§ 1. INTRODUCTION

In a recent paper, R. W. Richardson [7] has proved rigidity theorems for Lie and associative algebras, over either the real field or any algebraically closed field, which may be sketched as follows. Let $A$ be an algebra (of one of the above types), $B$ a subalgebra; so $A/B$ is a $B$-module. His main result states that, if $H^1(B, A/B) = 0$, then $B$ is a rigid subalgebra of $A$, i.e. all subalgebras whose underlying spaces are near that of $B$ are images of $B$ by an inner automorphism of $A$. Richardson gives numerous details, and considers also "equivalence groups" other than that of inner automorphisms 2).

In many other deformation problems a theorem on "Kuranishi families" 3) has been proved (cf. e.g. [4, 5, 6]) as well as a rigidity theorem, but so far not for subalgebras of an algebra. The analysis (resp. algebraic geometry) are well capable of handling the problem—the missing link was a set of relations sufficient to cut down the number of equations to be solved. In other deformation problems that link took the form of a Jacobi identity in a graded Lie algebra, or a variation on "$\delta^2 = 0".  

The above provides one reason for an analysis of the formalism surrounding subalgebras of an algebra, to which this paper is devoted. (Analytic or algebraic-geometric arguments have at most been sketched.) The crucial identity for deformations of subalgebras is supplied in Proposition 5.7; it is applied in § 6.

A second reason for an elaboration on the formalism is provided by a recent result [3], on the existence of a GLA structure on $H^*(B, A/B)$. We show here that this GLA structure supplies the primary obstruction to extending an infinitesimal deformation of a subalgebra into a finite one; cf. Proposition 6.1.

A third purpose of the paper is the presentation of a further unification of methods in dealing with the three cases under study (associative
algebras, commutative associative algebras and Lie algebras). This unification is accomplished by the formulation of a set of common properties ("axioms") for the graded systems associated with these cases. Thus the concept of "composition system" arose, which is somewhat more limited than the system considered in [3], and is a refinement of what Gerstenhaber [2] calls a pre-Lie ring. The question of composition systems other than the known ones presented itself, but was not directly investigated 1). However, it was impossible to resist the temptation to investigate in abstraceto a few properties of composition systems.

The formalism developed here was strongly influenced by a paper by A. Frölicher and the author [1] on almost complex and almost product structures on manifolds. The context may seem different from that at hand here, but the concepts carry over, anyhow. The formulas in this paper (especially sections 3 and 4) are only a selection from the older paper. Also, some modifications were necessary to accommodate fields of prime characteristic.

The composition systems are introduced in § 2. The deformation theory proper is dealt with in sections 6 and 8. Section 7 relates inner automorphisms of an algebra to the structure group of the ubiquitous $GLA(E, [,])$.

The standard references for this paper are [3, 5, 7]; of these, [3] 2) is the most direct background. Its definitions, notation and results will be assumed known.

§ 2. COMPOSITION SYSTEMS

Throughout this paper $V$ denotes a finite-dimensional vector space over an infinite field $K$. In the main part of the discussion, $\text{char } K \neq 2$; the modifications needed when $\text{char } K = 2$ are outlined separately. The hypothesis that $K$ is infinite is used repeatedly, as follows. Let $f(t)$ be a polynomial whose coefficients belong to a vector space $E_1$, and whose values lie, for infinitely many values from $K$, in a subspace $E_2$. Then the coefficients of $f(t)$ lie, in fact, in $E_2$.

A composition system associated with $V$ is a graded vector space $E(V) = \bigoplus_{n \geq 0} E^n(V)$ with a bilinear product (denoted $\circ$) and provided with a rational representation $\phi$ of $GL(V)$, subject to conditions $C_0$–$C_7$ below. Definitions and propositions are interspersed between them to facilitate a simple formulation. We write $E$ for $E(V)$ when circumstances permit.

$C_0$. Each $E^n$ is a subspace of $\text{Hom}_K (\otimes^n V, V)$; in particular, $E^0 = V$ and $E^1 = \text{Hom}_K (V, V)$.

Definition. The elements of $E^n$ have degree $n$; their reduced degree is $n - 1$.

1) A composition system for algebras satisfying an identity $(ab)c - a(bc) = -(ba)c - b(ac)$ has meanwhile been found. (Added in proof.)

2) Note that the unnumbered formulas for $[f, g]^{-}$ in [3] have a minus sign too many. The numbered ones are, however, correct.
C.1. The product \( \circ \) is compatible with the reduced grading: \( E^{n+1} \circ E^{m+1} \subset E^{n+m+1} \). In particular, \( E^0 \circ E^n = \{0\} \).

C.2. \( f \circ g = f \circ g \) if \( f \in E^1 \).

C.3. \( (f \circ g)(x_1, ..., x_n) = \sum_{i=1}^n f(x_1, ..., x_{i-1}, g(x_i), x_{i+1}, ..., x_n) \) if \( f \in E^n \) and \( g \in E^1 \).

C.4. \( (f \circ g) \circ h = f \circ (g \circ h) \) if \( f \in E^1 \).

C.5. \( (f \circ g) \circ h - f \circ (g \circ h) = (-1)^{(m-1)(p-1)}\{(f \circ h) \circ g - f \circ (h \circ g)\} \) if \( f \in E^n \), \( g \in E^m \), \( h \in E^p \).

C.6. If \( W \) is any subspace of \( V \), if the values of \( g \in E^m \) lie in \( W \), and if \( f(x_1, ..., x_n) = 0 \) whenever \( x_i \in W \) for any \( i \), \( 1 \leq i \leq n \), then \( f \circ g = 0 \).

Remark. Condition \( C_5 \) is a special case of \( C_4 \) when \( n = 1 \), and follows from \( C_2 \) and \( C_3 \) when \( m = p = 1 \). Condition \( C_6 \) for \( m = 1 \) follows from \( C_3 \).

More generally, \( C_6 \) is certainly valid when \( f \circ g \) can be expressed as a finite sum in which each term involves evaluation of \( f \) on at least one value of \( g \). It is unknown to the author if \( C_6 \) is a consequence of the other 7 conditions.

**Proposition 2.1.** The product \([,]_0\) on \( E \) given by

\[
[f, g]_0 = g \circ f - (-1)^{(m-1)(n-1)}f \circ g
\]

for \( f \in E^n \), \( g \in E^m \), gives \( E \) a GLA structure with respect to the reduced grading. The restriction of \([,]_0\) to \( E^1 \) is the opposite of the usual Lie algebra structure of \( gl(V) \). For each \( f \in E^1 \), the map \( \delta_f = [f, ,]_0 \) is a derivation on \( E \) with respect to \( \circ \) of degree zero:

\[
\delta_f(g \circ h) = \delta_f g \circ h + g \circ \delta_f h.
\]

Proof. A write-out of the last statement shows six terms, four of which cancel by \( C_5 \), and two by \( C_4 \). The statement on \( gl(V) \) is a matter of a simple write-out and \( C_2 \). The Jacobi identity for \([,]_0\) follows by triple application of \( C_5 \); cf. e.g. [5]; the other GLA properties follow easily, too. See [5] for a systematic discussion of GLA's.

