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ON THE GLOBAL DIMENSION OF SOME FILTERED ALGEBRAS (II)

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Introduction

Let k be a commutative field of char. 0. Let $\mathfrak{g}$ be a Lie algebra over k. Let f be a k-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$. We set $\mathfrak{g}(f) = T(\mathfrak{g})/U_f(\mathfrak{g})$, where $T(\mathfrak{g})$ denotes the tensor algebra of the vector space $\mathfrak{g}$ and $U_f(\mathfrak{g})$ the two sided ideal of $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y] - f(x, y)$ for $x, y \in \mathfrak{g}$. It is known [14] that $\mathfrak{g}(f)$ is a filtered k-algebra whose associated graded is isomorphic to a polynomial algebra over k and that every filtered k-algebra with this property is isomorphic to one such.

In this paper we determine [Section 2, Theorem 2.7] the global dimension of $\mathfrak{g}(f)$ where $\mathfrak{g}$ is a finite dimensional solvable Lie algebra over k and deduce some interesting results. This paper is a sequel to the author's previous paper of the same title [2].

In Section 1 we prove some results which are used in the proof of the main theorem.

1

Let A be a ring and d be a derivation of A. The Ore-extension $A[X; d]$ is the ring generated by A and an indeterminate X satisfying the relation $Xa - aX = d(a)$ for all $a \in A$. It is easy to see that any element b of $A[X; d]$ is of the form $\sum_{0 \leq i \leq n} X^i a_i$ with $a_i \in A$ and $a_n \neq 0$ and that such expression is unique. We call n to be the *degree of* b and a_n *the leading coefficient of* b.

Now we make following remarks regarding $A[X; d]$.

Remark 1. If A is left (resp. right) noetherian then $A[X; d]$ is also left (resp. right) noetherian.

Remark 2. $A[X; d]$ is A-free as a left as well as a right module.

Remark 3. $\text{l.gl.dim}\, A \leq \text{l.gl.dim}\, A[X;d] \leq 1 + \text{l.gl.dim}\, A$ if $\text{l.gl.dim}\, A < \infty$ [see 5, Proposition 3].

Let T be a multiplicatively closed subset of A contained in its centre such that $1 \in T$ and no element of T is a zero divisor in A. Then as a subset of $A[X;d]$, T has the properties: 1) no element of T is a zero divisor in $A[X;d]$, 2) T is left (resp. right) permutable, i.e. for $s \in T$ and $b \in A[X;d]$ there exist $t \in T$ and $c \in A[X;d]$ such that $sc = bt$ (resp. $cs = tb$).

Therefore from [15, Proposition 15.1] it follows that the left ring of fractions of $A[X;d]$ with respect to T exists and that it is isomorphic to $A_T[X;d']$ where A_T is the localisation of A with respect to T and d' is the derivation of A_T induced by d. We get similar results for the right ring of fractions of $A[X;d]$ with respect to T.

We denote both the left and right ring of fractions of $A[X;d]$ with respect to T by $A[X;d]_T$. Then by Remark 2 it follows that $A[X;d]_T$ is flat as a left and as a right $A[X;d]$-module.

Let $\mathfrak{a}$ be a left ideal of A. Let for $a \in A$, $(\mathfrak{a}:a) = \{b/b \in A, ba \in \mathfrak{a}\}$. Then $(\mathfrak{a}:a)$ is also a left ideal of A and $(\mathfrak{a}:a) = A$ if and only if $a \in \mathfrak{a}$. If $\mathfrak{a}$ is a maximal left ideal then so also $(\mathfrak{a}:a)$ for $a \notin \mathfrak{a}$. Moreover the map $\phi : A/(\mathfrak{a}:a) \to A/\mathfrak{a}$ given by $\phi(\bar{b}) = \overline{ba}$ is an isomorphism of A-modules.

For the sake of simplicity of notation throughout this section we write B (resp. B_T) for $A[X;d]$ (resp. $A[X;d]_T$.

With the above notation we prove the following

Proposition 1.1. *Let A be a ring which contains Q. Let d be a derivation of A. Let a be an element of the centre of A such that $Aa + Ad(a) = A$. Let $\mathfrak{a}'$ be a left ideal of B which contains a. Then $\mathfrak{a}' = B\mathfrak{a}$ where $\mathfrak{a} = \mathfrak{a}' \cap A$.*

Proof. Since $\mathfrak{a} \subset \mathfrak{a}'$ we have $B\mathfrak{a} \subset \mathfrak{a}'$. If $B\mathfrak{a} \neq \mathfrak{a}'$ then there exists an element $b \in \mathfrak{a}'$ such that $b \notin B\mathfrak{a}$ and is of smallest degree with such property. Let $b = \sum_{0 \leq i \leq n} X^i a_i$, $a_n \neq 0$. Since every element of $B\mathfrak{a}$ is of the form $\sum_{0 \leq j \leq m} X^j c_j$, $c_j \in \mathfrak{a}$ for $0 \leq j \leq m$, by choice of b we get $a_n \notin \mathfrak{a}$. Since $a \in \mathfrak{a}'$ we have $b' = (X^n a_n)a - a\sum_{0 \leq i \leq n} X^i a_i \in \mathfrak{a}'$. But $b' = X^{n-1}(nd(a)a_n - aa_{n-1}) +$ terms of smaller degree. Since degree $b' \leq n-1 <$ degree b, $b' \in B\mathfrak{a}$. Therefore $nd(a)a_n - aa_{n-1} \in \mathfrak{a}$. But $a \in \mathfrak{a} = \mathfrak{a}' \cap A$. Therefore $nd(a)a_n \in \mathfrak{a}$, i.e. $d(a)a_n \in \mathfrak{a}$. Since $Aa + Ada = A$ there exist c and $c' \in A$ such that $ca + c'd(a) = 1$. This shows that $a_n = c(aa_n) + c'(d(a)a_n) \in \mathfrak{a}$, which is a contradiction. Therefore $B\mathfrak{a} = \mathfrak{a}'$. Hence the result.

Proposition 1.2. *Let K be an algebraically closed field of* char. 0. *Let A be a K-algebra. Let d be a K-derivation of A. Let a be an element of the centre of A such that $d(a) = 1$. Let $\mathfrak{b}$ be a proper left ideal of B. If for some $b' \notin \mathfrak{b}$ $(\mathfrak{b}:b') \cap K[a] \neq 0$ then there exists $b \notin \mathfrak{b}$. such that $a - \lambda \in (\mathfrak{b}:b)$ for some $\lambda \in K$.*

Proof. $d(a) = 1$ implies that a is transcendental over K. Therefore $K[a]$ is a polynomial algebra over K in one variable. Let $0 \neq f \in (\mathfrak{b}:b') \cap K[a]$. We prove the result by induction on $\deg f$ where $\deg f$ denotes the degree of f as an element of $K[a]$.

Since $\mathfrak{b}$ is a proper left ideal and $f \neq 0$, $\deg f \geq 1$. Let $\deg f = 1$. Then $f = \alpha a + \beta$ with $\alpha, \beta \in K$ and $\alpha \neq 0$. Then by taking $\lambda = -\alpha^{-1}\beta$ and $b' = b$ we get the required result.

Assume the result for $\deg f \leq m-1$. Let $\deg f = m > 1$. Then since K is algebraically closed there exists $\alpha \in K$ such that $f = (a-\alpha)f'$, $f' \in K[a]$ and $\deg f' = m-1$. If $f' \notin (\mathfrak{b} : b')$ then by taking $b = f'b'$ we get $b \notin \mathfrak{b}$ and $a - \alpha \in (\mathfrak{b} : b)$. If $f' \in (\mathfrak{b} : b')$ then since $\deg f' = m-1$ by our induction hypothesis there exist $b \notin \mathfrak{b}$ and $\lambda \in K$ such that $a - \lambda \in (\mathfrak{b} : b)$.

This completes the proof of Proposition 1.2.

