the number of idempotents in finite $\mathcal{P}_X$ is not even known. Nevertheless, we are still able to describe the idempotent generated subsemigroup, and also the submonoid generated by the idempotents and units, because the regular $*$-semigroup structure of $\mathcal{P}_X$ will allow us to concentrate on a far more manageable set of idempotents. Moreover, our proof will allow us to deduce Howie's results from [16].

The article is organized as follows. In Section 2 we define the partition monoids $\mathcal{P}_X$, and outline some of their basic properties. Preliminary results concerning the idempotents of $\mathcal{P}_X$, and the semigroup $\mathbb{E}(\mathcal{P}_X)$ generated by them, are proved in Section 3. A description of finite $\mathbb{E}(\mathcal{P}_X)$ is given in Section 4. In Sections 5 and 6, we introduce certain parameters we call the *collapse*, *cocollapse*, *defect*, *codefect* and *shift*. In Section 7 we describe the elements of infinite $\mathcal{I}_X$ that are products of idempotents from $\mathcal{P}_X$. A description of infinite $\mathbb{E}(\mathcal{P}_X)$ is given in Section 8, and in Section 9 we describe the submonoid $\mathbb{F}(\mathcal{P}_X)$ generated by the idempotents and units; both descriptions are in terms of the parameters introduced in the previous sections. In Section 10, we calculate the intersections of $\mathbb{E}(\mathcal{P}_X)$ and $\mathbb{F}(\mathcal{P}_X)$ with the submonoids $\mathcal{T}_X$, $\mathcal{I}_X$ and $\mathcal{J}_X$. Finally, in Section 11 we prove an embedding result concerning idempotent generated regular $*$-semigroups.

2. Preliminaries

Let X be a set, and X' a disjoint set in one–one correspondence with X via a mapping $X \to X' : x \mapsto x'$. If $A \subseteq X$ we will write $A' = \{a' \mid a \in A\}$. A *partition on* X is a collection of pairwise disjoint nonempty subsets of $X \cup X'$ whose union is $X \cup X'$; these subsets are called the *blocks* of the partition. The *partition monoid* on X is the set $\mathcal{P}_X$ of all partitions on X, with a natural associative binary operation defined below. A block A of a partition $\alpha \in \mathcal{P}_X$ is said to be a *transversal block* if $A \cap X \neq \emptyset \neq A \cap X'$, or otherwise an *upper* (respectively, *lower*) *nontransversal block* if $A \cap X' = \emptyset$ (respectively, $A \cap X = \emptyset$). If $\alpha \in \mathcal{P}_X$, we will write

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i \in I,\, j \in J,\, k \in K}$$

to indicate that α has transversal blocks $A_i \cup B_i'$ $(i \in I)$, upper nontransversal blocks C_j $(j \in J)$, and lower nontransversal blocks D_k' $(k \in K)$. The indexing sets I, J, K will sometimes be implied rather

[2] For reasons we explain later, we prefer to use the notation $\mathcal{J}_X$ for the dual symmetric inverse monoid, rather than the standard $\mathcal{I}_X^*$.

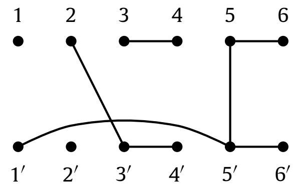

Fig. 1. A graphical representation of a partition (see text for further explanation).

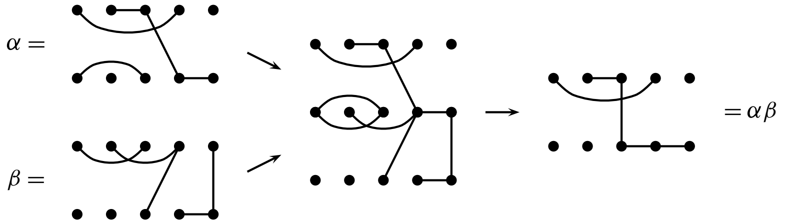

Fig. 2. Calculating the product of two partitions α, β.

than explicit, for brevity; if they are distinct, they will generally be assumed to be disjoint. Sometimes we will use slight variants of this notation, but it should always be clear what is meant.

A partition may be represented as a graph on the vertex set $X \cup X'$; edges are included so that the connected components of the graph correspond to the blocks of the partition. Of course such a graphical representation is not unique, but we regard two such graphs as equivalent if they have the same connected components. We think of the vertices from X (respectively, X') as being the *upper vertices* (respectively, *lower vertices*), explaining our use of these words in relation to the nontransversal blocks. An example is given in Fig. 1 for the partition $\alpha = \{\{1\}, \{2, 3', 4'\}, \{3, 4\}, \{5, 6, 1', 5', 6'\}, \{2'\}\} \in \mathcal{P}_X$, where $X = \{1, 2, 3, 4, 5, 6\}$.

The rule for multiplication of partitions is best described in terms of the graphical representations. Let $\alpha, \beta \in \mathcal{P}_X$. Consider now a third set X'', disjoint from both X and X', and in bijection with both sets via the maps $X \to X'' : x \mapsto x''$ and $X' \to X'' : x' \mapsto x''$. Let $\overline{\alpha}$ be the graph obtained from (a graph representing) α simply by changing the label of each lower vertex x' to x''. Similarly, let $\overline{\beta}$ be the graph obtained from β by changing the label of each upper vertex x to x''. Consider now the graph $\Gamma(\alpha, \beta)$ on the vertex set $X \cup X' \cup X''$ obtained by joining $\overline{\alpha}$ and $\overline{\beta}$ together so that each lower vertex x'' of $\overline{\alpha}$ is identified with the corresponding upper vertex x'' of $\overline{\beta}$. (Note that this new graph may contain multiple edges.) Then $\alpha\beta \in \mathcal{P}_X$ is defined to be the partition that satisfies the property that $x, y \in X \cup X'$ belong to the same block of $\alpha\beta$ if and only if there is a path from x to y in $\Gamma(\alpha, \beta)$. An example calculation (with X finite) is given in Fig. 2. (See also [22] for an equivalent formulation of the product; there $\mathcal{P}_X$ was denoted $\mathcal{CS}_X$, and called the *composition semigroup* on X.)

This product is easily checked to be associative, and so gives $\mathcal{P}_X$ the structure of a monoid; the identity element is the partition $\{\{x, x'\} \mid x \in X\}$, which we denote by 1. The partition monoid $\mathcal{P}_X$ is not inverse (it is easy to find non-commuting idempotents), but it is regular. In fact, more can be said. We define a map $^* : \mathcal{P}_X \to \mathcal{P}_X : \alpha \mapsto \alpha^*$ where α^* is the result of "turning α upside-down". More precisely:

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array}\right) \quad \Rightarrow \quad \alpha^* = \left(\begin{array}{c|c} B_i & D_k \\ \hline A_i & C_j \end{array}\right).$$

The next lemma is proved easily, and collects the basic properties of the * map that we will need. Essentially it states that $\mathcal{P}_X$ is a *regular* $*$*-semigroup*; see [18,25,26] for example. (Note that these kinds of semigroups are sometimes called *regular involution semigroups*, or even *special* $*$*-semigroups*.)

Lemma 1. *Let $\alpha, \beta \in \mathcal{P}_X$. Then*

$$(\alpha^*)^* = \alpha, \qquad \alpha\alpha^*\alpha = \alpha, \qquad \alpha^*\alpha\alpha^* = \alpha^*, \qquad (\alpha\beta)^* = \beta^*\alpha^*.$$

Next we record some notation and terminology. With this in mind, let $\alpha \in \mathcal{P}_X$. For $x \in X \cup X'$, we denote the block of α containing x by $[x]_\alpha$. The *domain* and *codomain* of α are defined to be the following subsets of X:

$$\operatorname{dom}(\alpha) = \{x \in X \mid [x]_\alpha \cap X' \neq \emptyset\},$$
$$\operatorname{codom}(\alpha) = \{x \in X \mid [x']_\alpha \cap X \neq \emptyset\}.$$

We also define the *kernel* and *cokernel* of α to be the following equivalences on X:

$$\ker(\alpha) = \{(x, y) \in X \times X \mid [x]_\alpha = [y]_\alpha\},$$
$$\operatorname{coker}(\alpha) = \{(x, y) \in X \times X \mid [x']_\alpha = [y']_\alpha\}.$$

Basic properties include formulae such as

$$\operatorname{dom}(\alpha^*) = \operatorname{codom}(\alpha), \qquad \ker(\alpha^*) = \operatorname{coker}(\alpha),$$
$$\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}(\alpha), \qquad \ker(\alpha) \subseteq \ker(\alpha\beta),$$
$$\operatorname{codom}(\alpha\beta) \subseteq \operatorname{codom}(\beta), \qquad \operatorname{coker}(\beta) \subseteq \operatorname{coker}(\alpha\beta).$$

We may now describe a number of important submonoids of $\mathcal{P}_X$. Write $\Delta = \{(x, x) \mid x \in X\}$ for the trivial equivalence on X (that is, the equality relation), and let

- $\mathcal{I}_X = \{\alpha \in \mathcal{P}_X \mid \ker(\alpha) = \operatorname{coker}(\alpha) = \Delta\}$,
- $\mathcal{J}_X = \{\alpha \in \mathcal{P}_X \mid \operatorname{dom}(\alpha) = \operatorname{codom}(\alpha) = X\}$, and
- $\mathcal{T}_X = \{\alpha \in \mathcal{P}_X \mid \operatorname{dom}(\alpha) = X \text{ and } \operatorname{coker}(\alpha) = \Delta\}$.

It is easy to see that these are submonoids of $\mathcal{P}_X$, and are isomorphic to the symmetric inverse monoid $\mathscr{I}_X$, the dual symmetric inverse monoid $\mathscr{I}_X^*$, and the full transformation semigroup $\mathscr{T}_X$ (respectively). See [10,14,17,20–22,27] for further information on these monoids. In this way, we will regard $\mathscr{I}_X$ and $\mathcal{I}_X$ as one and the same thing, and so too for the other two monoids. So, for example, if $\alpha \in \mathcal{T}_X$ and $x \in X$, we will sometimes write $x\alpha$ for "the image of x under α"; i.e. the unique element of X that satisfies $(x\alpha)' \in [x]_\alpha$. Note that here we prefer to use the notation $\mathcal{J}_X$ for (the isomorphic copy of) the dual symmetric inverse monoid in order to avoid confusion with the * map introduced above. The monoids $\mathcal{I}_X$ and $\mathcal{J}_X$ are closed under the * map, but $\mathcal{T}_X$ is sent to

- $\mathcal{T}_X^* = \{\alpha \in \mathcal{P}_X \mid \operatorname{codom}(\alpha) = X \text{ and } \ker(\alpha) = \Delta\}$,

an anti-isomorphic copy of $\mathcal{T}_X$. The intersection of all four of these monoids is the set

- $\mathcal{S}_X = \{\alpha \in \mathcal{P}_X \mid \ker(\alpha) = \operatorname{coker}(\alpha) = \Delta \text{ and } \operatorname{dom}(\alpha) = \operatorname{codom}(\alpha) = X\}$,

and is isomorphic to (and will be identified with) the symmetric group $\mathscr{S}_X$, and is easily seen to be the group of units of $\mathcal{P}_X$.

3. Idempotents and projections

As usual, we denote by $E(\mathcal{P}_X)$ the set of all idempotents of $\mathcal{P}_X$ (i.e. those partitions α that satisfy $\alpha^2 = \alpha$), and write $\mathbb{E}(\mathcal{P}_X) = \langle E(\mathcal{P}_X)\rangle$ for the subsemigroup of $\mathcal{P}_X$ generated by its idempotents. Note that by "generated" we will always mean "generated as a semigroup". An idempotent partition α is called a *projection* if it additionally satisfies $\alpha^* = \alpha$. (Note that a partition satisfying $\alpha^* = \alpha$ need not be an idempotent.) Write $P(\mathcal{P}_X)$ for the set of all projections of $\mathcal{P}_X$, and $\mathbb{P}(\mathcal{P}_X) = \langle P(\mathcal{P}_X)\rangle$ for the subsemigroup of $\mathcal{P}_X$ generated by its projections. The next lemma is true for any regular $*$-semigroup; the proof may be found in [18].

