On the Conjecture of Meinardus on Rational Approximation of $e^x$

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This paper is concerned with uniform approximation of $e^x$ on the interval $[-1, +1]$ by $(m, n)$-degree rationals, i.e., by rational functions whose numerator and denominator have degree $m$ and $n$, respectively. Several years ago, Meinardus [1, p. 168] conjectured that the norm of the error function for the best approximation is asymptotically

$$\frac{m! \ n!}{2^{m+n}(m+n)! \ (m+n+1)!} \quad \text{as} \quad m+n \to \infty. \quad (1)$$

Recently, Newman [3] has proved that the degree of approximation is indeed better than 8 times the conjectured value. Here we will establish a lower bound by applying de la Vallée-Poussin's theorem to the rational function constructed in [3]. We will show that the error function oscillates $n + m + 1$ times by evaluating a winding number.

Let

$$p(z) = \int_0^\infty t^n(t+z)^m e^{-t} \, dt, \quad q(z) = \int_0^\infty (t-z)^n t^m e^{-t} \, dt.$$ 

Then $p/q$ is the $(m, n)$-degree Padé approximant to $e^z$. Following the evaluation in [3, p. 234] we get

$$q(z) e^z - p(z) = \int_0^\infty (t-z)^n t^m e^{-t+z} \, dt - \int_0^\infty t^n(t+z)^m e^{-t} \, dt$$

$$= \int_0^z (t-z)^n t^m e^{-t} \, dt$$

$$= z^{m+n+1} \int_0^1 (u-1)^n u^m e^{(1-u)z} \, du.$$
Hence, for $|z| \leq \frac{1}{2}$,

$$|q(z) e^z - p(z)| \geq |z|^{m+n+1} \Re \int_0^1 (1-u)^n u^m e^{(1-u)z} \, du$$

$$= |z|^{m+n+1} \int_0^1 (1-u)^n u^m e^{(1-u)\Re z} \cos((1-u) \Im z) \, du$$

$$\geq |z|^{m+n+1} \int_0^1 (1-u)^n u^m \, du \, e^{-1/2} \cos \frac{z}{2}$$

$$\geq \frac{7}{8} e^{-1/2} |z|^{m+n+1} \frac{m! n!}{(m+n+1)!}. \quad (2)$$

Observe that this is just $7/(8e)$ times the upper bound for $|qe^z - p|$ given in [3].

Next, an upper bound for $q(z)$, $|z| \leq \frac{1}{2}$, is derived:

$$|q(z)| \leq \int_0^\infty (t + \frac{1}{2})^n t^m e^{-t} \, dt$$

$$\leq e^{1/2} \int_{-1/2}^{\infty} (t + \frac{1}{2})^{n+m} e^{-t} \, dt$$

$$= e^{1/2} (m+n)!.$$ \quad (3)

By combining (2) and (3) we get

$$|e^z - p(z)/q(z)| \geq \frac{7}{16e} \frac{2^{-m-n}m! n!}{(m+n)! (m+n+1)!}, \quad |z| = \frac{1}{2}. \quad (4)$$

Given $x \in [-1, +1]$, put $z = (x + iy)/2$ with $x^2 + y^2 = 1$. Obviously, $e^x = e^\frac{x}{2} e^\frac{y}{2}$. The crucial point is Newman's detection that $R(x) = p(z)/[q(z) q(z)]$ is an $(m,n)$-degree rational function in the variable $x$.

Put $a = e^x$, $b = p(z)/q(z)$. Then the error $e^x - R(x)$ is just $\bar{a}a - \bar{b}b$. It will be treated by using the formula

$$\bar{a}a - \bar{b}b = 2 \Re \bar{a}(a - b) - |a - b|^2, \quad a, b \in \mathbb{C}. \quad (5)$$

From (4) we get the estimate for the first term

$$|e^z [e^z - p/q]| \geq \frac{7}{16e^{3/2}} \frac{2^{-m-n}m! n!}{(m+n)! (m+n+1)!}, \quad |z| = \frac{1}{2}. \quad (6)$$
Denote by \( \arg w \) the argument of the complex number \( w \). Then

\[
\arg \left( e^x \left[ e^x - \frac{p(z)}{q(z)} \right] \right) = \arg \left( e^{-x} \left[ e^x - \frac{p(z)}{q(z)} \right] \right)
\]

\[
= \arg \left\{ \frac{e^{-x}}{q(z)} \left[ q(z) e^x - p(z) \right] \right\}. \tag{7}
\]

For short, let \( h(z) \) denote the function within the braces in (7).

Since \( \frac{p}{q} \) is the Padé approximation, \( z = 0 \) is a zero of \( q e^z - p \) of multiplicity \( n + m + 1 \). Moreover, \( q(z) \neq 0 \) for \( |z| \leq \frac{1}{2} \) is easily checked with the techniques in [3, p. 235]. Consequently, \( h \) has the winding number \( n + m + 1 \) for the circle \( |z| = \frac{1}{2} \). Hence, when an entire circuit has been completed, \( \arg(h(z)) \) is increased by \( (n + m + 1) 2\pi \). The argument is increased by \( (n + m + 1) \pi \) as \( z \) traverses the upper half of the circle, because \( h(x) \) is real for \( x \) on the real line. It follows by the same arguments as in [1, pp. 38–39] that \( h \) attains real values on \( n + m + 2 \) points \( z_k = (x_k + iy_k)/2 \) with \( +1 = x_1 > x_2 > \cdots > x_{n+m+2} = -1 \) and that the sign changes between any pair of consecutive \( x \)'s. The same is true for \( e^x \left[ e^x - \frac{p}{q} \right] \). Referring to (5) we have

\[
\min_{1 \leq k \leq n + m + 2} \left| e^{z_k} e^x - \frac{p(z_k)}{q(z_k)} \right| \geq \min_{|z| = 1/2} 2 \left| e^x \left[ e^x - \frac{p(z)}{q(z)} \right] \right| - \max_{|z| = 1/2} \left| e^x - \frac{p(z)}{q(z)} \right|^2
\]

\[
\geq \frac{7}{8e^{3/2}} \frac{2^{-m-n}m!n!}{(m+n)! (m+n+1)!} \left\{ 1 - \frac{\text{const}}{2^{m+n}(m+n+1)!} \right\}. \tag{8}
\]

From the theorem of de la Vallée-Poussin [1, p. 147] it is known that the expression in (8) is a lower bound for the distance of \( e^x \) from the \((m, n)\)-degree rational functions. The gap between the upper bound in [2] and the lower bound is roughly a factor \( e^{5/2}/((2 - e^{1/2}) \cos \frac{1}{2}) < 40 \).

If \( m - n \), one gets better estimates for the constants from the result in [2].

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REFERENCES