Proposition 1.3. *Let A be a ring which is left and right noetherian. Let* $\text{l.gl.dim}\, A < \infty$. *Let d be a derivation of A. Let $\mathfrak{a}$ be a left ideal of B such that* $\text{l.gl.dim}\, B = \text{hd}_B B/\mathfrak{a}$. *Let T be a multiplicatively closed subset of A contained in its centre such that $1 \in T$ and no element of T is a zero divisor in A. If for every $b \notin \mathfrak{a}$ $(\mathfrak{a} : b) \cap T = \emptyset$ then* $\text{l.gl.dim}\, B = \text{l.gl.dim}\, B_T$.

Proof. Since A is left and right noetherian and $\text{l.gl.dim}\, A < \infty$, by our earlier remarks, it follows that B as well as B_T are left and right noetherian and have finite left global dimension. Therefore by [1, Theorem 1] there exists a left ideal $\mathfrak{b}$ of B_T such that $\text{l.gl.dim}\, B_T = \text{hd}_{B_T} B_T/\mathfrak{b}$. But since B_T is a left ring of fractions of B there exists a left ideal $\mathfrak{b}'$ of B such that $B_T/\mathfrak{b} \simeq B_T \otimes_B B/\mathfrak{b}'$ as B_T-modules.

For a ring R and a left module N let $\text{w.dim}_R N$ denote the weak dimension of N. If R is left noetherian and N is finitely generated then $\text{hd}_R N = \text{w.dim}_R N$ [3, Chapter VI]. Therefore

$$\text{l.gl.dim}\, B_T = \text{hd}_{B_T} B_T/\mathfrak{b} = \text{w.dim}_{B_T} B_T/\mathfrak{b} = \text{w.dim}_{B_T} B_T \otimes_B B/\mathfrak{b}'.$$

Since B_T is B-flat as a right B-module we get

$$\text{w.dim}_{B_T} B_t \otimes_B B/\mathfrak{b}' \leq \text{w.dim}_B B/\mathfrak{b}' = \text{hd}_B B/\mathfrak{b}' \leq \text{l.gl.dim}\, B.$$

Therefore $\text{l.gl.dim}\, B_T \leq \text{l.gl.dim}\, B$.

Now since $(\mathfrak{a} : b) \cap T = \emptyset$ for all $b \notin \mathfrak{a}$, the mapping $\psi : B/\mathfrak{a} \to B_T \otimes_B B/\mathfrak{a}$ given by $\psi(\bar{x}) = 1 \otimes \bar{x}$ is a monomorphism. Therefore, since $\text{l.gl.dim}\, B = \text{hd}_B B/\mathfrak{a} = \text{w.dim}_B B/\mathfrak{a}$ we get $\text{l.gl.dim}\, B = \text{w.dim}_B B_T \otimes_B B/\mathfrak{a}$. But B_T is B-flat as a left module. Therefore

$$\text{w.dim}_B B_T \otimes_B B/\mathfrak{a} \leq \text{w.dim}_{B_T} B_T \otimes_B B/\mathfrak{a} = \text{hd}_{B_T} B_T \otimes_B B/\mathfrak{a} \leq \text{l.gl.dim}\, B_T.$$

This shows that $\text{l.gl.dim}\, B \leq \text{l.gl.dim}\, B_T$. Hence the equality.

This completes the proof of Proposition 1.3.

Let $\mathfrak{g}$ be a Lie algebra over a field k of char. 0. Let f be a k-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$ [14, p. 532]. Let θ be an element of $\text{Hom}_k(\mathfrak{g}, k)$.

Definition. A subalgebra h of $\mathfrak{g}$ is said to be *f-subordinate to θ* if for every $h_1, h_2 \in h$ we have $\theta[h_1, h_2] + f(h_1, h_2) = 0$.

Remark 4. From the definition it follows that if h is a subalgebra of $\mathfrak{g}$ then the restriction of f to $h \times h$ is a coboundary if and only if there exists $\theta \in \mathrm{Hom}_K(\mathfrak{g}, k)$ such that h is f-subordinate to θ. Therefore if a subalgebra h is f-subordinate to θ then $h(f)$ is isomorphic to $h(0)$ [14, Theorem 3.1]. But $h(0)$ is nothing but the usual enveloping algebra of the Lie algebra h. Therefore $\mathrm{l.gl.dim}\, h(0) = \dim_k h$ [3, p. 283, Theorem 8.2]. Moreover the map $\theta : h \to k$ defines an $h(f)$-module structure denoted by $k(\theta, h)$ on k such that $\mathrm{hd}_{h(f)} k(\theta, h) = \dim_k h = \mathrm{l.gl.dim}\, h(f)$. Since $\mathfrak{g}(f)$ is $h(f)$-free as a right as well as a left module and contains $h(f)$ as a direct summand, from [8, Lemma 1] it follows that

$$\mathrm{l.gl.dim}\, \mathfrak{g}(f) \geq \mathrm{hd}_{\mathfrak{g}(f)}\, \mathfrak{g}(f) \underset{h(f)}{\otimes} k(\theta, h) = \mathrm{hd}_{h(f)}\, k(\theta, h) = \dim_k h.$$

On the other hand from [13, Theorem 1] we get $\dim_k \mathfrak{g} \geq \mathrm{l.gl.dim}\, \mathfrak{g}(f)$. Therefore we always have inequality $\dim_k \mathfrak{g} \geq \mathrm{l.gl.dim}\, \mathfrak{g}(f) \geq \dim_k h$ for a subalgebra h of $\mathfrak{g}$ for which the restriction of f to $h \times h$ is a coboundary.

2

We begin this section with the following theorem.

Theorem 2.1. *Let k be an algebraically closed field of* char.0. *Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over k. Let f be a k-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$. Then there exists θ in* $\mathrm{Hom}_k(\mathfrak{g}, k)$ *and a subalgebra h of $\mathfrak{g}$ such that*

I) *h is f-subordinate to θ*
II) $\mathrm{l.gl.dim}\, \mathfrak{g}(f) = \dim_k h$.

For the proof of this theorem we require some lemmas. In the first two lemmas (i.e. Lemma 2.2 and Lemma 2.3) k, $\mathfrak{g}$, and f are as in the statement of Theorem 2.1.

Let $x \in \mathfrak{g}$ be such that $k \cdot x$ is an ideal of $\mathfrak{g}$. Let $\mathfrak{g}_1 = \{z / z \in \mathfrak{g}, [x, z] = 0\}$, $\mathfrak{g}' = \{z / z \in \mathfrak{g}, f(x, z) = 0\}$.

Lemma 2.2. *If $\mathfrak{g}'$ is a subspace of $\mathfrak{g}$ of codimension 1 then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$ if $\mathfrak{g}' \subset \mathfrak{g}_1$.*

Proof. It is easy to see that $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ of codim ≤ 1. If $\mathrm{codim}\, \mathfrak{g}_1 = 1$ then since $\mathrm{codim}\, \mathfrak{g}' = 1$, $\mathfrak{g}' \subset \mathfrak{g}_1$ implies that $\mathfrak{g}' = \mathfrak{g}_1$. Therefore $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$.

If $\mathrm{codim}\, \mathfrak{g}_1 = 0$ then $\mathfrak{g}_1 = \mathfrak{g}$. This means that x is an element of the centre of $\mathfrak{g}$. But then for any $z, w \in \mathfrak{g}$ we have

$$f(x,[z,w]) = f(x,[z,w]) + f(w,[x,z]) + f(z,[w,x]) = 0.$$

Therefore $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}'$. Hence $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$.

This completes the proof of Lemma 2.2.

Lemma 2.3. *If $\mathfrak{g}'$ is not an ideal of $\mathfrak{g}$ then $\mathfrak{g}' \cap \mathfrak{g}_1$ is an ideal of $\mathfrak{g}_1$ of* codim 1 *and $\mathfrak{g}_1 = \mathfrak{g}' \cap \mathfrak{g}_1 \oplus k \cdot w$ where $w \in \mathfrak{g}_1$ be such that $f(w,x) = 1$ and the adjoint action of w on $\mathfrak{g}_1$ is nilpotent.*

Proof. Since $\mathfrak{g}'$ is not an ideal of $\mathfrak{g}$, by Lemma 2.2 we have $\mathfrak{g}' \not\subset \mathfrak{g}_1$. Therefore $\mathfrak{g}_1 \subsetneq \mathfrak{g}$ and $\mathfrak{g}' \cap \mathfrak{g}_1$ is a subspace of $\mathfrak{g}_1$ of codim 1. Applying Lemma 2.2 again to x, $\mathfrak{g}_1$ and f we get that $\mathfrak{g}_1 \cap \mathfrak{g}'$ is an ideal of $\mathfrak{g}_1$ of codim 1.