Remark. The action of \( E^1 \) on \( E \) by \( \delta_f \), \( f \in E^1 \), is a representation of the Lie algebra structure of \( E^1 \). It is the infinitesimal version of the right action of \( GL(V) \) given by

\[
(g(x))g(x_1, ..., x_m) = \alpha^{-1}g(\alpha x_1, ..., \alpha x_m), \quad \alpha \in GL(V),
\]

which is defined for \( g \in Hom_K(\otimes^m V, V) \). The fact that \( \delta_f g \in E \) when \( f \in E^1 \) and \( g \in E \) says that \( E \) is a subspace of \( Hom_K(\otimes V, V) \) which is stable under the infinitesimal action induced by the rational representation \( g \). Similarly, the derivation property of \( \delta_f \) with respect to \( \circ \) says

1) If \( f \in E^1 \) we often write \( fg \) instead of \( f \circ g \) or \( f \circ g \).
that, infinitesimally at least, the representation $\varrho$ consists of automorphisms of $E$. When $\text{char } K = 0$ this implies the following statement. When $\text{char } K \neq 0$ we have to state it separately:

C7. $E$ is a stable subspace of $\text{Hom}_K (\otimes V, V)$ under the representation $\varrho$ of $GL(V)$, and $\varrho(GL(V))$ is a group of automorphisms (called inner) for the product $\circ$ on $E$.

Remark. $\varrho$ is the product of two commuting representations $\varrho_1$ and $\varrho_2$, given, respectively, by

$$
\begin{align*}
(\varrho_1(x)g)(x_1, \ldots, x_m) &= g(x x_1, \ldots, x x_m), \\
(\varrho_2(x)g)(x_1, \ldots, x_m) &= x^{-1}g(x_1, \ldots, x_m).
\end{align*}
$$

Evidently, $E$ is stable under $\varrho_2$, in view of $C_2$, and because $GL(V)$ is a subset of $E^1$. Then, by $C_7$, also $\varrho_1$ leaves $E$ invariant.

Let $f \in E^1$, and let $t \in K$ be such that $I + tf$ is invertible, hence $I + tf \in GL(V)$. Let $g \in E^m$; then

$$
\varrho_1(I + tf)g = g + tg \circ f + t^2q^0(f)g + \ldots,
$$

where the right side is a polynomial in $t$ of degree $m$. The coefficient of $t^2$, denoted $q^0(f)g$, is given by

$$
(2.1) \quad q^0(f)g(x_1, \ldots, x_m) = \sum_{i < j} g(x_1, \ldots, x_{i-1}, f(x_i), x_{i+1}, \ldots, x_{j-1}, f(x_j), x_{j+1}, \ldots, x_m).
$$

Clearly, $q^0(f)g = 0$ when $m < 1$. When $\text{char } K \neq 2$ we have

$$
(2.2) \quad q^0(f)g = \frac{1}{2}([g \circ f] \circ f - g \circ (f \circ f)).
$$

That shows that, in this case, by $C_1$, $q^0(f)g \in E$. If $\text{char } K = 2$, we must invoke $C_7$ to show that $q^0(f)g \in E$. Indeed, by $C_7$ the values of the polynomial $q^0(I + tf)g$ belong to $E$ for almost all $t$, hence so do the coefficients of the powers of $t$; in particular the coefficient of $t^2$.

In order to further complete the system when $\text{char } K = 2$, one should extend the definition of $q^0(f)g$ to all $f \in E^n$ with $n$ odd, such as to generalize $\frac{1}{2}([g \circ f] \circ f - g \circ (f \circ f))$. Axioms for this operation would include formulas for $q^0(f_1 + f_2)g$, commutator formulas for $q^0(f_1)$ and $\circ f_2$, and for $q^0(f_1)$ and $q^0(f_2)$. This whole task is left to the reader, who is referred to [5], § 2, where a similar extension is discussed.

The existence of $q^0(f)g$ in the associative, commutative and Lie cases is immediate from formulas (2.3) and (4.3) in [3], where the right-hand sides contain a factor 2 when $g = h$ and $m + 1 = p + 1$ is odd. In (2.3) this is at once obvious; in (4.3) we only have to observe that the sums with $\sigma(o) < \sigma(m+1)$ and with $\sigma(o) > \sigma(m+1)$ are equal.

If $(E, \circ )$ is any composition system, then there is a homomorphism $\mathcal{S}$ of it into the composition system of section 4 of [3], which we denote
here \((E_{\text{alt}}, \wedge)\). The map \(\mathcal{S}\) generalizes that of section 4 of [3], and does, in fact, associate with any algebra (in the sense of section 4) a Lie algebra.

Let \(f \in E^{n+1}\); define \(\mathcal{S}f\) by

\[(\mathcal{S}f)(x_0, \ldots, x_n) = (\ldots (f \circ x_0 \circ x_1 \ldots) \circ x_n).\]

By \(C_5\), with \(g = x \in E^0\), \(h = y \in E^0\) we have \((f \circ x) \circ y = -(f \circ y) \circ x\); so \(\mathcal{S}f \in \text{Hom}_K (A^{n+1}V, V) = E_{\text{alt}}^{n+1}\).

Proposition 2.2. The map \(\mathcal{S} : E \rightarrow E_{\text{alt}}\) is a homomorphism of the composition systems \((E, \circ)\) and \((E_{\text{alt}}, \wedge)\).

Proof. Let \(h = x \in E^0\) in \(C_5\); since \(x \circ g = 0\) (by \(C_1\)) we have, for \(f \in E^{n+1}, g \in E^{m+1}\)

\[(f \circ g) \circ x = f \circ (g \circ x) + (-1)^m (f \circ x) \circ g.\]

Repeated application of this rule gives

\[\ldots (f \circ g) \circ x_0 \ldots \circ x_{n+m} = (-1)^{mn} \sum' sg\sigma (\ldots (f \circ x_{\sigma(0)} \ldots \circ x_{\sigma(n-1)}) \circ (\ldots (g \circ x_{\sigma(n)}) \ldots \circ x_{\sigma(n+m)}),\]

where \(\sum'\) extends over the permutations \(\sigma\) of \(\{0, \ldots, m+n\}\) for which \(\sigma(0) < \ldots < \sigma(n-1)\) and \(\sigma(n) < \ldots < \sigma(n+m)\). The left side equals

\[\mathcal{S}(f \circ g)(x_0, \ldots, x_{n+m});\]

the right side can be written as

\[(-1)^{mn} \sum' sg\sigma (\mathcal{S}f)(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}, \mathcal{S}g)(x_{\sigma(n)}, \ldots, x_{\sigma(m+n)}),\]

which, due to the skew-symmetry of \(\mathcal{S}f\) and \(\mathcal{S}g\), may be written as

\[\sum'' sg\sigma (\mathcal{S}f)((\mathcal{S}g)(x_{\sigma(0)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n)}),\]

where \(\sum''\) extends over \(\sigma(0) < \ldots < \sigma(m)\) and \(\sigma(m+1) < \ldots < \sigma(m+n)\). Hence we find

\[\mathcal{S}(f \circ g) = \mathcal{S}f \wedge \mathcal{S}g,\]

which proves the homomorphism property.

