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Asymptotic models for the topological sensitivity of the energy functional

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ABSTRACT

The refined method of matched asymptotic expansions is used for constructing asymptotic models for the topological sensitivity of the energy functional with respect to the creation of several small holes in the geometrical domain.

1. Introduction

Topological sensitivity analysis [1,2] deals with the asymptotic expansion of a given shape functional $J(\Omega, v)$ with respect to the creation of a small hole ω_ε^0 with the center at a given point x^0 in the geometrical domain Ω. In the case of a Neumann boundary condition imposed on the boundary $\partial\omega_\varepsilon^0$ of the hole, the topological asymptotics for the energy functional has the form (see, e.g., [3])

$$J(\Omega_\varepsilon, u^\varepsilon) = J(\Omega, v^0) + \mathfrak{T}_\omega^0(x^0)|\omega_\varepsilon^0| + o(|\omega_\varepsilon^0|).$$

Here, u^ε is a solution of the boundary value problem defined on the domain $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon^0}$, $|\omega_\varepsilon^0|$ is the Lebesgue measure of ω_ε^0 (area if $\Omega \subset \mathbb{R}^2$ or volume if $\Omega \subset \mathbb{R}^3$), v^0 is the solution of the boundary value problem defined on the original domain Ω, $\mathfrak{T}_\omega^0(x^0)$ is the topological derivative, i.e.,

$$\mathfrak{T}_\omega^0(x^0) = \lim_{\varepsilon\to 0+} \frac{J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0)}{|\omega_\varepsilon^0|}.$$

The topological derivative $\mathfrak{T}_\omega^0(x^0)$ determines whether a change of topology via nucleation of the small hole ω_ε^0 at the point x^0 in the interior of the domain Ω would result in improving the value of the shape functional $J(\Omega, v)$ or not.

The idea of *topological sensitivity* was introduced in [4] (the so-called *characteristic function*) in the framework of the bubble method for topology and shape optimization. The topological derivative concept was generalized for nucleation of cavities of arbitrary shape as well as for different boundary conditions imposed on $\partial\omega_\varepsilon(x^0)$ and state differential equations defined on $\Omega_\varepsilon \subset \mathbb{R}^n$ [2]. Also, various approaches were suggested for calculating the topological derivative [2].

By definition, we put

$$\mathfrak{T}_\omega^\varepsilon(x^0) = \frac{J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0)}{|\omega_\varepsilon^0|}.$$

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We shall say that $\mathfrak{T}_\omega^\varepsilon(x^0)$ is the *topological sensitivity* of the shape functional $J(\Omega, v)$ with respect to the internal topological variation. The aim of the *asymptotic modelling* of the topological sensitivity is to obtain an asymptotic representation for $\mathfrak{T}_\omega^\varepsilon(x^0)$. Using asymptotic analysis, we obtain the asymptotic formula

$$\mathfrak{T}_\omega^\varepsilon(x^0) = \mathfrak{S}_\omega^\varepsilon(x^0) + o(\mathfrak{T}_\omega^0(x^0)), \quad \varepsilon \to 0,$$

where $\mathfrak{S}_\omega^\varepsilon(x^0)$ is an asymptotic model for the topological sensitivity.

In [5], the case of a finite number of circular holes was treated by means of the so-called topological gradient which contains the topological derivatives evaluated at the centers of holes. For the modelling of the internal multiple topological variations in [6] two new approaches were developed using the self-adjoint extensions of differential operators and the variational formulation with point asymptotic conditions in a functional space with separated asymptotics. A number of related inverse problems have been considered in [7].

In the present work the third approach based on the refined method of matched asymptotic expansions [8,9] in the form [9] is proposed. The asymptotic analysis performed in the work is formal. Estimates for the proposed approximations in weighted Hölder spaces [2] and weighted Sobolev spaces [10] were derived in the context of shape optimization. In Section 3, the methodology of simultaneous multiple changes of topology is proposed for refinement of the bubble method. In Section 4, the case of the Dirichlet boundary conditions imposed on the boundaries of cavities is considered with an example where the optimal solution from the direct optimization algorithm (based on the topological derivative) is known explicitly. It is shown that the asymptotic model allows us to improve the solution.