Lemma 2. *We have*

(i) $P(\mathcal{P}_X) = \{\alpha\alpha^* \mid \alpha \in \mathcal{P}_X\}$,
(ii) $E(\mathcal{P}_X) = \{\alpha\beta \mid \alpha, \beta \in P(\mathcal{P}_X)\}$, *and*
(iii) $\mathbb{E}(\mathcal{P}_X) = \mathbb{P}(\mathcal{P}_X)$.

Remark 3. The proof of (ii) makes use of (i) and the fact that $\alpha = (\alpha\alpha^*)(\alpha^*\alpha)$ for any idempotent $\alpha \in E(\mathcal{P}_X)$.

This lemma allows us to concentrate on the more manageable set of projections. In contrast to the idempotents, the projections of $\mathcal{P}_X$ are rather easy to describe. The proof of the following is routine, and is omitted.

Lemma 4. *A partition is a projection if and only if it is of the form*

$$\begin{pmatrix} A_i & C_j \\ A_i & C_j \end{pmatrix}_{i\in I,\, j\in J}.$$

As usual, for a subset $Y \subseteq X$ we write $\mathrm{id}_Y \in \mathcal{I}_X$ for the restriction of the identity function to Y. That is,

$$\mathrm{id}_Y = \begin{pmatrix} y & x \\ y & x \end{pmatrix}_{y\in Y,\, x\in X\setminus Y}.$$

Here we use shorthand notation, and write x instead of $\{x\}$, etc.

A *quotient* of X is a collection $\mathbf{Y}$ of pairwise disjoint nonempty subsets of X whose union is X. (We refrain from calling such a $\mathbf{Y}$ a partition of X for obvious reasons.) We write $\mathbf{Y} \preccurlyeq X$ to indicate that $\mathbf{Y}$ is a quotient of X. For $\mathbf{Y} \preccurlyeq X$, we write $\mathrm{id}_{\mathbf{Y}} \in \mathcal{J}_X$ for the restriction of the identity function to $\mathbf{Y}$. That is,

$$\mathrm{id}_{\mathbf{Y}} = \begin{pmatrix} Y \\ Y \end{pmatrix}_{Y\in\mathbf{Y}}.$$

The shorthand notation used here indicates that there are no nontransversal blocks.

Suppose now that $\mathbf{Y} \preccurlyeq X$ and $Z \subseteq X$ is such that each block of $\mathbf{Y}$ intersects Z in exactly one point. Such a set Z is called a *cross-section* of $\mathbf{Y}$. For $z \in Z$, write Y_z for the block of $\mathbf{Y}$ that contains z. We write $\varepsilon_{\mathbf{Y},Z}$ for the partition

$$\varepsilon_{\mathbf{Y},Z} = \begin{pmatrix} Y_z & \emptyset \\ z & x \end{pmatrix}_{z\in Z,\, x\in X\setminus Z}.$$

Note that in fact $\varepsilon_{\mathbf{Y},Z} \in \mathcal{T}_X$. We will also write $\eta_{Z,\mathbf{Y}} = \varepsilon^*_{\mathbf{Y},Z} \in \mathcal{T}^*_X$. That is,

$$\eta_{Z,\mathbf{Y}} = \begin{pmatrix} z & x \\ Y_z & \emptyset \end{pmatrix}_{z\in Z,\, x\in X\setminus Z}.$$

The results contained in the following lemma are well-known; proofs may be found in various places, including [10,17,20].

Lemma 5. *We have*

- $E(\mathcal{I}_X) = \{\mathrm{id}_Y \mid Y \subseteq X\}$,
- $E(\mathcal{J}_X) = \{\mathrm{id}_{\mathbf{Y}} \mid \mathbf{Y} \preccurlyeq X\}$,
- $E(\mathcal{T}_X) = \{\varepsilon_{\mathbf{Y},Z} \mid \mathbf{Y} \preccurlyeq X,\ Z \text{ a cross-section of } \mathbf{Y}\}$, *and*
- $E(\mathcal{T}_X^*) = \{\eta_{Z,\mathbf{Y}} \mid \mathbf{Y} \preccurlyeq X,\ Z \text{ a cross-section of } \mathbf{Y}\}$.

We are now ready to prove the main preliminary result of this section. Apart from being interesting in its own right, the first equality in particular will be crucial to our methods in later sections.

Proposition 6. *We have* $\mathbb{E}(\mathcal{P}_X) = \langle E(\mathcal{I}_X) \cup E(\mathcal{J}_X)\rangle = \langle E(\mathcal{T}_X) \cup E(\mathcal{T}_X^*)\rangle$.

Proof. Consider a projection

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline A_i & C_j \end{array}\right)_{i\in I,\ j\in J}.$$

Put $\mathbf{Y} = X/\ker(\alpha) = \{A_i \mid i \in I\} \cup \{C_j \mid j \in J\}$ and $Z = \mathrm{dom}(\alpha) = \bigcup_{i\in I} A_i$. Then $\alpha = \mathrm{id}_{\mathbf{Y}}\mathrm{id}_Z\mathrm{id}_{\mathbf{Y}}$ which, together with Lemma 2, establishes the first equality.

The second equality follows from the first, together with the observation that

$$\eta_{Z,\mathbf{Y}}\varepsilon_{\mathbf{Y},Z} = \mathrm{id}_Z \quad \text{and} \quad \varepsilon_{\mathbf{Y},Z}\eta_{Z,\mathbf{Y}} = \mathrm{id}_{\mathbf{Y}}$$

whenever Z is a cross-section of $\mathbf{Y} \preccurlyeq X$. □

The next result shows that no non-trivial permutation may be expressed as a product of idempotents, and will be useful on a number of occasions.

Lemma 7. *We have* $\mathcal{S}_X \cap \mathbb{E}(\mathcal{P}_X) = \{1\}$.

Proof. Let $\pi \in \mathcal{S}_X \cap \mathbb{E}(\mathcal{P}_X)$ and write $\pi = \rho_1 \cdots \rho_k$, where $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$ and k is minimal. Now $\Delta = \ker(\pi) = \ker(\rho_1 \cdots \rho_k) \supseteq \ker(\rho_1) \supseteq \Delta$, which implies that $\ker(\rho_1) = \Delta$, and so $\rho_1 \in E(\mathcal{I}_X)$. A similar argument, considering the domain, shows that $\rho_1 \in E(\mathcal{J}_X)$. But $E(\mathcal{I}_X) \cap E(\mathcal{J}_X) = \{1\}$ by Lemma 5, so $\rho_1 = 1$. Now, if $k \geqslant 2$, then we would have $\pi = 1(\rho_2 \cdots \rho_k) = \rho_2 \cdots \rho_k$, contradicting the minimality of k. It follows that $k = 1$, and $\pi = \rho_1 = 1$. □

4. Finite $\mathbb{E}(\mathcal{P}_X)$

In this section we describe $\mathbb{E}(\mathcal{P}_X)$ in the case that X is finite. In fact, our description is exactly analogous to Howie's description of the semigroup generated by the idempotents of a finite full transformation semigroup, which we now state.

Theorem 8. *(See Howie [16, Theorem I].) Let X be a finite set. Then*

$$\mathbb{E}(\mathcal{T}_X) = \{1\} \cup (\mathcal{T}_X \setminus \mathcal{S}_X).$$

The following result may also be found in [8], where it was deduced from more general results concerning presentations. For completeness, we provide a different, and more direct proof, relying on Theorem 8.

Theorem 9. *Let X be a finite set. Then*

$$\mathbb{E}(\mathcal{P}_X) = \{1\} \cup (\mathcal{P}_X \setminus \mathcal{S}_X).$$

Proof. Without loss of generality, we may assume $X = \{1, \dots, n\}$. By Lemma 7, it suffices to show that $\mathcal{P}_X \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{P}_X)$, so suppose we are given $\alpha \in \mathcal{P}_X \setminus \mathcal{S}_X$, and write

$$\alpha = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_i & C_1 & \cdots & C_j \\ \hline B_1 & \cdots & B_i & D_1 & \cdots & D_k \end{array}\right).$$

Put $\varepsilon = \mathrm{id}_{\{1,\dots,i\}} \in E(\mathcal{I}_X)$, and let

$$\tau_1 = \begin{pmatrix} A_1 & \cdots & A_i & C_1 & \cdots & C_j \\ 1 & \cdots & i & i+1 & \cdots & i+j \end{pmatrix} \in \mathcal{T}_X,$$

$$\tau_2 = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & i+k \\ B_1 & \cdots & B_i & D_1 & \cdots & D_k \end{pmatrix} \in \mathcal{T}_X^*.$$

We have not indicated the lower (respectively, upper) nontransversal singleton blocks in τ_1 (respectively, τ_2). It is clear that $\alpha = \tau_1 \varepsilon \tau_2$. If $\ker(\alpha) \neq \Delta$ and $\mathrm{coker}(\alpha) \neq \Delta$, then $\tau_1 \in \mathcal{T}_X \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{T}_X)$ and $\tau_2 \in \mathcal{T}_X^* \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{T}_X^*)$, so $\alpha \in \mathbb{E}(\mathcal{P}_X)$ in this case. If $\ker(\alpha) \neq \Delta$ and $\mathrm{coker}(\alpha) = \Delta$, then $\tau_2 \in \mathcal{S}_X$, and so $\alpha = \tau_1\tau_2(\tau_2^{-1}\varepsilon\tau_2) \in \mathbb{E}(\mathcal{P}_X)$ since $\tau_1\tau_2 \in \mathcal{T}_X \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{T}_X)$ and $\tau_2^{-1}\varepsilon\tau_2 \in E(\mathcal{I}_X)$. A dual argument covers the case in which $\ker(\alpha) = \Delta$ and $\mathrm{coker}(\alpha) \neq \Delta$. For the final case, suppose $\ker(\alpha) = \Delta = \mathrm{coker}(\alpha)$. (In this case, $\alpha \in \mathcal{I}_X \setminus \mathcal{S}_X$.) So $\alpha = \eta\tau$, where $\eta = \tau_1\varepsilon\tau_1^{-1} \in E(\mathcal{I}_X)$, and $\tau = \tau_1\tau_2 \in \mathcal{S}_X$. If $\eta = \mathrm{id}_\emptyset$, then $\alpha = \eta \in E(\mathcal{P}_X)$. Otherwise, choose $x \in X \setminus \mathrm{dom}(\eta)$ and $y \in \mathrm{dom}(\eta)$, and let $\sigma \in E(\mathcal{T}_X)$ act as the identity on $X \setminus \{x\}$, and map x to y. Clearly $\eta = \eta\sigma$, and so $\alpha = \eta(\sigma\tau) \in \mathbb{E}(\mathcal{P}_X)$, since $\sigma\tau \in \mathcal{T}_X \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{T}_X)$. □

Remark 10. It is also possible to deduce Theorem 8 from Theorem 9, or indeed from [6, Theorem 50].

As a corollary, we may deduce the following result found in [8].

Theorem 11. *Let $X = \{1, \dots, n\}$. For $1 \leqslant i \leqslant n$ and $1 \leqslant r < s \leqslant n$, let*

$$\varepsilon_i = \mathrm{id}_{X\setminus\{i\}} \quad \textit{and} \quad \eta_{rs} = \mathrm{id}_{\mathbf{Y}_{rs}},$$

where $\mathbf{Y}_{rs} \preccurlyeq X$ has $\{r, s\}$ as its only non-trivial block. Then $\mathcal{P}_X \setminus \mathcal{S}_X$ is generated by the set

$$\{\varepsilon_i \mid 1 \leqslant i \leqslant n\} \cup \{\eta_{rs} \mid 1 \leqslant r < s \leqslant n\}.$$

Proof. It follows from Proposition 6 and Theorem 9 that $\{1\} \cup (\mathcal{P}_X \setminus \mathcal{S}_X)$ is generated by $E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. It is well-known that $E(\mathcal{I}_X)$ and $E(\mathcal{J}_X)$ are generated (as monoids) by the ε_i and η_{rs} respectively; see [24, p. 115] and [9, Theorem 2]. The result now follows from the fact that the identity $1 \in \mathcal{P}_X$ cannot be expressed as the product of non-trivial idempotents. □

Remark 12. Defining relations were given for the above generating set in [8, Theorem 46], and it was also shown [8, Theorem 19] that this set has minimal cardinality among all generating sets for $\mathcal{P}_X \setminus \mathcal{S}_X$.

