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SCIENCE DIRECT®
J. Differential Equations 227 (2006) 258–281
Journal of Differential Equations
www.elsevier.com/locate/jde

Coupled nonlinear Schrödinger systems with potentials

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Received 1 June 2005; revised 25 August 2005

Available online 10 October 2005

Abstract

Coupled nonlinear Schrödinger systems describe some physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photorefractive media in optics and Bose–Einstein condensates. In this paper, we study the existence of concentrating solutions of a singularly perturbed coupled nonlinear Schrödinger system, in presence of potentials. We show how the location of the concentration points depends strictly on the potentials.
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MSC: 35B40; 35J50; 35Q55

Keywords: Coupled nonlinear Schrödinger systems; Concentrating solutions

1. Introduction

Very recently, different authors focused their attention on coupled nonlinear Schrödinger systems which describe physical phenomena such as the propagation in birefringent optical fibers, Kerr-like photorefractive media in optics and Bose–Einstein condensates.

First of all, let us recall that, in the last twenty years, motivated by the study of the propagation of pulse in nonlinear optical fiber, the nonlinear Schrödinger equation,

$$-\Delta u + u = u^3 \quad \text{in } \mathbb{R}^3,$$

has been faced by many authors. It has been proved the existence of the least energy solution (ground state solution), which is radial with respect to some point, positive and exponentially

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doi:10.1016/j.jde.2005.09.002

decaying with its first derivatives at infinity. Moreover, there are also many papers about the semiclassical states for the nonlinear Schrödinger equation with the presence of potentials

$$-\varepsilon^2 \Delta u + V(x)u = u^3 \quad \text{in } \mathbb{R}^3,$$

giving sufficient and necessary conditions to the existence of solutions concentrating in some points, and recently, in set with nonzero dimension (see, e.g., [4–8,13,15,16,20,21,27,28,30,31]).

However, by I.P. Kaminow [19], we know that single-mode optical fibers are not really "single-mode," but actually bimodal due to the presence of birefringence. This birefringence can deeply influence the way in which an optical evolves during the propagation along the fiber. Indeed, it can occur that the linear birefringence makes a pulse split in two, while nonlinear birefringent traps them together against splitting. C.R. Menyuk [25,26] showed that the evolution of two orthogonal pulse envelopes in birefringent optical fibers is governed by the following coupled nonlinear Schrödinger system:

$$\begin{cases} i\phi_t + \phi_{xx} + |\phi|^2\phi + \beta|\psi|^2\phi = 0, \\ i\psi_t + \psi_{xx} + |\psi|^2\psi + \beta|\phi|^2\psi = 0, \end{cases} \tag{1.1}$$

with β positive constant depending on the anisotropy of the fibers. System (1.1) is also important for industrial applications in fiber communications systems [17] and all-optical switching devices [18]. If one seeks for standing wave solutions of (1.1), namely solutions of the form

$$\phi(x,t) = e^{iw_1^2 t} u(x) \quad \text{and} \quad \psi(x,t) = e^{iw_2^2 t} v(x),$$

then (1.1) becomes

$$\begin{cases} -u_{xx} + u = |u|^2 u + \beta|v|^2 u & \text{in } \mathbb{R}, \\ -v_{xx} + w^2 v = |v|^2 v + \beta|u|^2 v & \text{in } \mathbb{R}, \end{cases} \tag{1.2}$$

with $w^2 = w_2^2/w_1^2$. Finally, we want to recall that (1.2) describes also other physical phenomena, such as Kerr-like photorefractive media in optics (cf. [1,10]).

Problem (1.2), in a more general situation and also in higher dimension, has been studied by R. Cipolatti and W. Zumpichiatti [11,12]. By concentration compactness arguments, they prove the existence and the regularity of the ground state solutions $(u,v) \neq (0,0)$. Later on, in two very recent papers, T.C. Lin and J. Wei [22] and L.A. Maia, E. Montefusco and B. Pellacci [24] deal with problem (1.2), also in the multidimensional case, and, among other results, they prove the existence of least energy solutions of the type (u,v), with $u, v > 0$. Moreover, T.C. Lin and J. Wei [22] prove that, if $\beta < 0$, then the ground state solution for (1.2) does not exist. We refer to all these papers and to references therein for more complete informations about (1.2).

Another motivation to the study of coupled Schrödinger systems arises from the Hartree–Fock theory for the double condensate, that is a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (cf. [14]). Indeed, these phenomena are governed by the following system:

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega, \\ -\varepsilon^2 \Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^3$. Physically, u and v represent the corresponding condensate amplitudes, $\varepsilon^2 = \hbar^2/(2m)$, with $\hbar$ the Planck constant and m the atom mass. Moreover $\mu_j = -(N_j - 1)U_{jj}$ and $\beta = -N_2 U_{12}$, with $N_j \geqslant 1$ a fixed number of atoms in the hyperfine state $|j\rangle$, and $U_{ij} = 4\pi(\hbar^2/m)a_{ij}$, where a_{jj}'s and a_{12} are the intraspecies and interspecies scattering lengths. Besides, by E. Timmermans [29], we infer that $\mu_j = \mu_j(x)$ represents a chemical potential. For more informations about (1.3), see [22,23] and references therein.

T.C. Lin and J. Wei, in [23], studied problem (1.3) with $\lambda_1, \lambda_2, \mu_1, \mu_2$ positive constants and they proved that if $\beta < \sqrt{\mu_1\mu_2}$, for ε sufficiently small, (1.3) has a least energy solution $(u_\varepsilon, v_\varepsilon)$. Moreover, they distinguished two cases: the attractive case and the repulsive one. In the attractive case, which occurs whenever $\beta > 0$, u_ε and v_ε concentrate respectively in Q_ε and Q'_ε, with

$$\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to 0, \quad \text{as } \varepsilon \to 0.$$

Precisely they proved that

$$(Q_\varepsilon, \partial\Omega) \to \max_{Q\in\Omega} d(Q, \partial\Omega), \qquad d\big(Q'_\varepsilon, \partial\Omega\big) \to \max_{Q\in\Omega} d(Q, \partial\Omega).$$

In the repulsive case, that is when $\beta < 0$, the concentration points Q_ε and Q'_ε satisfy the following condition:

$$\varphi\big(Q_\varepsilon, Q'_\varepsilon\big) \to \max_{(Q,Q')\in\Omega^2} \varphi(Q, Q'),$$

where

$$\varphi(Q, Q') = \min\big\{\sqrt{\lambda_1}|Q - Q'|, \sqrt{\lambda_2}|Q - Q'|, \sqrt{\lambda_1}d(Q, \partial\Omega), \sqrt{\lambda_2}d(Q', \partial\Omega)\big\}.$$

In particular,

$$\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to \infty, \quad \text{as } \varepsilon \to 0.$$

Motivated by these results and by the fact that we know that μ_j may be not constants (cf. [29]), in this paper we study the following problem:

$$\begin{cases} -\varepsilon^2\Delta u + J_1(x)u = J_2(x)u^3 + \beta uv^2 & \text{in } \Omega, \\ -\varepsilon^2\Delta v + K_1(x)v = K_2(x)v^3 + \beta u^2 v & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

with $\Omega \subset \mathbb{R}^3$, possibly unbounded and with smooth boundary, and with $\beta < 0$, namely in the repulsive case. We will show that the presence of the potentials change drastically the situation with respect to the case with positive constants for what concerns the location of peaks, but, in some sense, not the repulsive nature of the problem. In fact, with suitable assumptions on the potentials, for ε sufficiently small, we will find solutions $(u_\varepsilon, v_\varepsilon)$ of $(\mathcal{P}_\varepsilon)$, even if not of least energy, concentrating respectively on Q_ε and Q'_ε which tend toward the same point, determined by the potentials, as $\varepsilon \to 0$, but with the property that the distance between them divided by ε diverges (see Remark 1.2).

Up to our knowledge, in this paper we give a first existence result of concentrating solutions for problem $(\mathcal{P}_\varepsilon)$, in presence of potentials.

On the potentials J_i and K_i we will do the following assumptions:

(J) for $i = 1, 2$, $J_i \in C^1(\Omega, \mathbb{R})$, J_i and DJ_i are bounded; moreover,

$$J_i(x) \geqslant C > 0 \quad \text{for all } x \in \Omega;$$

(K) for $i = 1, 2$, $K_i \in C^1(\Omega, \mathbb{R})$, K_i and DK_i are bounded; moreover,

$$K_i(x) \geqslant C > 0 \quad \text{for all } x \in \Omega.$$

Without lost of generality, we can suppose that there exists $\varepsilon_0 > 0$, such that $\Omega_0 := \Omega \cap (\Omega - \varepsilon_0 e_1) \neq \emptyset$, where $e_1 = (1, 0, 0)$.

Let us introduce an auxiliary function which will play a crucial role in the study of $(\mathcal{P}_\varepsilon)$. Let $\Gamma : \Omega_0 \to \mathbb{R}$ be a function defined by

$$\Gamma(Q) = J_1(Q)^{\frac{1}{2}} J_2(Q)^{-1} + K_1(Q)^{\frac{1}{2}} K_2(Q)^{-1}. \tag{1.4}$$

Let us observe that by (J) and (K), Γ is well defined.