§ 3. SUBSPACES AND QUOTIENT SPACES

Suppose \(V\) is a finite-dimensional vector space, and \(E = E(V)\) a composition system. Now, let \(W\) be a subspace of \(V\). We will show that there is a unique composition system \(E(W)\) associated with \(W\). The construction will show that \(E(W)\) depends only on \(E(V)\) and the dimension of \(W\); i.e. if \(W_1\) and \(W_2\) are subspaces of \(V\), and if there is \(\alpha \in GL(V)\) which sends \(W_1\) into \(W_2\), then induces an isomorphism between \(E(W_1)\) and \(E(W_2)\). If \(W\) is a subspace of \(V\), then also a unique composition system \(E(V/W)\) is determined, with similar properties.
In order to deal with both cases at once, we take complementary subspaces \( W \) and \( W' \) in \( V \). (The construction will show, again, that in this situation \( E(W) \) and \( E(V/W') \) are isomorphic.) We use the direct sum decomposition \( V = W + W' \) through the projection operator \( P \) onto \( W \) with kernel \( W' \), and \( Q = I - P \), which is the complementary projection operator. We first show that \( P \) and \( Q \) give rise to a decomposition of the \( E^n \).

Let \( f \in \text{Hom}_K (\otimes^n V, V) \); then we have

\[
    f(x_1, \ldots, x_n) = f(Px_1 + Qx_1, \ldots, Px_n + Qx_n).
\]

Upon using the multilinearity of \( f \) we can express this as a sum of \( 2^n \) terms. For each \( p, 0 < p < n \), the number of terms with \( p \) entries containing \( P \) and \( n - p \) entries containing \( Q \) is \( \binom{n}{p} \). These terms together constitute

\[
    \sum_{p=0}^{n} \left( \prod_{p \cdot n} \right) f(x_1, \ldots, x_n),
\]

and we have, obviously,

\[
    f = \sum_{p=0}^{n} \left( \prod_{p \cdot n} \right) f.
\]

The functions \( \prod_{p \cdot n} f \) are again multilinear, and clearly, the maps

\[
    \prod_{p \cdot n} : \text{Hom}_K (\otimes^n V, V) \to \text{Hom}_K (\otimes^n V, V)
\]

just defined are linear. Also, each \( \prod_{p \cdot n} \) is a projection operator, and the product of any two distinct ones vanishes. The images \( E_{p \cdot n} = \prod_{p \cdot n} E^n \) are subspaces of \( \text{Hom}_K (\otimes^n V, V) \).

We show that they belong to \( E^n \)—it follows then that they produce a direct sum decomposition of \( E^n \).

**Lemma 3.1.** If \( 0 < p < n \), then \( E_{p \cdot n} \) is a subspace of \( E^n \).

**Proof.** Let \( t \neq -1 \), then \( (I + tP) \in GL(V) \), and \( E \) is stable under \( q_1(I + tP) \)—see the Remark following \( C_7 \). For \( f \in \text{Hom}_K (\otimes^n V, V) \) we have

\[
    (q_1(I + tP)f)(x_1, \ldots, x_n) = f((I + tP)x_1, \ldots, (I + tP)x_n).
\]

If we replace \( f \) by any one of the terms constituting \( \prod_{p \cdot n} f \), then \( p \) of the entries after the action of \( q_1(I + tP) \) are of the form \( (I + tP)Px_i = (1 + t)Px_i \), and \( n - p \) of the entries are of the form \( (I + tP)Qx_j = Qx_j \). Hence, on each such term, \( q_1(I + tP) \) acts by multiplication by \( (1 + t)^p \), and we see

\[
    q_1(I + tP) \prod_{p \cdot n} f = (1 + t)^p \prod_{p \cdot n} f.
\]

The same reasoning shows that \( q_1(I + tP) \) and \( \prod_{p \cdot n} \) commute. Hence follows

\[
    q_1(I + tP)f = \sum_{p=0}^{n} (1 + t)^p \prod_{p \cdot n} f;
\]
that is, \( \prod_{p, n-p} f \) is the coefficient of the \( p \)-th power of the expansion of \( q_1(I + tP)f \) in terms of \((1 + t)\). If \( f \in E^n \), then by \( C_7, q_1(I + tP)f \in E^n \), and hence also the coefficients of the powers of \((1 + t)\) belong to \( E^n \).

We now construct, for each \( n \), some auxiliary spaces. \( E^n(V; W) \) is the space of those \( f \) in \( E^n \) whose restriction to \( W \) (more precisely, to \( \otimes^n W \)) has its values in \( W \); in other words, these elements of \( E^n \) are candidates for producing multilinear maps of \( W \) into \( W \). The actual restrictions to \( W \) of the elements of \( E^n(V; W) \) constitute \( E(W) \); the elements of \( E^n(V; W) \) whose restrictions to \( W \) are the zero map, constitute \( F^n(V; W) \). Then \( E^n(W) \) can be identified with \( E^n(V; W)/F^n(V; W) \). We prove below that \( E(V; W) \) is a subalgebra of \( E(V) \), and that \( F(V; W) \) is a two-sided ideal in \( E(V; W) \). Thus \( E(W) \) inherits a composition product. (The verification of the axioms for this product is left to the reader.)

In the same manner, \( E(V; W') \) is the space of those \( f \in E^n \) with the property that \( f(x_1, \ldots, x_n) \in W' \) whenever at least one of the entries \( x_1, \ldots, x_n \) belongs to \( W' \). If \( \pi \) denotes the natural projection, of \( V \) to \( V/W' \), then \( E(V/W) \) consists of the \( \tilde{f} \in \text{Hom}_K (\otimes^n(V/W'), V/W') \) which are \( \pi \)-related to some \( f \in E^n(V; W') \), that is,
\[
\tilde{f}(\pi x_1, \ldots, \pi x_n) = \pi f(x_1, \ldots, x_n).
\]

To each \( f \in E^n(V; W') \) there is exactly one such \( \tilde{f} \); these \( \tilde{f} \) constitute \( E^n(V/W) \). The elements of \( E^n(V; W') \) which give rise to the zero element of \( E^n(V/W') \) form \( F^n(V; W') \); they have the property that all their values lie in \( W' \). We prove below that \( E(V; W') \) is a subalgebra of \( E(V) \), and that \( F(V; W') \) is a two-sided ideal in \( E(V; W') \). Thus \( E(V/W') \) inherits a composition product. (The verification of the axioms for this product is left to the reader.)

**Proposition 3.2.** The spaces \( E(V; W) \) and \( E(V; W') \) are subalgebras of \( E(V) \); the spaces \( F(V; W) \) resp. \( F(V; W') \) are two-sided ideals in \( E(V; W) \) resp. \( E(V; W') \).