2. Multiple changes of the topology class

Let $\mathbb{R}^2 \supset \omega^j$ be a domain on the plane of stretched coordinates (ξ_1^j, ξ_2^j). For sufficiently small $\varepsilon > 0$ it is always possible for any fixed different points $x^1, \dots, x^N \in \Omega$ to remove N small sets $\omega_\varepsilon^j(x^j) = \{x \mid \varepsilon^{-1}(x - x^j) \in \omega^j\}$ from Ω, obtaining the N-connected singularly perturbed domain Ω_ε.

In the domain Ω_ε we consider the following mixed boundary value problem:

$$-\Delta_x u^\varepsilon(x) = f(x), \quad x \in \Omega_\varepsilon; \qquad u^\varepsilon(x) = 0, \quad x \in \partial\Omega; \tag{1}$$

$$\partial_n u^\varepsilon(x) = 0, \quad x \in \partial\omega_\varepsilon^j(x^j)\ (j = 1, 2, \dots, N). \tag{2}$$

Using the refined method of matched expansions in the form [9], we take the sum

$$\mathcal{V}^\varepsilon(x) = v^0(x) + \varepsilon^2 C_\alpha^j G^{(\alpha)}(x^j, x) \tag{3}$$

as the outer asymptotic representation of the solution $u^\varepsilon(x)$. Here, the summation is performed over doubly repeated indices $j = 1, 2, \dots, N$ and $\alpha = 1, 2$; $G^{(k)}(x^j, x) = (x_k - x_k^j)(2\pi|x - x^j|)^{-2} + g^{(k)}(x^j, x)$ are singular solutions to the homogeneous Dirichlet problem in the domain Ω.

The inner asymptotic representations have the form $\mathcal{W}^{\varepsilon j}(\xi^j) = \text{const} + \varepsilon w^{1j}(\xi^j)$, where $\xi_i^j = \varepsilon^{-1}(x_i - x_i^j)$ are the stretched coordinates, while for the function $w^{1j}(\xi^j)$ we obtain the following refined matching asymptotic condition:

$$w^{1j}(\xi^j) = \xi^j \left(\nabla_x v_j^0 + \varepsilon^2 \nabla g_j^{(\bullet)} C^j + \varepsilon^2 \sum_{m \neq j} \nabla G_{mj}^{(\bullet)} C^m \right) + O\left(|\xi^j|^{-1}\right). \tag{4}$$

Here, $\nabla g_j^{(\bullet)}$ and $\nabla G_{mj}^{(\bullet)}$ are the symmetric 2×2 matrices with elements $\partial_{x_l} g^{(k)}(x^j, x^j)$ and $\partial_{x_l} G^{(k)}(x^m, x^j)$ $(k, l = 1, 2)$, respectively, $C^j = (C_1^j, C_2^j)^\top$; $\nabla_x v_j^0 = \nabla_x v^0(x^j)$.

According to (4), we obtain

$$w^{1j}(\xi^j) = \left(\nabla_x v_j^0 + \varepsilon^2 \nabla g_j^{(\bullet)} C^j + \varepsilon^2 \sum_{m \neq j} \nabla G_{mj}^{(\bullet)} C^m \right) \mathcal{Y}^j(\xi^j). \tag{5}$$

Here, $\mathcal{Y}^j = (\mathcal{Y}^{1,j}, \mathcal{Y}^{2,j})^\top$, $\mathcal{Y}^{l,j}(\xi^j) = \xi_l^j + \mathcal{Y}_0^{l,j}(\xi^j)$ are special solutions to the exterior Neumann problem in the domain $\mathbb{R}^2 \setminus \overline{\omega}_j$ that admit the expansions $\mathcal{Y}_0^{l,j}(\xi^j) = (2\pi|\xi^j|^2)^{-2} m_{\alpha l}^j \xi_\alpha^j + O(|\xi^j|^{-2})$ as $|\xi^j| \to \infty$. The symmetric positive definite 2×2 matrix $m^j = \|m_{kl}^j\|$ is called the polarization matrix of the cavity $\omega_\varepsilon^j(x^j)$ (see, e.g., [11,7]); $\tilde{m}^j = |\omega^j|^{-1} m^j$ is the *reduced polarization matrix*.