Our main result is:

Theorem 1.1. *Suppose* (J) *and* (K) *and* $\beta < 0$. *Let* $Q_0 \in \Omega_0$ *be an isolated local strict minimum or maximum of* Γ. *There exists a* $\bar{\varepsilon} > 0$ *such that if* $0 < \varepsilon < \bar{\varepsilon}$, *then* $(\mathcal{P}_\varepsilon)$ *possesses a solution* $(u_\varepsilon, v_\varepsilon)$ *such that* u_ε *concentrates in* Q_ε *with* $Q_\varepsilon \to Q_0$, *as* $\varepsilon \to 0$, *and* v_ε *concentrates in* Q'_ε *with* $Q'_\varepsilon \to Q_0$, *as* $\varepsilon \to 0$.

Remark 1.2. Let us observe that, by the proof, it will be clear that, even if

$$\left|Q_\varepsilon - Q'_\varepsilon\right| \to 0, \quad \text{as } \varepsilon \to 0,$$

we have

$$\frac{|Q_\varepsilon - Q'_\varepsilon|}{\varepsilon} \to \infty, \quad \text{as } \varepsilon \to 0.$$

Let us present how Theorem 1.1 becomes in some particular situations.

Let $H : \Omega \to \mathbb{R}$ satisfies the assumption:

(H) $H \in C^1(\Omega, \mathbb{R})$, H and DH are bounded; moreover,

$$H(x) \geqslant C > 0 \quad \text{for all } x \in \Omega.$$

Corollary 1.3. *Suppose* (H) *and* $\beta < 0$. *Suppose, moreover, that we are in one of the following situations*:

- *all the potentials J_i and K_i coincide with H;*
- *there exists $i_0 = 1, 2$ such that $J_{i_0} \equiv H$ and $K_{i_0} \equiv H$, for $i = i_0$, while J_i and K_i are constant for $i \neq i_0$;*
- *all the potentials J_i and K_i are constant, except only one, which coincides with H.*

Let $Q_0 \in \Omega_0$ be an isolated local strict minimum or maximum of H. There exists $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$, then $(\mathcal{P}_\varepsilon)$ possesses a solution $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates in Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, and v_ε concentrates in Q'_ε with $Q'_\varepsilon \to Q_0$, as $\varepsilon \to 0$.

Remark 1.4. If, instead of β constant, we consider $\beta \in C^1(\Omega, \mathbb{R})$, bounded and bounded above by a negative constant, then we have exactly the same results.

Finally, we want to observe that we can treat also a more general problem than $(\mathcal{P}_\varepsilon)$. Let us consider, indeed,

$$\begin{cases} -\varepsilon^2 \Delta u + J_1(x)u = J_2(x)u^{2p-1} + \beta u^{p-1}v^p & \text{in } \Omega, \\ -\varepsilon^2 \Delta v + K_1(x)v = K_2(x)v^{2p-1} + \beta u^p v^{p-1} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\bar{\mathcal{P}}_\varepsilon$}$$

with $\Omega \subset \mathbb{R}^N$, possibly unbounded and with smooth boundary, with $\beta < 0$ and

$$2 < 2p < 2^* = \begin{cases} +\infty & \text{if } N = 1, 2, \\ \frac{2N}{N-2} & \text{if } N \geqslant 3. \end{cases} \tag{1.5}$$

Also in this case, without lost of generality, we can suppose that there exists $\varepsilon_0 > 0$, such that $\bar{\Omega}_0 := \Omega \cap (\Omega - \varepsilon_0 \bar{e}_1) \neq \emptyset$, where $\bar{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$.

Let us define now $\bar{\Gamma} : \bar{\Omega}_0 \to \mathbb{R}$ be a function defined by

$$\bar{\Gamma}(Q) = J_1(Q)^{\frac{p}{p-1} - \frac{N}{2}} J_2(Q)^{-\frac{1}{p-1}} + K_1(Q)^{\frac{p}{p-1} - \frac{N}{2}} K_2(Q)^{-\frac{1}{p-1}}.$$

In this case, Theorem 1.1 becomes:

Theorem 1.5. *Assume* (1.5) *and suppose* (J) *and* (K) *and $\beta < 0$. Let $Q_0 \in \bar{\Omega}_0$ be an isolated local strict minimum or maximum of $\bar{\Gamma}$. There exists $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon < \bar{\varepsilon}$, then $(\bar{\mathcal{P}}_\varepsilon)$ possesses a solution $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates in Q_ε with $Q_\varepsilon \to Q_0$, as $\varepsilon \to 0$, and v_ε concentrates in Q'_ε with $Q'_\varepsilon \to Q_0$, as $\varepsilon \to 0$.*

Remark 1.6. Let us observe that, if $p = 2$ and $N = 3$, then Theorem 1.1 is nothing else than a particular case of Theorem 1.5. Nevertheless, since problem $(\mathcal{P}_\varepsilon)$ is more natural and more important by a physical point of view, we prefer to present Theorem 1.1 as our main result and to prove it directly, showing how, with slight modifications, the proof of Theorem 1.5 follows.

Theorem 1.1 will be proved as a particular case of a multiplicity result in Section 5 (see Theorem 5.1). The proof of the theorem relies on a finite dimensional reduction, precisely on the perturbation technique developed in [2,3,7]. In Section 2 we give some preliminary lemmas and some estimates which will be useful in Section 3 and Section 4, where we perform the Lyapunov–Schmidt reduction, making also the asymptotic expansion of the finite dimensional functional. Finally, in Section 5, we give also a short proof of Theorem 1.5.

Notations.

- We denote $\Omega_0 := \Omega \cap (\Omega - \varepsilon_0 e_1)$, where $e_1 = (1,0,0)$ and ε_0 is sufficiently small such that $\Omega_0 \neq \emptyset$.
- If $r > 0$ and $x_0 \in \mathbb{R}^3$, $B_r(x_0) := \{x \in \mathbb{R}^3\colon |x - x_0| < r\}$. We denote with B_r the ball of radius r centered in the origin.
- If $u : \mathbb{R}^3 \to \mathbb{R}$ and $P \in \mathbb{R}^3$, we set $u_P := u(\cdot - P)$.
- If $\varepsilon > 0$, we set $\Omega_\varepsilon := \Omega/\varepsilon = \{x \in \mathbb{R}^3\colon \varepsilon x \in \Omega\}$.
- We denote $\mathcal{H}_\varepsilon = H_0^1(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon)$.
- If there is no misunderstanding, we denote with $\|\cdot\|$ and with $(\cdot \mid \cdot)$ respectively the norm and the scalar product both of $H_0^1(\Omega_\varepsilon)$ and of $\mathcal{H}_\varepsilon$. While we denote with $\|\cdot\|_{\mathbb{R}^3}$ and with $(\cdot \mid \cdot)_{\mathbb{R}^3}$ respectively the norm and the scalar product of $H^1(\mathbb{R}^3)$.
- With C_i and c_i, we denote generic positive constants, which may also vary from line to line.

2. Some preliminary

Performing the change of variable $x \mapsto \varepsilon x$, problem $(\mathcal{P}_\varepsilon)$ becomes:

$$\begin{cases} -\Delta u + J_1(\varepsilon x)u = J_2(\varepsilon x)u^3 + \beta u v^2 = 0 & \text{in } \Omega_\varepsilon, \\ -\Delta v + K_1(\varepsilon x)v = K_2(\varepsilon x)v^3 + \beta u^2 v = 0 & \text{in } \Omega_\varepsilon, \\ u, v > 0 & \text{in } \Omega_\varepsilon, \\ u = v = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.1}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Of course if (u, v) is a solution of (2.1), then $(u(\cdot/\varepsilon), v(\cdot/\varepsilon))$ is a solution of $(\mathcal{P}_\varepsilon)$.

Solutions of (2.1) will be found in

$$\mathcal{H}_\varepsilon = H_0^1(\Omega_\varepsilon) \times H_0^1(\Omega_\varepsilon),$$

endowed with the following norm:

$$\|(u, v)\|_{\mathcal{H}_\varepsilon}^2 = \|u\|_{H_0^1(\Omega_\varepsilon)}^2 + \|v\|_{H_0^1(\Omega_\varepsilon)}^2, \quad \text{for all } (u, v) \in \mathcal{H}_\varepsilon.$$

If there is no misunderstanding, we denote with $\|\cdot\|$ and with $(\cdot \mid \cdot)$ respectively the norm and the scalar product both of $H^1(\Omega_\varepsilon)$ and of $\mathcal{H}_\varepsilon$.