**Proof.** In terms of the previously-discussed decompositions we have
\[
E^n(V; W) = \bigoplus_{0 \leq p \leq n} PE^{p,n-p} \oplus \bigoplus_{0 \leq p < n} QE^{p,n-p}
\]
\[
F^n(V; W) = \bigoplus_{0 \leq p \leq n} PE^{p,n-p} \oplus \bigoplus_{0 \leq p < n} QE^{p,n-p}
\]
\[
E^n(V; W') = PE^{n,0} \oplus \bigoplus_{0 \leq p < n} QE^{p,n-p}
\]
\[
F^n(V; W') = \bigoplus_{0 \leq p \leq n} QE^{p,n-p}.
\]
We have written \( PE^{p,n-p} \) for the set of \( P \circ f \), with \( f \in E^{p,n-p} \). Proposition 3.2 is now an immediate consequence of the following.
Lemma 3.3. The following inclusions hold:

\[ PE^{p,n-p} \subseteq PE^{q,m-q} \subseteq PE^{p+q-1,n+m-p-q} \]
\[ QE^{p,n-p} \subseteq PE^{q,m-q} \subseteq PE^{p+q-1,n+m-p-q} \]
\[ PE^{p,n-p} \subseteq QE^{q,m-q} \subseteq PE^{p+q,n+m-p-q-1} \]
\[ QE^{p,n-p} \subseteq QE^{q,m-q} \subseteq PE^{p+q,n+m-p-q-1} \]
\[ PE^{n,0} \subseteq QE^{m,q-n} = \{0\} \]
\[ QE^{n,0} \subseteq QE^{m,q-n} = \{0\}. \]

Proof. The last two formulas follow from \( E^{n,0} \) vanish whenever any entry is from \( W' \), while \( Q \) has all its values in \( W' \). The number of the other identities can be halved by the observation that \( C_4 \) implies \( P \circ (f \circ g) = (P \circ f) \circ g \) and similarly for \( Q \). Hence the following inclusions remain to be shown.

\[ E^{p,n-p} \subseteq PE^{q,m-q} \subseteq E^{p+q-1,n+m-p-q} \]
\[ E^{p,n-p} \subseteq QE^{q,m-q} \subseteq E^{p+q,n+m-p-q-1} \]

Let \( f \in E^n, g \in E^m, \alpha \in GL(V) \). Then by \( C_7 \) we have

\[ g(\alpha)f \circ g(\alpha)g = g(\alpha)(f \circ g). \]

Furthermore (see the remark following \( C_7 \)) \( q(x) - q_1(x)q_2(x) \). By \( C_4 \) we also have \( q_2(x)(f \circ g) = (q_2(x)f) \circ g \). Therefore,

\[ q_1(x)(f \circ g) = q_1(x)f \circ q_1(x)q_2(x)g. \]

If \( \alpha = I + tP \) and \( t \neq -1 \), then \( q_2(x^{-1})(P \circ g) = (I + tP) \circ P \circ g = (1 + t)P \circ g \), and \( q_2(x^{-1})(Q \circ g) = (I + tP) \circ Q \circ g = Q \circ g \). Hence,

\[ q_2(I + tP)(P \circ g) = (1 + t)^{-1}P \circ g, \]
\[ q_2(I + tP)(Q \circ g) = Q \circ g. \]

We now use the known actions of \( q_1 \) and \( q_2 \) in (3.1) and obtain

\[ \sum_{r=0}^{n+m-1} (1+t)^r \prod_{r,n+m-r-1} (f \circ g) = \]
\[ = \left( \sum_{p=0}^n (1+t)^p \prod_{p,n-p} f \circ \left( \sum_{q=0}^m (1+t)^{q-1} \prod_{q,m-q} P g + \sum_{q=0}^m (1+t)^{q-1} \prod_{q,m-q} Q g \right) \right). \]

We now equate the coefficients of equal powers of \( (1+t) \) and find

\[ \prod_{r,n+m-r-1} (f \circ g) = \]
\[ = \sum_{p,q-r+1} \prod_{p,n-p} f \circ \prod_{q,m-q} P g + \sum_{p,q-r} \prod_{p,n-p} f \circ \prod_{q,m-q} Q g. \]

By replacing \( g \) by \( P \circ g \) resp. \( Q \circ g \) we obtain the desired inclusions.

Remark. Properties (a), (b) of [3], § 6 are simple corollaries of Proposition 4.2. Property (c) translates into the following.
Proposition 4.4. If \( f \in E^{p,n-p} \) for \( p < n \), and if \( g \in E^m \), then
\[
\prod_{n+m-1,0} (g \circ f) = 0.
\]

Proof. By the above formula, we have
\[
\prod_{n+m-1,0} (g \circ f) = \prod_{m,0} g \circ \prod_{n-1,1} Qf + \prod_{m,0} \prod_{n,0} Pj + \prod_{m-1,1} \prod_{n,0} g \circ \prod_{n,0} Qj.
\]
Now the first term on the right vanishes by lemma 4.3 (last two formulas), and the second and third terms vanish by the hypothesis on \( f \).

As, thus, the hypotheses of § 6 of [3] are valid, so are the conclusions.

§ 4. Algebra

The subject-proper of this paper is a study of algebras, i.e. of associative algebras, commutative associative algebras and Lie algebras. The efforts of [3] and the preceding two sections have produced a unified framework for these three cases. In fact, it is possible that other types of algebras exist that would also fit into the same framework. (The question of actual existence of such types has not been investigated.) Therefore, we shall use, in this paper, the term “algebra” for any pair \( (V, \mu) \) consisting of a finite-dimensional vector space \( V \), and a map \( \mu \in \text{Hom}_K (V \otimes V, V) \) which satisfies \( \mu \circ \mu = 0 \) in some composition system \( E \) associated with \( V \). \( E \) is then the “type” of the algebra. The map \( \mathcal{S} \) of section 2 assigns to each algebra a Lie algebra; cf. Proposition 2.2. It assigns to an associative algebra its Lie algebra of commutators, and to a Lie algebra, itself. Any deformation theory associated with \( (V, \mu) \) is to be with respect to the (fixed) type \( E \). For deformations of \( \mu \), the theory of [5] is directly applicable. In this paper we study deformations of subalgebras of \( (V, \mu) \). Subspaces (and quotient spaces) in which products are induced are again algebras of a related type:

Theorem 4.1. Suppose that \( A=(V, \mu) \) is an algebra with respect to a composition system \( E(V) \). Let \( W \) be a subspace of \( V \), and \( E(W) \) the composition system associated with \( W \) in virtue of its being a subspace of \( V \). Let, further, \( \mu(W \times W) \subset W \). Then the restriction \( \mu^* \) of to \( W \) defines \( B=(W, \mu^*) \) as an algebra (subalgebra) with respect to \( E(W) \). Further: let \( W' \) be a subspace of \( V \), and \( E(V/W') \) the associated composition system. Let, further, \( \mu(W' \times V) \subset W' \) and \( \mu(V \times W') \subset W' \). Then the bilinear map \( \mu' \) of \( V/W' \) into itself, thus induced, defines \( C=(V/W', \mu') \) as a (quotient) algebra with respect to \( E(V/W') \).

Proof. One only needs to observe that \( \mu \) belongs to \( E(V; W) \) resp. \( E(V; W') \), and that \( \mu^* \) resp. \( \mu' \) are the images of \( \mu \) in \( E(W) \) resp. \( E(V/W') \). The fact that the latter are quotient algebras of the former then implies \( \mu^* \circ \mu^* = 0 \) resp. \( \mu' \circ \mu' = 0 \).

As we showed at the end of § 3, the axioms for a composition system imply those of § 6 of [3]. We are thus justified in using all and any results of [3], and shall do so most freely, sometimes without reference.
The products $[ , ]^0$, $[ , ]^\nu$ and $[ , ]$ of [3] are supplemented by quadratic operations (cf. § 2' of [5]) $Q^0$, $Q^\nu$ resp. $Q$, which are particularly useful when char $K = 2$. They are defined for $f$ of odd degree (odd reduced degree in the first case), by

$$Q^0(f) = f \circ f,$$
$$Q^\nu(f) = q^0(f)\mu,$$
$$Q(f) = q^0(f)\mu - f \circ \delta f.$$  

As the latter two depend on a definition of $q^0(f)$, we shall use them only for $f \in E^1$; cf. (2.3).