As a result of the matching procedure, we obtain the equations $(j = 1, 2, \dots, N)$

$$\left(I - \varepsilon^2 m^j \nabla g_j^{(\bullet)} \right) C^j - \varepsilon^2 \sum_{m \neq j} m^j \nabla G_{mj}^{(\bullet)} C^m = m^j \nabla_x v_j^0. \tag{6}$$

Replacing $u^\varepsilon(x)$ with its asymptotic representation (3) in the integral

$$J(\Omega_\varepsilon, u^\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla_x u^\varepsilon(x)|^2 \, dx = \frac{1}{2} \int_{\Omega_\varepsilon} f(x) u^\varepsilon(x) \, dx,$$

and denoting $v^0(x^j) f(x^j)$ by f_0^j, we find

$$2J(\Omega_\varepsilon, u^\varepsilon) = 2J(\Omega, v^0) - \varepsilon^2 f_0^j |\omega^j| + \varepsilon^2 \nabla_x v_j^{0\top} C^j + O(\varepsilon^3). \tag{7}$$

Eq. (6) is used to determine the columns of coefficients $C^j = (C_1^j, C_2^j)^\top$.

3. Asymptotic model for the topological sensitivity

We put $\sum_j |\omega_\varepsilon^j(x^j)| = \varepsilon^2|\Omega|$ and introduce the design dimensionless variables $\chi_j = |\omega^j||\Omega|^{-1}$ such that $\sum_j \chi_j = 1$. By virtue of (6) and (7), we derive the following asymptotic model for the topological sensitivity of the energy functional with respect to the multiple variations of the topology:

$$\frac{J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0)}{\sum\limits_j |\omega_\varepsilon^j(x^j)|} \simeq \mathfrak{S}_\chi(x^1, \ldots, x^N), \quad \varepsilon \to 0;$$

$$2\mathfrak{S}_\chi(x^1, \ldots, x^N) = -\chi_j f_0^j + \nabla_x v_j^{0\top} C^j. \tag{8}$$

The columns C^j are determined from the equations ($j = 1, 2, \ldots, N$)

$$\left(|\Omega|^{-1} I - \varepsilon^2 \chi_j \tilde{m}^j \nabla g_j^{(\bullet)}\right) C^j - \varepsilon^2 \chi_j \sum_{m \neq j} \tilde{m}^j \nabla G_{mj}^{(\bullet)} C^m = \chi_j \tilde{m}^j \nabla_x v_j^0. \tag{9}$$

Consider the problem of the simultaneous choice of the optimal topology configuration: For given $N \geq 2$, $x^1, \ldots, x^N$, $\tilde{m}^1, \ldots, \tilde{m}^N$, and $\varepsilon \in (0, \varepsilon_0)$ find

$$\min_{\chi \in \mathbb{R}^N} \mathfrak{S}_\chi(x^1, \ldots, x^N) \tag{10}$$

subject to the following conditions:

$$\chi_j \geq 0 \quad (j = 1, 2, \ldots, N), \qquad \sum_{j=1}^N \chi_j = 1. \tag{11}$$

Notice that the map of the topology derivative $\mathfrak{T}_\omega^0(x)$, $x \in \Omega$, (used in the bubble method [4]) does not contain necessary information for formulating any substantial variational problem of optimal multiple changes of the topology.

Theorem. *There exists $\varepsilon_1 > 0$ such that, for every real number ε between 0 and ε_1, the problem (10) and (11) has a solution.*

It is a consequence of the fact that the function $\chi \longmapsto \mathfrak{S}_\chi(x^1, \ldots, x^N)$ defined by explicit formulas (8) and (9) is continuous over the compact subset Σ of $\mathbb{R}^N$, defined by the restrictions (11).

From the coefficients of Eq. (9) we set up a $2N \times 2N$ matrix $\mathbb{G}$ with diagonal 2×2 blocks $\mathbb{G}_{jj} = \chi_j \tilde{m}^j \nabla_x g_j^{(\bullet)}$ and non-diagonal 2×2 blocks ($j \neq m$) $\mathbb{G}_{jm} = \chi_j \tilde{m}^j \nabla_x G_{mj}^{(\bullet)}$. Suppose C and B are columns composed of the columns C^j and $\chi_j \tilde{m}^j \nabla_x v^0(x^j)$ ($j, m = 1, 2, \ldots, N$). Then, denoting the $2N \times 2N$ identity matrix by $\mathbb{I}$, we find the solution of the system (9) in the form $C = \left(|\Omega|^{-1}\mathbb{I} - \varepsilon^2 \mathbb{G}\right)^{-1} B$. Hence, there exists $\varepsilon_1 > 0$ such that the function $\chi \longmapsto C(\chi)$ is continuous on Σ.