5. Collapse and defect

In contrast to the finite case, $\mathcal{P}_X \setminus \mathcal{S}_X$ is not closed under multiplication if X is infinite; indeed we have $(\mathcal{P}_X \setminus \mathcal{S}_X)^2 = \mathcal{P}_X$. This therefore makes infinite $\mathbb{E}(\mathcal{P}_X)$ harder to describe. As in Howie's treatment of infinite $\mathbb{E}(\mathcal{T}_X)$ in [16], we make use of a number of parameters, which we define in this section and the next. In [16], the collapse, defect and shift of a mapping from $\mathcal{T}_X$ were defined, and the infinitary elements of infinite $\mathbb{E}(\mathcal{T}_X)$—those that move an infinite number of points—were characterized as those mappings that are uniform in the sense of having equal (and infinite) collapse, defect and shift. In this section we introduce the collapse and defect (and dually, the cocollapse and codefect) of a partition, and show that a member of $\mathbb{E}(\mathcal{P}_X)$ is uniform in the sense that the sum of its collapse and defect is equal to the sum of its cocollapse and codefect. Note that these definitions, although similar, are not identical to those in [16]. In Section 8 we characterize the infinitary members of $\mathbb{E}(\mathcal{P}_X)$ as those infinitary partitions that satisfy this uniformity condition and also a further condition on their shift; we postpone the definition of shift until the next section, where we also formally introduce the concept of an infinitary partition (for now it is sufficient to note that the definition is analogous to that of infinitary transformations). The definitions and results of the current section are valid regardless of whether X is finite or infinite, so we do not make any assumptions on the cardinality of X at present.

Consider a partition $\alpha \in \mathcal{P}_X$, and write

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i\in I,\, j\in J,\, k\in K}.$$

The *collapse*, *cocollapse*, *defect*, and *codefect* of α are defined to be the cardinals

$$\operatorname{col}(\alpha) = \sum_{i\in I}\big(|A_i| - 1\big), \qquad \operatorname{def}(\alpha) = \sum_{j\in J}|C_j|,$$
$$\operatorname{cocol}(\alpha) = \sum_{i\in I}\big(|B_i| - 1\big), \qquad \operatorname{codef}(\alpha) = \sum_{k\in K}|D_k|.$$

Note that these definitions have similar meanings to the various parameters introduced in [16] (see also [15]) in the context of the full transformation semigroup $\mathcal{T}_X$. However, there are certain differences here (even when $\alpha \in \mathcal{T}_X$), so the reader should not be tempted to draw too many comparisons.

$$\begin{aligned}
\operatorname{def}(\alpha) = \operatorname{cocol}(\alpha) = 0 &\quad \text{if } \alpha \in \mathcal{T}_X,\\
\operatorname{codef}(\alpha) = \operatorname{col}(\alpha) = 0 &\quad \text{if } \alpha \in \mathcal{T}_X^*,\\
\operatorname{col}(\alpha) = \operatorname{cocol}(\alpha) = 0 &\quad \text{if } \alpha \in \mathcal{I}_X,\\
\operatorname{def}(\alpha) = \operatorname{codef}(\alpha) = 0 &\quad \text{if } \alpha \in \mathcal{J}_X.
\end{aligned}$$

As alluded to above, we will be particularly interested in various sums of these parameters. With this in mind, we define the *singularity* and *cosingularity* of $\alpha \in \mathcal{P}_X$ to be

$$\operatorname{sing}(\alpha) = \operatorname{col}(\alpha) + \operatorname{def}(\alpha), \qquad \operatorname{cosing}(\alpha) = \operatorname{cocol}(\alpha) + \operatorname{codef}(\alpha).$$

Our main goal in this section is to show that $\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha)$ for all $\alpha \in \mathbb{E}(\mathcal{P}_X)$; see Proposition 15 below. The proof of Proposition 15 uses induction, and the following two lemmas serve to establish the inductive step. On a number of occasions during the proofs, we will make use of the fact that whenever $\{A_i \mid i \in I\}$ is a collection of pairwise disjoint nonempty sets, we have the equality

$$\sum_{i\in I} |A_i| = \sum_{i\in I} \big(|A_i| - 1\big) + |I|.$$

Lemma 13. *Suppose $\alpha \in \mathcal{P}_X$ and $\rho \in E(\mathcal{I}_X)$. Then*

$$\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha) \quad \Rightarrow \quad \operatorname{sing}(\alpha\rho) = \operatorname{cosing}(\alpha\rho).$$

Proof. Let $\rho = \mathrm{id}_Y$, where $Y \subseteq X$, and write

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array} \right)_{i\in I,\, j\in J,\, k\in K}.$$

Consider the decomposition $I = I_1 \cup I_2$, where

$$I_1 = \{i \in I \mid B_i \cap Y \neq \emptyset\} \quad \text{and} \quad I_2 = \{i \in I \mid B_i \cap Y = \emptyset\}.$$

Then

$$\alpha\rho = \left(\begin{array}{c|c} A_{i_1} & A_{i_2}, C_j \\ \hline B_{i_1} \cap Y & D_k \cap Y, x \end{array} \right)_{i_1\in I_1,\, i_2\in I_2,\, j\in J,\, k\in K,\, x\in X\setminus Y}.$$

(Note that some of the $D_k \cap Y$ may be empty.) We then have

$$\operatorname{col}(\alpha\rho) = \sum_{i_1\in I_1} \big(|A_{i_1}| - 1\big) \quad \text{and}$$

$$\operatorname{def}(\alpha\rho) = \sum_{i_2\in I_2} |A_{i_2}| + \sum_{j\in J} |C_j| = \sum_{i_2\in I_2} \big(|A_{i_2}| - 1\big) + |I_2| + \sum_{j\in J} |C_j|.$$

Thus,

$$\operatorname{col}(\alpha\rho) + \operatorname{def}(\alpha\rho) = \sum_{i\in I} \big(|A_i| - 1\big) + \sum_{j\in J} |C_j| + |I_2| = \operatorname{col}(\alpha) + \operatorname{def}(\alpha) + |I_2|.$$

That is,

$$\operatorname{sing}(\alpha\rho) = \operatorname{sing}(\alpha) + |I_2|. \tag{13.1}$$

On the other hand, we have

$$\operatorname{cocol}(\alpha\rho) = \sum_{i_1\in I_1} \big(|B_{i_1} \cap Y| - 1\big) \quad \text{and} \quad \operatorname{codef}(\alpha\rho) = \sum_{k\in K} |D_k \cap Y| + |X \setminus Y|.$$

Now

$$X \setminus Y = \Big((X \setminus Y) \cap \bigcup_{k\in K} D_k\Big) \cup \Big((X \setminus Y) \cap \bigcup_{i\in I} B_i\Big)$$
$$= \bigcup_{k\in K} \big((X \setminus Y) \cap D_k\big) \cup \bigcup_{i\in I} \big((X \setminus Y) \cap B_i\big).$$

Since all the unions are disjoint unions, it follows that

$$\operatorname{codef}(\alpha\rho) = \sum_{k\in K} |D_k \cap Y| + \sum_{k\in K} \big|(X \setminus Y) \cap D_k\big| + \sum_{i\in I} \big|(X \setminus Y) \cap B_i\big|$$
$$= \sum_{k\in K} |D_k| + \sum_{i_1\in I_1} \big|(X \setminus Y) \cap B_{i_1}\big| + \sum_{i_2\in I_2} |B_{i_2}|.$$

Thus,

$$\operatorname{cocol}(\alpha\rho) + \operatorname{codef}(\alpha\rho) = \sum_{i_1\in I_1} \big(|B_{i_1} \cap Y| - 1\big) + \sum_{i_1\in I_1} \big|(X \setminus Y) \cap B_{i_1}\big| + \sum_{i_2\in I_2} |B_{i_2}| + \sum_{k\in K} |D_k|$$
$$= \sum_{i_1\in I_1} \big(|B_{i_1}| - 1\big) + \sum_{i_2\in I_2} \big(|B_{i_2}| - 1\big) + |I_2| + \sum_{k\in K} |D_k|$$
$$= \sum_{i\in I} \big(|B_i| - 1\big) + \sum_{k\in K} |D_k| + |I_2|$$
$$= \operatorname{cocol}(\alpha) + \operatorname{codef}(\alpha) + |I_2|.$$

That is,

$$\operatorname{cosing}(\alpha\rho) = \operatorname{cosing}(\alpha) + |I_2|. \tag{13.2}$$

Clearly Eqs. (13.1) and (13.2) give us the implication in the statement of the lemma. □

Lemma 14. *Suppose $\alpha \in \mathcal{P}_X$ and $\rho \in E(\mathcal{J}_X)$. Then*

$$\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha) \quad \Rightarrow \quad \operatorname{sing}(\alpha\rho) = \operatorname{cosing}(\alpha\rho).$$

Proof. We first show that we may replace ρ by a simpler idempotent, thus reducing the complexity of the task. Now $\rho = \mathrm{id}_{\mathbf{Y}}$ for some $\mathbf{Y} \preccurlyeq X$. Let ε be the equivalence relation on X corresponding to $\mathbf{Y}$. That is, two elements of X are ε-related if and only if they belong to the same block of $\mathbf{Y}$, and we have $\mathbf{Y} = X/\varepsilon$. Put $\eta = \operatorname{coker}(\alpha)$, and let $\mathbf{Z} = X/\eta$. Clearly we have $\alpha = \alpha\mathrm{id}_{\mathbf{Z}}$. So $\alpha\rho = \alpha\mathrm{id}_{\mathbf{Z}}\mathrm{id}_{\mathbf{Y}} = \alpha\mathrm{id}_{\mathbf{W}}$, where $\mathbf{W} = X/(\varepsilon \vee \eta)$; here $\varepsilon \vee \eta$ denotes the least equivalence on X containing both ε and η. Let us write

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array}\right)_{i\in I,\, j\in J,\, k\in K},$$

and $\mathbf{W} = \{W_l \mid l \in L\}$. To avoid notational ambiguity later in the proof, we will assume that I and L are disjoint, and also that L does not contain the symbols 1 and 2. Note that $\mathbf{Z} = \{B_i \mid i \in I\} \cup \{D_k \mid k \in K\}$.

Now, since $\eta \subseteq \varepsilon \vee \eta$, every block of $\mathbf{Z}$ is contained in a block of $\mathbf{W}$. So we have functions $\beta : I \to L$ and $\delta : K \to L$ defined by $B_i \subseteq W_{i\beta}$ and $D_k \subseteq W_{k\delta}$. For $l \in L$, define $I_l = l\beta^{-1}$ and $K_l = l\delta^{-1}$. Also put

$$L_1 = \mathrm{im}(\beta), \qquad L_2 = L \setminus \mathrm{im}(\beta), \qquad K_1 = L_1\delta^{-1}, \qquad K_2 = L_2\delta^{-1}.$$

For $l \in L_1$, write $A_l = \bigcup_{i \in I_l} A_i$. We then have

$$\alpha\rho = \alpha \mathrm{id}_{\mathbf{W}} = \begin{pmatrix} A_{l_1} & C_j \\ W_{l_1} & W_{l_2} \end{pmatrix}_{l_1 \in L_1,\, j \in J,\, l_2 \in L_2}.$$

Now, for a fixed $l \in L_1$, we have

$$|A_l| = \sum_{i \in I_l} |A_i| = \sum_{i \in I_l} \big(|A_i| - 1\big) + |I_l|,$$

so that

$$\begin{aligned} \mathrm{col}(\alpha\rho) &= \sum_{l \in L_1} \big(|A_l| - 1\big) \\ &= \sum_{l \in L_1} \left(\sum_{i \in I_l} \big(|A_i| - 1\big) + |I_l| - 1 \right) \\ &= \sum_{i \in I} \big(|A_i| - 1\big) + \sum_{l \in L_1} \big(|I_l| - 1\big) \\ &= \mathrm{col}(\alpha) + \sum_{l \in L_1} \big(|I_l| - 1\big). \end{aligned}$$

Since we also have $\mathrm{def}(\alpha\rho) = \mathrm{def}(\alpha)$, it follows that

$$\mathrm{sing}(\alpha\rho) = \mathrm{sing}(\alpha) + \sum_{l \in L_1} \big(|I_l| - 1\big). \tag{14.1}$$