Solutions of (2.1) are critical points of the functional $f_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$, defined as

$$\begin{aligned} f_\varepsilon(u, v) = {} & \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} J_1(\varepsilon x)u^2 - \frac{1}{4}\int_{\Omega_\varepsilon} J_2(\varepsilon x)u^4 \\ & + \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} K_1(\varepsilon x)v^2 - \frac{1}{4}\int_{\Omega_\varepsilon} K_2(\varepsilon x)v^4 - \frac{\beta}{2}\int_{\Omega_\varepsilon} u^2 v^2. \end{aligned}$$

If we define $f_\varepsilon^J : H_0^1(\Omega_\varepsilon) \to \mathbb{R}$ and $f_\varepsilon^K : H_0^1(\Omega_\varepsilon) \to \mathbb{R}$ as

$$f_\varepsilon^J(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} J_1(\varepsilon x)u^2 - \frac{1}{4}\int_{\Omega_\varepsilon} J_2(\varepsilon x)u^4,$$

$$f_\varepsilon^K(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} K_1(\varepsilon x)v^2 - \frac{1}{4}\int_{\Omega_\varepsilon} K_2(\varepsilon x)v^4,$$

we have

$$f_\varepsilon(u,v) = f_\varepsilon^J(u) + f_\varepsilon^K(v) - \frac{\beta}{2}\int_{\Omega_\varepsilon} u^2v^2.$$

Furthermore, for any fixed $Q \in \Omega$, we define the two functionals $F_Q^J : H^1(\mathbb{R}^3) \to \mathbb{R}$ and $F_Q^K : H^1(\mathbb{R}^3) \to \mathbb{R}$, as follows:

$$F_Q^J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} J_1(Q)u^2 - \frac{1}{4}\int_{\mathbb{R}^3} J_2(Q)u^4,$$

$$F_Q^K(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^3} K_1(Q)v^2 - \frac{1}{4}\int_{\mathbb{R}^3} K_2(Q)v^4.$$

The solutions of (2.1) will be found near (U^Q, V^Q), properly truncated, where U^Q is the unique solution of

$$\begin{cases} -\Delta u + J_1(Q)u = J_2(Q)u^3 & \text{in } \mathbb{R}^3,\\ u > 0 & \text{in } \mathbb{R}^3,\\ u(0) = \max_{\mathbb{R}^3} u, \end{cases} \tag{2.2}$$

and V_Q is the unique solution of

$$\begin{cases} -\Delta v + K_1(Q)v = K_2(Q)v^3 & \text{in } \mathbb{R}^3,\\ v > 0 & \text{in } \mathbb{R}^3,\\ v(0) = \max_{\mathbb{R}^3} v, \end{cases} \tag{2.3}$$

for an appropriate choice of $Q \in \Omega_0$. It is easy to see that

$$U^Q(x) = \sqrt{J_1(Q)/J_2(Q)} \cdot W\big(\sqrt{J_1(Q)} \cdot x\big), \tag{2.4}$$

$$V^Q(x) = \sqrt{K_1(Q)/K_2(Q)} \cdot W\big(\sqrt{K_1(Q)} \cdot x\big), \tag{2.5}$$

where W is the unique solution of

$$\begin{cases} -\Delta z + z = z^3 & \text{in } \mathbb{R}^3,\\ z > 0 & \text{in } \mathbb{R}^3,\\ z(0) = \max_{\mathbb{R}^3} z, \end{cases} \tag{2.6}$$

which is radially symmetric and decays exponentially at infinity with its first derivatives (cf. [16, 20]).

For all $Q \in \Omega_0$, we set $Q' = Q'(\varepsilon, Q) = Q + \sqrt{\varepsilon} e_1 \in \Omega$ and moreover we call $P = P(\varepsilon, Q) = Q/\varepsilon \in \Omega_\varepsilon$ and $P' = P'(\varepsilon, Q) = Q'/\varepsilon \in \Omega_\varepsilon$. Let us observe that

$$|P - P'| = \frac{1}{\sqrt{\varepsilon}} \to \infty, \quad \text{as } \varepsilon \to 0. \tag{2.7}$$

Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that

$$\begin{aligned} \chi(x) &= 1, \quad \text{for } |x| \leqslant \varepsilon^{-1/4}; \\ \chi(x) &= 0, \quad \text{for } |x| \geqslant 2\varepsilon^{-1/4}; \\ 0 \leqslant \chi(x) &\leqslant 1, \quad \text{for } \varepsilon^{-1/4} \leqslant |x| \leqslant 2\varepsilon^{-1/4}; \\ \left|\nabla \chi(x)\right| &\leqslant 2\varepsilon^{1/4}, \quad \text{for } \varepsilon^{-1/4} \leqslant |x| \leqslant 2\varepsilon^{-1/4}. \end{aligned} \tag{2.8}$$

We denote

$$U_P(x) := \chi(x - P) U^Q(x - P), \tag{2.9}$$

$$V_{P'}(x) := \chi(x - P') V^Q(x - P'). \tag{2.10}$$

Let us observe that $(U_P, V_{P'}) \in \mathcal{H}_\varepsilon$. For Q varying in Ω_0, $(U_P, V_{P'})$ describes a 3-dimensional manifold, namely,

$$Z^\varepsilon = \left\{ (U_P, V_{P'}) \colon\ Q \in \Omega_0 \right\}. \tag{2.11}$$

Remark 2.1. Of course, if $\Omega = \mathbb{R}^3$, then $\Omega_0 = \mathbb{R}^3$ and we do not need to truncate U^Q and V^Q. In this case, we would have simply $U_P = U^Q(\cdot - P)$ and $V_{P'} = V^Q(\cdot - P')$.

First of all let us give the following estimate which will be very useful in the sequel.

Lemma 2.2. *For all $Q \in \Omega_0$ and for all ε sufficiently small, if $Q' = Q + \sqrt{\varepsilon} e_1$, $P = Q/\varepsilon \in \Omega_\varepsilon$ and $P' = Q'/\varepsilon \in \Omega_\varepsilon$, then*

$$\int_{\Omega_\varepsilon} U_P^2 V_{P'}^2 = o(\varepsilon). \tag{2.12}$$

Proof. Let us start observing that, since

$$|P - P'| = \varepsilon^{-1/2} > 4\varepsilon^{-1/4},$$

we infer that

$$B_{2\varepsilon^{-1/4}}(P) \cup B_{2\varepsilon^{-1/4}}(P') = \emptyset.$$

Therefore, by the definitions of (2.9) and (2.10) and by the exponential decay of U_P and $V_{P'}$, we get

$$\int_{\Omega_\varepsilon} U_P^2 V_{P'}^2 \leqslant \int_{B_{2\varepsilon^{-1/4}}(P)\cup B_{2\varepsilon^{-1/4}}(P')} \left(U^Q\right)^2(x-P)\left(V^Q\right)^2(x-P')$$
$$\leqslant c_1 \int_{\mathbb{R}^3\setminus B_{2\varepsilon^{-1/4}}(P')} \left(V^Q\right)^2(x-P') + c_2 \int_{\mathbb{R}^3\setminus B_{2\varepsilon^{-1/4}}(P)} \left(U^Q\right)^2(x-P) = o(\varepsilon).$$

This concludes the proof. □

In the next lemma we show that the 3-dimensional manifold Z_ε, defined in (2.11), is actually a manifold of almost critical points of f_ε.

Lemma 2.3. *For all $Q \in \Omega_0$ and for all ε sufficiently small, if $Q' = Q + \sqrt{\varepsilon}e_1$, $P = Q/\varepsilon \in \Omega_\varepsilon$ and $P' = Q'/\varepsilon \in \Omega_\varepsilon$, then*

$$\left\|\nabla f_\varepsilon(U_P, V_{P'})\right\| = O\left(\varepsilon^{1/2}\right). \tag{2.13}$$

Proof. For all $(u, v) \in \mathcal{H}_\varepsilon$, we have

$$\left(\nabla f_\varepsilon(U_P, V_{P'}) \mid (u,v)\right) = \int_{\Omega_\varepsilon} \left[\nabla U_P \cdot \nabla u + J_1(\varepsilon x)U_P u - J_2(\varepsilon x)U_P^3 u\right]$$
$$+ \int_{\Omega_\varepsilon} \left[\nabla V_{P'} \cdot \nabla v + K_1(\varepsilon x)V_{P'} v - K_2(\varepsilon x)V_{P'}^3 v\right]$$
$$- \beta \int_{\Omega_\varepsilon} U_P V_{P'}^2 u - \beta \int_{\Omega_\varepsilon} U_P^2 V_{P'} v. \tag{2.14}$$

Let us study the first integral of the right-hand side of (2.14). By the exponential decay of U^Q and recalling that U^Q is solution of (2.2), we get

$$\int_{\Omega_\varepsilon} \left[\nabla U_P \cdot \nabla u + J_1(\varepsilon x)U_P u - J_2(\varepsilon x)U_P^3 u\right]$$
$$= \int_{(\Omega-Q)/\varepsilon\cap B_{\varepsilon^{-1/4}}} \left[\nabla U^Q \cdot \nabla u_{-P} + J_1(\varepsilon x+Q)U^Q u_{-P}\right]$$
$$- \int_{(\Omega-Q)/\varepsilon\cap B_{\varepsilon^{-1/4}}} J_2(\varepsilon x+Q)\left(U^Q\right)^3 u_{-P} + o(\varepsilon)$$
$$= \int_{\mathbb{R}^3} \left[\nabla U^Q \cdot \nabla u_{-P} + J_1(\varepsilon x+Q)U^Q u_{-P} - J_2(\varepsilon x+Q)\left(U^Q\right)^3 u_{-P}\right] + o(\varepsilon)$$

$$= \int_{\mathbb{R}^3} \left[\nabla U^Q \cdot \nabla u_{-P} + J_1(Q) U^Q u_{-P} - J_2(Q) \left(U^Q\right)^3 u_{-P}\right]$$
$$+ \int_{\mathbb{R}^3} \left(J_1(\varepsilon x + Q) - J_1(Q)\right) U^Q u_{-P}$$
$$- \int_{\mathbb{R}^3} \left(J_2(\varepsilon x + Q) - J_2(Q)\right)\left(U^Q\right)^3 u_{-P} + o(\varepsilon)$$
$$= \int_{\mathbb{R}^3} \left(J_1(\varepsilon x + Q) - J_1(Q)\right) U^Q u_{-P}$$
$$- \int_{\mathbb{R}^3} \left(J_2(\varepsilon x + Q) - J_2(Q)\right)\left(U^Q\right)^3 u_{-P} + o(\varepsilon). \tag{2.15}$$