§ 5. Subalgebras

In this section we deduce relations between the $\delta$-operator on $E(B, A/B)$ and the bracket $[,]$, which are similar to those of § 6 of [3], but show more detail. The underlying space of $A$ is $V$, the product map of $A$ is $\mu$, and $W$ and $W'$ are complementary subspaces of $V$, with corresponding projections $P$ and $Q$. When the image of $\mu|W$ lies in $W$, the subalgebra thus arising is denoted $B$. We set

$$T(x, y) = P\mu(Qx, Qy), \quad T'(x, y) = Q\mu(Px, Py),$$

so $T = \prod_{1,1}^0 \mu$, $T' = \prod_{2,0}^0 \mu$. Clearly, $W$ is a subalgebra if $T' = 0$; then the product on $W$ is given by $\prod_{2,0} \mu$, and the $B$-module structure of $V/W$ (which is isomorphic to $W'$ under the natural projection $\pi: V \to V/W$) is determined by $\prod_{1,1} \mu$.

Proposition 5.1. The following identity holds:

$$\frac{1}{2}[P, P] = \frac{1}{2}[Q, Q] = T + T'.$$

Corollary 5.2. $W$ is a subalgebra of $A$ if and only if $Q \circ [Q, Q] = 0$.

Proof. By a write-out:

$$\frac{1}{2}[P, P](x, y) = (P + Q)\mu(Px, Py) + P\mu(Px + Qx, Py + Qy) - P\mu(Px, Py) - P\mu(Px + Qx, Py) - P\mu(Qx, Qy).$$

We now use $PP = P$, insert $(P + Q)$ in a number of places, and cancel terms:

$$\frac{1}{2}[P, P](x, y) = (P + Q)\mu(Px, Py) + P\mu(Px + Qx, Py + Qy) - P\mu(Px, Py) - P\mu(Px + Qx, Py) - P\mu(Qx, Qy).$$

The map $f \to (\pi \circ f)|W$ yields $E(W, V/W)$ as the image of $E(V)$, cf. § 6 of [3]. The restriction of this map to $\bigoplus QE^n.0$ is denoted $\tau$; it is a vector space isomorphism.
Proposition 5.3. Let \( f \in QE^{n,0} \); then

\[
\{Q, f\} = \{f \circ (P \prod_{2,0} \mu) + (-1)^n (Q \prod_{1,0} \mu) \circ f\} - \{(-1)^n T \circ f\} - \{f \circ T\},
\]

where the terms in braces belong to, respectively, \( QE^{n+1,0} \), \( PE^{n,1} \) and \( QE^{n-1,2} \). When \( W \) is a subalgebra, then the \( \tau \)-image of the first term is exactly \(-\delta \tau f\):

\[
\delta \tau f = \tau \{-f \circ (P \prod_{2,0} \mu) - (-1)^n (Q \prod_{1,0} \mu) \circ f\}.
\]

Proof. By straightforward write-out, and decomposition of \( \mu \). We use the formulas of Lemma 3.3 without individual reference.

\[
\{Q, f\} = -\{P, f\} = (-1)^n \{(\mu \circ P) \circ f - \mu \circ (P \circ f)\} + \nonumber\]

\[
+ f \circ (P \circ \mu - \mu \circ P) - P \circ (f \circ \mu + (-1)^n \mu \circ f) = \nonumber\]

\[
= (-1)^n (Q \prod_{1,1} \mu) \circ f + (-1)^n (P \prod_{1,1} \mu) \circ f - \nonumber\]

\[
- f \circ (P \prod_{2,0} \mu + P \prod_{1,1} \mu + P \prod_{1,1} \mu - 2P \prod_{2,0} \mu - P \prod_{1,1} \mu) + \nonumber\]

\[
+ (-1)^{n+1} (P \prod_{1,1} \mu) \circ f + (-1)^{n+1} (P \prod_{1,1} \mu) \circ f = \nonumber\]

\[
= (-1)^n (Q \prod_{1,1} \mu) \circ f + f \circ (P \prod_{2,0} \mu) - f \circ T + (-1)^{n+1} T \circ f.
\]

The statement on \( \delta \tau f \) is evident by consideration of the construction of \( \delta \) on \( E(B, A/B) \); cf. § 6 of [3].

There is no GLA structure on \( E(B, A/B) \) corresponding to \([.,] \). However, we shall write \([\tau f, \tau g] \) for \( \tau (Q \prod_{n+m,0} [f, g]) \) when \( f \in QE^{n,0} \) and \( g \in QE^{m,0} \). Then \([.,]\) induces the GLA structure on \( H^*(B, A/B) \), which was established in § 6 of [3].

Proposition 5.4. The following identity holds:

\[
\prod_{3,0} [Q, Q \circ [Q, Q]] = 0.
\]

Proof. We have \( Q \circ [Q, Q] = 2T' \), and \([Q, [Q, Q]] = 0 \) by the Jacobi identity, so \([Q, T + T'] = 0 \), cf. Proposition 5.1. It follows that \([Q, T] = -\{Q, T\} = [P, T]\). A decomposition of the right side by Proposition 5.3, with \( W \) and \( W' \), and \( P \) and \( Q \) interchanged, yields that \([P, T]\) has components only in the spaces \( PE^{0,3} \), \( QE^{1,2} \) and \( PE^{2,1} \). In particular, there is no component in \( E^{3,0} \). That completes the proof.

Proposition 5.5. Let \( f \in QE^{n,0} \) and \( g \in QE^{m,0} \), then

\[
[f, g] = \tau^{-1} [\tau f, \tau g] + (-1)^{m+1} (T \circ g) \circ f + \{(-g \circ T) \circ f + (-1)^{mn} f \circ (T \circ g)\},
\]

where the terms on the right belong to, respectively, \( QE^{n+m,0} \), \( PE^{n+m,0} \) and \( QE^{n+m-1,1} \).
Proof. We use the expressions (5.4,6) of [3], and then decompose, deleting at once all terms that vanish on account of Lemma 3.3.

\[ [f, g] = \delta g \circ f + (-1)^n \delta(g \circ f) + (-1)^{mn+m+1} f \circ g = \]
\[ = (-g \circ \mu - (-1)^m \mu \circ g) \circ f + (-1)^{mn+m+1}((-P \prod_{\mu} + Q \prod_{\mu}) \circ g) \circ f + \]
\[ + (-1)^{mn} f \circ ((P \prod_{\mu} + Q \prod_{\mu}) \circ g) = \]
\[ = \{ (-g \circ (P \prod_{\mu}) \circ f + (-1)^{m+1}((P \prod_{\mu}) \circ g) \circ f + \}
\[ + (-1)^{mn} f \circ ((P \prod_{\mu}) \circ g) + \]
\[ + (-1)^{m+1}(T \circ g) \circ f - (g \circ T) \circ f + (-1)^{mn} (T \circ g). \]

The term in braces is equal to \( \tau^{-1}[\tau f, \tau g]^{r} \) by virtue of its being the component of \([f, g]\) in \( Q E^{m+n,0} \).