Next, note that for fixed $l \in L_1$,

$$|W_l| = \sum_{i \in I_l} |B_i| + \sum_{k \in K_l} |D_k| = \sum_{i \in I_l} \big(|B_i| - 1\big) + |I_l| + \sum_{k \in K_l} |D_k|,$$

so that

$$\begin{aligned} \mathrm{cocol}(\alpha\rho) &= \sum_{l \in L_1} \big(|W_l| - 1\big) \\ &= \sum_{l \in L_1} \left(\sum_{i \in I_l} \big(|B_i| - 1\big) + \sum_{k \in K_l} |D_k| + |I_l| - 1 \right) \\ &= \sum_{i \in I} \big(|B_i| - 1\big) + \sum_{k \in K_1} |D_k| + \sum_{l \in L_1} \big(|I_l| - 1\big). \end{aligned}$$

We also have

$$\operatorname{codef}(\alpha\rho) = \sum_{l\in L_2} |W_l| = \sum_{l\in L_2}\sum_{k\in K_l} |D_k| = \sum_{k\in K_2} |D_k|.$$

It follows that

$$\begin{aligned}
\operatorname{cocol}(\alpha\rho) + \operatorname{codef}(\alpha\rho) &= \sum_{i\in I}\big(|B_i|-1\big) + \sum_{k_1\in K_1} |D_{k_1}| + \sum_{k_2\in K_2} |D_{k_2}| + \sum_{l\in L_1}\big(|I_l|-1\big) \\
&= \sum_{i\in I}\big(|B_i|-1\big) + \sum_{k\in K} |D_k| + \sum_{l\in L_1}\big(|I_l|-1\big) \\
&= \operatorname{cocol}(\alpha) + \operatorname{codef}(\alpha) + \sum_{l\in L_1}\big(|I_l|-1\big).
\end{aligned}$$

That is,

$$\operatorname{cosing}(\alpha\rho) = \operatorname{cosing}(\alpha) + \sum_{l\in L_1}\big(|I_l|-1\big). \tag{14.2}$$

Again, Eqs. (14.1) and (14.2) give us the required implication. □

We are now ready to prove the main result of this section.

Proposition 15. *If* $\alpha \in \mathbb{E}(\mathcal{P}_X)$, *then* $\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha)$.

Proof. By Proposition 6, $\alpha = \rho_1 \cdots \rho_k$ for some $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. The $k = 1$ case follows immediately from Lemma 5. Lemmas 13 and 14 provide the inductive step. □

Remark 16. If X is finite, then all elements of $\mathcal{P}_X$ satisfy the identity in Proposition 15, as may easily be checked by directly evaluating both sides of the identity in terms of the general form

$$\alpha = \left(\begin{array}{ccc|ccc} A_1 & \cdots & A_i & C_1 & \cdots & C_j \\ \hline B_1 & \cdots & B_i & D_1 & \cdots & D_k \end{array}\right)$$

of an element $\alpha \in \mathcal{P}_X$. This is not the case for infinite $\mathcal{P}_X$; indeed, even if $\alpha \in \mathcal{T}_X$ (or $\mathcal{I}_X$ or $\mathcal{J}_X$), the discrepancy between $\operatorname{sing}(\alpha)$ and $\operatorname{cosing}(\alpha)$ could be any cardinal from 0 to $|X|$.

The following will also be of use on a number of occasions.

Lemma 17. *If* $\alpha, \beta \in \mathbb{E}(\mathcal{P}_X)$, *then* $\operatorname{sing}(\alpha\beta) \geqslant \max(\operatorname{sing}(\alpha), \operatorname{sing}(\beta))$.

Proof. Write $\beta = \rho_1 \cdots \rho_k$, where $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. By Eqs. (13.1) and (14.1), and a simple induction on k, we see that $\operatorname{sing}(\alpha\beta) \geqslant \operatorname{sing}(\alpha)$. Using Proposition 15 and the inequality just established, we also have

$$\operatorname{sing}(\alpha\beta) = \operatorname{cosing}(\alpha\beta) = \operatorname{sing}\big((\alpha\beta)^*\big) = \operatorname{sing}\big(\beta^*\alpha^*\big) \geqslant \operatorname{sing}\big(\beta^*\big) = \operatorname{cosing}(\beta) = \operatorname{sing}(\beta).$$

This completes the proof. □

6. Shift and warp

In this section, we identify two more parameters associated to a partition: namely, the shift and warp. The shift will play a key part in our description of infinite $\mathbb{E}(\mathcal{P}_X)$ in Section 8, and the warp will allow us to distinguish those partitions that behave like finite ones. Again, the definitions do not require X to be infinite, so we do not assume it is. Note however that some of the assumptions in the statements of results will imply X is infinite.

Let $\alpha \in \mathcal{P}_X$ and write

$$\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\, j\in J,\, k\in K}.$$

The *shift set* of α (with respect to the choice of indexing sets) is defined to be

$$\mathrm{Sh}(\alpha) = \{i \in I \mid A_i \cap B_i = \emptyset\}.$$

The *shift* of α, denoted $\mathrm{sh}(\alpha)$, is defined to be the cardinality of $\mathrm{Sh}(\alpha)$, and is independent of the indexing set used.

Lemma 18. *Suppose $\alpha \in \mathcal{P}_X$ and $\rho = \mathrm{id}_Y \in E(\mathcal{I}_X)$. Then* $\mathrm{sh}(\alpha\rho) \leqslant \mathrm{sh}(\alpha) + |X \setminus Y|$.

Proof. Write

$$\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\, j\in J,\, k\in K}.$$

As in the proof of Lemma 13, the transversal blocks of $\alpha\rho$ are all of the form $A_i \cup (B_i \cap Y)'$, where $i \in I$ is such that $B_i \cap Y \neq \emptyset$. Such an $i \in I$ can only belong to $\mathrm{Sh}(\alpha)$ if either

(i) $A_i \cap B_i = \emptyset$ and $B_i \cap Y \neq \emptyset$, or
(ii) $A_i \cap B_i \neq \emptyset$ but $A_i \cap (B_i \cap Y) = \emptyset$.

Clearly there are at most $\mathrm{sh}(\alpha)$ values of i that satisfy (i). Now (ii) holds if and only if $\emptyset \neq A_i \cap B_i \subseteq X \setminus Y$, so there are at most $|X \setminus Y|$ values of i that satisfy (ii). The result follows. □

Lemma 19. *Suppose $\alpha \in \mathcal{P}_X$ and $\rho \in E(\mathcal{J}_X)$. Then* $\mathrm{sh}(\alpha\rho) \leqslant \mathrm{sh}(\alpha)$.

Proof. Again, write

$$\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\, j\in J,\, k\in K}.$$

As in the proof of Lemma 14, we may assume that $\rho = \mathrm{id}_{\mathbf{W}}$ where $\mathbf{W} = \{W_l \mid l \in L\}$ is such that every block of $\mathrm{coker}(\alpha)$ is contained in some block of $\mathbf{W}$. Again we assume that $I \cap L = \emptyset$. From the same proof, we also know that every transversal block of $\alpha\rho$ is of the form $A_l \cup W_l'$ for some $l \in L$, where $A_l = \bigcup_{i\in I_l} A_i$ with $I_l = \{i \in I \mid B_i \subseteq W_l\}$ nonempty. If such an l belongs to $\mathrm{Sh}(\alpha\rho)$, then for any $i \in I_l$, we have $A_i \cap B_i \subseteq A_l \cap W_l = \emptyset$, so that $I_l \subseteq \mathrm{Sh}(\alpha)$. The result follows. □

Before we move on to the next series of lemmas, we make some further definitions. The *warp set* of a partition $\alpha \in \mathcal{P}_X$ is

$$\mathrm{Warp}(\alpha) = \big\{x \in X \;\big|\; [x]_\alpha \neq \{x, x'\}\big\}.$$

(If $Y \subseteq X$, the natural embedding $\mathcal{P}_Y \hookrightarrow \mathcal{P}_X$ maps $\mathcal{P}_Y$ isomorphically onto the set $\{\alpha \in \mathcal{P}_X \mid \mathrm{Warp}(\alpha) \subseteq Y\}$.) We will also write $\mathrm{warp}(\alpha)$ for the cardinality of $\mathrm{Warp}(\alpha)$, and call this cardinal the *warp* of α. We will call α *finitary* or *infinitary* according to whether $\mathrm{warp}(\alpha) < \aleph_0$ or $\mathrm{warp}(\alpha) \geqslant \aleph_0$.

Lemma 20. *If $\alpha \in \mathcal{P}_X$ is finitary and $\rho \in E(\mathcal{I}_X)$ infinitary, then* $\mathrm{codef}(\alpha\rho) \geqslant \aleph_0$.

Proof. It is easy to see that $\mathrm{codef}(\gamma\delta) \geqslant \mathrm{codef}(\delta)$ for all $\gamma, \delta \in \mathcal{P}_X$, and the result follows. □

Lemma 21. *If $\alpha \in \mathcal{P}_X$ is finitary and $\rho \in E(\mathcal{J}_X)$ infinitary, then* $\mathrm{cocol}(\alpha\rho) \geqslant \aleph_0$.

Proof. Again we assume that $\rho = \mathrm{id}_{\mathbf{W}}$, where $\mathbf{W} = \{W_l \mid l \in L\}$ is such that every block of $\mathrm{coker}(\alpha)$ is contained in some W_l. Since ρ is infinitary, either

(i) there exists some $l \in L$ with $|W_l| \geqslant \aleph_0$, or
(ii) there exist infinitely many $l_1, l_2, \ldots \in L$ with $|W_{l_1}|, |W_{l_2}|, \ldots \geqslant 2$.

If (i) holds, then $\alpha\rho$ contains a transversal block of the form $Z \cup W_l'$, where Z is some subset of $W_l \cup \mathrm{Warp}(\alpha)$, and it follows that $\mathrm{cocol}(\alpha\rho) \geqslant |W_l| - 1 \geqslant \aleph_0$. If (ii) holds, then $\mathrm{Warp}(\alpha)$ intersects only finitely many of the sets $W_{l_1}, W_{l_2}, \ldots$, so we may suppose that $W_{l_r}, W_{l_{r+1}}, \ldots$ all have empty intersection with $\mathrm{Warp}(\alpha)$, for some $r \geqslant 1$. Then $\alpha\rho$ contains the transversal blocks $W_{l_t} \cup W_{l_t}'$ for $t = r, r+1, \ldots$ and so, in particular, we have $\mathrm{cocol}(\alpha\rho) \geqslant (|W_{l_r}| - 1) + (|W_{l_{r+1}}| - 1) + \cdots \geqslant \aleph_0$. □

Lemma 22. *If $\alpha \in \mathbb{E}(\mathcal{P}_X)$ is infinitary, then* $\mathrm{sing}(\alpha) \geqslant \aleph_0$.

Proof. By Proposition 6, $\alpha = \rho_1 \cdots \rho_k$ for some $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. If $k = 1$, then we are done, after consulting Lemma 5, so suppose $k \geqslant 2$. If $\rho_1 \cdots \rho_{k-1}$ is infinitary, then Lemma 17 and an induction hypothesis give $\mathrm{sing}(\alpha) \geqslant \mathrm{sing}(\rho_1 \cdots \rho_{k-1}) \geqslant \aleph_0$. If $\rho_1 \cdots \rho_{k-1}$ is finitary, then ρ_k is not. If $\rho_k \in E(\mathcal{I}_X)$, then by Proposition 15 and Lemma 20 we have

$$\mathrm{sing}(\alpha) = \mathrm{cosing}(\alpha) \geqslant \mathrm{codef}(\alpha) = \mathrm{codef}\big((\rho_1 \cdots \rho_{k-1})\rho_k\big) \geqslant \aleph_0,$$

and a similar calculation, invoking Lemma 21, yields the same conclusion if $\rho_k \in E(\mathcal{J}_X)$. □

We are now ready to prove the main result of this section.

Proposition 23. *If $\alpha \in \mathbb{E}(\mathcal{P}_X)$ is infinitary, then* $\mathrm{sing}(\alpha) \geqslant \mathrm{sh}(\alpha)$.