Moreover, from the assumption DJ_i bounded, we infer that

$$\left|J_i(\varepsilon x + Q) - J_i(Q)\right| \leqslant c_1 \varepsilon |x|,$$

and so,

$$\int_{\mathbb{R}^3} \left(J_1(\varepsilon x + Q) - J_1(Q)\right) U^Q u_{-P} \leqslant \|u\| \left(\int_{\mathbb{R}^3} \left|J_1(\varepsilon x + Q) - J_1(Q)\right|^2 \left|U^Q\right|^2 \right)^{1/2}$$
$$\leqslant c_1 \|u\| \left(\int_{\mathbb{R}}^{3} \varepsilon^2 |x|^2 \left|U^Q\right|^2 \right)^{1/2} = O(\varepsilon)\|u\|. \tag{2.16}$$

Analogously,

$$\int_{\mathbb{R}^3} \left(J_2(\varepsilon x + Q) - J_2(Q)\right)\left(U^Q\right)^3 u_{-P} = O(\varepsilon)\|u\|. \tag{2.17}$$

Therefore, by (2.15)–(2.17), we infer

$$\int_{\Omega_\varepsilon} \left[\nabla U_P \cdot \nabla u + J_1(\varepsilon x) U_P u - J_2(\varepsilon x) U_P^3 u\right] = O(\varepsilon)\|u\|. \tag{2.18}$$

Similarly, since V^Q is solution of (2.3), we get

$$\int_{\Omega_\varepsilon} \left[\nabla V_{P'} \cdot \nabla v + K_1(\varepsilon x) V_{P'} v - K_2(\varepsilon x) V_{P'}^3 v\right]$$
$$= \int_{\mathbb{R}^3} \left(K_1(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_1(Q)\right) V^Q v_{-P'}$$

$$-\int_{\mathbb{R}^3} \big(K_2(\varepsilon x + Q + \sqrt{\varepsilon} e_1) - K_2(Q)\big)\big(V^Q\big)^3 v_{-P'} + o(\varepsilon). \tag{2.19}$$

Therefore, from the assumption DK_i bounded, we infer that

$$\big|K_i\big(\varepsilon x + Q + \sqrt{\varepsilon} e_1\big) - K_i(Q)\big| \leqslant c_2\sqrt{\varepsilon}\big|\sqrt{\varepsilon} x + e_1\big|,$$

and so,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(K_1\big(\varepsilon x + Q + \sqrt{\varepsilon} e_1\big) - K_1(Q)\big) V^Q v_{-P'} \\
&\quad \leqslant \|v\| \bigg(\int_{\mathbb{R}^3} \big|K_1\big(\varepsilon x + Q + \sqrt{\varepsilon} e_1\big) - K_1(Q)\big|^2 |V^Q|^2\bigg)^{1/2} \\
&\quad \leqslant c_2\|v\| \bigg(\int_{\mathbb{R}}^{3} \varepsilon\big|\sqrt{\varepsilon} x + e_1\big|^2 \big|V^Q\big|^2\bigg)^{1/2} = O\big(\varepsilon^{1/2}\big)\|v\|.
\end{aligned} \tag{2.20}$$

Analogously,

$$\int_{\mathbb{R}^3} \big(K_2\big(\varepsilon x + Q + \sqrt{\varepsilon} e_1\big) - K_2(Q)\big)\big(V^Q\big)^3 v_{-P'} = O\big(\varepsilon^{1/2}\big)\|v\|. \tag{2.21}$$

Therefore, by (2.19)–(2.21), we infer

$$\int_{\Omega_\varepsilon} \big[\nabla V_{P'} \cdot \nabla v + K_1(\varepsilon x) V_{P'} v - K_2(\varepsilon x) V_{P'}^3 v\big] = O\big(\varepsilon^{1/2}\big)\|v\|. \tag{2.22}$$

Let us study the last two terms of (2.14). Arguing as in Lemma 2.2, we get

$$\bigg|\int_{\Omega_\varepsilon} U_P V_{P'}^2 u\bigg| \leqslant c_3 \bigg(\int_{\Omega_\varepsilon} U_P^{4/3} V_{P'}^{8/3}\bigg)^{3/4} \|u\| = o(\varepsilon)\|u\| \tag{2.23}$$

and

$$\bigg|\int_{\Omega_\varepsilon} U_P^2 V_{P'} v\bigg| = o(\varepsilon)\|v\|. \tag{2.24}$$

Now the conclusion of the proof easily follows by (2.14), (2.18), (2.22)–(2.24). □

3. Invertibility of $D^2 f_\varepsilon$ on $(T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$

In this section we will show that $D^2 f_\varepsilon$ is invertible on $(T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$, where $T_{(U_P,V_{P'})}Z^\varepsilon$ denotes the tangent space to Z^ε at the point $(U_P, V_{P'})$.

Let

$$L_{\varepsilon,Q} : \big(T_{(U_P,V_{P'})}Z^\varepsilon\big)^\perp \to \big(T_{(U_P,V_{P'})}Z^\varepsilon\big)^\perp$$

denote the operator defined by setting

$$\big(L_{\varepsilon,Q}(h,h') \mid (k,k')\big) = D^2 f_\varepsilon(U_P, V_{P'})\big[(h,h'),(k,k')\big].$$

Lemma 3.1. *Given $\mu > 0$, there exists $C > 0$ such that, for ε small enough and for all $Q \in \Omega_0$ with $|Q| \leqslant \mu$, one has that*

$$\big\|L_{\varepsilon,Q}(h,h')\big\| \geqslant C\big\|(h,h')\big\|, \quad \forall (h,h') \in \big(T_{(U_P,V_{P'})}Z^\varepsilon\big)^\perp. \tag{3.1}$$

Proof. First of all, let us observe that, for all $(h,h'),(k,k') \in \mathcal{H}_\varepsilon$, we have

$$\begin{aligned} &D^2 f_\varepsilon(u,v)\big[(h,h'),(k,k')\big] \\ &\quad = D^2 f_\varepsilon^J(u)[h,k] + D^2 f_\varepsilon^K(v)[h',k'] \\ &\qquad - \beta \int_{\Omega_\varepsilon} v^2 hk - 2\beta \int_{\Omega_\varepsilon} uvhk' - 2\beta \int_{\Omega_\varepsilon} uvh'k - \beta \int_{\Omega_\varepsilon} u^2 h'k'. \end{aligned} \tag{3.2}$$

By (2.4), if we set $a(Q) = \sqrt{J_1(Q)/J_2(Q)}$ and $b(Q) = \sqrt{J_1(Q)}$, we have that $U^Q(x) = a(Q)W(b(Q)x)$ and so $U_P(x) = \chi(x-P)a(\varepsilon P)W(b(\varepsilon P)(x-P))$. Therefore, we have

$$\begin{aligned} \partial_{P_i} U_P(x) &= \partial_{P_i}\big(\chi(x-P)U^Q(x-P)\big) \\ &= -U^Q(x-P)\partial_{x_i}\chi(x-P) + \chi(x-P)\partial_{P_i}U^Q(x-P) \\ &= -U^Q(x-P)\partial_{x_i}\chi(x-P) + \varepsilon\chi(x-P)\partial_{P_i}a(\varepsilon P)W\big(b(\varepsilon P)(x-P)\big) \\ &\quad + \varepsilon\chi(x-P)a(\varepsilon P)\partial_{P_i}a(\varepsilon P)\nabla W\big(b(\varepsilon P)(x-P)\big)\cdot(x-P) \\ &\quad - \chi(x-P)a(\varepsilon P)b(\varepsilon P)(\partial_{x_i}W)\big(b(\varepsilon P)(x-P)\big). \end{aligned}$$

Hence

$$\partial_{P_i} U_P(x) = -\partial_{x_i} U_P(x) + O(\varepsilon). \tag{3.3}$$

Analogously, we can prove that

$$\partial_{P_i} V_{P'}(x) = \partial_{P_i'} V_{P'}(x) = -\partial_{x_i} V_{P'}(x) + O(\varepsilon). \tag{3.4}$$

We recall that

$$T_{(U_P,V_{P'})}Z^\varepsilon = \operatorname{span}_{\mathcal{H}_\varepsilon}\big\{(\partial_{P_1}U_P, \partial_{P_1}V_{P'}), (\partial_{P_2}U_P, \partial_{P_2}V_{P'}), (\partial_{P_3}U_P, \partial_{P_3}V_{P'})\big\}.$$

We set

$$\mathcal{V}_\varepsilon = \operatorname{span}_{\mathcal{H}_\varepsilon}\big\{(U_P, V_{P'}), (\partial_{x_1} U_P, \partial_{x_1} V_{P'}), (\partial_{x_2} U_P, \partial_{x_2} V_{P'}), (\partial_{x_3} U_P, \partial_{x_3} V_{P'})\big\}.$$

