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Existence of Solutions of Abstract Nonlinear Second-Order Neutral Functional Integrodifferential Equations

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Abstract—Sufficient conditions for existence of mild solutions for abstract second-order neutral functional integrodifferential equations are established by using the theory of strongly continuous cosine families of operators and the Schaefer theorem.

Keywords—Neutral functional integrodifferential equation, Strongly continuous cosine operators, Schaefer fixed-point theorem.

1. INTRODUCTION

In this paper, we are concerned with the abstract Cauchy problem for the nonlinear second-order neutral functional integrodifferential equation

$$\frac{d}{dt}\left[x'(t) - g(t, x_t)\right] = Ax(t) + \int_0^t F\left(t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau))\, d\tau\right) ds, \quad t \in (0, T), \qquad x_0 = \phi, \quad x'(0) = y_0 \in X, \tag{1}$$

where A is the infinitesimal generator of the strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators in a Banach space X, $f : [0,T] \times [0,T] \times C \times X \to X$, $F : [0,T] \times [0,T] \times C \times X \times X \to X$, and $g : [0,T] \times C \to X$ are given functions and $\phi \in C = C([-r, 0], X)$.

Several papers have appeared for the existence of solutions of first-order neutral functional differential equations in Banach spaces [1–4]. There seems to be little known about the solvability of the nonlinear second-order neutral equations in abstract spaces. Recently, Balachandran

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and Marshal Anthoni [5,6] studied the existence problem for both Volterra integrodifferential equations and neutral differential equations in Banach spaces. Ntouyas [7] and Ntouyas and Tsamatos [8] established the existence of solutions for semilinear second-order delay differential equations. In many cases, it is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order equations. A useful tool for the study of abstract second-order differential equations is the theory of strongly continuous cosine families. We refer to the papers [9,10] for a detailed discussion of cosine family theory. Second-order equations which appear in a variety of physical problems can be found in [11,12]. The purpose of this paper is to study the existence of mild solutions for second-order neutral functional integrodifferential equations in Banach spaces using the Schaefer fixed-point theorem.

2. PRELIMINARIES

Let X be a Banach space with norm $|\cdot|$ and let $T > 0$ be a real number. By C we denote the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the sup-norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

Also for $x \in C([-r,T],X)$, we have $x_t \in C$ for $t \in J = [0,T]$, and $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

DEFINITION 2.1. *(See [9].) A one-parameter family $C(t)$, $t \in R$, of bounded linear operators in the Banach space X is called a strongly continuous cosine family iff*

(i) *$C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s,t \in R$;*
(ii) *$C(0) = I$;*
(iii) *$C(t)x$ is continuous in t on R for each fixed $x \in X$.*

Define the associated sine family $S(t)$, $t \in R$, by

$$S(t)x = \int_0^t C(s)x\,ds, \qquad x \in X, \quad t \in R.$$

Assume the following conditions on A.

(H_1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from X into itself, and the adjoint operator A^* is densely defined; i.e., $\overline{D(A^*)} = X^*$ (see [13]).

The infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, is the operator $A : X \to X$ defined by

$$Ax = \left.\frac{d^2}{dt^2}C(t)x\right|_{t=0}, \qquad x \in D(A),$$

where $D(A) = \{x \in X : C(t)x$ is twice continuously differentiable in $t\}$.

Define $E = \{x \in X : C(t)x$ is once continuously differentiable in $t\}$.

To establish our main theorem, we need the following lemmas.

LEMMA 2.1. *(See [9].) Let (H_1) hold. Then*

(i) *there exist constants $M \geq 1$ and $\omega \geq 0$ such that $|C(t)| \leq Me^{\omega|t|}$ and $|S(t) - S(t^*)| \leq M|\int_t^{t^*} e^{\omega|s|}\,ds|$ for $t, t^* \in R$;*
(ii) *$S(t)X \subset E$ and $S(t)E \subset D(A)$ for $t \in R$;*
(iii) *$\frac{d}{dt}C(t)x = AS(t)x$ for $x \in E$ and $t \in R$;*
(iv) *$\frac{d^2}{dt^2}C(t)x = AC(t)x$ for $x \in D(A)$ and $t \in R$.*