Proof. By Proposition 6 we have $\alpha = \rho_1 \cdots \rho_k$ for some $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. If $k = 1$, then $\mathrm{sh}(\alpha) = 0$ and we are done, so suppose $k \geqslant 2$. For simplicity, write $\alpha' = \rho_1 \cdots \rho_{k-1}$ and $\rho = \rho_k$. There are two cases:

(i) α' is finitary, and
(ii) α' is infinitary.

We consider case (i) first. Since α' is finitary, ρ is infinitary. There are two subcases:

(a) $\rho = \mathrm{id}_Y$ for some $Y \subseteq X$, and
(b) $\rho = \mathrm{id}_{\mathbf{Y}}$ for some $\mathbf{Y} \preccurlyeq X$.

Suppose first that (a) holds. Since ρ is infinitary, we have $|X \setminus Y| \geqslant \aleph_0$. By Lemma 18, we then have $\mathrm{sh}(\alpha) = \mathrm{sh}(\alpha'\rho) \leqslant \mathrm{sh}(\alpha') + |X \setminus Y| = |X \setminus Y|$, since $\mathrm{sh}(\alpha') < \aleph_0$. By Lemma 17 we also have $\mathrm{sing}(\alpha) = \mathrm{sing}(\alpha'\rho) \geqslant \max(\mathrm{sing}(\alpha'), \mathrm{sing}(\rho)) = \mathrm{sing}(\rho) = |X \setminus Y|$. Next suppose that (b) holds. By Lemma 19 we have $\mathrm{sh}(\alpha) = \mathrm{sh}(\alpha'\rho) \leqslant \mathrm{sh}(\alpha') < \aleph_0$ and, since ρ is infinitary, $\mathrm{sing}(\alpha) = \mathrm{sing}(\alpha'\rho) \geqslant$

$\max(\mathrm{sing}(\alpha'), \mathrm{sing}(\rho)) = \mathrm{sing}(\rho) \geqslant \aleph_0$. We now consider case (ii). Since α' is infinitary, an induction hypothesis tells us that $\mathrm{sh}(\alpha') \leqslant \mathrm{sing}(\alpha')$. Again we consider the two subcases:

(a) $\rho = \mathrm{id}_Y$ for some $Y \subseteq X$, and
(b) $\rho = \mathrm{id}_{\mathbf{Y}}$ for some $\mathbf{Y} \preccurlyeq X$.

Suppose first that (a) holds. If $\mathrm{sh}(\alpha) < \aleph_0$, then we are done since $\mathrm{sing}(\alpha) \geqslant \aleph_0$ by Lemma 22. Otherwise, $\mathrm{sh}(\alpha) = \mathrm{sh}(\alpha'\rho) \leqslant \mathrm{sh}(\alpha') + |X \setminus Y| = \max(\mathrm{sh}(\alpha'), |X \setminus Y|)$, by Lemma 18, since at least one of $\mathrm{sh}(\alpha')$ or $|X \setminus Y|$ is infinite (as $\mathrm{sh}(\alpha)$ is). On the other hand, $\mathrm{sing}(\alpha) = \mathrm{sing}(\alpha'\rho) \geqslant \max(\mathrm{sing}(\alpha'), \mathrm{sing}(\rho)) \geqslant \max(\mathrm{sh}(\alpha'), |X \setminus Y|)$, by Lemma 17, since $\mathrm{sing}(\alpha') \geqslant \mathrm{sh}(\alpha')$ and $\mathrm{sing}(\rho) = |X \setminus Y|$. Finally, if (b) holds, then by Lemmas 17 and 19, we have $\mathrm{sh}(\alpha) = \mathrm{sh}(\alpha'\rho) \leqslant \mathrm{sh}(\alpha') \leqslant \mathrm{sing}(\alpha') \leqslant \mathrm{sing}(\alpha'\rho) = \mathrm{sing}(\alpha)$, and the proof is complete. □

Remark 24. If $\alpha \in \mathbb{E}(\mathcal{P}_X)$ is finitary, it is possible for $\mathrm{sh}(\alpha) > \mathrm{sing}(\alpha)$. For example if $X = \{1, \ldots, n\}$, then

$\alpha =$  $\in \mathcal{I}_X \setminus \mathcal{S}_X \subseteq \mathbb{E}(\mathcal{P}_X)$

satisfies $\mathrm{sh}(\alpha) - \mathrm{sing}(\alpha) = n - 2$. (This example extends to infinite X by allowing α to act as above on a finite subset of X, and as the identity elsewhere.)

7. Partial bijections

It is our next goal to determine which infinitary members of $\mathcal{I}_X$ belong to $\mathbb{E}(\mathcal{P}_X)$. Throughout this section we will assume X is infinite, and we remind the reader that we are identifying $\mathcal{I}_X$ with the symmetric inverse monoid on X. In particular, if $\alpha \in \mathcal{I}_X$ and $x \in \mathrm{dom}(\alpha)$, we will write $x\alpha$ for the image of x under α. We will also write $\alpha^* = \alpha^{-1}$, since the * map sends an element of $\mathcal{I}_X$ to its inverse mapping.

Since $\ker(\alpha)$ is trivial for all $\alpha \in \mathcal{I}_X$, the transversal blocks of α are naturally indexed by the elements of $\mathrm{dom}(\alpha)$. This allows us to regard the shift set of α as a subset of X and so, just for this section, we write

$$\mathrm{Sh}(\alpha) = \{x \in \mathrm{dom}(\alpha) \mid x\alpha \neq x\}.$$

Of course $\mathrm{sh}(\alpha) = |\mathrm{Sh}(\alpha)|$ has the same meaning as in the original definition.

It will also be useful to introduce a new parameter associated to a partial bijection $\alpha \in \mathcal{I}_X$. The *fail set* of α is defined to be

$$\mathrm{Fail}(\alpha) = X \setminus (\mathrm{dom}(\alpha) \cup \mathrm{codom}(\alpha)).$$

That is, $\mathrm{Fail}(\alpha)$ is the set of all points $x \in X$ for which $\{x\}$ and $\{x'\}$ are both singleton blocks of α. We also write $\mathrm{fail}(\alpha)$ for the cardinality of $\mathrm{Fail}(\alpha)$, and call this cardinal the *failure* of α.

The familiar notion of a cycle decomposition of a permutation has a natural extension to the elements of $\mathcal{I}_X$. We give a basic overview here; a more thorough treatment (though limited to finite X) may be found in [21]. In the case of a permutation $\alpha \in \mathcal{S}_X$, there are two types of orbits:

(i) finite cycles, $(x_1, x_2, \ldots, x_n)$, and
(ii) infinite cycles, $(\ldots, x_{-1}, x_0, x_1, x_2, \ldots)$.

These indicate that

(i) $x_1 \overset{\alpha}{\longmapsto} x_2 \overset{\alpha}{\longmapsto} \cdots \overset{\alpha}{\longmapsto} x_n \overset{\alpha}{\longmapsto} x_1$, and
(ii) $\cdots \overset{\alpha}{\longmapsto} x_{-1} \overset{\alpha}{\longmapsto} x_0 \overset{\alpha}{\longmapsto} x_1 \overset{\alpha}{\longmapsto} x_2 \overset{\alpha}{\longmapsto} \cdots,$

respectively. In the case of a partial bijection $\alpha \in \mathcal{I}_X$, there are five orbit types. In addition to the two types of cycles above, we also have the following three types of *trails*:

(iii) finite trails $[x_1, x_2, \ldots, x_n]$,
(iv) right-infinite trails $[x_0, x_1, x_2, \ldots)$, and
(v) left-infinite trails $(\ldots, x_{-2}, x_{-1}, x_0]$.

These indicate that

(iii) $x_1 \xmapsto{\alpha} x_2 \xmapsto{\alpha} \cdots \xmapsto{\alpha} x_n$ with $x_1 \notin \mathrm{codom}(\alpha)$ and $x_n \notin \mathrm{dom}(\alpha)$,
(iv) $x_0 \xmapsto{\alpha} x_1 \xmapsto{\alpha} x_2 \xmapsto{\alpha} \cdots$ with $x_0 \notin \mathrm{codom}(\alpha)$, and
(v) $\cdots \xmapsto{\alpha} x_{-2} \xmapsto{\alpha} x_{-1} \xmapsto{\alpha} x_0$ with $x_0 \notin \mathrm{dom}(\alpha)$,

respectively. Note that $x \in \mathrm{Fail}(\alpha)$ if and only if α has $[x]$ in its cycle-trail decomposition, while $x \in \mathrm{dom}(\alpha) \setminus \mathrm{Sh}(\alpha)$ if and only if α has (x) in its cycle-trail decomposition; these are called *trivial trails* and *trivial cycles*, respectively, and all others are called *non-trivial*. Each $\alpha \in \mathcal{I}_X$ is determined uniquely by its cycle-trail decomposition.

Lemma 25. *Suppose $\alpha \in \mathcal{I}_X$ satisfies* $\mathrm{fail}(\alpha) \geqslant \mathrm{sh}(\alpha)$. *Then $\alpha = \beta\gamma\delta$ for some $\gamma, \delta \in E(\mathcal{T}_X)$, where $\beta = \mathrm{id}_{\mathrm{dom}(\alpha)} \in E(\mathcal{I}_X)$.*

Proof. Since $\mathrm{fail}(\alpha) \geqslant \mathrm{sh}(\alpha)$, we may fix some injective map

$$\phi : \mathrm{Sh}(\alpha) \to \mathrm{Fail}(\alpha).$$

Consider the cycle-trail decomposition of α, and suppose that the non-trivial cycles and trails of types (i) through (v) above are indexed by sets I, J, K, L, M respectively. We will assume that these indexing sets are pairwise disjoint (but some or even all of them may be empty). Let $Q = I \cup J \cup K \cup L \cup M$ and, for $q \in Q$, denote by c_q the cycle or trail corresponding to q. We will define a family $\{X_q \mid q \in Q\}$ of pairwise disjoint subsets of X, and construct γ, δ by defining their restrictions to each X_q, and stipulating that they both act as the identity on $X \setminus \bigcup_{q \in Q} X_q$. If $q \in Q$ and ξ is one of α, β, γ, δ, we will write ξ_q for the restriction $\xi|_{X_q}$ of ξ to X_q. We now consider five separate cases, according to the type of c_q.

(i) Let $i \in I$, and write $c_i = (x^i_1, x^i_2, \ldots, x^i_{n_i})$. For $r \in \{1, \ldots, n_i\}$ put $y^i_r = x^i_r\phi$, noting that $x^i_r \in \mathrm{Sh}(\alpha)$ for all r. Let

$$X_i = \{x^i_1, \ldots, x^i_{n_i}\} \cup \{y^i_1, \ldots, y^i_{n_i}\}.$$

We then define

$$\gamma_i = \left(\begin{array}{c|c|c|c} x^i_1, y^i_1 & x^i_2, y^i_2 & \cdots & x^i_{n_i}, y^i_{n_i} \\ y^i_1 & y^i_2 & \cdots & y^i_{n_i} \end{array} \right),$$

and

$$\delta_i = \left(\begin{array}{c|c|c|c} y^i_1, x^i_2 & y^i_2, x^i_3 & \cdots & y^i_{n_i}, x^i_1 \\ x^i_2 & x^i_3 & \cdots & x^i_1 \end{array} \right).$$

In the shorthand notation here, we have only indicated the transversal blocks of γ_i and δ_i; all other blocks are understood to be nontransversal lower singleton blocks (so that γ_i and δ_i both

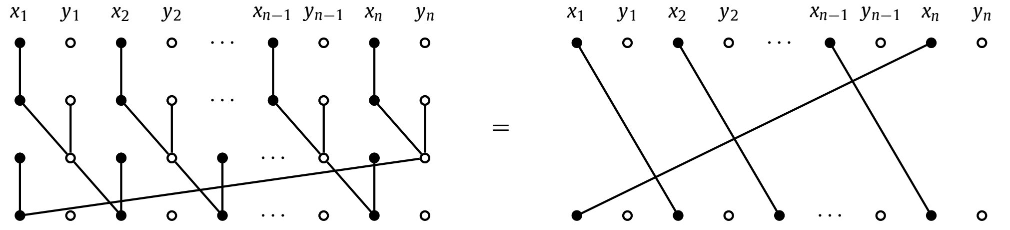

Fig. 3. A finite cycle as a product of three idempotents (see text for further explanation).

belong to $\mathcal{T}_{X_i}$). In Fig. 3, we show that $\alpha_i = \beta_i\gamma_i\delta_i$; in the diagram, we write n for n_i and x_r, y_r for x_r^i, y_r^i; note that $\beta_i = \mathrm{id}_{\{x_1^i,\dots,x_{n_i}^i\}}$.