Substituting (11) in (8), we establish that the function $\mathfrak{S}_\chi(x^1, \ldots, x^N)$ is also continuous on Σ. To complete the proof it remains to apply Weierstrass's theorem. Since Σ is closed and bounded, the set of minima of the function $\mathfrak{S}_\chi(x^1, \ldots, x^N)$ over Σ is nonempty and compact. □

Observe that the asymptotic model (8) for the topological sensitivity $\mathfrak{S}_\chi(x^1, \ldots, x^N)$ in view of (9) can be regarded as a Padé approximant. A relation between the Padé approximation and the refined method of matched asymptotic expansions was established in [9]. This relation explains a surprising increasing of accuracy of such asymptotic representations (see, e.g., [12]).

4. Asymptotic model in the case of Dirichlet boundary conditions

In the domain Ω_ε with the inclusions $\omega_\varepsilon^1(x^1), \ldots, \omega_\varepsilon^N(x^N)$ defined in Section 2, we consider the following singularly perturbed Dirichlet problem:

$$-\Delta_x u^\varepsilon(x) = f(x), \quad x \in \Omega_\varepsilon; \qquad u^\varepsilon(x) = 0, \quad x \in \partial\Omega; \tag{12}$$

$$u^\varepsilon(x) = 0, \quad x \in \partial\omega_\varepsilon^j(x^j) \ (j = 1, 2, \ldots, N). \tag{13}$$

The asymptotics of $u^\varepsilon(x)$ was obtained in [13], with the help of the method of matched asymptotic expansions. According to [13,14], we have

$$2\left(J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0)\right) = c_j v^0(x^j) + O(\varepsilon|\log\varepsilon|^{-1}),$$

where the coefficients c_j are determined from the following system of equations:

$$\frac{c_j}{2\pi} \ln \frac{r_j}{\varepsilon r_\infty^j} + \sum_{m \neq j} c_m G(x^m, x^j) = -v^0(x^j) \quad (j = 1, 2, \ldots, N). \tag{14}$$

Here, $G(x^j, x)$ is Green's function, r^j is the harmonic radius of the domain Ω with respect to the point x^j, r^j_∞ is the outer conformal radius of the domain $\overline{\omega}^j$ [11].

Now, we put

$$J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0) \simeq \mathfrak{S}^\varepsilon_{\omega^1,\dots,\omega^N}(x^1, \dots, x^N), \quad \varepsilon \to 0, \tag{15}$$

where

$$2\mathfrak{S}^\varepsilon_{\omega^1,\dots,\omega^N}(x^1, \dots, x^N) = c_j v^0(x^j). \tag{16}$$

Notice that in this case the variational problem (10) with the dimensional design variables r^j_∞ $(j = 1, 2, \dots, N)$ is formulated for the *singular* function (16). This leads to the fact that, in general, the optimal solutions will be positive, i.e., $r^j_\infty > 0$ $(j = 1, 2, \dots, N)$. In other words, the inner boundary of the optimal domain Ω_ε 'wants' to have more small components. Note that this property can be regarded as a consequence of the well known non-existence of optimal shapes [15].

On the other hand, the topological derivative is defined by (see, e. g., [1])

$$\mathfrak{T}^0(x^1) = \lim_{\varepsilon\to 0+} \frac{J(\Omega_\varepsilon, u^\varepsilon) - J(\Omega, v^0)}{2\pi(\log\varepsilon)^{-1}} = \frac{1}{2}\left[v^0(x^1)\right]^2, \tag{17}$$

whereas formula (16) gives

$$\mathfrak{S}^\varepsilon_{\omega^1}(x^1) = -\pi\left[v^0(x^1)\right]^2\left(\ln\frac{r_1}{\varepsilon r^1_\infty}\right)^{-1}. \tag{18}$$

Observe that the simple asymptotic model (18) in contrast to (17) provides much more information, since it depends on the integral characteristic εr^1_∞ of the inclusion $\omega^1_\varepsilon(x^1)$ and the characteristic r_1 of the location of the point $x^1 \in \Omega$.