Since $\mathfrak{g}_1 \subsetneq \mathfrak{g}$ and $k \cdot x$ is an ideal of $\mathfrak{g}$, there exists $y \in \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus k \cdot y$ and $[y,x] = x$. Since $\mathfrak{g}' \neq \mathfrak{g}_1$ there exists $w' \in \mathfrak{g}_1$ such that $f(w',x) = 1$. Therefore

$$1 = f(w',x) = f(w',[y,x]) = -f(x,[w',y]) - f(y,[x,w']) = f([w',y],x).$$

Let $w = [w',y]$. Since $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ of codim 1 $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}_1$. Since $\mathfrak{g}$ is solvable and $w \in [\mathfrak{g},\mathfrak{g}]$ the adjoint action of w on $\mathfrak{g}$ and therefore on $\mathfrak{g}_1$ is nilpotent.

Hence the result.

Remark 5. If $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$ of codim 1 then $\mathfrak{g}' \subset \mathfrak{g}_1$ and $\mathfrak{g} = \mathfrak{g}' \oplus k \cdot y$. Therefore x is an element of the centre of $\mathfrak{g}'(f)$ and $\mathfrak{g}(f)$ is the Ore-extension of $\mathfrak{g}'(f)$ with respect to the derivation d induced by y. If $\mathfrak{g}_1 = \mathfrak{g}$ then we can choose y such that $f(y,x) = 1$. Then $d(x) = 1$. If $\mathfrak{g}_1 \subsetneq \mathfrak{g}$ then we can choose y such that $[y,x] = x$ and then $d(x) = x + \lambda$ where $\lambda = f(y,x)$. Therefore $d(x+\lambda) = x + \lambda$.

Lemma 2.4. *Let D be a Dedekind domain of* char. 0. *Let L be its quotient field. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over L with a basis $(x_1, x_2, \ldots, x_n)$. Let f be a L-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$. Let $\theta \in$* $\mathrm{Hom}_L(\mathfrak{g}, L)$ *and h be a subalgebra of $\mathfrak{g}$ such that h is f-subordinate to θ. Then there exists a discrete valuation ring R with $D \subset R \subset L$ such that*

(1) $[z,w] \in \mathfrak{g}_R$ *for all* $z, w \in \mathfrak{g}_R$

2) $f_R(z,w) \in R$ *for all* $z, w \in \mathfrak{g}_R$

3) $\theta(\mathfrak{g}_R) \subset R$

where $\mathfrak{g}_R = \sum_{1 \leq i \leq n} R x_i$, $f_R = f \mid \mathfrak{g}_R \times \mathfrak{g}_R$.

Proof. Since L is the quotient field of D and $\mathfrak{g}$ is finite dimensional over L there exists $0 \neq s \in D$ such that $s \cdot [x_i, x_j] \in \sum_{1 \leq i \leq n} D x_i$, $s \cdot f(x_i, x_j) \in D$ and $s \cdot \theta(x_i) \in D$ for all i, j, $1 \leq i, j \leq n$. Let m be a maximal ideal of D such that $s \notin m$. Then by taking $R = D_m$ we get the required result.

Lemma 2.5. *Let L, $\mathfrak{g}$, f, $\mathfrak{g}_R$, h, θ be as in the statement of Lemma 2.4. Let K be the residue field of R. Let $\bar{\mathfrak{g}} = K \otimes_R \mathfrak{g}_R$, $\bar{f} = I_K \otimes_R f_R$. Then there exist $\bar{\theta} \in$* $\mathrm{Hom}_K(\bar{\mathfrak{g}}, K)$ *and a subalgebra $\bar{h}$ of $\bar{\mathfrak{g}}$ such that $\bar{h}$ is $\bar{f}$-subordinate to $\bar{\theta}$ and* $\dim_K \bar{h} = \dim_L h$.

Proof. Let $h' = \mathfrak{g}_R \cap h$. Since h' is a R-submodule of $\mathfrak{g}_R$ and R is a discrete valuation ring h' is a free R-module of rank r. Since $L \otimes_R h' \simeq h$ as L-vector spaces $r = \dim_L h$.

Let $\mathfrak{a}$ be the maximal ideal of R. Then $h' = \mathfrak{g}_R \cap h$ implies $\mathfrak{a}h' = \mathfrak{a}\mathfrak{g}_R \cap h'$. This shows that the map $i; K \otimes_R h' \to K \otimes_R \mathfrak{g}_R\,(=\bar{\mathfrak{g}})$ given by $i(\lambda \otimes x) = \lambda \otimes x$ is a monomorphism. We identify $K \otimes_R h'$ with its image in $K \otimes_R \mathfrak{g}_R$ under the mapping i. Let $\bar{\theta} : \bar{\mathfrak{g}} \to K$ be the map given by $\bar{\theta}(\lambda \otimes x) = \lambda \eta \theta(x)$ where $\eta : R \to K$ is the canonical map. It is easy to see that $\bar{\theta}$ is well defined and K-linear. Let $\bar{h} = K \otimes_R h'$. We claim that $\bar{h}$ is f-subordinate to $\bar{\theta}$.

Let $u, v \in \bar{h}$. Then there exist $z, w \in h'$ such that $u = 1 \otimes z$, $v = 1 \otimes w$. Therefore

$$\begin{aligned}\bar{f}(u,v) + \bar{\theta}[u,v] &= \bar{f}(1\otimes z, 1\otimes w) + \bar{\theta}[1\otimes z, 1\otimes w] = 1 \otimes f(z,w) + \eta\theta[z,w] \\ &= \eta f(z,w) + \eta\theta[z,w] = \eta(f(z,w) + \theta[z,w]) = 0.\end{aligned}$$

Thus $\bar{h}$ is $\bar{f}$-subordinate to $\bar{\theta}$. Since h' is R-free of rank r we have $\dim_K \bar{h}\,(= K \otimes_R h') = r = \dim_L h$. Hence the result.

Thus the proof of Lemma 2.5 is complete.

Proof of Theorem 2.1. We will prove the result by induction on $\dim_k \mathfrak{g}$. Let $\dim_k \mathfrak{g} = 1$. Then $\mathfrak{g}(f)$ is a polynomial algebra $k[x]$ in one variable over k. Let $\theta : \mathfrak{g} \to k$ be the map given by $\theta(x) = 0$ where $\mathfrak{g} = k \cdot x$. Then $\mathfrak{g}$ is f-subordinate to θ and $\text{l.gl.dim}\, \mathfrak{g}(f) = \text{gl.dim}\, k[x] = 1 = \dim_k \mathfrak{g}$.

Assume the result for $\dim_k \mathfrak{g} \leq n-1$. Let $\dim \mathfrak{g} = n$.

Since $\mathfrak{g}(f)$ is left and right noetherian and of finite left global dimension, by [2, Proposition 1.1] there exists a maximal left ideal $\mathfrak{a}$ of $\mathfrak{g}(f)$ such that $\text{l.gl.dim}\, \mathfrak{g}(f) = \text{hd}_{\mathfrak{g}(f)}\, \mathfrak{g}(f)/(\mathfrak{a} : a)$ for all $a \notin \mathfrak{a}$. Since $\mathfrak{g}$ is solvable and k-algebraically closed there exists $x \in \mathfrak{g}$ such that $k \cdot x$ is a non zero ideal of $\mathfrak{g}$. Let $\mathfrak{g}_1 = \{w / w \in \mathfrak{g}, [x,w] = 0\}$, $\mathfrak{g}' = \{w / w \in \mathfrak{g}, f(x,w) = 0\}$.