(ii) Let $j \in J$, and write $c_j = (\dots, x_{-1}^j, x_0^j, x_1^j, x_2^j, \dots)$. For $r \in \mathbb{Z}$ put $y_r^j = x_r^j\phi$ and let

$$X_j = \{x_r^j \mid r \in \mathbb{Z}\} \cup \{y_r^j \mid r \in \mathbb{Z}\}.$$

We then define

$$\gamma_j = \left(\begin{array}{c|c|c|c|c} \cdots & x_{-1}^j, y_{-1}^j & x_0^j, y_0^j & x_1^j, y_1^j & \cdots \\ \cdots & y_{-1}^j & y_0^j & y_1^j & \cdots \end{array}\right),$$

and

$$\delta_j = \left(\begin{array}{c|c|c|c|c} \cdots & y_{-1}^j, x_0^j & y_0^j, x_1^j & y_1^j, x_2^j & \cdots \\ \cdots & x_0^j & x_1^j & x_2^j & \cdots \end{array}\right).$$

The reader is invited to check diagrammatically that $\alpha_j = \beta_j\gamma_j\delta_j$.

(iii) Let $k \in K$, and write $c_k = [x_1^k, x_2^k, \dots, x_{n_k}^k]$. For $r \in \{1, \dots, n_k - 1\}$ put $y_r^k = x_r^k\phi$ (noting that $x_{n_k} \notin \mathrm{Sh}(\alpha)$) and let

$$X_k = \{x_1^k, \dots, x_{n_k}^k\} \cup \{y_1^k, \dots, y_{n_k-1}^k\}.$$

We then define

$$\gamma_k = \left(\begin{array}{c|c|c|c|c} x_1^k, y_1^k & x_2^k, y_2^k & \cdots & x_{n_k-1}^k, y_{n_k-1}^k & x_{n_k}^k \\ y_1^k & y_2^k & \cdots & y_{n_k-1}^k & x_{n_k}^k \end{array}\right),$$

and

$$\delta_k = \left(\begin{array}{c|c|c|c|c} x_1^k & y_1^k, x_2^k & y_2^k, x_3^k & \cdots & y_{n_k-1}^k, x_{n_k}^k \\ x_1^k & x_2^k & x_3^k & \cdots & x_{n_k}^k \end{array}\right).$$

Again, the reader may check that $\alpha_k = \beta_k\gamma_k\delta_k$.

The details for cases (iv) and (v) are left for the reader to supply. So far we have defined γ, δ on the set $Y = \bigcup_{q\in Q} X_q$, and we have seen that $\beta|_Y \cdot \gamma|_Y \cdot \delta|_Y = \alpha|_Y$. Now put $Z = X \setminus Y$. We complete the definition of γ, δ by allowing them to act as the identity on Z. We clearly have $\beta|_Z \cdot \gamma|_Z \cdot \delta|_Z = \beta|_Z = \mathrm{id}_{Z\cap\mathrm{dom}(\alpha)} = \alpha|_Z$, and this completes the proof. □

Remark 26. There are two reasons why the statement of the previous lemma (and also the next) involves generators from $E(\mathcal{T}_X)$ rather than the simpler projections from $E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. First, it shortens the factorizations; see also Remarks 29 and 31. Secondly, it will be convenient to have these statements available when we deduce Howie's description of infinite $\mathbb{E}(\mathcal{T}_X)$ in Section 10. The lemmas could be rephrased to involve only generators from $E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$, since if $\gamma \in E(\mathcal{T}_X)$ then, as in Remark 3, $\gamma = (\gamma\gamma^*)(\gamma^*\gamma)$, with $\gamma\gamma^* \in E(\mathcal{J}_X)$ and $\gamma^*\gamma \in E(\mathcal{I}_X)$.

Lemma 27. *Suppose $\alpha \in \mathcal{I}_X$ satisfies*

$$\operatorname{def}(\alpha) = \operatorname{codef}(\alpha) \geqslant \max(\aleph_0, \operatorname{sh}(\alpha)), \tag{27.1}$$

$$\operatorname{sh}(\alpha) > \operatorname{fail}(\alpha). \tag{27.2}$$

Then $\alpha = \beta\gamma\delta\varepsilon$ for some $\gamma, \delta, \varepsilon \in E(\mathcal{T}_X)$, where $\beta = \operatorname{id}_{\operatorname{dom}(\alpha)} \in E(\mathcal{I}_X)$.

Proof. Put $A = \operatorname{dom}(\alpha)$ and $B = \operatorname{codom}(\alpha)$. We first show that

$$|A \setminus B| = |B \setminus A|. \tag{27.3}$$

To do this, suppose for the moment that (27.3) does not hold. Without loss of generality, suppose $|A \setminus B| > |B \setminus A|$. We have the disjoint unions

$$X \setminus A = \operatorname{Fail}(\alpha) \cup (B \setminus A), \tag{27.4}$$

$$X \setminus B = \operatorname{Fail}(\alpha) \cup (A \setminus B). \tag{27.5}$$

Now by (27.1), we have $|X \setminus A| = \operatorname{def}(\alpha) = \operatorname{codef}(\alpha) = |X \setminus B| \geqslant \aleph_0$. Since the unions in (27.4) and (27.5) are disjoint, $\operatorname{fail}(\alpha) + |B \setminus A| = |X \setminus A| = |X \setminus B| = \operatorname{fail}(\alpha) + |A \setminus B|$ and, since this cardinal is infinite, this then implies that

$$\max(\operatorname{fail}(\alpha), |B \setminus A|) = \max(\operatorname{fail}(\alpha), |A \setminus B|). \tag{27.6}$$

Next we claim that

$$\operatorname{fail}(\alpha) \geqslant |B \setminus A|. \tag{27.7}$$

Indeed, if (27.7) did not hold, then $\operatorname{fail}(\alpha) < |B \setminus A|$ which, together with (27.6), implies that $|B \setminus A| = \max(\operatorname{fail}(\alpha), |A \setminus B|)$, contradicting $|A \setminus B| > |B \setminus A|$. This establishes (27.7) which, together with (27.1), gives $\operatorname{sh}(\alpha) \leqslant \operatorname{def}(\alpha) = |X \setminus A| = \operatorname{fail}(\alpha) + |B \setminus A| = \operatorname{fail}(\alpha)$, contradicting (27.2). This completes the proof that (27.3) holds.

We now return to the main proof. Put $B_1 = (A \cap B)\alpha \subseteq B$. Now, $B \setminus B_1 = A\alpha \setminus (A \cap B)\alpha = (A \setminus B)\alpha$, so it follows by (27.3) that

$$|B \setminus B_1| = |B \setminus A|. \tag{27.8}$$

Choose and fix a bijection $\phi : B \setminus B_1 \to B \setminus A$, and define $\pi : B \to B$ by

$$b\pi = \begin{cases} b\alpha^{-1} & \text{if } b \in B_1, \\ b\phi & \text{if } b \in B \setminus B_1. \end{cases}$$

Clearly $\pi \in \mathcal{S}_B$, but we think of π as being a member of $\mathcal{I}_X$, with $\operatorname{Fail}(\pi) = X \setminus B$. Note that $\operatorname{Sh}(\alpha) = (A \setminus B) \cup \{x \in A \cap B \mid x\alpha \neq x\}$, and $\operatorname{Sh}(\pi) \subseteq (B \setminus B_1) \cup \{y \in B_1 \mid y\alpha^{-1} \neq y\}$. Now $\{y \in B_1 \mid y\alpha^{-1} \neq y\}$ is the image of $\{x \in A \cap B \mid x\alpha \neq x\}$ under α, so these two sets have the same cardinality. Since $A \setminus B$ and $B \setminus B_1$ also have the same cardinality by (27.3) and (27.8), it follows by (27.1) that $\text{sh}(\pi) \leqslant \text{sh}(\alpha) \leqslant \text{codef}(\alpha) = |X \setminus B| = \text{fail}(\pi)$. In particular, Lemma 25 tells us that $\pi = \text{id}_B\delta\varepsilon$ for some $\delta, \varepsilon \in E(\mathcal{T}_X)$. Now $\text{codom}(\pi^{-1}) = B$, so $\pi^{-1}\text{id}_B = \pi^{-1}$. It follows that $\alpha = (\alpha\pi^{-1})\pi = (\alpha\pi^{-1})\text{id}_B\delta\varepsilon = (\alpha\pi^{-1})\delta\varepsilon$, so the proof will be complete if we can show that $\alpha\pi^{-1} = \text{id}_A\gamma$ for some $\gamma \in E(\mathcal{T}_X)$. Let us write $\alpha' = \alpha\pi^{-1}$. Now $\text{dom}(\alpha') = A$ and $\text{codom}(\alpha') = B$. We also have $x\alpha' = x$ for all $x \in A \cap B$. This shows that $\text{Sh}(\alpha') = A \setminus B$, and that α' consists only of trivial cycles and trails together with trails of the form $[x, x\alpha']$ with $x \in A \setminus B$. Put

$$\gamma = \begin{pmatrix} x, x\alpha' & y \\ x\alpha' & y \end{pmatrix}_{x \in A \setminus B,\, y \in (A \cap B) \cup \text{Fail}(\alpha)}.$$

Then $\gamma \in E(\mathcal{T}_X)$, and it is easy to see diagrammatically that $\alpha' = \text{id}_A\gamma$. This completes the proof. □

Since every $\alpha \in \mathcal{I}_X$ obviously satisfies one of $\text{fail}(\alpha) \geqslant \text{sh}(\alpha)$ or $\text{sh}(\alpha) > \text{fail}(\alpha)$, the next result is an immediate consequence of Lemmas 25 and 27.

Proposition 28. *Suppose* $\alpha \in \mathcal{I}_X$ *satisfies* $\text{def}(\alpha) = \text{codef}(\alpha) \geqslant \max(\aleph_0, \text{sh}(\alpha))$. *Then* $\alpha \in \mathbb{E}(\mathcal{P}_X)$.

Remark 29. We have actually shown a bit more than this. In fact, if $\alpha \in \mathcal{I}_X$ satisfies $\text{def}(\alpha) = \text{codef}(\alpha) \geqslant \max(\aleph_0, \text{sh}(\alpha))$, then α may be expressed as the product of at most four elements from $E(\mathcal{I}_X) \cup E(\mathcal{T}_X)$. By contrast, if $\alpha \in \mathcal{I}_X$ contains only trivial cycles and a single non-trivial finite trail of length n then, as the reader might like to check as an exercise, any expression of α as a product of idempotents from $E(\mathcal{P}_X)$ must involve at least n idempotents.

8. Infinite $\mathbb{E}(\mathcal{P}_X)$

We now have all the information required to describe infinite $\mathbb{E}(\mathcal{P}_X)$. Recall that we say a partition $\alpha \in \mathcal{P}_X$ is finitary if $\text{warp}(\alpha) < \aleph_0$. We will write $\mathcal{P}_X^{\text{fin}}$ for the set of all finitary partitions. Since $\text{Warp}(\alpha\beta)$ is clearly contained in $\text{Warp}(\alpha) \cup \text{Warp}(\beta)$, we see that $\mathcal{P}_X^{\text{fin}}$ is a submonoid of $\mathcal{P}_X$. We also write $\mathcal{T}_X^{\text{fin}}$, $\mathcal{S}_X^{\text{fin}}$, etc. for the set of all finitary transformations, permutations, etc. It is clear that $\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}$ is a subsemigroup of $\mathcal{P}_X$; in fact, it is a union of many isomorphic copies of finite $\mathcal{P}_Y \setminus \mathcal{S}_Y$.