LEMMA 2.2. *(See [9].) Let (H_1) hold, let $v : R \to X$ such that v is continuously differentiable, and let $q(t) = \int_0^t S(t-s)v(s)\,ds$. Then q is twice continuously differentiable and for $t \in R$, $q(t) \in D(A)$,*

$$q'(t) = \int_0^t C(t-s)v(s)\,ds \qquad \text{and} \qquad q''(t) = Aq(t) + v(t).$$

SCHAEFER'S FIXED-POINT THEOREM. *(See [14].) Let S be a normed linear space. Let $\Phi : S \to S$ be a completely continuous operator; that is, it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\xi(\Phi) = \{x \in S : x = \lambda\Phi x \text{ for some } 0 < \lambda < 1\}.$$

Then either $\xi(F)$ is unbounded or Φ has a fixed point.

DEFINITION 2.2. *A continuous function $x : [-r, T] \to X$, $T > 0$, is called a mild solution of problem (1) if $x_0 = \phi$, and if it satisfies the integral equation*

$$x(t) = C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,x_s)\,ds + \int_0^t S(t-s)\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds, \qquad t \in J.$$

We make the following assumptions.

- (H_2) *$C(t)$, $t > 0$ is compact.*
- (H_3) *$g : J \times C \to X$ is completely continuous and for any bounded set K in $C([-r,T],X)$, the set $\{t \to g(t,x_t) : x \in K\}$ is equicontinuous in $C([0,T]),X)$.*
- (H_4) *There exist constants c_1 and c_2 such that*

$$|g(t,\phi)| \le c_1\|\phi\| + c_2, \qquad t \in J, \quad \phi \in C.$$

- (H_5) *The function $f(t,s,.,.) : C \times X \to X$ is continuous for each $t,s \in J$.*
- (H_6) *The function $f(.,.,x,y) : J \times J \to X$ is strongly measurable for each $x \in C$ and $y \in X$.*
- (H_7) *There exists a continuous function $h : J \times J \to [0,\infty)$ such that*

$$|f(t,s,x,y)| \le h(t,s)\Omega_0(\|x\| + |y|), \qquad t,s \in J, \quad x \in C, \quad \text{and} \quad y \in X,$$

  where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.
- (H_8) *The function $F(t,s,.,.,.) : C \times X \times X \to X$ is continuous for each $t,s \in J$.*
- (H_9) *The function $F(.,.,x,y,z) : J \to X$ is strongly measurable for each $x \in C$, $y \in X$, and $z \in X$.*
- (H_{10}) *For every positive constant k, there exists $\alpha_k \in L^1(J)$ such that*

$$\sup_{\|x\|,|y|,|z|\le k}\left|\int_0^t F(t,s,x,y,z)\,ds\right| \le \alpha_k(t), \qquad \text{for } t \in J \text{ a.e.}$$

- (H_{11}) *There exists a continuous function $l : J \times J \to [0,\infty)$ such that*

$$|F(t,s,x,y,z)| \le l(t,s)\Omega(\|x\| + |y| + |z|), \qquad t \in J, \quad x \in C, \quad y,z \in X,$$

  where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function and

$$\int_0^T m(s)\,ds < \int_c^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)} < \infty,$$

  where

$$m(t) = \max\left\{c_1\left[Mc_1 + M + M^*\right], M(c_1T + T + 1)\int_0^t l(t,s)\,ds, h(t,t)\right\},$$
$$M = \sup\{|C(t)| : t \in J\}, \qquad M^* = \sup\{|AS(t)| : t \in J\},$$
$$c = (M + M^* + c_1)\|\phi\| + (1+T)M\{|y_0| + c_1\|\phi\| + c_2\} + (M + M^*)c_2T + c_2.$$

3. MAIN RESULT

THEOREM 3.1. *Suppose* (H_1)–(H_{11}) *hold. Then the IVP (1) has at least one mild solution on* $[-r, T]$.