**Proposition 5.6.** Let \( f \in Q E^{1,0} \), and let \( W \) be a subalgebra. Then

\[ T'(f) = \frac{1}{2}(Q+f) \circ [Q+f, Q+f] \]

belongs to \( Q E^{2,0} \), and

\[ \tau T'(f) = -\delta f + \frac{1}{2} [\tau f, \tau f]^{r} + \tau(f \circ q^{0}(f) T). \]

**Proof:** by a write-out and application of the preceding propositions.

\[ \frac{1}{2}(Q+f) \circ [Q+f, Q+f] = \]
\[ = \frac{1}{2} f \circ [Q, Q] + Q \circ [Q, f] + \frac{1}{2} Q \circ [f, f] + f \circ [Q, f] + \frac{1}{2} f \circ [f, f] = \]
\[ = (-T^{-1} \delta f + f \circ (-T^{-1} \delta f + f \circ (-T^{-1} \delta f + f \circ ((T \circ f) \circ f) = \]
\[ = -\tau^{-1}(-\delta f + \frac{1}{2} [\tau f, \tau f]^{r}) + (T \circ q^{0}(f) T. \]

**Proposition 5.7.** The expression \( T''(f) \) of the preceding proposition satisfies the identity

\[ -\delta T''(f) + [\tau f, \tau T''(f)]^{r} + \tau (-T'(f) \circ q^{0}(f) T + f \circ ((T \circ f) \circ T'(f)) \} = 0. \]

**Proof.** By proposition 5.4 we have \( \prod_{2,0} [Q+f, T''(f)] = 0 \), where \( \prod_{2,0} \) is the decomposition with respect to \((P-f, Q+f)\). By the previous proposition, \( T''(f) \in Q E^{2,0} \); by the decomposition formulas we have

\[ [Q+f, T''(f)] = [Q, T''(f)] + [f, T''(f)] = \]
\[ = -\tau^{-1} \delta T''(f) + T \circ T''(f) - T''(f) \circ T + \tau^{-1}[\tau f, \tau T''(f)]^{r} - \]
\[ - (T \circ f) \circ T'(f) - (T'(f) \circ T) \circ f + f \circ (T \circ T'(f)). \]
Application of \( \prod_{a,0} F \) to an element \( F \) of \( E^3 \) yields

\[
(\prod_{a,0} F)(x, y, z) = F(Px - f x, Py - f y, Pz - f z),
\]
hence

\[
\prod_{a,0} F = \prod_{a,0} F - (\prod_{a,1} F) \circ f + q^0(f) \prod_{a,1} F \cdot \frac{1}{3!} (\prod_{a,2} F(\circ f))^3
\]

(the last term can be simplified to have no denominator, but such a term does not appear above, anyhow). After determination of the subspace to which each term belongs (by Proposition 3.2), application of the proper \( f \)-operator, and using (5.1) of [3], we find

\[
0 = -\tau^{-1} \delta(T'(f) + \tau^{-1}[\tau f, \tau T'(f)] + (T \circ f) \circ T'(f) + T'(f) \circ q^0(f) T -
-(T \circ f) \circ T'(f) - 2T'(f) \circ q^0(f) T + \tau \circ ((T \circ f) \circ T'(f)).
\]

Cancellation and combination of terms, followed by application of \( \tau \) yields the result.

§ 6. DEFORMATIONS OF A SUBALGEBRA

Let \( W \) and \( W_1 \) both be subspaces of a finite-dimensional vector space \( V \), complementary to the subspace \( W' \). We then have projection operators \( (P, Q) \) for \( (W, W') \), and projection operators \( (P_1, Q_1) \) for \( (W_1, W') \). Let \( x \in V \); then \( Px - P_1 x \) is parallel to \( W' \), and vanishes when \( x \in W' \). This implies that \( P - P_1 \in QE^{1.0} \). It is easy to show that, conversely, if \( f \in QE^{1.0} \), then \( P_1 = P - f \) is a projection operator with kernel \( W' \). (Its complementary operator is \( Q_1 = Q + f \).) Thus we have a one-to-one correspondence between the subspaces of \( V \) complementary to \( W' \) and the elements of \( QE^{1.0} \).

RICHARDSON [7] has shown that this correspondence is a coordinate map of a neighborhood of \( W \) in the Grassmannian.

The deformation problem for subalgebras of \( A = (V, \mu) \) is the determination of all those subspaces \( W_1 \) near \( W \) (in the topology of the Grassmannian) which are subalgebras of \( V \). (The equivalence problem is discussed in § 8.) \( W \) is assumed to be a subalgebra, \( B \).

Thus, we have to find all solutions close to zero of the equation \( (Q + f) \circ \circ [Q + f, Q + f] = 0 \), with \( f \in QE^{1.0} \); cf. corollary 5.2. In order to transfer the problem to \( E(B, A|B) \), we set \( f = \tau^{-1} \varphi \), with \( \varphi \in E^1(B, A|B) \). By Proposition 5.6 the deformation equation then becomes

\[
\delta \varphi - \frac{1}{2} [\varphi, \varphi]^3 - \varphi \circ q^0(\varphi) T = 0. 1)
\]

To solve this equation we assume that the underlying field \( K \) is \( \mathbb{R} \) or \( \mathbb{C} \), or more generally, that it is algebraically closed, so the implicit function

1) In the spirit of § 6 of [3] we have used \( \circ \) (and \( q^0 \), by implication) for elements in quotient spaces when in those spaces an operation is induced by \( \circ \) (resp. \( q^0 \)). In this case, \( \varphi \circ q^0(\varphi) T \) thus stands for \( \tau(f \circ q^0(f) T) \).
theorem or its algebraic-geometric variants may be applied. We shall only sketch the solution, as its technical details are exactly the same as those used in previous deformation problems, cf. [5] and [7].

We choose a "Hodge" decomposition: $B^n$ denotes the subspace of $\delta$-coboundaries in $E^n(B, A/B)$; $H^n$ a subspace complementary to $B^n$ in the space $Z^n$ of $\delta$-cochains, and $C^n$ a subspace of $E^n(B, A/B)$ complementary to $Z^n$. The projections on these factors are denoted $\pi_B$, $\pi_H$, $\pi_Z$, $\pi_C$. The deformation equation (6.1) splits into three equations through the successive application of $\pi_B$, $\pi_H$ and $\pi_C$:

\[
\begin{align*}
\text{(a)} & \quad \delta\varphi + \pi_B\{-\frac{1}{2}[\varphi, \varphi]^T - \varphi \circ q^0(\varphi)T\} = 0, \\
\text{(b)} & \quad \pi_H\{-\frac{1}{2}[\varphi, \varphi]^T - \varphi \circ q^0(\varphi)T\} = 0, \\
\text{(c)} & \quad \pi_C\{-\frac{1}{2}[\varphi, \varphi]^T - \varphi \circ q^0(\varphi)T\} = 0.
\end{align*}
\]

To solve (6.2a) locally, we set $\varphi = z + u$, where $z \in Z^1$ and $u \in C^1$. Then the equation is

\[
\delta u + \pi_B\{-\frac{1}{2}[z + u, z + u]^T - (z + u) \circ q^0(z + u)T\} = 0.
\]

The left side defines a map $F: Z^1 \times C^1 \to B^2$, whose derivative $D_2F(0, 0) = \delta|C^1$ is an isomorphism. In the cases $K = R$ or $K = C$ the implicit function theorem produces a map $\Phi$ defined on a neighborhood $N(Z^1)$ of the origin of $Z^1$ into $C^1$ such that all "small" solutions of (6.2a) are of the form $\varphi = z + \Phi(z)$. In the case when $K$ is algebraically closed, the set of solutions has a simple point at $(0, 0)$ and the tangent space to it is $Z^1$; see Lemma 19.1 of [5] for details in a very similar case.