Theorem 30. *Let X be an infinite set. Then*

$$\mathbb{E}(\mathcal{P}_X) = \{1\} \cup \left(\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}\right) \cup \left\{\alpha \in \mathcal{P}_X \;\middle|\; \text{sing}(\alpha) = \text{cosing}(\alpha) \geqslant \max\left(\aleph_0, \text{sh}(\alpha)\right)\right\}.$$

Proof. Suppose first that $\alpha \in \mathbb{E}(\mathcal{P}_X)$. By Proposition 15 we have $\text{sing}(\alpha) = \text{cosing}(\alpha)$. By Proposition 6, we have $\alpha = \rho_1 \cdots \rho_k$ for some $\rho_1, \ldots, \rho_k \in E(\mathcal{I}_X) \cup E(\mathcal{J}_X)$. If all the ρ_i are finitary, then so too is α and so, by Lemma 7, we have $\alpha \in \{1\} \cup (\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}})$. If some ρ_i is infinitary, then $\text{sing}(\alpha) = \text{sing}((\rho_1 \cdots \rho_{i-1})\rho_i(\rho_{i+1} \cdots \rho_k)) \geqslant \text{sing}(\rho_i) \geqslant \aleph_0$, by Lemma 17. It follows that α is infinitary, and so we also have $\text{sing}(\alpha) \geqslant \text{sh}(\alpha)$, by Proposition 23. This completes the proof of the forward set containment.

The remainder of the proof will be devoted to establishing the reverse set containment. Obviously $1 \in \mathbb{E}(\mathcal{P}_X)$. Next, note that if $\alpha \in \mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}$, then α belongs to an isomorphic copy of $\mathcal{P}_Y \setminus \mathcal{S}_Y$ for some finite subset $Y \subseteq X$. Since $\mathcal{P}_Y \setminus \mathcal{S}_Y \subseteq \mathbb{E}(\mathcal{P}_Y)$ by Theorem 9, it quickly follows that $\alpha \in \mathbb{E}(\mathcal{P}_X)$. Finally, suppose $\alpha \in \mathcal{P}_X$ satisfies

$$\text{sing}(\alpha) = \text{cosing}(\alpha) \geqslant \max\left(\aleph_0, \text{sh}(\alpha)\right),$$

and write

$$\alpha = \begin{pmatrix} A_i & C_j \\ B_i & D_k \end{pmatrix}_{i\in I,\, j\in J,\, k\in K}.$$

For each $i \in I$, choose and fix elements $a_i \in A_i$ and $b_i \in B_i$. We make these choices in such a way that $a_i = b_i$ if $A_i \cap B_i \neq \emptyset$. Put

$$A = \{a_i \mid i \in I\} \quad \text{and} \quad B = \{b_i \mid i \in I\},$$

and define

$$\beta = \begin{pmatrix} a_i & x \\ b_i & y \end{pmatrix}_{i\in I,\, x\in X\setminus A,\, y\in X\setminus B}.$$

Note that in fact $\beta \in \mathcal{I}_X$. Put $\varepsilon_1 = \mathrm{id}_A$ and $\varepsilon_2 = \mathrm{id}_B$, and also

$$\eta_1 = \begin{pmatrix} A_i & C_j \\ A_i & C_j \end{pmatrix} \quad \text{and} \quad \eta_2 = \begin{pmatrix} B_i & D_k \\ B_i & D_k \end{pmatrix}.$$

It is clear that $\beta = \varepsilon_1 \alpha \varepsilon_2$ and $\alpha = \eta_1 \beta \eta_2$. Since ε_1, ε_2, η_1, η_2 are themselves idempotents (indeed, projections), it follows that $\alpha \in \mathbb{E}(\mathcal{P}_X)$ if and only if $\beta \in \mathbb{E}(\mathcal{P}_X)$. By construction, we immediately have

$$\mathrm{sh}(\beta) = \mathrm{sh}(\alpha), \qquad \mathrm{def}(\beta) = \mathrm{sing}(\alpha), \qquad \mathrm{codef}(\beta) = \mathrm{cosing}(\alpha).$$

It follows that $\mathrm{def}(\beta) = \mathrm{codef}(\beta) \geqslant \max(\aleph_0, \mathrm{sh}(\beta))$, and Proposition 28 then tells us that $\beta \in \mathbb{E}(\mathcal{P}_X)$. This completes the proof. □

Remark 31. By the above proof and also Lemmas 2(ii), 25 and 27, we see that any infinitary member of $\mathbb{E}(\mathcal{P}_X)$ is the product of at most five idempotents from $\mathcal{P}_X$.

9. Idempotents and units

If X is finite, $\mathcal{P}_X$ is generated by its units and idempotents; indeed, this follows immediately from Theorem 9, and was also proved in [7]. This is also the case with finite $\mathcal{T}_X$ [1] and $\mathcal{I}_X$ [28], but not $\mathcal{J}_X$ [5,10]. It is our goal in this section to describe the submonoid of infinite $\mathcal{P}_X$ generated by its idempotents and units. It turns out that this submonoid may be easily described in terms of the parameters we have already introduced.

As usual, if M is a monoid, we write $E(M)$ for the set of idempotents of M, and $G(M)$ for the group of units of M. We write

$$\mathbb{F}(M) = \langle E(M) \cup G(M) \rangle$$

for the submonoid of M generated by its idempotents and units, and we call $\mathbb{F}(M)$ the *factorizable part of* M. This extends the standard use of the word factorizable in the context of inverse monoids; see for example [3,4]. The next lemma justifies our use of the word; the proof is straightforward, and is omitted.

Lemma 32. *Let M be a monoid. Then we have $\mathbb{F}(M) = \mathbb{E}(M)G(M) = G(M)\mathbb{E}(M)$.*

Theorem 33. *For any set X we have*

$$\mathbb{F}(\mathcal{P}_X) = \{\alpha \in \mathcal{P}_X \mid \operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha)\}.$$

If X is finite, then both of these sets are equal to $\mathcal{P}_X$.

Proof. The statements for finite X follow from Theorem 9 and Remark 16. From now on we assume X is infinite. To establish the forward set containment, suppose $\alpha \in \mathbb{F}(\mathcal{P}_X)$. By Lemma 32, we have $\alpha = \beta\gamma$ for some $\beta \in \mathbb{E}(\mathcal{P}_X)$ and $\gamma \in G(\mathcal{P}_X) = \mathcal{S}_X$. Now, if

$$\beta = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array}\right),$$

then

$$\alpha = \beta\gamma = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i\gamma & D_k\gamma \end{array}\right),$$

so that clearly $\operatorname{sing}(\alpha) = \operatorname{sing}(\beta)$ and $\operatorname{cosing}(\alpha) = \operatorname{cosing}(\beta)$. By Proposition 15, we have $\operatorname{sing}(\beta) = \operatorname{cosing}(\beta)$, and it follows that $\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha)$.

To establish the reverse set containment, suppose $\alpha \in \mathcal{P}_X$ satisfies $\operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha)$, and write

$$\alpha = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i & D_k \end{array}\right)_{i\in I,\, j\in J,\, k\in K}.$$

For each $i \in I$, choose and fix $a_i \in A_i$ and $b_i \in B_i$. Put $A = \{a_i \mid i \in I\}$ and $B = \{b_i \mid i \in I\}$, noting that $|X \setminus A| = \operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha) = |X \setminus B|$. Fix any bijection $\phi : X \setminus B \to X \setminus A$, and define $\gamma \in \mathcal{S}_X$ by

$$x\gamma = \begin{cases} a_i & \text{if } x = b_i \text{ for some } i \in I, \\ x\phi & \text{if } x \in X \setminus B. \end{cases}$$

Since $\gamma \in \mathcal{S}_X$, we have $\alpha = (\alpha\gamma)\gamma^{-1}$, so the proof will be complete if we can show that $\alpha\gamma \in \mathbb{E}(\mathcal{P}_X)$. Observe that

$$\alpha\gamma = \left(\begin{array}{c|c} A_i & C_j \\ \hline B_i\gamma & D_k\gamma \end{array}\right)_{i\in I,\, j\in J,\, k\in K}.$$

Since $a_i = b_i\gamma \in A_i \cap B_i\gamma$ for all $i \in I$, it follows that $\operatorname{sh}(\alpha\gamma) = 0$. Now

$$\operatorname{sing}(\alpha\gamma) = \operatorname{sing}(\alpha) = \operatorname{cosing}(\alpha) = \operatorname{cosing}(\alpha\gamma).$$

If $\operatorname{sing}(\alpha\gamma) \geqslant \aleph_0$, then Theorem 30 immediately implies $\alpha\gamma \in \mathbb{E}(\mathcal{P}_X)$. On the other hand, if $\operatorname{sing}(\alpha\gamma) < \aleph_0$, then this together with $\operatorname{sh}(\alpha\gamma) = 0$ implies that $\alpha\gamma$ is finitary, and so $\alpha\gamma \in \{1\} \cup (\mathcal{P}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}})$, and again we conclude that $\alpha\gamma \in \mathbb{E}(\mathcal{P}_X)$ by Theorem 30. This completes the proof. □

10. Intersections

If T is a subsemigroup of a semigroup S, it is of course possible for $\mathbb{E}(S) \cap T \neq \mathbb{E}(T)$ to hold; that is, for the idempotents of S to generate more of T than do the idempotents of T itself. Indeed, by consulting Theorems 9 and 30, it is easy to see that this is the case when S is $\mathcal{P}_X$ and T is one of $\mathcal{I}_X$ or $\mathcal{J}_X$. We will see shortly that this is not the case when $S = \mathcal{P}_X$ and $T = \mathcal{T}_X$. We now give the precise statements. It will be convenient to consider finite and infinite X separately. We will not state the corresponding results for $\mathcal{T}_X^*$; they are simply dual to those for $\mathcal{T}_X$.

Theorem 34. *Let X be a finite set. Then*

- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{I}_X = \{1\} \cup (\mathcal{I}_X \setminus \mathcal{S}_X)$,
- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{J}_X = \{1\} \cup (\mathcal{J}_X \setminus \mathcal{S}_X)$,
- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X = \mathbb{E}(\mathcal{T}_X) = \{1\} \cup (\mathcal{T}_X \setminus \mathcal{S}_X)$.

Proof. The statements follow directly from Theorems 8 and 9. □

Remark 35. Note that $\mathbb{E}(\mathcal{I}_X) = E(\mathcal{I}_X)$ and $\mathbb{E}(\mathcal{J}_X) = E(\mathcal{J}_X)$ for arbitrary X, since $\mathcal{I}_X$ and $\mathcal{J}_X$ are inverse monoids. It follows that $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{I}_X = \mathbb{E}(\mathcal{I}_X)$ if and only if $|X| \leqslant 1$, while $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{J}_X = \mathbb{E}(\mathcal{J}_X)$ if and only if $|X| \leqslant 2$.

Theorem 36. *Let X be an infinite set. Then*

- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{I}_X = \{1\} \cup (\mathcal{I}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}) \cup \{\alpha \in \mathcal{I}_X \mid \text{def}(\alpha) = \text{codef}(\alpha) \geqslant \max(\aleph_0, \text{sh}(\alpha))\}$,
- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{J}_X = \{1\} \cup (\mathcal{J}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}) \cup \{\alpha \in \mathcal{J}_X \mid \text{col}(\alpha) = \text{cocol}(\alpha) \geqslant \max(\aleph_0, \text{sh}(\alpha))\}$,
- $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X = \mathbb{E}(\mathcal{T}_X) = \{1\} \cup (\mathcal{T}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}) \cup \{\alpha \in \mathcal{T}_X \mid \text{col}(\alpha) = \text{codef}(\alpha) \geqslant \max(\aleph_0, \text{sh}(\alpha))\}$.