PROOF. Consider the space $Z = C([-r,T],X) \cap C^1(J,X)$ with the norm

$$\|x\|^* = \max\{\|x\|_r, \|x'\|_0\},$$

where

$$\|x\|_r = \sup\{|x(t)| : -r \le t \le T\}, \qquad \|x'\|_0 = \sup\{|x'(t)| : 0 \le t \le T\}.$$

To prove the existence of a mild solution of the IVP (1), we have to apply the Schaefer fixed-point theorem for the nonlinear operator equation

$$x(t) = \lambda \Phi x(t), \qquad 0 < \lambda < 1$$

where the operator $\Phi : Z \to Z$ is defined by

$$\begin{aligned} \Phi x(t) &= C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s) g(s, x_s)\, ds \\ &+ \int_0^t S(t-s) \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta))\, d\theta\right) d\tau\, ds, \qquad t \in J. \end{aligned} \tag{2}$$

Then we have, for $t \in J$,

$$\begin{aligned} |x(t)| &\le M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\} + Mc_1 \int_0^t \|x_s\|\, ds \\ &+ MT \int_0^t \int_0^s l(s,\tau)\Omega\left(\|x_\tau\| + |x'(\tau)| + \int_0^\tau h(\tau,\theta)\Omega_0\left(\|x_\theta\| + |x'(\theta)|\right) d\theta\right) d\tau\, ds. \end{aligned}$$

Consider the function q defined by

$$q(t) = \sup\{|x(s)| : -r \le s \le t\}, \qquad t \in J.$$

Let $t^* \in [-r, t]$ be such that $q(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have, for $t \in J$,

$$\begin{aligned} q(t) &\le M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\} + Mc_1 \int_0^t q(s)\, ds \\ &+ MT \int_0^t \int_0^s l(s,\tau)\Omega\left(q(\tau) + |x'(\tau)| + \int_0^\tau h(\tau,\theta)\Omega_0\left(q(\theta) + |x'(\theta)|\right) d\theta\right) d\tau\, ds. \end{aligned}$$

If $t^* \in [-r, 0]$, then $q(t) = \|\phi\|$ and the previous inequality holds since $M \ge 1$.

Denoting by $v(t)$ the right-hand side of the above inequality, we have

$$q(t) \le v(t), \qquad t \in J, \qquad v(0) = M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\}$$

and for $t \in J$,

$$\begin{aligned} v'(t) &= Mc_1 q(t) + MT \int_0^t l(t,s)\Omega\left(q(s) + |x'(s)| + \int_0^s h(s,\tau)\Omega_0\left(q(\tau) + |x'(\tau)|\right) d\tau\right) ds \\ &\le Mc_1 v(t) + MT \int_0^t l(t,s)\Omega\left(v(s) + |x'(s)| + \int_0^s h(s,\tau)\Omega_0\left(v(\tau) + |x'(\tau)|\right) d\tau\right) ds. \end{aligned}$$

By

$$x'(t) = \lambda AS(t)\phi(0) + \lambda C(t)[y_0 - g(0,\phi)] + \lambda g(t,x_t) + \lambda \int_0^t AS(t-s)g(s,x_s)\,ds$$
$$+ \lambda \int_0^t C(t-s) \int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)\,d\tau\,ds, \qquad t \in J,$$

we obtain

$$|x'(t)| \le M^*\|\phi\| + M\{|y_0| + c_1\|\phi\| + c_2\} + c_1\|x_t\| + c_2 + M^*\left\{c_2T + c_1\int_0^t \|x_s\|\,ds\right\}$$
$$+ M\int_0^t\int_0^s l(s,\tau)\Omega\left(q(\tau) + |x'(\tau)| + \int_0^\tau h(\tau.\theta)\Omega_0\left(q(\theta) + |x'(\theta)|\right)\,d\theta\right)\,d\tau\,ds.$$

Denoting by $r(t)$ the right-hand side of the above inequality, we have for $t \in J$,

$$|x'(t)| \le r(t),$$
$$r(0) = M^*\|\phi\| + M\{|y_0| + c_1\|\phi\| + c_2\} + c_1\|\phi\| + c_2 + M^*c_2T,$$

and

$$r'(t) \le c_1v'(t) + M^*c_1v(t)$$
$$+ M\int_0^t l(t,s)\Omega\left(v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\,d\tau\right)\,ds$$
$$\le c_1\left\{Mc_1v(t) + MT\int_0^t l(t,s)\Omega\left(v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\,d\tau\right)\,ds\right\}$$
$$+ M^*c_1v(t) + M\int_0^t l(t,s)\Omega\left(v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\,d\tau\right)\,ds.$$