We divide the proof in following four cases:

Case 1: $\mathfrak{g} = \mathfrak{g}' = \mathfrak{g}_1$. Then x will be an element of the centre of $\mathfrak{g}(f)$.

Let $M = \mathfrak{g}(f)/\mathfrak{a}$ and let $I = \text{ann}\, M$. Then since M is a simple left $\mathfrak{g}(f)$-module and x an element of the centre of $\mathfrak{g}(f)$ by [10, p. 171] we get $x - \lambda \in I$ for some $\lambda \in k$.

Let $\tilde{\mathfrak{g}} = \mathfrak{g}/(x)$. Let $\alpha : \mathfrak{g} \to k$, be the k-linear map given by

$$\alpha(x_1) = \lambda$$

$$\alpha(x_i) = 0, \quad 2 \leq i \leq n$$

where $x = x_1, x_2, x_3, \ldots, x_n$ is a k-basis of $\mathfrak{g}$. Let $\tilde{f} : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to k$ be the map defined by

$$\tilde{f}(\tilde{z}, \tilde{w}) = f(z,w) + \alpha[z,w].$$

Then $\tilde{\mathfrak{g}}$ is a solvable Lie algebra, $\tilde{f}$ a k-valued 2-cocycle on the 'standard complex' for $\tilde{\mathfrak{g}}$ such that $\tilde{\mathfrak{g}}(\tilde{f}) = \mathfrak{g}(f)/(x_1 - \lambda)$.

Since $x_1 - \lambda \in I = \text{ann}\, M$, we can regard M as a $\tilde{\mathfrak{g}}(\tilde{f})$-module. Since $x_1 - \lambda$ is an element of the centre of $\mathfrak{g}(f)$ which is neither a unit nor a divisor of zero and

$\mathrm{hd}_{\tilde{g}(f)} M \leq \mathrm{l.gl.dim}\, \tilde{g}(f) \leq \mathrm{Dim}_K \tilde{g} < \infty$, by Kaplansky's Theorem [6, p. 172, Theorem 3]; $\mathrm{hd}_{\tilde{g}(f)} M = \mathrm{hd}_{g(f)} M - 1 = \mathrm{l.gl.dim}\, g(f) - 1$. But since $\mathrm{l.gl.dim}\, \tilde{g}(\tilde{f}) < \infty$ we always have $\mathrm{l.gl.dim}\, \tilde{g}(\tilde{f}) \leq \mathrm{l.gl.dim}\, g(f) - 1$ [6, p. 173, Theorem 4]. Therefore $\mathrm{l.gl.dim}\, \tilde{g}(f) = \mathrm{l.gl.dim}\, g(f) - 1$. (One can easily prove Kaplansky's Theorems [6, p. 172, Theorem 3 and p. 173, Theorem 4] for an element $x - \lambda \in g(f)$ which is neither a unit nor a divisor of zero and which is such that $g(f)(x - \lambda) = (x - \lambda)g(f)$. Therefore our conclusions remain valid for such element $x - \lambda$ even though it may not be an element of the centre of $g(f)$. This fact we have used in the proof of the theorem for case 3).

Since $\dim_k \tilde{g} = n - 1$, by our induction hypothesis there exists a subalgebra $\tilde{h}$ of $\tilde{g}$ and an element $\tilde{\theta}$ of $\mathrm{Hom}_k(\tilde{g}, k)$ such that (I) $\tilde{h}$ is $\tilde{f}$-subordinate to $\tilde{\theta}$, (II) $\mathrm{l.gl.dim}\, \tilde{g}(\tilde{f}) = \dim_k \tilde{h}$.

Let h be a subalgebra of g such that $x_1 \in h$ and $h/(x_1) = \tilde{h}$. Let $\theta : g \to k$ be the k-linear map such that $\theta(x_1) = \lambda$ and $\theta(x_i) = \tilde{\theta}(\bar{x}_i)$ for $2 \leq i \leq n$ where $\bar{x}_i$ denotes the image of x_i in $\tilde{g}(= g/(x_1))$ under the canonical mapping $\eta : g \to \tilde{g}$.

Then $\tilde{h}$ is $\tilde{f}$-subordinate $\tilde{\theta}$ implies that h is f-subordinate to θ and $\dim_k h = \dim \tilde{h} + 1 = \mathrm{l.gl.dim}\, \tilde{g}(\tilde{f}) + 1 = \mathrm{l.gl.dim}\, g(f)$.

Case 2: $g_1 = g$, $g' \subsetneq g$. Then from Lemma 2.2 it follows that g' is an ideal of g of codim 1. Let $g = g' \oplus ky$ with $y \in g$ be such that $f(y, x) = 1$. Then $g(f)$ is the Ore-extension of $g'(f)$ with respect to the derivation d induced by y and $x - \lambda$ is an element of the centre of $g'(f)$ with $d(x - \lambda) = 1$ for every $\lambda \in k$. We claim that $\mathrm{l.gl.dim}\, g'(f) = \mathrm{l.gl.dim}\, g(f)$.

If for some $b' \notin \mathfrak{a}$ $(\mathfrak{a} : b') \cap k[x] \neq 0$ then by Proposition 1.2 we get an element $b \notin \mathfrak{a}$ and an element $\lambda \in k$ such that $x - \lambda \in (\mathfrak{a} : b)$. But then by Proposition 1.1 we have $\mathfrak{a}'' = g(f)\mathfrak{a}'$ where $\mathfrak{a}' = \mathfrak{a}'' \cap g'(f)$ and $\mathfrak{a}'' = (\mathfrak{a} : b)$. Therefore $g(f)/\mathfrak{a}'' \simeq g(f) \otimes_{g'(f)} g'(f)/\mathfrak{a}'$. Since $g(f)$ is $g'(f)$-free as a left as well as a right module and contains $g'(f)$ as a direct summand, from [8, Lemma 1] it follows that

$$\mathrm{l.gl.dim}\, g'(f) \leq \mathrm{l.gl.dim}\, g(f) = \mathrm{hd}_{g(f)}\, g(f)/(\mathfrak{a} : b)$$

$$= \mathrm{hd}_{g(f)}\, g(f) \underset{g'(f)}{\otimes} g'(f)/\mathfrak{a}' \leq \mathrm{hd}_{g'(f)}\, g'(f)/\mathfrak{a}' \leq \mathrm{l.gl.dim}\, g'(f).$$

Therefore we have $\mathrm{l.gl.dim}\, g(f) = \mathrm{l.gl.dim}\, g'(f)$.

Suppose for all $b \notin \mathfrak{a}(\mathfrak{a} : b) \cap k[x] = 0$. Let $T = k[x] - \{0\}$. Then T is a multiplicatively closed set contained in the centre of $g'(f)$ such that no element of T is a zero divisor of $g'(f)$ (in fact $g'(f)$ itself is without proper divisors of zero). Therefore from Proposition 1.3 it follows that $\mathrm{l.gl.dim}\, g(f) = \mathrm{l.gl.dim}\, g(f)_T = \mathrm{l.gl.dim}\, g'(f)_T[X; d'] \leq 1 + \mathrm{l.gl.dim}\, g'(f)_T$ where d' is the derivation of $g'(f)_T$ induced by the derivation d of $g'(f)$.

Let $x = x_1, x_2, \ldots, x_{n-1}$ be a k-basis of g'. Let K be the quotient field of $k[x]$ (note that since $d(x) = 1$, $k[x]$ is a polynomial algebra over k). Let $g'' = K \otimes_k g'/(x)$. Let $\beta : g' \to k[x_1](= k[x])$ be the k-linear map given by

$$\beta(x_1) = x_1$$

$$\beta(x_i) = 0, \qquad 2 \leqslant i \leqslant n-1.$$

Let $f'' : \mathfrak{g}'' \times \mathfrak{g}'' \to K$ be the map defined by

$$f''(1 \otimes \bar{z}, 1 \otimes \bar{w}) = f(z, w) + \beta[z, w].$$

Then $\mathfrak{g}''$ is a solvable Lie algebra over K, f'' a K-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}''$ such that $\mathfrak{g}''(f'') \simeq \mathfrak{g}'(f)_T$.