By (3.3) and (3.4), therefore it suffices to prove Eq. (3.1) for all $(h, h') \in \operatorname{span}_{\mathcal{H}_\varepsilon}\{(U_P, V_{P'}), (\phi, \phi')\}$, where (ϕ, ϕ') is orthogonal to $\mathcal{V}_\varepsilon$. Precisely we shall prove that there exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$ small enough, one has

$$\big(L_{\varepsilon,Q}(U_P, V_{P'}) \mid (U_P, V_{P'})\big) \leqslant -C_1 < 0, \tag{3.5}$$

$$\big(L_{\varepsilon,Q}(\phi, \phi') \mid (\phi, \phi')\big) \geqslant C_2 \big\|(\phi, \phi')\big\|^2, \quad \text{for all } (\phi, \phi') \perp \mathcal{V}_\varepsilon. \tag{3.6}$$

Proof of (3.5). By (3.2), we get

$$\begin{aligned} &D^2 f_\varepsilon(U_P, V_{P'})\big[(U_P, V_{P'}), (U_P, V_{P'})\big] \\ &\quad = D^2 f_\varepsilon^J(U_P)[U_P, U_P] + D^2 f_\varepsilon^K(V_{P'})[V_{P'}, V_{P'}] - 6\beta \int_{\Omega_\varepsilon} U_P^2 V_{P'}^2. \end{aligned} \tag{3.7}$$

Let us study the first term of the right-hand side of (3.7),

$$\begin{aligned} &D^2 f_\varepsilon^J(U_P)[U_P, U_P] \\ &\quad = \int_{\Omega_\varepsilon} |\nabla U_P|^2 + \int_{\Omega_\varepsilon} J_1(\varepsilon x) U_P^2 - 3 \int_{\Omega_\varepsilon} J_2(\varepsilon x) U_P^4 \\ &\quad = \int_{(\Omega - Q)/\varepsilon \cap B_{\varepsilon^{-1/4}}} \big[\big|\nabla U^Q\big|^2 + J_1(\varepsilon x + Q)\big(U^Q\big)^2 - 3 J_2(\varepsilon x + Q)\big(U^Q\big)^4\big] + o(\varepsilon) \\ &\quad = \int_{\mathbb{R}^3} \big[\big|\nabla U^Q\big|^2 + J_1(Q)\big(U^Q\big)^2 - 3 J_2(Q)\big(U^Q\big)^4\big] \\ &\qquad + \int_{\mathbb{R}^3} \big(J_1(\varepsilon x + Q) - J_1(Q)\big)\big(U^Q\big)^2 - 3 \int_{\mathbb{R}^3} \big(J_2(\varepsilon x + Q) - J_2(Q)\big)\big(U^Q\big)^4 + o(\varepsilon) \\ &\quad = -2 \int_{\mathbb{R}^3} J_2(Q)\big(U^Q\big)^4 + O(\varepsilon) \\ &\quad = -2 J_1(Q)^{\frac{1}{2}} J_2(Q)^{-1} \int_{\mathbb{R}^3} W^4 + O(\varepsilon) \leqslant -c_1. \end{aligned}$$

In a similar way it is possible to prove that

$$D^2 f_\varepsilon^K(V_{P'})[V_{P'}, V_{P'}] \leqslant -c_2.$$

Finally, by Lemma 2.2, we know that

$$\int_{\Omega_\varepsilon} U_P^2 V_{P'}^2 = o(\varepsilon),$$

and so Eq. (3.5) is proved.

Proof of (3.6). Recalling the definition of χ (see (2.8)), we set $\chi_1 := \chi$ and $\chi_2 := 1 - \chi_1$. Given $(\phi, \phi') \perp \mathcal{V}_\varepsilon$, let us consider the functions

$$\phi_i(x) = \chi_i(x - P)\phi(x), \quad i = 1, 2; \tag{3.8}$$

$$\phi_i'(x) = \chi_i(x - P')\phi'(x), \quad i = 1, 2. \tag{3.9}$$

With calculations similar to those of [7], we have

$$\|\phi\|^2 = \|\phi_1\|^2 + \|\phi_2\|^2 + \underbrace{2\int_{\Omega_\varepsilon} \chi_1\chi_2\big(\phi^2 + |\nabla\phi|^2\big)}_{I_\phi} + O\big(\varepsilon^{1/4}\big)\|\phi\|^2, \tag{3.10}$$

$$\|\phi'\|^2 = \big\|\phi_1'\big\|^2 + \big\|\phi_2'\big\|^2 + \underbrace{2\int_{\Omega_\varepsilon} \chi_1\chi_2\big((\phi')^2 + |\nabla\phi'|^2\big)}_{I_{\phi'}} + O\big(\varepsilon^{1/4}\big)\|\phi'\|^2. \tag{3.11}$$

We need to evaluate the three terms in the equation below:

$$\begin{aligned}\big(L_{\varepsilon,Q}(\phi,\phi') \mid (\phi,\phi')\big) = {} & \big(L_{\varepsilon,Q}\big(\phi_1,\phi_1'\big) \bigm| \big(\phi_1,\phi_1'\big)\big) + \big(L_{\varepsilon,Q}\big(\phi_2,\phi_2'\big) \bigm| \big(\phi_2,\phi_2'\big)\big) \\ & + 2\big(L_{\varepsilon,Q}\big(\phi_1,\phi_1'\big) \bigm| \big(\phi_2,\phi_2'\big)\big).\end{aligned} \tag{3.12}$$

Let us start with $(L_{\varepsilon,Q}(\phi_1,\phi_1') \mid (\phi_1,\phi_1'))$. Since $\beta < 0$, we get

$$\begin{aligned}\big(L_{\varepsilon,Q}\big(\phi_1,\phi_1'\big) \bigm| \big(\phi_1,\phi_1'\big)\big) = {} & D^2 f_\varepsilon^J(U_P)[\phi_1,\phi_1] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_1',\phi_1'\big] \\ & - 4\beta\int_{\Omega_\varepsilon} U_P V_{P'}\phi_1\phi_1' - \beta\int_{\Omega_\varepsilon} U_P^2 {\phi_1'}^2 - \beta\int_{\Omega_\varepsilon} V_{P'}^2\phi_1^2 \\ > {} & D^2 f_\varepsilon^J(U_P)[\phi_1,\phi_1] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_1',\phi_1'\big] \\ & - 4\beta\int_{\Omega_\varepsilon} U_P V_{P'}\phi_1\phi_1'.\end{aligned} \tag{3.13}$$

Arguing as in Lemma 2.2, we know that

$$\int_{\Omega_\varepsilon} U_P V_{P'}\phi_1\phi_1' = o(\varepsilon). \tag{3.14}$$

Therefore we need only to study the first two terms of the right-hand side of (3.13). For simplicity, we can assume that $Q=\varepsilon P$ is the origin $\mathcal{O}$. In this case, we recall that we denote with $U^{\mathcal{O}}$ the unique solution of (2.2) whenever $Q=\mathcal{O}$, while we denote with $U_{\mathcal{O}}$ the truncation of $U^{\mathcal{O}}$, namely $U_{\mathcal{O}}=\chi U^{\mathcal{O}}$, where χ is defined in (2.8). We have

$$
\begin{aligned}
D^2 f_\varepsilon^J(U_{\mathcal{O}})[\phi_1,\phi_1] &= \int_{\Omega_\varepsilon} \left[|\nabla\phi_1|^2 + J_1(\varepsilon x)\phi_1^2 - 3J_2(\varepsilon x)U_{\mathcal{O}}^2\phi_1^2\right] \\
&= \int_{\mathbb{R}^3} \left[|\nabla\phi_1|^2 + J_1(\varepsilon x)\phi_1^2 - 3J_2(\varepsilon x)\left(U^{\mathcal{O}}\right)^2\phi_1^2\right] + o(\varepsilon)\|\phi\|^2 \\
&= D^2F^{J(\mathcal{O})}\left(U^{\mathcal{O}}\right)[\phi_1,\phi_1] + \int_{\mathbb{R}^3} \left(J_1(\varepsilon x) - J_1(\mathcal{O})\right)\phi_1^2 \\
&\quad - 3\int_{\mathbb{R}^3} \left(J_2(\varepsilon x) - J_2(\mathcal{O})\right)\left(U^{\mathcal{O}}\right)^2\phi_1^2 + o(\varepsilon)\|\phi\|^2 \\
&\geqslant D^2F^{J(\mathcal{O})}\left(U^{\mathcal{O}}\right)[\phi_1,\phi_1] - c_3\varepsilon\int_{\mathbb{R}^3} |x|\phi_1^2 + O(\varepsilon)\|\phi\|^2 \\
&= D^2F^{J(\mathcal{O})}\left(U^{\mathcal{O}}\right)[\phi_1,\phi_1] + O\left(\varepsilon^{3/4}\right)\|\phi\|^2,
\end{aligned}
$$

therefore

$$
D^2 f_\varepsilon^J(U_{\mathcal{O}})[\phi_1,\phi_1] \geqslant D^2F^{J(\mathcal{O})}\left(U^{\mathcal{O}}\right)[\phi_1,\phi_1] + O\left(\varepsilon^{3/4}\right)\|\phi\|^2. \tag{3.15}
$$

We recall that ϕ is orthogonal to

$$
\mathcal{V}_\varepsilon^U = \operatorname{span}_{H_0^1(\Omega_\varepsilon)}\{U_{\mathcal{O}}, \partial_{x_1}U_{\mathcal{O}}, \partial_{x_2}U_{\mathcal{O}}, \partial_{x_3}U_{\mathcal{O}}\}.
$$