Let $u(t) = v(t) + r(t)$, $t \in J$. Then $u(0) = c$, and

$$u'(t) = v'(t) + r'(t)$$
$$\le c_1\left[Mc_1 + M + M^*\right]v(t)$$
$$+ M(c_1T + T + 1)\int_0^t l(t,s)\Omega\left(v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\,d\tau\right)\,ds$$
$$\le c_1\left[Mc_1 + M + M^*\right]u(t)$$
$$+ M(c_1T + T + 1)\int_0^t l(t,s)\Omega\left(u(s) + \int_0^s h(s,\tau)\Omega_0(u(\tau))\,d\tau\right)\,ds, \qquad t \in J.$$

Let $w(t) = u(t) + \int_0^t h(t,s)\Omega_0(u(s))\,ds$, $t \in J$. Then $w(0) = c$, and for $t \in J$,

$$w'(t) = u'(t) + h(t,t)\Omega_0(u(t))$$
$$\le c_1\left[Mc_1 + M + M^*\right]w(t)$$
$$+ M(c_1T + T + 1)\int_0^t l(t,s)\Omega(w(s))\,ds + h(t,t)\Omega_0(w(t))$$
$$\le c_1[Mc_1 + M + M^*]w(t)$$
$$+ M(c_1T + T + 1)\Omega(w(t))\int_0^t l(t,s)\,ds + h(t,t)\Omega_0(w(t)).$$

This implies

$$\int_{w(0)}^{w(t)} \frac{ds}{s + \Omega(s) + \Omega_0(s)} \le \int_0^T m(s)\,ds < \int_c^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)}, \qquad t \in J.$$

This inequality implies that there is a constant K such that $w(t) \le K$, $t \in J$. Then

$$|x(t)| \le v(t) \le K, \qquad t \in J,$$
$$|x'(t)| \le r(t) \le K, \qquad t \in J,$$

and hence,

$$\|x\|^* = \max\{\|x\|_r, \|x'\|_0\} \le K,$$

where K depends on the constants T, N and on the functions Ω_0 and Ω.

We shall now prove that the operator $\Phi : Z \to Z$ defined by

$$\begin{aligned}(\Phi x)(t) &= \phi(t), \qquad -r \le t \le 0,\\ (\Phi x)(t) &= C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,x_s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds, \qquad t \in J,\end{aligned} \tag{3}$$

is a completely continuous operator.

Let $B_k = \{x \in Z : \|x\|^* \le k\}$ for some $k \ge 1$. We first show that Φ maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \le T$,

$$\begin{aligned}&|(\Phi x)(t_1) - (\Phi x)(t_2)| \\ &\le |[C(t_1) - C(t_2)]\phi(0)| + |[S(t_1) - S(t_2)][y_0 - g(0,\phi)]| \\ &\quad + \left|\int_0^{t_1}[C(t_1-s) - C(t_2-s)]g(s,x_s)\,ds\right| + \left|\int_{t_1}^{t_2} C(t_2-s)g(s,x_s)\,ds\right| \\ &\quad + \left|\int_0^{t_1}[S(t_1-s) - S(t_2-s)]\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds\right| \\ &\quad + \left|\int_{t_1}^{t_2} S(t_2-s)\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds\right| \\ &\le |C(t_1) - C(t_2)|\|\phi\| + |S(t_1) - S(t_2)|\{|y_0| + c_1\|\phi\| + c_2\} \\ &\quad + \int_0^{t_1}|C(t_1-s) - C(t_2-s)|\{c_1\|x_s\| + c_2\}\,ds \\ &\quad + \int_{t_1}^{t_2}|C(t_2-s)|\{c_1\|x_s\| + c_2\}\,ds + \int_0^{t_1}|S(t_1-s) - S(t_2-s)|\alpha_k(s)\,ds \\ &\quad + \int_{t_1}^{t_2}|S(t_2-s)|\alpha_k(s)\,ds,\end{aligned} \tag{4}$$