Let Ω be the algebraic closure of K. Let $\mathfrak{g}''_\Omega = \Omega \otimes_K \mathfrak{g}''$, $f''_\Omega = I_\Omega \otimes_K f''$. Then $\mathfrak{g}''_\Omega$ is a solvable Lie algebra over Ω of dim $n-2$, f''_Ω a Ω-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}''_\Omega$ such that $\mathfrak{g}''_\Omega(f''_\Omega) = \Omega \otimes_K \mathfrak{g}''(f'')$. Since $\dim_\Omega \mathfrak{g}''_\Omega = n-2$, by our induction hypothesis there exist a subalgebra h' of $\mathfrak{g}''_\Omega$ and an element $\theta'' \in \operatorname{Hom}_\Omega(\mathfrak{g}''_\Omega, \Omega)$ such that I) h' is f''_Ω-subordinate to θ'', II) $\text{l.gl.dim}\, \mathfrak{g}''_\Omega(f''_\Omega) = \dim_\Omega h'$. Since $\mathfrak{g}''$ is finite dimensional over K and f'' is completely determined by its values on a K-basis of $\mathfrak{g}'' \times \mathfrak{g}''$, there exist a finite extension L of K and a subalgebra h of $\mathfrak{g}''_L$ such that $\theta(\mathfrak{g}''_L) \subset L$ and $\Omega \otimes_L h = h'$ where $\mathfrak{g}''_L = L \otimes_K \mathfrak{g}''$ and $\theta = \theta'' | \mathfrak{g}''_L$. This implies that h is f''_L-subordinate to θ where $f''_L = I_L \otimes_K f''$ and $\text{l.gl.dim}\, \mathfrak{g}''_\Omega(f''_\Omega) = \dim_\Omega h' = \dim_L h = \text{l.gl.dim}\, h(f''_L) \leqslant \text{l.gl.dim}\, \mathfrak{g}''_L(f''_L)$. But $\mathfrak{g}''_L(f''_L) \simeq L \otimes_K \mathfrak{g}''(f'')$ and L is a finite separable extension. Therefore by [4, p. 74] we have $\text{l.gl.dim}\, \mathfrak{g}''_L(f''_L) = \text{l.gl.dim}\, \mathfrak{g}''(f'')$. Since $\mathfrak{g}''_\Omega(f''_\Omega)$ is $\mathfrak{g}''(f'')$-free as a left and a right module and contains $\mathfrak{g}''(f'')$ as a direct summand, by [8, Lemma 1] we have $\text{l.gl.dim}\, \mathfrak{g}''(f'') \leqslant \text{l.gl.dim}\, \mathfrak{g}''_\Omega(f''_\Omega) = \dim h' = \dim_L h \leqslant \text{l.gl.dim}\, \mathfrak{g}''_L(f''_L) = \text{l.gl.dim}\, \mathfrak{g}''(f'')$. Therefore $\text{l.gl.dim}\, \mathfrak{g}'(f)_T = \text{l.gl.dim}\, \mathfrak{g}''(f'') = \text{l.gl.dim}\, \mathfrak{g}_L(f''_L) = \text{l.gl.dim}\, \mathfrak{g}''_\Omega(f''_\Omega)$.

Let D be the integral closure of $k[x]$ in L. Then since L is separable over K, D is a Dedekind domain. Then by Lemma 2.4 there exists a discrete valuation ring R with $D \subset R \subset L$ such that 1) $[u, v] \in \mathfrak{g}''_R$ for $u, v \in \mathfrak{g}''_R$, 2) $f''_R(u, v) \in R$, 3) $\theta(\mathfrak{g}''_R) \subset R$ where $\mathfrak{g}''_R = \sum_{2 \leqslant i \leqslant n-1} R(1 \otimes \bar{x}_i)$, $\bar{x}_i$ is the image of x_i in $\mathfrak{g}'/(x)$ under the canonical map $\eta : \mathfrak{g}' \to \mathfrak{g}'/(x)$ for $2 \leqslant i \leqslant n-1$, $f''_R = f''_L | \mathfrak{g}''_R \times \mathfrak{g}''_R$. From the construction of R and $\mathfrak{g}''_R$ it follows that the residue field of R is k and $\mathfrak{g}''_R \simeq R \otimes_k \mathfrak{g}'/(\bar{x})$.

Let $\bar{\mathfrak{g}} = k \otimes_R \mathfrak{g}''_R$, $\bar{f} = I_k \otimes_R f''_R$. Then it is easy to see that $\bar{\mathfrak{g}} \simeq k \otimes_R \mathfrak{g}''_R \simeq k \otimes_R R \otimes_k \mathfrak{g}'/(x) \simeq \mathfrak{g}'/(x)$ as Lie algebras over k and when we identify $\bar{\mathfrak{g}}$ with $\mathfrak{g}'/(x)$ then $\bar{f}(\bar{z}, \bar{w}) = f(z, w) + \eta'\beta[z, w]$ where $\beta : \mathfrak{g}' \to k[x]$ is the map as defined above and $\eta' : R \to k$ is the canonical map. From this it follows that $\bar{\mathfrak{g}}(\bar{f}) \simeq \mathfrak{g}'(f)/(x - \lambda)$ where $\lambda = \eta'(x)$.

Lemma 2.5 shows that there exists a subalgebra $\bar{h}$ of $\bar{\mathfrak{g}}$ and an element $\bar{\theta} \in \operatorname{Hom}_k(\bar{\mathfrak{g}}, k)$ such that $\bar{h}$ is $\bar{f}$-subordinate to $\bar{\theta}$ and $\dim_k \bar{h} = \dim_L h$. Therefore we have

$$\begin{aligned} \text{l.gl.dim}\, \mathfrak{g}'(f)_T &= \text{l.gl.dim}\, \mathfrak{g}''(f'') = \text{l.gl.dim}\, \mathfrak{g}''_L(f''_L) = \dim_L h \\ &= \dim_k \bar{h} = \text{l.gl.dim}\, \bar{h}(\bar{f}) \leqslant \text{l.gl.dim}\, \bar{\mathfrak{g}}(\bar{f}) = \text{l.gl.dim}\, \mathfrak{g}'(f)/(x - \lambda). \end{aligned}$$

Since $\text{l.gl.dim}\, \mathfrak{g}'(f)/(x - \lambda) \leqslant \dim \bar{\mathfrak{g}} < \infty$ and $x - \lambda$ is an element of the centre of $\mathfrak{g}'(f)$ which is neither a unit nor a divisor of zero, by Kaplansky's Theorem [6, p. 173, Theorem 4] we have $\text{l.gl.dim}\,\mathfrak{g}'(f)/(x-\lambda) \leq \text{l.gl.dim}\,\mathfrak{g}'(f) - 1 < \text{l.gl.dim}\,\mathfrak{g}'(f)$. This shows that

$$\text{l.gl.dim}\,\mathfrak{g}(f) = \text{l.gl.dim}\,\mathfrak{g}(f)_T \leq 1 + \text{l.gl.dim}\,\mathfrak{g}'(f)_T < 1 + \text{l.gl.dim}\,\mathfrak{g}'(f).$$

But since $\mathfrak{g}(f)$ is the Ore-extension of $\mathfrak{g}'(f)$ and $\text{l.gl.dim}\,\mathfrak{g}'(f) < \infty$, by our Remark 3 we have $\text{l.gl.dim}\,\mathfrak{g}'(f) \leq \text{l.gl.dim}\,\mathfrak{g}(f)$. Therefore we get $\text{l.gl.dim}\,\mathfrak{g}'(f) = \text{l.gl.dim}\,\mathfrak{g}(f)$.

Thus our claim that $\text{l.gl.dim}\,\mathfrak{g}(f) = \text{l.gl.dim}\,\mathfrak{g}'(f)$ if $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}' \subsetneqq \mathfrak{g}$ is proved.