Proof. The statements concerning $\mathcal{I}_X$ and $\mathcal{J}_X$ follow immediately from Theorem 30 and the fact that $\text{col}(\alpha) = \text{cocol}(\alpha) = 0 = \text{def}(\beta) = \text{codef}(\beta)$ for $\alpha \in \mathcal{I}_X$ and $\beta \in \mathcal{J}_X$. Similarly, it is also clear that

$$\mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X = \{1\} \cup \left(\mathcal{T}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}\right) \cup \left\{\alpha \in \mathcal{T}_X \;\middle|\; \text{col}(\alpha) = \text{codef}(\alpha) \geqslant \max\left(\aleph_0, \text{sh}(\alpha)\right)\right\},$$

so it remains to prove that $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X = \mathbb{E}(\mathcal{T}_X)$. Suppose $\alpha \in \mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X$. If α is finitary then, by Theorem 30, either $\alpha = 1$ or α belongs to an isomorphic copy of $\mathcal{T}_Y \setminus \mathcal{S}_Y$ for some finite subset $Y \subseteq X$, and it follows quickly by Theorem 8 that $\alpha \in \mathbb{E}(\mathcal{T}_X)$. Next suppose α is infinitary, and write

$$\alpha = \begin{pmatrix} A_i \\ b_i \end{pmatrix}_{i \in I}.$$

(Again we have not indicated the elements of $X \setminus \text{codom}(\alpha)$.) As in the proof of Theorem 30, choose and fix elements $a_i \in A_i$ for each $i \in I$, making the choices so that $a_i = b_i$ if $b_i \in A_i$, and put $A = \{a_i \mid i \in I\}$. We then have $\alpha = \eta\beta$, where

$$\eta = \begin{pmatrix} A_i \\ A_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$

(Note that η was denoted η_1 in the proof of Theorem 30, and that the partition η_2 from that proof is the identity since $\text{coker}(\alpha) = \Delta$.) By Lemmas 25 and 27, $\beta = \text{id}_A \rho_1 \rho_2 \rho_3$ for some $\rho_1, \rho_2, \rho_3 \in E(\mathcal{T}_X)$, where possibly $\rho_3 = 1$. It follows that $\alpha = \eta\beta = (\eta \text{id}_A)\rho_1\rho_2\rho_3 \in \mathbb{E}(\mathcal{T}_X)$, since

$$\eta \text{id}_A = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

clearly belongs to $E(\mathcal{T}_X)$. Thus, $\mathbb{E}(\mathcal{P}_X) \cap \mathcal{T}_X \subseteq \mathbb{E}(\mathcal{T}_X)$, and the reverse set containment is obvious. □

Remark 37. We have in fact shown that the infinitary elements of $\mathbb{E}(\mathcal{T}_X)$ can be expressed as the product of at most four idempotents from $\mathcal{T}_X$; see also [16] for a different argument.

As an application of the previous result, we now deduce Howie's description [16] of infinite $\mathbb{E}(\mathcal{T}_X)$. In order to state the theorem in its original form, we first make a number of definitions. Let $\alpha \in \mathcal{T}_X$, and define sets

$$C(\alpha) = \bigcup_{\substack{x \in \operatorname{codom}(\alpha) \\ |x\alpha^{-1}| \geqslant 2}} x\alpha^{-1}, \qquad Z(\alpha) = X \setminus \operatorname{codom}(\alpha), \qquad S(\alpha) = \{x \in X \mid x\alpha \neq x\}.$$

The cardinals $|C(\alpha)|$, $|Z(\alpha)|$, $|S(\alpha)|$ were called the collapse, defect and shift of α in [16]; in order to avoid confusion, we will not use those names for these cardinals here.

Theorem 38. *(See Howie [16, Theorem III].) Let X be an infinite set. Then*

$$\mathbb{E}(\mathcal{T}_X) = \{1\} \cup \left(\mathcal{T}_X^{\text{fin}} \setminus \mathcal{S}_X^{\text{fin}}\right) \cup \left\{\alpha \in \mathcal{T}_X \;\middle|\; \left|C(\alpha)\right| = \left|Z(\alpha)\right| = \left|S(\alpha)\right| \geqslant \aleph_0\right\}.$$

Proof. Let $\alpha \in \mathcal{T}_X$. We begin with a number of observations. First, we clearly have

$$\left|Z(\alpha)\right| = \operatorname{codef}(\alpha). \tag{38.1}$$

Now write

$$\alpha = \begin{pmatrix} A_i \\ b_i \end{pmatrix}_{i \in I}.$$

Consider the decomposition $I = I_1 \cup I_2$, where

$$I_1 = \left\{i \in I \;\middle|\; |A_i| \geqslant 2\right\} \quad \text{and} \quad I_2 = \left\{i \in I \;\middle|\; |A_i| = 1\right\}.$$

Then

$$\left|C(\alpha)\right| = \sum_{i \in I_1} \left(|A_i| - 1\right) + |I_1| = \operatorname{col}(\alpha) + |I_1|. \tag{38.2}$$

Notice also that $\operatorname{col}(\alpha) = \sum_{i \in I_1} (|A_i| - 1) \geqslant |I_1|$, since $|A_i| \geqslant 2$ for all $i \in I_1$. Together with (38.2), this implies that

$$\left|C(\alpha)\right| \geqslant \aleph_0 \quad \Leftrightarrow \quad \operatorname{col}(\alpha) \geqslant \aleph_0 \quad \Rightarrow \quad \left|C(\alpha)\right| = \operatorname{col}(\alpha). \tag{38.3}$$

Next, consider the decomposition $I = I_3 \cup I_4$, where

$$I_3 = \{i \in I \mid b_i \in A_i\} \quad \text{and} \quad I_4 = \{i \in I \mid b_i \notin A_i\}.$$

Then

$$\left|S(\alpha)\right| = \sum_{i_3 \in I_3} \left(|A_{i_3}| - 1\right) + \sum_{i_4 \in I_4} |A_{i_4}|$$

$$= \sum_{i_3 \in I_3} \big(|A_{i_3}| - 1\big) + \sum_{i_4 \in I_4} \big(|A_{i_4}| - 1\big) + |I_4|$$
$$= \mathrm{col}(\alpha) + \mathrm{sh}(\alpha). \tag{38.4}$$

We now return to the main proof. By Theorem 36, it suffices to show that α satisfies

$$\mathrm{col}(\alpha) = \mathrm{codef}(\alpha) \geqslant \max\big(\aleph_0, \mathrm{sh}(\alpha)\big) \tag{38.5}$$

if and only if it satisfies

$$\big|C(\alpha)\big| = \big|Z(\alpha)\big| = \big|S(\alpha)\big| \geqslant \aleph_0. \tag{38.6}$$

Suppose first that (38.5) holds, and write $\mu = \mathrm{col}(\alpha) \geqslant \aleph_0$. By (38.1) and (38.5) we have $|Z(\alpha)| = \mu$. Next, note that (38.3) implies that $|C(\alpha)| = \mu$. Finally, (38.4) implies that $|S(\alpha)| = \mu + \mathrm{sh}(\alpha) = \mu$, since $\mu \geqslant \aleph_0$ and $\mu \geqslant \mathrm{sh}(\alpha)$ by (38.5). This completes the proof that (38.5) implies (38.6). Conversely, suppose (38.6) holds, and write $\nu = |C(\alpha)| \geqslant \aleph_0$. Now (38.1), (38.3) and (38.6) imply $\mathrm{codef}(\alpha) = \mathrm{col}(\alpha) = \nu$, while (38.4) and (38.6) give $\nu = |S(\alpha)| = \mathrm{col}(\alpha) + \mathrm{sh}(\alpha) \geqslant \mathrm{sh}(\alpha)$. This shows that (38.6) implies (38.5), and the proof is complete. □

We now calculate the intersections of $\mathbb{F}(\mathcal{P}_X)$ with the submonoids $\mathcal{I}_X$, $\mathcal{J}_X$, $\mathcal{T}_X$. It will be convenient to consider them one-by-one.

Theorem 39. *For any set X we have*

$$\mathbb{F}(\mathcal{P}_X) \cap \mathcal{I}_X = \mathbb{F}(\mathcal{I}_X) = \big\{\alpha \in \mathcal{I}_X \;\big|\; \mathrm{def}(\alpha) = \mathrm{codef}(\alpha)\big\}.$$

If X is finite, then all three sets are equal to $\mathcal{I}_X$.

Proof. Let Ω denote the set on the right-hand side. It is immediate from Theorem 33 that $\mathbb{F}(\mathcal{P}_X) \cap \mathcal{I}_X = \Omega$. It is easy to see, and it is proved in [3], that $\mathbb{F}(\mathcal{I}_X) = \Omega$, and that $\mathbb{F}(\mathcal{I}_X) = \mathcal{I}_X$ for finite X. □

The corresponding result for $\mathcal{J}_X$ also follows quickly from Theorem 33.

Theorem 40. *For any set X we have*

$$\mathbb{F}(\mathcal{P}_X) \cap \mathcal{J}_X = \big\{\alpha \in \mathcal{J}_X \;\big|\; \mathrm{col}(\alpha) = \mathrm{cocol}(\alpha)\big\}.$$

If X is finite, then both sets are equal to $\mathcal{J}_X$.

Remark 41. An element of $\mathcal{J}_X$ belongs to $\mathbb{F}(\mathcal{J}_X)$ if and only if each block A satisfies $|A \cap X| = |A \cap X'|$; see [9,10], where these elements were called *uniform*. Thus, $\mathbb{F}(\mathcal{P}_X) \cap \mathcal{J}_X$ is not equal to $\mathbb{F}(\mathcal{J}_X)$ unless $|X| \leqslant 2$.

The submonoid $\mathbb{F}(\mathcal{T}_X)$ generated by the idempotents and units of $\mathcal{T}_X$ was described in [13], but not in [16]; the finite case is classical [1].

Theorem 42. *(See Higgins, Howie and Ruškuc [13].) For any set X we have*

$$\mathbb{F}(\mathcal{P}_X) \cap \mathcal{T}_X = \mathbb{F}(\mathcal{T}_X) = \big\{\alpha \in \mathcal{T}_X \;\big|\; \mathrm{col}(\alpha) = \mathrm{codef}(\alpha)\big\}.$$

If X is finite, then all three sets are equal to $\mathcal{T}_X$.

Proof. As mentioned above, the finite case is known, so suppose X is infinite. Denote the set on the right-hand side by Ω. It follows immediately from Theorem 33 that $\mathbb{F}(\mathcal{P}_X)\cap\mathcal{T}_X=\Omega$. Next, note that we clearly have $\mathbb{F}(\mathcal{T}_X)\subseteq\mathbb{F}(\mathcal{P}_X)$. The proof will therefore be complete if we can show that $\mathbb{F}(\mathcal{P}_X)\cap\mathcal{T}_X\subseteq\mathbb{F}(\mathcal{T}_X)$, so suppose $\alpha\in\mathbb{F}(\mathcal{P}_X)\cap\mathcal{T}_X$. By Lemma 32, $\alpha=\beta\pi$ for some $\beta\in\mathbb{E}(\mathcal{P}_X)$ and $\pi\in\mathcal{S}_X$. Now $\alpha\in\mathcal{T}_X$ implies $\beta=\alpha\pi^{-1}\in\mathcal{T}_X$, so in fact $\beta\in\mathbb{E}(\mathcal{P}_X)\cap\mathcal{T}_X=\mathbb{E}(\mathcal{T}_X)$ by Theorem 36. Thus $\alpha=\beta\pi\in\mathbb{E}(\mathcal{T}_X)\mathcal{S}_X=\mathbb{F}(\mathcal{T}_X)$. □

11. Embeddings

A consequence of Howie's description of $\mathbb{E}(\mathcal{T}_X)$ is that every semigroup S embeds in an idempotent generated regular semigroup T, with T finite if S is [16, Theorems II and IV]. This result can be extended, by showing that T can in fact be taken to be a regular $*$-semigroup.

Theorem 43. *Let S be a semigroup. Then S embeds in some idempotent generated regular $*$-semigroup T. If S is finite, then we may take T to be finite too.*

Proof. Let $\phi:S\to\mathcal{T}_X$ be any embedding (for example the Cayley representation), with X finite if S is finite. If S is finite let $Y=\{y\}$, where $y\notin X$. Otherwise, let Y be a set disjoint from X and of the same cardinality as X. In either case put $Z=X\cup Y$. It is clear that the map

$$\psi:\mathcal{T}_X\to\mathcal{P}_Z:\alpha\mapsto\alpha\cup\big\{\{y\}\ \big|\ y\in Y\big\}\cup\big\{\{y'\}\ \big|\ y\in Y\big\}$$

is an embedding. For any $\alpha\in\mathcal{T}_X$, $\alpha\psi$ belongs to $\mathcal{P}_Z\setminus\mathcal{S}_Z$ and, if S is infinite, we have $\operatorname{sing}(\alpha\psi)=\operatorname{cosing}(\alpha\psi)=|Z|$. We see then, by Theorems 9 and 30 as appropriate, that $\operatorname{im}(\psi)$ is contained in $\mathbb{E}(\mathcal{P}_Z)$. The proof is now complete, since we have shown that S embeds in $T=\mathbb{E}(\mathcal{P}_Z)$, the latter clearly being a regular $*$-semigroup. □

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