and similarly,

$$\begin{aligned}&|(\Phi x)'(t_1) - (\Phi x)'(t_2)| \\ &\le |A(S(t_1) - S(t_2))\phi(0)| + |[C(t_1) - C(t_2)][y_0 - g(0,\phi)]| \\ &\quad + |g(t_1,x_{t_1}) - g(t_2,x_{t_2})| + \left|\int_0^{t_1} A(S(t_1-s) - S(t_2-s))g(s,x_s)\,ds\right| \\ &\quad + \left|\int_{t_1}^{t_2} AS(t_2-s)g(s,x_s)\,ds\right| + \left|\int_0^{t_1}[C(t_1-s) - C(t_2-s)]\right. \\ &\quad \left.\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds\right| \\ &\quad + \left|\int_{t_1}^{t_2} C(t_2-s)\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\,ds\right|\end{aligned} \tag{5}$$

$$\begin{aligned}
&\leq |A(S(t_1) - S(t_2))| \|\phi\| + |[C(t_1) - C(t_2)]| \{|y_0| + c_1\|\phi\| + c_2\} \\
&\quad + |g(t_1, x_{t_1}) - g(t_2, x_{t_2})| + \int_0^{t_1} |A(S(t_1 - s) - S(t_2 - s))| \{c_1\|x_s\| + c_2\} \, ds \\
&\quad + \int_{t_1}^{t_2} |AS(t_2 - s)| \{c_1\|x_s\| + c_2\} \, ds + \int_0^{t_1} |C(t_1 - s) - C(t_2 - s)| \alpha_k(s) \, ds \\
&\quad + \int_{t_1}^{t_2} |C(t_2 - s)| \alpha_k(s) \, ds. \qquad \text{((5) cont.)}
\end{aligned}$$

The right-hand sides of (4) and (5) are independent of $y \in B_k$ and tend to zero as $t_2 - t_1 \to 0$, since $C(t), S(t)$ are uniformly continuous for $t \in J$ and the compactness of $C(t)$, $S(t)$ for $t > 0$ imply the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$ (see [10]).

Thus, Φ maps B_k into an equicontinuous family of functions. It is easy to see that the family ΦB_k is uniformly bounded.

Next we show $\overline{\Phi B_k}$ is compact. Since we have shown ΦB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that Φ maps B_k into a precompact set in X.

Let $0 < t \leq T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}
(\Phi_\epsilon x)(t) &= C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^{t-\epsilon} C(t - s) g(s, x_s) \, ds \\
&\quad + \int_0^{t-\epsilon} S(t - s) \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta\right) d\tau \, ds, \qquad t \in J.
\end{aligned}$$

Since $C(t)$, $S(t)$ are compact operators, the set $Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$\begin{aligned}
|(\Phi x)(t) - (\Phi_\epsilon x)(t)| &\leq \int_{t-\epsilon}^t |C(t - s) g(s, x_s)| \, ds \\
&\quad + \int_{t-\epsilon}^t \left| S(t - s) \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta\right) d\tau \right| ds \\
&\leq \int_{t-\epsilon}^t |C(t - s)| \{c_1\|x_s\| + c_2\} \, ds \\
&\quad + \int_{t-\epsilon}^t |S(t - s)| \alpha_k(s) \, ds \to 0, \qquad \text{as } \epsilon \to 0,
\end{aligned}$$

and

$$\begin{aligned}
|(\Phi x)'(t) - (\Phi_\epsilon x)'(t)| &\leq |g(t, x_t) - C(\epsilon) g(t - \epsilon, x_{t-\epsilon})| + \int_{t-\epsilon}^t |AS(t - s) g(s, x_s)| \, ds \\
&\quad + \int_{t-\epsilon}^t \left| C(t - s) \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta\right) d\tau \right| ds \\
&\leq |g(t, x_t) - C(\epsilon) g(t - \epsilon, x_{t-\epsilon})| + \int_{t-\epsilon}^t |AS(t - s)| \{c_1\|x_s\| + c_2\} \, ds \\
&\quad + \int_{t-\epsilon}^t |C(t - s)| \alpha_k(s) \, ds \to 0, \qquad \text{as } \epsilon \to 0.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Phi x)(t) : x \in B_k\}$. Hence, the set $\{(\Phi x)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $\Phi : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer q such that $|x_n(t)| \leq q$, $|x_n'(t)| \leq q$ for all n and $t \in J$, so $|x(t)| \leq q$, $|x'(t)| \leq q$, and $x, x' \in Z$. By (H_5) and (H_8),