Since $\dim_k \mathfrak{g}' = n-1$ by our induction hypothesis there exist a subalgebra h of $\mathfrak{g}'$ and an element $\theta' \in \text{Hom}_k(\mathfrak{g}', k)$ such that I) h is f-subordinate to θ', II) $\text{l.gl.dim}\,\mathfrak{g}'(f) = \dim_k h$. Let $\theta = \mathfrak{g} \to k$ be the k-linear map such that $\theta(y) = 0$ and $\theta \mid \mathfrak{g}' = \theta'$. Then it is easy to see that h is f-subordinate to θ also and $\text{l.gl.dim}\,\mathfrak{g}(f) = \text{l.gl.dim}\,\mathfrak{g}'(f) = \dim_k h$.

Case 3: $\mathfrak{g}_1 \subset \mathfrak{g}'$, $\mathfrak{g}_1 \subsetneqq \mathfrak{g}$. Then $\mathfrak{g}_1$ is an ideal of $\mathfrak{g}$ of codim 1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus k \cdot y$ where $y \in \mathfrak{g}$ be such that $[y, x] = x$. Let $f(x, y) = \lambda$. Then $\mathfrak{g}(f)$ is the Ore-extension of $\mathfrak{g}_1(f)$ with respect to the derivation d induced by y, $x - \lambda$ an element of the centre of $\mathfrak{g}_1(f)$ with $d(x-\lambda) = x - \lambda$.

If $\text{l.gl.dim}\,\mathfrak{g}(f) = \text{l.gl.dim}\,\mathfrak{g}_1(f)$ then the proof of the theorem for case 2 shows that there exists an element $\theta \in \text{Hom}_k(\mathfrak{g}, k)$ and a subalgebra h of $\mathfrak{g}$ such that I) h is f-subordinate to θ, II) $\text{l.gl.dim}\,\mathfrak{g}(f) = \dim_k h$.

If $\text{l.gl.dim}\,\mathfrak{g}(f) > \text{l.gl.dim}\,\mathfrak{g}_1(f)$ then the proof of the theorem for case 2 and Proposition 1.1 shows that there exists an element $b \in \mathfrak{g}(f)$ such that $b \notin \mathfrak{a}$ and $x - \lambda \in (\mathfrak{a} : b)$. Since $x - \lambda$ is an element of the centre of $\mathfrak{g}_1(f)$ and $d(x-\lambda) = x - \lambda$ we have $\mathfrak{g}(f)x - \lambda = (x-\lambda)\mathfrak{g}(f)$. Therefore $x - \lambda \in (\mathfrak{a} : b)$ implies that $\mathfrak{g}(f)x - \lambda \in I$ where I is the greatest two sided ideal of $\mathfrak{g}(f)$ contained in $(\mathfrak{a} : b)$. It is easy to see that I is also the greatest two sided ideal of $\mathfrak{g}(f)$ contained in $\mathfrak{a}$ and $I = \text{ann}\, M$ where $M = \mathfrak{g}(f)/\mathfrak{a}$. Then the proof of the theorem for case 1 shows that there exists an element $\theta \in \text{Hom}_k(\mathfrak{g}, k)$ and a subalgebra h of $\mathfrak{g}$ such that I) h is f-subordinate to θ, II) $\text{l.gl.dim}\,\mathfrak{g}(f) = \dim_k h$.

Case 4: $\mathfrak{g}_1 \subsetneqq \mathfrak{g}$, $\mathfrak{g}' \subsetneqq \mathfrak{g}$, $\mathfrak{g}_1 \neq \mathfrak{g}'$. Then $\mathfrak{g}'$ is not an ideal of $\mathfrak{g}$. Therefore by Lemma 2.3 we have $\mathfrak{g}_1 = \mathfrak{g}_1 \cap \mathfrak{g}' \oplus k \cdot w$, $f(w, x) = 1$ and the adjoint action of w on $\mathfrak{g}_1$ is nilpotent. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus ky$ with $[y, x] = x$ and $f(y, x) = 0$. Then $\mathfrak{g}(f)$ is the Ore-extension of $\mathfrak{g}_1(f)$ with respect to the derivation d of $\mathfrak{g}_1(f)$ induced by y.

Let $\mathfrak{g}'' = \mathfrak{g}_1 \cap \mathfrak{g}'$. Let d' be the derivation of $\mathfrak{g}''(f)$ induced by w. Then $\mathfrak{g}_1(f) = \mathfrak{g}''(f)[X, d']$. Since the adjoint action of w on $\mathfrak{g}''$ is nilpotent it follows that d' is a locally nilpotent derivation of $\mathfrak{g}''(f)$. Since x is an element of the centre $\mathfrak{g}''(f)$ and $d'(x) = 1$ it follows from [9, p. 78] that there exists an isomorphism

$$\psi : \mathfrak{g}_1(f) \to \mathfrak{g}''(f)/(x) \otimes_k A_1(k)$$

$$\psi(x_i) = \bar{x}_i \otimes 1 + \overline{d'(x_i)} \otimes X_1 + \overline{d'^2(x_i)} \otimes \frac{X_1^2}{2!} + \cdots \text{ for } 1 \leq i \leq n-2$$

$$\psi(x_{n-1}) = 1 \otimes Y_1$$

where $x = x_1, x_2, \ldots, x_{n-1} = w$ is a k-basis of $\mathfrak{g}_1$ and $A_1(k)$ is the Weyl algebra $k[X_1, Y_1]$ of index 1 with coefficients in k, i.e. $A_1(k)$ is the k-algebra generated by X_1 and Y_1 with the relation $Y_1X_1 - X_1Y_1 = 1$.

Let $\tilde{d}$ be the k-derivation of $\mathfrak{g}''(f)/(x_1)\otimes_k A_1(k)$ induced by d through the isomorphism ψ. Then $\mathfrak{g}(f) = \mathfrak{g}_1(f)[X; d] \simeq \mathfrak{g}''(f)/(x_1)\otimes_k A_1(k)[X; \tilde{d}]$. Now every element b of $g''(f)/(x)\otimes_k A_1(k)$ has the unique expression of the type

$$b = \sum_{i+j\geq 0} a_{ij} \otimes X_1^i Y_1^j, \qquad a_{ij} \in \mathfrak{g}''(f)/(x).$$

We define a k-derivation d_0 on $\mathfrak{g}''(f)/(x)\otimes_k A_1(k)$ as follows

$$d_0(a\otimes 1) = a_{00}\otimes 1 \quad \text{if } \tilde{d}(a\otimes 1) = a_{00}\otimes 1 + \sum_{i+j>0} a_{ij}\otimes X_1^i Y_1^j$$

$$d_0(1\otimes X_1) = d_0(1\otimes Y_1) = 0.$$

Then from [7, Lemma 2.15] it follows that there exists an element g of $\mathfrak{g}''(f)/(x)\otimes_k A_1(k)$ such that $\tilde{d}(b) - d_0(b) = bg - gb$ for all $b \in \mathfrak{g}''(f)/(x)\otimes_k A_1(k)$. Therefore we have

$$\begin{aligned}\mathfrak{g}(f) &\simeq \mathfrak{g}''(f)/(x)\otimes_k A_1(k)[X, \tilde{d}] \simeq \mathfrak{g}''(f)/(x)\otimes_k A_1(k)[X; d_0]\\ &\simeq \mathfrak{g}''(f)/(x)[X; d_0]\otimes_k A_1(k).\end{aligned}$$

Let $a \in \mathfrak{g}''(f)$ then

$$\bar{a}\otimes 1 = \psi\left(a - d'(a)x + \frac{d'^2(a)x^2}{2!} - \frac{d'^3(a)x^3}{3!} + \cdots\right).$$

Therefore

$$\begin{aligned}\tilde{d}(\bar{a}\otimes 1) &= \psi(d(a) - d(d'(a)x - d'(a)x) + \frac{d(d'^2(a))x^2}{2!} + d'^2(a)x^2 + \cdots)\\ &= \overline{d(a)}\otimes 1 + \text{terms of the type } \sum_{i+j>0} a_{ij}\otimes X_1^i Y_1^j.\end{aligned}$$

This shows that $d_0(\bar{a}\otimes 1) = \overline{d(a)}\otimes 1$.