In the real and complex cases the left side of (6.2b), with $\varphi$ replaced by $z + \Phi(z)$, is denoted $\Omega(z)$; evidently, $\Omega$ is an analytic map of $N(Z^1)$ into $H^2$; which is the obstruction map. The set of zeros of $\Omega$ corresponds to, and parametrizes (by $\varphi = z + \Phi(z)$) all "small" solutions of (6.2a, b). In the algebraic case ($K$ algebraically closed) the argument is more delicate, but is in essence the same as in Lemma 23.1 and Theorem 23.4 of [5].

As to the third equation, (6.2c), we show it is a consequence of the first two in a sufficiently small neighborhood of 0. The argument is in essence the same as in Lemma 19.1 of [5]; the important thing is the basic identity, which in this case is given in Proposition 5.7. We write it as

\[
\delta Y + L(\varphi)Y = 0,
\]

where $Y$ is short for $\tau T'(\tau^{-1}\varphi)$, and where $L(\varphi)$ is a linear operator acting on $Y$. $L(\varphi)$ is a polynomial in $\varphi$, and $L(0) = 0$.

The hypothesis that the first two equations, (6.2a, b), are satisfied, means $Y \in C^2$. Now $\delta$, restricted to $C^2$, is an injection into $E^2(B, A/B)$. Hence, for small $\varphi$, also $\delta + L(\varphi)$ is an injection. Thus, for all solutions $\varphi$ of (6.2a, b) in a neighborhood of 0, (6.3) implies $Y = 0$.

The bracket $[,]$ induces in $H^*(B, A/B)$ a GLA structure, cf. § 6 of [3].
That it provides the primary obstruction to extend infinitesimal deformations (i.e. solutions of the linearized version of (6.1)) is seen thus.

**Proposition 6.1.** Let $\varphi = t\varphi_1 + t^2\varphi_2 + \ldots$ be a formal power series of elements of $E^1(B, A/B)$. Then $\varphi$ satisfies (6.1) mod $t^2$ if and only if $\varphi_1$ is a cocycle. In order that $\varphi_2$ exist such that $\varphi$ satisfies (6.1) mod $t^3$ it is necessary and sufficient that the cocycle $[\varphi_1, \varphi_1]$ in $E^2(B, A/B)$ be a coboundary.

**Proof.** (6.1) implies

$$t\delta \varphi_1 + t^2(\delta \varphi_2 - \frac{1}{2}[\varphi_1, \varphi_1]) \equiv 0 \pmod{t^3}.$$  

The coefficients of $t$ resp $t^2$ are to vanish to satisfy (6.1) mod $t^3$. Now, $[\varphi_1, \varphi_1] = \tau[\tau^{-1}\varphi_1, \tau^{-1}\varphi_1]$, and by Proposition 6.1 of [3] this implies that $\frac{1}{2}[\varphi_1, \varphi_1]$ is a cocycle. In order that it be of the form $\delta \varphi_2$, it must be a coboundary.

§ 7. **INNER AUTOMORPHISMS AND STRUCTURE GROUPS.**

Two subalgebras of an algebra $A$ shall be called equivalent if there is an inner automorphism of $A$ which sends the one into the other. This section serves to clarify the notions of inner automorphism and structure group; the latter is a slight generalization of the former.

In a unitary associative algebra $A$ an inner automorphism is any map $b \rightarrow a^{-1}ba$ where $a$ is an element of the group of units of $A$. More generally, without the assumption of unitariness, an inner automorphism is defined as follows. Let $I$ be the identity map of the underlying space $V$ of $A$, and $l_a$ the map $b \rightarrow l_ab = ab = \mu(a, b)$; similarly, $b \rightarrow r_ab = ba = \mu(b, a)$. (Note that $l_a$ and $r_a$ commute.) Restrict now the subscripts $a$ in $l_a$ and $r_a$ to elements $a$ for which $I + l_a \in GL(V)$ and $I + r_a \in GL(V)$; the set of all those $a$, which we denote $G$, forms a neighborhood of 0. The maps

$$g_a: b \rightarrow (I + l_a)^{-1}(I + r_a)b$$  

are the inner automorphisms $^1$). They form a group $G_0$, as

$$g_ag_c = g_{a+c+ca}, \quad (g_a)^{-1} = g_{(I-r_a)^{-1}a}.$$  

This suggests also a group structure on $G$; we define the product by $(a, b) \rightarrow a + b + ba$. The map $a \rightarrow g_a$ is a homomorphism of $G$ onto $G_0$; it is an isomorphism if $G$ contains no central elements of $A$ other than 0. $G$ is the structure group, $G_0$ the group of inner automorphisms. $G_0$ is a subgroup of $GL(V)$; $G$ is algebraic.

The infinitesimal actions of $G$ and $G_0$ on $V$ are given by $b \rightarrow (r_a - l_a)b = ba - ab$; that is, by inner derivations. They can be brought into the following form:

$$b \rightarrow ba - ab = -(\mu \circ a)(b) = \delta a(b) = \delta_ab.$$  

$^1$) Note that this group—like all others in this paper—acts on the right. This is a change from previous work, and is due to the change in sign in $[,]^0$.  


Here, $\delta_{a}$ is a special case of $\delta_{f}$, $f \in E^{1}$; cf. § 2. The commutators give (cf. § 5 of [3])

$$[\delta_{ba}, \delta_{ac}] = \delta_{[ba, ac]} = \delta_{[a, c]}.$$ 

Hence, the Lie algebra of $G$ is the subalgebra $E^{0}$ of $(E, [,])$, while that of $G^{0}$ is $E^{0}/Z^{0}$; note that $Z^{0}$ is the center of $A$.

By its definition, $G$ acts through $G^{0}$ as a subgroup of $GL(V)$, so (with some abuse of notation) the representations $\rho$, $\rho_{1}$ and $\rho_{2}$ can be evaluated on its elements. In particular, $\rho(g)\mu = \mu$ for $g \in G$. Furthermore, $\rho(G)$ is a group of automorphisms of the composition system $(E, \circ)$. Combining these two facts, we see that $\rho(G)$ is a group of automorphisms of $(E, [,])$. $G$ is a structure group of $(E, [,])$, and $(E, \rho, G)$ becomes an algebraic or analytic graded Lie algebra in the sense of § 9, 10 of [5].

In the case of commutative algebras the above holds trivially, as $l_{a} = r_{a}$, so $G^{0}$ is the identity, and $G$ is abelian.

In the case of a Lie algebra the situation is different. While the adjoint representation produces infinitesimal inner automorphisms, it does not always produce a structure group or group of inner automorphisms if the base field is of prime characteristic. One is therefore forced to assume that a group with the (now obvious) desired properties exists. For $G$ the Lie algebra should be $E^{0}$, for $G^{0}$ it should be $E^{0}$ modulo its center.