Moreover by [9], we know that if $\tilde\phi$ is orthogonal to $\mathcal{V}$ with

$$
\mathcal{V}^U = \operatorname{span}_{H^1(\mathbb{R}^3)}\left\{U^{\mathcal{O}}, \partial_{x_1}U^{\mathcal{O}}, \partial_{x_2}U^{\mathcal{O}}, \partial_{x_3}U^{\mathcal{O}}\right\},
$$

then the fact that $U^{\mathcal{O}}$ is a Mountain Pass critical point of $F^{J(\mathcal{O})}$ implies that

$$
D^2F^{J(\mathcal{O})}\left(U^{\mathcal{O}}\right)[\tilde\phi,\tilde\phi] > c_4\|\tilde\phi\|_{\mathbb{R}^3}^2 \quad \text{for all } \tilde\phi \perp \mathcal{V}^U. \tag{3.16}
$$

We can write $\phi_1 = \xi + \zeta$, where $\xi \in \mathcal{V}^U$ and $\zeta \perp \mathcal{V}^U$. More precisely,

$$
\xi = \left(\phi_1 \mid U^{\mathcal{O}}\right)_{\mathbb{R}^3} U^{\mathcal{O}}\left\|U^{\mathcal{O}}\right\|_{\mathbb{R}^3}^{-2} + \sum_{i=1}^{3}\left(\phi_1 \mid \partial_{x_i}U^{\mathcal{O}}\right)_{\mathbb{R}^3}\partial_{x_i}U^{\mathcal{O}}\left\|\partial_{x_i}U^{\mathcal{O}}\right\|_{\mathbb{R}^3}^{-2}.
$$

Let us calculate $(\phi_1 \mid U^{\mathcal{O}})_{\mathbb{R}^3}$. By the exponential decay of $U^{\mathcal{O}}$ and since $\phi \perp \mathcal{V}_\varepsilon^U$, we have

$$\begin{aligned}
(\phi_1 \mid U^{\mathcal{O}})_{\mathbb{R}^3} &= \int_{\mathbb{R}^3} \nabla\phi_1 \cdot \nabla U^{\mathcal{O}} + \int_{\mathbb{R}^3} \phi_1 U^{\mathcal{O}} \\
&= \int_{\Omega_\varepsilon} \nabla\phi_1 \cdot \nabla U_{\mathcal{O}} + \int_{\Omega_\varepsilon} \phi_1 U_{\mathcal{O}} + o(\varepsilon)\|\phi\| \\
&= \int_{\Omega_\varepsilon} \nabla\phi \cdot \nabla U_{\mathcal{O}} + \int_{\Omega_\varepsilon} \phi U_{\mathcal{O}} + o(\varepsilon)\|\phi\| = o(\varepsilon)\|\phi\|.
\end{aligned}$$

In a similar way, we can prove also that $(\phi_1 \mid \partial_{x_i} U^{\mathcal{O}})_{\mathbb{R}^3} = o(\varepsilon)\|\phi\|$, and so

$$\|\xi\|_{\mathbb{R}^3} = o(\varepsilon)\|\phi\|, \tag{3.17}$$

$$\|\zeta\|_{\mathbb{R}^3} = \|\phi_1\| + o(\varepsilon)\|\phi\|. \tag{3.18}$$

Let us estimate $D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\phi_1, \phi_1]$. We get:

$$\begin{aligned}
D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\phi_1, \phi_1] = {} & D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\zeta, \zeta] + 2D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\zeta, \xi] \\
& + D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\xi, \xi].
\end{aligned} \tag{3.19}$$

By (3.16) and (3.18), since $\zeta \perp \mathcal{V}^U$, we know that

$$D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\zeta, \zeta] > c_3\|\zeta\|^2_{\mathbb{R}^3} = c_3\|\phi_1\|^2 + o(\varepsilon)\|\phi\|^2,$$

while, by (3.17) and straightforward calculations, we have

$$D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\zeta, \xi] = o(\varepsilon)\|\phi\|^2, \qquad D^2 F^{J(\mathcal{O})}(U^{\mathcal{O}})[\xi, \xi] = o(\varepsilon)\|\phi\|^2.$$

By these last two estimates, (3.19) and (3.15), we can say that

$$D^2 f_\varepsilon^J(U_{\mathcal{O}})[\phi_1, \phi_1] > c_4\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2.$$

Hence, in the general case, we infer that, for all $Q \in \Omega_0$ with $|Q| \leqslant \mu$,

$$D^2 f_\varepsilon^J(U_P)[\phi_1, \phi_1] > c_4\|\phi_1\|^2 + O(\varepsilon^{3/4})\|\phi\|^2, \tag{3.20}$$

and, analogously,

$$D^2 f_\varepsilon^K(V_{P'})[\phi_1', \phi_1'] > c_5\|\phi_1'\|^2 + O(\varepsilon^{1/2})\|\phi'\|^2. \tag{3.21}$$

By (3.13), (3.14), (3.20) and (3.21), we can say that

$$\big(L_{\varepsilon,Q}(\phi_1, \phi_1') \mid (\phi_1, \phi_1')\big) > c_6\big\|(\phi_1, \phi_1')\big\|^2 + O(\varepsilon^{1/2})\big\|(\phi, \phi_1')\big\|^2. \tag{3.22}$$

Let us now evaluate $(L_{\varepsilon,Q}(\phi_2, \phi_2') \mid (\phi_2, \phi_2'))$. Arguing as in Lemma 2.2, since $\beta < 0$ and using the definition of χ_i and the exponential decay of U_P and of $V_{P'}$, we easily get

$$\begin{aligned}\big(L_{\varepsilon,Q}\big(\phi_2,\phi_2'\big)\mid\big(\phi_2,\phi_2'\big)\big) &= D^2 f_\varepsilon^J(U_P)[\phi_2,\phi_2] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_2',\phi_2'\big] \\ &\quad - 4\beta\int_{\Omega_\varepsilon} U_P V_{P'}\phi_2\phi_2' - \beta\int_{\Omega_\varepsilon} U_P^2 {\phi_2'}^2 - \beta\int_{\Omega_\varepsilon} V_{P'}^2\phi_2^2 \\ &\geqslant D^2 f_\varepsilon^J(U_P)[\phi_2,\phi_2] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_2',\phi_2'\big] + o(\varepsilon)\big\|(\phi,\phi')\big\|^2 \\ &\geqslant c_7\big\|\big(\phi_2,\phi_2'\big)\big\|^2 + o(\varepsilon)\big\|(\phi,\phi')\big\|^2. \end{aligned} \tag{3.23}$$

Let us now study $(L_{\varepsilon,Q}(\phi_1,\phi_1') \mid (\phi_2,\phi_2'))$. Arguing as in Lemma 2.2, we get

$$\begin{aligned}\big(L_{\varepsilon,Q}\big(\phi_1,\phi_1'\big)\mid\big(\phi_2,\phi_2'\big)\big) &= D^2 f_\varepsilon^J(U_P)[\phi_1,\phi_2] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_1',\phi_2'\big] \\ &\quad - 2\beta\int_{\Omega_\varepsilon} U_P V_{P'}\phi_1\phi_2' - 2\beta\int_{\Omega_\varepsilon} U_P V_{P'}\phi_2\phi_1' \\ &\quad - \beta\int_{\Omega_\varepsilon} U_P^2\phi_1'\phi_2' - \beta\int_{\Omega_\varepsilon} V_{P'}^2\phi_1\phi_2 \\ &= D^2 f_\varepsilon^J(U_P)[\phi_1,\phi_2] + D^2 f_\varepsilon^K(V_{P'})\big[\phi_1',\phi_2'\big] \\ &\quad - \beta\int_{\Omega_\varepsilon} U_P^2\phi_1'\phi_2' - \beta\int_{\Omega_\varepsilon} V_{P'}^2\phi_1\phi_2 + o(\varepsilon)\big\|(\phi,\phi')\big\|^2. \end{aligned} \tag{3.24}$$

Using the definition of χ_i and the exponential decay of U_P and of $V_{P'}$, we easily get

$$D^2 f_\varepsilon^J(U_P)[\phi_1,\phi_2] \geqslant c_8 I_\phi + O\big(\varepsilon^{1/4}\big)\|\phi\|^2, \tag{3.25}$$

$$D^2 f_\varepsilon^K(V_{P'})\big[\phi_1',\phi_2'\big] \geqslant c_9 I_{\phi'} + O\big(\varepsilon^{1/4}\big)\|\phi'\|^2, \tag{3.26}$$

where I_ϕ and $I_{\phi'}$ are defined, respectively in (3.10) and (3.11). Moreover, by the definition of χ (see (2.8)), and by the definitions of ϕ_i and ϕ_i' (see (3.8) and (3.9)),

$$\phi_1(x)\phi_2(x) = \chi(x-P)\big(1-\chi(x-P)\big)\phi^2(x) \geqslant 0, \quad \text{for all } x\in\mathbb{R}^3,$$

and so, also

$$\phi_1'(x)\phi_2'(x) \geqslant 0, \quad \text{for all } x\in\mathbb{R}^3.$$

Therefore

$$-\beta\int_{\Omega_\varepsilon} U_P^2\phi_1'\phi_2' - \beta\int_{\Omega_\varepsilon} V_{P'}^2\phi_1\phi_2 \geqslant 0. \tag{3.27}$$

By (3.24)–(3.27), we infer

$$\big(L_{\varepsilon,Q}\big(\phi_1,\phi_1'\big)\mid\big(\phi_2,\phi_2'\big)\big) \geqslant c_{10}(I_\phi + I_{\phi'}) + O\big(\varepsilon^{1/4}\big)\big\|(\phi,\phi')\big\|^2. \tag{3.28}$$

Hence, by (3.12), (3.22), (3.23), (3.28) and recalling (3.10) and (3.11), we get

$$\big(L_{\varepsilon,Q}(\phi,\phi') \mid (\phi,\phi')\big) \geqslant c_{11}\big\|(\phi,\phi')\big\|^2 + O\big(\varepsilon^{1/4}\big)\big\|(\phi,\phi')\big\|^2.$$

This completes the proof of the lemma. □

4. The finite dimensional reduction

By means of the Lyapunov–Schmidt reduction, the existence of critical points of f_ε can be reduced to the search of critical points of an auxiliary finite dimensional functional.