It is easy to see that $\mathfrak{g}''$ is an ideal of $\mathfrak{g}$. Let $\bar{\mathfrak{g}} = \mathfrak{g}''\oplus k\cdot y$. Then $\bar{\mathfrak{g}}(f)$ is the Ore-extension of $\mathfrak{g}''(f)$ with respect to the derivation $\bar{d}$ induced by y. Since $[y, x] = x$, $f(y, x) = 0$, f induces 2-cocycle $\tilde{f}$ on $\tilde{\mathfrak{g}} = \bar{\mathfrak{g}}/(x)$ such that $\tilde{\mathfrak{g}}(\tilde{f}) \simeq \bar{\mathfrak{g}}(f)/(x) \simeq \mathfrak{g}''(f)/(x)[X; d_0]$. Let η be an abelian Lie algebra of dim 2 over k generated by X_1, Y_1. Let f' be a 2-cocycle on η defined by $f'(X_1, Y_1) = 1$. Let $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}}\oplus\eta$, $\hat{f} = \tilde{f}\oplus f'$. Then one can see that $\hat{\mathfrak{g}}$ is a solvable Lie algebra over k, $\hat{f}$ a 2-cocycle on the 'standard complex' for $\hat{\mathfrak{g}}$ such that $\tilde{\mathfrak{g}}(\tilde{f})\otimes_k A_1(k) = \hat{\mathfrak{g}}(\hat{f})$. Since X_1 is an element of the centre of $\hat{\mathfrak{g}}$ and $\hat{f}(u, X_1) = 0$ for all $u \in \tilde{\mathfrak{g}}$, the proof of the theorem for case 2 shows that l.gl.dim $\hat{\mathfrak{g}}(\hat{f}) = 1 +$ l.gl.dim $\tilde{\mathfrak{g}}(\tilde{f})$. Therefore we have

$$\begin{aligned}\text{l.gl.dim}\,\mathfrak{g}(f) &= \text{l.gl.dim}\,\mathfrak{g}''(f)/(x_1)\otimes_k A_1(k)[X; \tilde{d}]\\ &= \text{l.gl.dim}\,\mathfrak{g}''(f)/(x_1)[X; d_0]\otimes_k A_1(k)\\ &= \text{l.gl.dim}\,\tilde{\mathfrak{g}}(\tilde{f})\otimes_k A_1(k) = \text{l.gl.dim}\,\hat{\mathfrak{g}}(\hat{f}) = 1 + \text{l.gl.dim}\,\tilde{\mathfrak{g}}(\tilde{f}).\end{aligned}$$

Since $\dim_k \tilde{g} = n - 2$ by induction hypothesis there exist an element $\tilde{\theta} \in \operatorname{Hom}_k(\tilde{g}, k)$ and a subalgebra $\tilde{h}$ of $\tilde{g}$ such that I) $\tilde{h}$ is $\tilde{f}$-subordinate to $\tilde{\theta}$, II) $\operatorname{l.gl.dim} \tilde{g}(\tilde{f}) = \dim_k \tilde{h}$.

Let h be a subalgebra of g such that $x \in h$ and $h/(x) = \tilde{h}$. Let $\theta : g \to k$ be the k-linear map such that $\theta(x) = \theta(w) = 0$ and $\bar{\theta} \mid \tilde{g}/(x) = \tilde{\theta}$ where $\bar{\theta} : g/(x) \to k$ is the map induced by θ. Then it is easy to see that I) h is f-subordinate to θ, II) $\operatorname{l.gl.dim} g(f) = 1 + \operatorname{l.gl.dim} \tilde{g}(\tilde{f}) = 1 + \dim_k \tilde{h} = \dim_k h$.

Thus the theorem is proved for $\dim_k g = n$.

This completes the proof of Theorem 2.1.

Now we state the main theorem.

Theorem 2.6. *Let k be an algebraically closed field of* char. 0. *Let g be a finite dimensional solvable Lie algebra over k. Let f be a k-valued 2-cocycle on the 'standard complex' for g. Let $(h_j)_{j\in J}$ be the family of subalgebras of g for which the restriction of f to $h_j \times h_j$ is a coboundary. Then* $\operatorname{l.gl.dim} g(f) = \sup_{j\in J} \dim_k h_j$.

Proof. By Remark 4 it follows that if h is a subalgebra of g such that the restriction of f to $h \times h$ is a coboundary then $\dim_k h = \operatorname{l.gl.dim} h(f) \leq \operatorname{l.gl.dim} g(f)$. Therefore we always have $\operatorname{l.gl.dim} g(f) \geq \sup_{j\in J} \dim_k h_j$.

Theorem 2.1 shows that there exist a subalgebra h of g and an element $\theta \in \operatorname{Hom}_k(g, k)$ such that h is f-subordinate to θ and $\operatorname{l.gl.dim} g(f) = \dim_k h$. But h is f-subordinate to θ implies that the restriction of f to $h \times h$ is a coboundary. Therefore $\operatorname{l.gl.dim} g(f) = \dim_k h \leq \sup_{j\in J} \dim_k h_j$. Hence the equality.

This completes the proof of Theorem 2.6.

Remark 6. The following example shows that Theorem 2.6 is no longer valid if we drop the assumption that k is algebraically closed.

Example. Let g be the solvable Lie algebra over the field $\mathbf{R}$ of real numbers with a basis (x, y, z) such that $[x, y] = z$, $[x, z] = -y$, $[y, z] = 0$. Let f be a $\mathbf{R}$-valued 2-cocycle on the 'standard complex' for g such that $f(y, z) = 1$, $f(x, y) = f(x, z) = 0$. Then it is easy to prove that $\operatorname{l.gl.dim} g(f) = 2$. Let h be a subalgebra of g of dim 2 with a basis (e_1, e_2). Let $e_1 = \alpha_1 x + \beta_1 y + r_1 z$, $e_2 = \alpha_2 x + \beta_2 y + r_2 z$. Then if $f \mid h \times h$ is a coboundary we get either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Assume $\alpha_1 \neq 0$. If $[e_1, e_2] = 0$, then $f \mid h \times h$ is a coboundary implies that $f(e_1, e_2) = 0$ and this in turn will imply that e_1 and e_2 are linearly dependant which is contradiction. Therefore, $[e_1, e_2] \neq 0$. But $[e_1, e_2] = \beta_3 y + r_3 z$. Assume $\beta_3 \neq 0$ and let $e_2' = y + rz$, $r = \beta_3^{-1} r_3$. Then $e_2' \in h$. But then $[e_1, e_2'] = \alpha_1 z - \alpha_1 r y \in h$ and since $\alpha_1 \neq 0$ this will imply that $e_3' = z - ry \in h$. Since $[e_2', e_3'] = 0$ we get $f(e_2', e_3') = 0$.

But $f(e_2', e_3') = 1 + r^2$. Since $1, r \in \mathbf{R}$ we get a contradiction showing that there does not exist a subalgebra h of g of dim 2 such that $f \mid h \times h$ is a coboundary.

But for a finite dimensional solvable Lie algebra over an arbitrary field of char. 0 we have the following

Theorem 2.7. *Let K be a field of* char. 0. *Let Ω be its algebraic closure. Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over K. Let f be a K-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$. Then* $\mathrm{l.gl.dim}\, \mathfrak{g}(f) = \mathrm{l.gl.dim}\, \mathfrak{g}_\Omega(f_\Omega)$ *where* $\mathfrak{g}_\Omega = \Omega \otimes_K \mathfrak{g}$, $f_\Omega = I_\Omega \otimes_K f$.