$$\int_0^t F\left(t, s, x_{ns}, x_n{}'(s), \int_0^s f(s, \tau, x_{n\tau}, x_n{}'(\tau))\, d\tau\right) ds \longrightarrow \int_0^t F\left(t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau))\, d\tau\right) ds$$

for each $t \in J$ and since

$$\left|\int_0^t F\left(t, s, x_{ns}, x_n{}'(s), \int_0^s f(s, \tau, x_{n\tau}, x_n{}'(\tau))\, d\tau\right) ds - \int_0^t F\left(t, s, x_s, x'(s), \int_0^s f\left(s, \tau, x_\tau, x'(\tau)\right)\, d\tau\right) ds\right| \le 2\alpha_q(t),$$

we have by the dominated convergence theorem,

$$\begin{aligned}
\|\Phi x_n - \Phi x\| &= \sup_{t\in J}\left|\int_0^t C(t-s)[g(s, x_{ns}) - g(s, x_s)]\, ds\right. \\
&\quad + \int_0^t S(t-s)\left[\int_0^s F\left(s, \tau, x_{n\tau}, x_n{}'(\tau), \int_0^\tau f\left(\tau, \theta, x_{n\theta}, x_n{}'(\theta)\right) d\theta\right) d\tau\right. \\
&\quad \left.\left. - \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f\left(\tau, \theta, x_\theta, x'(\theta)\right) d\theta\right) d\tau\right] ds\right| \\
&\le \int_0^T |C(t-s)[g(s, x_{ns}) - g(s, x_s)]|\, ds \\
&\quad + \int_0^T \left|S(t-s)\left[\int_0^s F\left(s, \tau, x_{n\tau}, x_n{}'(\tau), \int_0^\tau f\left(\tau, \theta, x_{n\theta}, x_n{}'(\theta)\right) d\theta\right) d\tau\right.\right. \\
&\quad \left.\left. - \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f\left(\tau, \theta, x_\theta, x'(\theta)\right) d\theta\right) d\tau\right] ds\right| \to 0, \qquad \text{as } n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Phi x_n)' - (\Phi x)'\| \\
&= \sup_{t\in J}\left|[g(t, x_{nt}) - g(t, x_t)] + \int_0^t AS(t-s)[g(s, x_{ns}) - g(s, x_s)]\, ds\right. \\
&\quad + \int_0^t C(t-s)\left[\int_0^s F\left(s, \tau, x_{n\tau}, x_n{}'(\tau), \int_0^\tau f\left(\tau, \theta, x_{n\theta}, x_n{}'(\theta)\right) d\theta\right) d\tau\right. \\
&\quad \left.\left. - \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f\left(\tau, \theta, x_\theta, x'(\theta)\right) d\theta\right) d\tau\right] ds\right| \\
&\le |g(t, x_{nt}) - g(t, x_t)|\, ds + \int_0^t |AS(t-s)[g(s, x_{ns}) - g(s, x_s)]|\, ds \\
&\quad + \int_0^T \left|C(t-s)\left[\int_0^s F\left(s, \tau, x_{n\tau}, x_n{}'(\tau), \int_0^\tau f(\tau, \theta, x_{n\theta}, x_n{}'(\theta))\, d\theta\right) d\tau\right.\right. \\
&\quad \left.\left. - \int_0^s F\left(s, \tau, x_\tau, x'(\tau), \int_0^\tau f\left(\tau, \theta, x_\theta, x'(\theta)\right) d\theta\right) d\tau\right] ds\right| \to 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

Thus, Φ is continuous. This completes the proof that Φ is completely continuous.

Obviously, the set $\xi(\Phi) = \{x \in Z : x = \lambda\Phi x,\ \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem the operator Φ has a fixed point in Z. This means that any fixed point of Φ is a mild solution of (1) on $[-r, T]$ satisfying $(\Phi x)(t) = x(t)$. Thus, IVP (1) has at least one mild solution on $[-r, T]$.