5. Example

Let $\Omega = \mathbb{B}_R(\mathcal{O})$ and $\omega^j = \mathbb{B}_R(\mathcal{O})$ be circular domains $(j = 1, 2, \dots, N)$. For the sake of simplicity, we put $f(x) \equiv f_0$. Then, $v^0(|x|) = 4^{-1}f_0(R^2 - |x|^2)$. We consider the following shape optimization problem: For given N, ω^j $(j = 1, 2, \dots, N)$, and $\varepsilon \in (0, 1)$ find

$$\min_{x^1,\dots,x^N\in\Omega} \mathfrak{S}_{\omega^1,\dots,\omega^N}(x^1, \dots, x^N) \tag{19}$$

subject to the following conditions:

$$\mathbb{B}_{\sqrt{\varepsilon R}}(x^j) \subset \Omega, \qquad \mathbb{B}_{\sqrt{\varepsilon R}}(x^j) \cap \mathbb{B}_{\sqrt{\varepsilon R}}(x^m) = \emptyset \quad (j \neq m). \tag{20}$$

The statement of the problem (19) and (20) is motivated by optimum design problems concerning the optimal supporting of a membrane. Note that in the problem (12) and (13) matching regions can be defined as $|x - x^j| = O(\sqrt{\varepsilon}r_j)$ for sufficiently small ε.

In view of the symmetry we have the following two admissible configurations for the particular case of four holes $(N = 4)$:

(a) $x^j_1 = \rho\cos(j\pi/2), x^j_2 = \rho\sin(j\pi/2)$ $(j = 1, 2, 3, 4)$;

(b) $x^j_1 = \rho\cos(j2\pi/3), x^j_2 = \rho\sin(j2\pi/3)$ $(j = 1, 2, 3)$, $x^4_1 = x^4_2 = 0$.

In the first case, $c_j = c_0$ $(j = 1, 2, 3, 4)$, where

$$c_0 = -\left(\frac{1}{2\pi}\ln\frac{R^2-\rho^2}{\varepsilon R^2} + \frac{2}{2\pi}\ln\frac{\sqrt{\rho^2+\rho^{-2}R^4}}{\sqrt{2}R} + \frac{1}{2\pi}\ln\frac{\rho+\rho^{-1}R^2}{2R}\right)^{-1} v^0(\rho). \tag{21}$$

In the second case, $c_j = c_1$ $(j = 1, 2, 3)$ and $c_4 = c_0$ are determined from the system

$$\frac{c_0}{2\pi}\ln\frac{1}{\varepsilon} + 3\frac{c_1}{2\pi}\ln\frac{R}{\rho} = -v^0(0),$$

$$\frac{c_1}{2\pi}\ln\frac{R^2-\rho^2}{\varepsilon R^2} + 2\frac{c_1}{2\pi}\ln\frac{\sqrt{\rho^2+\rho^{-2}R^4+R^2}}{\sqrt{3}R} + \frac{c_0}{2\pi}\ln\frac{R}{\rho} = -v^0(\rho).$$

In the first case, substituting (21) for c_j in (16), we get

$$-\frac{4\Lambda}{\pi f_0^2}\mathfrak{S}^\varepsilon_1(\rho) = 1 - \Lambda^{-1}\left(\frac{3}{2}\ln\Lambda + \frac{3}{2} + \ln 2 - \frac{3}{2}\ln 3\right) + O\left(\Lambda^{-2}\ln^2\Lambda\right). \tag{22}$$

In the second case we obtain

$$-\frac{4\Lambda}{\pi f_0^2}\mathfrak{S}^\varepsilon_2(\rho) = 1 - \Lambda^{-1}\left(\frac{3}{2}\ln\Lambda + \frac{3}{2} - \frac{3}{4}\ln 3\right) + O\left(\Lambda^{-2}\ln^2\Lambda\right). \tag{23}$$

Here, $\Lambda = |\ln\varepsilon|$ is a large parameter. Since $4\ln 2 < 3\ln 3$, it follows that the first case (22) is better than the second (23). This result is very important, since by the direct optimization algorithm [1] based on the topological derivative (17) one can obtain only the symmetric optimal solution with a circular inclusion at the origin.

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