Lemma 4.1. *Fix $\mu > 0$. For $\varepsilon > 0$ small enough and for all $Q \in \Omega_0$ with $|Q| \leqslant \mu$, there exists a unique $(w, w') = (w(\varepsilon, Q), w'(\varepsilon, Q)) \in \mathcal{H}_\varepsilon$ of class C^1 such that*

(1) $(w(\varepsilon, Q), w'(\varepsilon, Q)) \in (T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$;
(2) $\nabla f_\varepsilon(U_P + w, V_{P'} + w') \in T_{(U_P,V_{P'})}Z^\varepsilon$.

Moreover, the functional $\mathcal{A}_\varepsilon : \Omega_0 \to \mathbb{R}$, defined as

$$\mathcal{A}_\varepsilon(Q) := f_\varepsilon\big(U_{Q/\varepsilon} + w(\varepsilon, Q), V_{(Q+e_1\sqrt{\varepsilon})/\varepsilon} + w'(\varepsilon, Q)\big)$$

is of class C^1 and satisfies

$$\nabla\mathcal{A}_\varepsilon(Q_0) = 0 \quad\Longleftrightarrow\quad \nabla f_\varepsilon\big(U_{Q_0/\varepsilon} + w(\varepsilon, Q_0), V_{(Q_0+e_1\sqrt{\varepsilon})/\varepsilon} + w'(\varepsilon, Q_0)\big) = 0.$$

Proof. Let $\mathcal{P} = \mathcal{P}_{\varepsilon,Q}$ denote the projection onto $(T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$. We want to find a solution $(w, w') \in (T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$ of the equation

$$P\nabla f_\varepsilon\big(U_P + w, V_{P'} + w'\big) = 0.$$

One has that

$$\nabla f_\varepsilon(U_P + w, V_{P'} + w') = \nabla f_\varepsilon(U_P, V_{P'}) + D^2 f_\varepsilon(U_P, V_{P'})[w, w'] + R(U_P, V_{P'}, w, w')$$

with $\|R(U_P, V_{P'}, w, w')\| = o(\|(w, w')\|)$, uniformly with respect to $(U_P, V_{P'})$. Therefore, our equation is

$$L_{\varepsilon,Q}(w, w') + \mathcal{P}\nabla f_\varepsilon(U_P, V_{P'}) + \mathcal{P}R(U_P, V_{P'}, w, w') = 0. \tag{4.1}$$

According to Lemma 3.1, this is equivalent to

$$(w, w') = N_{\varepsilon,Q}(w, w'),$$

where

$$N_{\varepsilon,Q}(w, w') = -(L_{\varepsilon,Q})^{-1}\big(\mathcal{P}\nabla f_\varepsilon(U_P, V_{P'}) + \mathcal{P}R(U_P, V_{P'}, w, w')\big).$$

By (2.13), it follows that

$$\big\|N_{\varepsilon,Q}(w,w')\big\| = O\big(\varepsilon^{1/2}\big) + o\big(\big\|(w,w')\big\|\big). \tag{4.2}$$

Therefore it is easy to check that $N_{\varepsilon,Q}$ is a contraction on some ball in $(T_{(U_P,V_{P'})}Z^\varepsilon)^\perp$ provided that $\varepsilon > 0$ is small enough. Then there exists a unique (w,w') such that $(w,w') = N_{\varepsilon,Q}(w,w')$. Let us point out that we cannot use the Implicit Function Theorem to find $(w(\varepsilon,Q), w'(\varepsilon,Q))$, because the map $(\varepsilon,u,v) \mapsto \mathcal{P}\nabla f_\varepsilon(u,v)$ fails to be C^2. However, fixed $\varepsilon > 0$ small, we can apply the Implicit Function Theorem to the map $(Q,w,w') \mapsto \mathcal{P}\nabla f_\varepsilon(U_P + w, V_{P'} + w')$. Then, in particular, the function $(w(\varepsilon,Q), w'(\varepsilon,Q))$ turns out to be of class C^1 with respect to Q. Finally, it is a standard argument, see [2,3], to check that the critical points of $\mathcal{A}_\varepsilon(Q) = f_\varepsilon(U_P + w, V_{P'} + w')$ give rise to critical points of f_ε. $\square$

Remark 4.2. From (4.2) it immediately follows that

$$\big\|(w,w')\big\| = O\big(\varepsilon^{1/2}\big). \tag{4.3}$$

Let us now make the asymptotic expansion of the finite dimensional functional.

Theorem 4.3. *Fix $\mu > 0$ and let $Q \in \Omega_0$ with $|Q| \leqslant \mu$, $Q' = Q + \sqrt{\varepsilon}e_1$, $P = Q/\varepsilon \in \Omega_\varepsilon$ and $P' = Q'/\varepsilon \in \Omega_\varepsilon$. Suppose* (J) *and* (K)*. Then, for ε sufficiently small, we get*

$$\mathcal{A}_\varepsilon(Q) = f_\varepsilon\big(U_P + w(\varepsilon,Q), V_{P'} + w'(\varepsilon,Q)\big) = c_0\Gamma(Q) + o\big(\varepsilon^{1/4}\big), \tag{4.4}$$

where $\Gamma : \Omega_0 \to \mathbb{R}$ is defined in (1.4)*, namely*

$$\Gamma(Q) = J_1(Q)^{\frac{1}{2}} J_2(Q)^{-1} + K_1(Q)^{\frac{1}{2}} K_2(Q)^{-1}$$

and

$$c_0 := \frac{1}{2}\int_{\mathbb{R}^3} W^4 \tag{4.5}$$

with W the unique solution of (2.6).

Proof. We have

$$\begin{aligned}
\mathcal{A}_\varepsilon(Q) &= f_\varepsilon\big(U_P + w(\varepsilon,Q), V_{P'} + w'(\varepsilon,Q)\big) \\
&= \frac{1}{2}\int_{\Omega_\varepsilon} \big|\nabla(U_P + w)\big|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} J_1(\varepsilon x)(U_P + w)^2 - \frac{1}{4}\int_{\Omega_\varepsilon} J_2(\varepsilon x)(U_P + w)^4 \\
&\quad + \frac{1}{2}\int_{\Omega_\varepsilon} \big|\nabla(V_{P'} + w')\big|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} K_1(\varepsilon x)(V_{P'} + w')^2 - \frac{1}{4}\int_{\Omega_\varepsilon} K_2(\varepsilon x)(V_{P'} + w')^4 \\
&\quad - \frac{\beta}{2}\int_{\Omega_\varepsilon} (U_P + w)^2(V_{P'} + w')^2.
\end{aligned}$$

Therefore, by (4.3) and Lemma 2.2,

$$\begin{aligned}\mathcal{A}_\varepsilon(Q) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla U_P|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} J_1(\varepsilon x)U_P^2 - \frac{1}{4}\int_{\Omega_\varepsilon} J_2(\varepsilon x)U_P^4 \\ &\quad + \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla V_{P'}|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} K_1(\varepsilon x)V_{P'}^2 - \frac{1}{4}\int_{\Omega_\varepsilon} K_2(\varepsilon x)V_{P'}^4 + O\big(\varepsilon^{1/2}\big) \\ &= f_\varepsilon^J(U_P) + f_\varepsilon^K(V_{P'}) + O\big(\varepsilon^{1/2}\big). \end{aligned} \tag{4.6}$$

Let us study the first term of the right-hand side of (4.6),

$$\begin{aligned} f_\varepsilon^J(U_P) &= \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla U_P|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} J_1(\varepsilon x)U_P^2 - \frac{1}{4}\int_{\Omega_\varepsilon} J_2(\varepsilon x)U_P^4 \\ &= \frac{1}{2}\int_{(\Omega-Q)/\varepsilon\cap B_{\varepsilon^{-1/4}}} \Big[\big|\nabla U^Q\big|^2 + J_1(\varepsilon x+Q)\big(U^Q\big)^2 - \frac{1}{2}J_2(\varepsilon x+Q)\big(U^Q\big)^4\Big] + o(\varepsilon) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \big|\nabla U^Q\big|^2 + \frac{1}{2}\int_{\mathbb{R}^3} J_1(Q)\big(U^Q\big)^2 - \frac{1}{4}\int_{\mathbb{R}^3} J_2(Q)\big(U^Q\big)^4 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} \big(J_1(\varepsilon x+Q) - J_1(Q)\big)\big(U^Q\big)^2 \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} \big(J_2(\varepsilon x+Q) - J_2(Q)\big)\big(U^Q\big)^4 + o(\varepsilon) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} J_2(Q)\big(U^Q\big)^4 + o\big(\varepsilon^{1/4}\big) \\ &= \frac{1}{2}J_1(Q)^{\frac{1}{2}}J_2(Q)^{-1}\int_{\mathbb{R}^3} W^4 + o\big(\varepsilon^{1/4}\big). \end{aligned}$$