Proof. Since Ω is algebraically closed field of char. 0, by Theorem 2.6 we get a subalgebra h' of $\mathfrak{g}_\Omega$ such that the restriction of f_Ω to $h' \times h'$ is a coboundary and $\mathrm{l.gl.dim}\, \mathfrak{g}_\Omega(f_\Omega) = \dim_\Omega h'$. Since $\mathfrak{g}$ is finite dimensional over K and f is completely determined by its values on a K-basis of $\mathfrak{g} \times \mathfrak{g}$, it follows that there exists a finite extension L of K and a subalgebra h of $\mathfrak{g}_L$ such that the restriction of f_L to $h \times h$ is a coboundary and $\Omega \otimes_L h = h'$ where $\mathfrak{g}_L = L \otimes_K \mathfrak{g}$ and $f_L = I_L \otimes_K f$. Therefore we have

$$\mathrm{l.gl.dim}\, \mathfrak{g}_\Omega(f_\Omega) = \dim_\Omega h' = \dim_L h \leqslant \mathrm{l.gl.dim}\, \mathfrak{g}_L(f_L).$$

But $\mathfrak{g}_L(f_L) \simeq L \otimes_K \mathfrak{g}(f)$ and L is a finite separable extension of K. Therefore by [4, p. 74] we have $\mathrm{l.gl.dim}\, \mathfrak{g}_L(f_L) = \mathrm{l.gl.dim}\, \mathfrak{g}(f)$. Therefore $\mathrm{l.gl.dim}\, \mathfrak{g}_\Omega(f_\Omega) \leqslant \mathrm{l.gl.dim}\, \mathfrak{g}(f)$. But $\mathfrak{g}_\Omega(f_\Omega)$ is $\mathfrak{g}(f)$-free as a left and as a right module and contains $\mathfrak{g}(f)$ as a direct summand. Therefore by [8, Lemma 1] we have $\mathrm{l.gl.dim}\, \mathfrak{g}(f) \leqslant \mathrm{l.gl.dim}\, \mathfrak{g}_\Omega(f_\Omega)$.

Hence the equality.

This completes the proof of Theorem 2.7.

Remark 6. The following example shows that Theorem 2.6 is not true if $\mathfrak{g}$ is not solvable.

Example. Let K be an algebraically closed field of char. 0. Let $\mathfrak{g}$ be a Lie algebra over K of dim 5 such that its radical is abelian and of dim 2. Let $\mathfrak{g} = Z \oplus S$ be the Levi decomposition of $\mathfrak{g}$ where Z is the radical of $\mathfrak{g}$ and S is a semisimple sub-algebra of $\mathfrak{g}$. Let f be a K-valued 2-cocycle on the 'standard complex' for $\mathfrak{g}$ such that $f \mid \mathfrak{g} \times S = 0$ and $f \mid Z \times Z \neq 0$. Since $\dim Z = 2$, $f \mid Z \times Z \neq 0$ implies $Z(f) \simeq A_1(K)$. Let us assume that Z is a simple S-module. Since every element of S defines a Lie-algebra derivation of Z there exists a Lie-algebra homomorphism $\psi : S \to \mathrm{Der}_K(Z(f))$. Since $Z(f) \simeq A_1(k)$ and S is semisimple it follows that $\psi(S) \subset D$ where D denotes the Lie-algebra of inner derivations of $Z(f)$. From this it follows that $\mathrm{l.gl.dim}\, \mathfrak{g}(f) = \mathrm{l.gl.dim}\, Z(f) + \dim_K S = 1 + 3 = 4$.

Let h be a subalgebra of $\mathfrak{g}$ of dim 4. Suppose $f \mid h \times h$ is a coboundary it follows that $\dim_K h \cap Z = 1$ and $h + Z = \mathfrak{g}$. From this it follows that $h = h \cap Z \oplus S'$ where S' is a semisimple sub-Lie algebra of $\mathfrak{g}$ such that $S' \simeq \mathfrak{g}/Z$. Therefore $\mathfrak{g} = Z \oplus S'$ is another Levi decomposition of $\mathfrak{g}$. Let $\phi : S \to S'$ be a Lie-algebra isomorphism defined as follows $\phi(s) = s'$ if $s = z + s'$, $z \in Z$, $s' \in S'$. Then since Z is abelian, we have for $z \in Z$, $s \in S$, $[s, z] = [\phi(s), Z]$. Therefore Z is a simple S'-module. But $h \cap Z$ is an ideal of h and $S' \subset h$. Therefore $h \cap Z$ is a S'-module. Since $\dim_K h \cap Z = 1$ we get a proper non zero S'-submodule of Z which is a contradiction.

This shows that if h is a subalgebra of $\mathfrak{g}$ such that $f \mid h \times h$ is a coboundary then $\dim_K h < 4 = \text{l.gl.dim}\, \mathfrak{g}(f)$.

We refer to [11] for the definition of Krull dimension of a module over (not necessarily commutative) ring. For a ring A let *l.Kr.dim A* denote the Krull dimension of A when A is regarded as a left module over A.

We state a result which has been proved by Roos J.E. in [12].

Theorem of Roos. *Let A be a filtered noetherian ring whose associated graded ring is a commutative regular noetherian ring. Then* l.Kr.dim $A \leq$ l.gl.dim A.

As a consequence of the above theorem and Theorem 2.6 we get the following corollary.

Corollary 2.8. *Let* $\mathfrak{g}$, f, K *be as given in Theorem 2.6. Then* l.gl.dim $\mathfrak{g}(f)$ = l.Kr.dim $\mathfrak{g}(f)$.

Proof. By Theorem of Roos we have l.Kr.dim $\mathfrak{g}(f) \leq$ l.gl.dim $\mathfrak{g}(f)$.

By Theorem 2.6 we get a subalgebra h of $\mathfrak{g}$ such that the restriction of f to $h \times h$ is a coboundary and l.gl.dim $\mathfrak{g}(f) = \dim_K h$. Since $h(f)$ is isomorphic to the usual enveloping algebra of the solvable Lie algebra h, by [11, P. 713, (9)] we have $\dim_K h \leq$ l.Kr.dim $h(f)$.

But since $\mathfrak{g}(f)$ is $h(f)$-free as a right and as a left module and contains $h(f)$ as a direct summand, it is easy to see that l.Kr.dim $h(f) \leq$ l.Kr.dim $\mathfrak{g}(f)$. Therefore we have l.gl.dim $\mathfrak{g}(f) = \dim_K h \leq$ l.Kr.dim $h(f) \leq$ l.Kr.dim $\mathfrak{g}(f)$.

Hence the equality.

This completes the proof of Corollary 2.8.

References

[1] M. Auslander, On the Dimension of Modules and algebras (III). Global dimension, Nagoya Math. J. 9 (1955) 67–77.

[2] S.M. Bhatwadekar, On the Global Dimension of some Filtered Algebras, to appear in the J. of the London Math. Soc.

[3] H. Cartan and S. Eilenberg, Homological Algebra (Princeton University Press, 1956).

[4] S. Eilenberg, A. Rosenberg and D. Zelinsky, On the Dimension of Modules and Algebras, VIII, Nagoya Math. J. 12 (1957) 71–93.

[5] K.R. Goodearl, Global dimension of differential operator rings, Proc. Amer. Math. Soc. 45 (1974) 315–322.

[6] Irving Kaplansky, Fields and Rings, Chicago Lectures in Mathematics (Chicago University Press, 1969).

[7] J.C. McConnell, Representations of Solvable Lie algebras and the Gelfand–Kirillov Conjucture, to appear.
[8] M.P. Murthy and R. Sridharan, On the Global Dimension of some Algebras, Math. Zeitschr. 81 (1963) 108–111.
[9] Y. Nouaze' and P. Gabriel, Algebras de Lie Nilpotentes, J. of Algebra 6 (1967) 77–79.
[10] D. Quillen, On the Endomorphism Rings of a Simple Module over an Enveloping Algebra, Proc. Amer. Math. Soc. 21 (1969) 171–172.
[11] R. Rentschler and P. Gabriel, Sur la dimension des anneaux et ensembles Ordonnes, C.R.Acad. Sci. Paris 265, Ser. A (1967) 712–715.
[12] J.E. Ross, The Weyl algebras are Gorenstein rings, Generalisations and applications, to appear.
[13] Amit Roy, A note on filtered rings, Archiv der Mathematik 26 (1965) 421–427.
[14] Sridharan, Filtered Algebras and representations of Lie algebras, Trans. Amer. Math. Soc. 100 (1961) 530–550.
[15] B.O. Stenstörm, Ring and Modules of Quotients (Springer, Berlin, 1971).