For general algebras associated with composition systems, the same considerations hold. Inner derivations are maps of the type $\delta_{a}$, where $a \in E^{0}$. They are, indeed, infinitesimal automorphisms of $\mu$:

$$\delta_{a} \mu = [\delta a, \mu]^{0} = [-[\mu, a]^{0}, \mu]^{0} = \frac{1}{2}[a, [\mu, \mu]^{0}]^{0} = 0.$$ 

A group $G^{0}$ of inner automorphisms or a structure group $G$ do not necessarily exist (unless the base field is $R$ or $C$), as in the Lie case. $\rho(G^{0})$ would also be an automorphisms group of $(E, [,])$, and $G$ would be a structure group.

It is unknown if, conversely, a structure group $G$ of $(E, [,])$ would produce a structure group for $(V, \mu)$—and similarly for $G^{0}$.

§ 8. Locally complete families

Let $A$ again be an algebra, $A = (V, \mu)$, and $B = (W, \mu|W)$ a subalgebra. A (local) family of deformations of $B$ is a connected set in the Grassmannian of $V$, containing $W$, and such that each element of this set is a subalgebra of $A$. We assume that the base field $K$ is real, or that it is algebraically closed; furthermore, we assume that $A$ has a group $G^{0}$ of inner automorphisms, or a structure group $G$.

Then a family of deformations of $B$ is locally complete if its image under $G$ or $G^{0}$ (under the induced action on the Grassmannian) fills a neighborhood of $B$ in the set of all subalgebras of $A$. 
The family (in the real and complex cases)
\[\mathcal{F} = \{z + \Phi(z) | z \in \Omega^{-1}(0)\}, \quad \Omega : N(Z^1) \rightarrow \mathbb{H}^2,\]
defined in § 6 certainly parametrizes a locally complete family, as it consists of all subalgebras near \( B \). We proceed to show that the smaller family ("Kuranishi family")
\[\mathcal{K} = \{z + \Phi(z) | z \in \Omega^{-1}(0) \cap H^1\}\]
still is locally complete. The procedure is strictly analogous to the argument of Theorems 20.3 and 23.4 of [5], and only a slight and perfectly obvious modification of that of Richardson [7], § 6–9. We shall therefore only outline it, and only for the real and complex cases.

First we obtain a formula expressing the group action of \( GL(V) \) on a neighborhood of \( W \) in the Grassmannian. Let \( (P_1, Q_1) \) be a pair of complementary projection operators, such that the image of \( P_1 \) is complementary to \( W' \), and the image of \( Q_1 \) complementary to \( W \). Then \( P + Q_1 \) and \( Q + P_1 \) are invertible, and (cf. [7], lemma 2.1) \( \psi = -QP_1(P_1 + Q)^{-1} \in QE^{1,0} \); the image of \( P - \psi \) equals the image of \( P_1 \).

Now we take \( W_1 \) transversal to \( W' \); denote by \( (P_1, Q_1) \) the corresponding projection operators. Let \( g \in GL(V) \) be such that \( P + e(g)Q_1 \) and \( Q + e(g)P_1 \) are invertible. (There are neighborhoods \( U_1 \) of \( W \) in the Grassmannian and \( U_2 \) of \( I \) in \( GL(V) \) such that these conditions are met if \( W_1 \in U_1 \) and \( g \in U_2 \).) The procedure of the preceding paragraph supplies the coordinates of \( e(g)W_1 \).

We recall that \( P_1 = P - f, Q_1 = Q + f \), with \( f \in QE^{1,0} \), and denote by \( U_1' \) the neighborhood of \( 0 \) in \( QE^{1,0} \) such that the image of \( P - f \) belongs to \( U_1 \) if \( f \in U_1' \). The map \( \Psi : U_1' \times U_2 \rightarrow QE^{1,0} \) is defined by
\[\Psi(f, g) = -Q(e(g)(P - f))(e(g)(P - f) + Q)^{-1};\]
it is an analytic expression for the map \( (g, W_1) \rightarrow e(g)W_1 \).

**Lemma 8.1.** \( \Psi \) is a rational map; its partial derivatives at \((0, I)\) are
\[D_1 \Psi(0, I) = id, \quad D_2 \Psi(0, I) = -Q \prod_{i=1}^n.\]

**Proof.** To compute \( D_1 \Psi(0, I) \) we set \( g = I \) and determine the linear term in \( f \). To compute \( D_2 \Psi(0, I) \) we set \( f = 0, g = I + \gamma \) and determine the linear term in \( \gamma \). The calculations are routine.

The result on the derivatives confirms the perfectly obvious fact that \( \Psi \) is locally surjective. More important, it shows us how the domains of \( f \) and \( g \) can be restricted so that \( D \Psi(0, I) \) is still a surjection, which will imply that the restricted \( \Psi \) is still a local surjection. First of all, we limit \( g \) to the group of inner automorphisms (or take the structure group and compose its homomorphism into \( GL(V) \) with \( g \)), so it permutes
the subspaces of $V$ that are subalgebras. Then the range of $D_{2}\Psi(0,I)$ is $Q\prod_{1,0}^{1}\delta E^{0}$, which contains

$$Q\prod_{1,0}^{1}\delta^{-1}E^{0}(B, A/B) = \tau^{-1}E^{0}(B, A/B) = \tau^{-1}B.$$ 

Now we can restrict $f$ to elements of $\tau^{-1}(H^{1} \oplus C^{1})$, since then $D_{2}\Psi(0,I)$ is still a surjection. With these restrictions on the domain of $\Psi$, every subspace $W_{2}$ of $V$ near $W$ is the $g(G)$-transform of some subspace $W_{1}$ whose parameters belong to $\tau^{-1}(H^{1} + C^{1})$. Moreover, $W_{2}$ is a subalgebra if and only if $W_{1}$ is a subalgebra. Hence, every subalgebra parametrized by an element of $\tau\mathcal{F}$, for a sufficiently small $N(Z^{1})$, is equivalent to a subalgebra parametrized by an element in $\tau(\mathcal{F} \cap (H^{1} \oplus C^{1})) = \tau\mathcal{X}$. This sketches the proof of the local completeness of $\mathcal{X}$.

**Summary**

A composition system associated with a finite-dimensional vector space $V$ is a graded system $E = \oplus E^{n}$ of spaces of multilinear maps of $V$ into $V$, provided with a bilinear product $\otimes$ and a representation $g$ of $GL(V)$ on it, subject to axioms $C_{0} - C_{7}$. For each type of algebra (associative, commutative associative or Lie) structure on $V$ there is a compositions system; whether other similar composition systems exist, is unknown. The composition systems supply the means for a unified theory of cohomology and deformations in the three cases. Of special interest in this case are the deformations of subalgebras. The present formalism provides a crucial relation for the existence of "Kuranishi families" of deformations. The parameter space for a locally complete family of deformations of a subalgebra $B$ of an algebra $A$ is contained in $H^{1}(B, A/B)$; the obstruction space is $H^{2}(B, A/B)$. Results on rigidity were previously obtained by Richardson [7].

The University of Pennsylvania
Université de Genève

**REFERENCES**