Hence

$$f_\varepsilon^J(U_P) = \frac{1}{2}J_1(Q)^{\frac{1}{2}}J_2(Q)^{-1}\int_{\mathbb{R}^3} W^4 + o\big(\varepsilon^{1/4}\big). \tag{4.7}$$

Analogously,

$$f_\varepsilon^K(V_{P'}) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla V_{P'}|^2 + \frac{1}{2}\int_{\Omega_\varepsilon} K_1(\varepsilon x)V_{P'}^2 - \frac{1}{4}\int_{\Omega_\varepsilon} K_2(\varepsilon x)V_{P'}^4$$

$$
\begin{aligned}
&= \frac{1}{2} \int\limits_{(\Omega - Q')/\varepsilon \cap B_{\varepsilon^{-1/4}}} \left[\left|\nabla V^Q\right|^2 + K_1(\varepsilon x + Q')\left(V^Q\right)^2 \right] \\
&\quad - \frac{1}{4} \int\limits_{(\Omega - Q')/\varepsilon \cap B_{\varepsilon^{-1/4}}} K_2(\varepsilon x + Q')\left(V^Q\right)^4 + o(\varepsilon) \\
&= \frac{1}{2} \int\limits_{\mathbb{R}^3} \left|\nabla V^Q\right|^2 + \frac{1}{2} \int\limits_{\mathbb{R}^3} K_1(Q)\left(V^Q\right)^2 - \frac{1}{4} \int\limits_{\mathbb{R}^3} K_2(Q)\left(V^Q\right)^4 \\
&\quad + \frac{1}{2} \int\limits_{\mathbb{R}^3} \left(K_1\left(\varepsilon x + Q + \sqrt{\varepsilon} e_1\right) - K_1(Q)\right)\left(V^Q\right)^2 \\
&\quad - \frac{1}{4} \int\limits_{\mathbb{R}^3} \left(K_2\left(\varepsilon x + Q + \sqrt{\varepsilon} e_1\right) - K_2(Q)\right)\left(V^Q\right)^4 + o(\varepsilon) \\
&= \frac{1}{2} \int\limits_{\mathbb{R}^3} K_2(Q)\left(V^Q\right)^4 + o\left(\varepsilon^{1/4}\right) \\
&= \frac{1}{2} K_1(Q)^{\frac{1}{2}} K_2(Q)^{-1} \int\limits_{\mathbb{R}^3} W^4 + o\left(\varepsilon^{1/4}\right).
\end{aligned}
$$

Therefore

$$
f_\varepsilon^J(V_{P'}) = \frac{1}{2} K_1(Q)^{\frac{1}{2}} K_2(Q)^{-1} \int\limits_{\mathbb{R}^3} W^4 + o\left(\varepsilon^{1/4}\right). \tag{4.8}
$$

Now (4.4) follows immediately by (4.6)–(4.8). □

5. A multiplicity result and proofs of theorems

In this section we give the proofs of our theorems. First of all, let us prove Theorem 1.1 as an easy consequence of the following multiplicity result:

Theorem 5.1. *Let* (J) *and* (K) *hold and suppose* Γ *has a compact set* $X \subset \Omega_0$ *where* Γ *achieves a strict local minimum* (*respectively maximum*)*, in the sense that there exist* $\delta > 0$ *and a* δ*-neighborhood* $X_\delta \subset \Omega_0$ *of* X *such that*

$$
b := \inf\left\{\Gamma(Q):\ Q \in \partial X_\delta\right\} > a := \Gamma|_X \quad \left(\textit{respectively } \sup\left\{\Gamma(Q):\ Q \in \partial X_\delta\right\} < \Gamma|_X\right).
$$

Then there exists $\bar{\varepsilon} > 0$ *such that* $(\mathcal{P}_\varepsilon)$ *has at least* $\operatorname{cat}(X, X_\delta)$ *solutions that concentrate near points of* X_δ*, provided* $\varepsilon \in (0, \bar{\varepsilon})$*. Here* $\operatorname{cat}(X, X_\delta)$ *denotes the Lusternik–Schnirelman category of* X *with respect to* X_δ*.*

Proof. First of all, we fix $\mu > 0$ in such a way that $|Q| < \mu$ for all $Q \in X$. We will apply the finite dimensional procedure with such μ fixed.

We will treat only the case of minima, being the other one similar. We set $Y = \{Q \in X_\delta\colon \mathcal{A}_\varepsilon(Q) \leqslant c_0(a+b)/2\}$, being c_0 defined in (4.5). By (4.4) it follows that there exists $\bar{\varepsilon} > 0$ such that

$$X \subset Y \subset X_\delta, \tag{5.1}$$

provided $\varepsilon \in (0, \bar{\varepsilon})$. Moreover, if $Q \in \partial X_\delta$ then $\Gamma(Q) \geqslant b$ and hence

$$\mathcal{A}_\varepsilon(Q) \geqslant c_0 \Gamma(Q) + o\big(\varepsilon^{1/4}\big) \geqslant c_0 b + o\big(\varepsilon^{1/4}\big).$$

On the other side, if $Q \in Y$ then $\mathcal{A}_\varepsilon(Q) \leqslant c_0(a+b)/2$. Hence, for ε small, Y cannot meet ∂X_δ and this readily implies that Y is compact. Then $\mathcal{A}_\varepsilon$ possesses at least $\operatorname{cat}(Y, X_\delta)$ critical points in X_δ. Using (5.1) and the properties of the category, one gets

$$\operatorname{cat}(Y, Y) \geqslant \operatorname{cat}(X, X_\delta).$$

Moreover, by Lemma 4.1, we know that to critical points of $\mathcal{A}_\varepsilon$ there correspond critical points of f_ε and so solutions of (2.1). Let $Q_\varepsilon \in X$ be one of these critical points, if $Q'_\varepsilon = Q_\varepsilon + \sqrt{\varepsilon} e_1$, then

$$\big(u_\varepsilon^{Q_\varepsilon}, v_\varepsilon^{Q_\varepsilon}\big) = \big(U_{Q_\varepsilon/\varepsilon} + w(\varepsilon, Q_\varepsilon), V_{Q'_\varepsilon/\varepsilon} + w'(\varepsilon, Q_\varepsilon)\big)$$

is a solution of (2.1). Therefore

$$u_\varepsilon^{Q_\varepsilon}(x/\varepsilon) \simeq U_{Q_\varepsilon/\varepsilon}(x/\varepsilon) = U^{Q_\varepsilon}\left(\frac{x - Q_\varepsilon}{\varepsilon}\right),$$
$$v_\varepsilon^{Q_\varepsilon}(x/\varepsilon) \simeq V_{Q'_\varepsilon/\varepsilon}(x/\varepsilon) = V^{Q_\varepsilon}\left(\frac{x - Q'_\varepsilon}{\varepsilon}\right)$$

is a solution of $(\mathcal{P}_\varepsilon)$ and also the concentration result follows. □

Let us now give a short proof of Theorem 1.5.

Proof of Theorem 1.5. We need only to observe that, in this case, the solutions of $(\bar{\mathcal{P}}_\varepsilon)$ will be found near $(\bar{U}^Q, \bar{V}^Q)$, properly truncated, where $\bar{U}^Q$ is the unique solution of

$$\begin{cases} -\Delta u + J_1(Q)u = J_2(Q)u^{2p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(0) = \max_{\mathbb{R}^N} u, \end{cases}$$

and $\bar{V}_Q$ is the unique solution of

$$\begin{cases} -\Delta v + K_1(Q)v = K_2(Q)v^{2p-1} & \text{in } \mathbb{R}^N, \\ v > 0 & \text{in } \mathbb{R}^N, \\ v(0) = \max_{\mathbb{R}^N} v, \end{cases}$$

for an appropriate choice of $Q \in \bar{\Omega}_0$. It is easy to see that

$$\bar{U}^Q(x) = \big(J_1(Q)/J_2(Q)\big)^{1/(2p-2)} \cdot \bar{W}\big(\sqrt{J_1(Q)} \cdot x\big),$$
$$\bar{V}^Q(x) = \big(K_1(Q)/K_2(Q)\big)^{1/(2p-2)} \cdot \bar{W}\big(\sqrt{K_1(Q)} \cdot x\big),$$

where $\bar{W}$ is the unique solution of

$$\begin{cases} -\Delta z + z = z^{2p-1} & \text{in } \mathbb{R}^N, \\ z > 0 & \text{in } \mathbb{R}^N, \\ z(0) = \max_{\mathbb{R}^N} z. \end{cases}$$

At this point, we can repeat the previous arguments, with suitable modifications. □

Remark 5.2. Of course, the analogous of Theorem 5.1 holds also for problem $(\bar{\mathcal{P}}_\varepsilon)$.

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