4. EXAMPLE

Consider the following partial differential equation:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} z(y,t) - \mu\left(t, z(y,t-r)\right)\right) &= \frac{\partial^2}{\partial y^2} z(y,t) + \int_0^t \sigma\left(t, s, z(y,s-r), \frac{\partial}{\partial s} z(y,s),\right. \\ &\left. \int_0^s \eta\left(s,\tau,z(y,\tau-r),\frac{\partial}{\partial \tau} z(y,\tau)\right) d\tau\right) ds, \\ z(0,t) = z(\pi,t) &= 0, \qquad \text{for } t > 0, \\ z(y,t) &= \phi(y,t), \qquad \text{for } -r \le t \le 0, \\ z_t(y,0) &= z_1(y), \qquad t \in J = [0,T], \quad \text{for } 0 < y < \pi, \end{aligned} \tag{6}$$

where ϕ is continuous and the functions μ, σ, η are defined below.

Let $X = L^2[0,\pi]$ and let $A : X \to X$ be defined by

$$Aw = w'', \qquad w \in D(A),$$

where $D(A) = \{w \in X : w, w' \text{ are absolutely continuous}, w'' \in X, w(0) = w(\pi) = 0\}$. Then, $Aw = \sum_{n=1}^{\infty} -n^2(w,w_n)w_n$, $w \in D(A)$, where $w_n(s) = \sqrt{2/\pi}\sin ns$, $n = 1,2,3,\dots$ is the orthogonal set of eigenvalues of A.

It can be easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, in X and is given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w,w_n)w_n, \qquad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(w,w_n)w_n, \qquad w \in X.$$

Let $g : J \times C \to X$ be defined by

$$g(t,u)(y) = \mu(t,u(y)), \qquad u \in C, \quad y \in [0,\pi],$$

where $\mu : J \times [0,\pi] \to [0,\pi]$ is continuous and strongly measurable. Also there exist positive constants c_1 and c_2 such that

$$\|\mu(t,\phi)\| \le c_1\|\phi\| + c_2.$$

Let $f : J \times J \times C \times X \to X$ be defined by

$$f(t,s,u,v)(y) = \eta(t,s,u(y),v(y)), \qquad u \in C, \quad v \in X, \quad y \in [0,\pi],$$

where $\eta : J \times J \times [0,\pi] \times [0,\pi] \to [0,\pi]$ is continuous and strongly measurable. Also, the function η satisfies the following condition: there exists a continuous function $\hat{q} : J \times J \to [0,\infty)$ such that

$$\|\eta(t,s,x,y)\| \le \hat{q}(t,s)\Omega(\|x\| + |y|), \qquad t \in J, \quad x \in C, \quad y \in X,$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

Let $F : J \times J \times C \times X \times X \to X$ be defined by

$$F(t,s,u,v,w)(y) = \sigma(t,s,u(y),v(y),w(y)), \qquad u \in C, \quad v \in X, \quad w \in X, \quad y \in [0,\pi],$$

where $\sigma : J \times J \times [0,\pi] \times [0,\pi] \times [0,\pi] \to [0,\pi]$ is continuous and strongly measurable.

Further, the function σ satisfies the following condition: there exists a continuous function $\hat{p}: J \times J \to [0,\infty)$ such that

$$\|\sigma(t,s,x,y,z)\| \le \hat{p}(t,s)\Omega(\|x\| + |y| + |z|), \qquad t \in J, \quad x \in C, \quad y,z \in X,$$

where $\Omega_2 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function such that

$$\int_0^T \hat{m}(s)\,ds < \int_c^\infty \frac{ds}{s + \Omega_2(s) + \Omega_1(s)} < \infty,$$

where

$$\hat{m}(t) = \max\left\{c_1\left[Mc_1 + M + M^*\right], M(c_1T + T + 1)\int_0^t \hat{p}(t,s)\,ds, \hat{q}(t,t)\right\},$$

and c is a known constant.

With this choice of A, g, f, and F, (1) is an abstract formulation of (6). Furthermore, all the conditions stated in the above theorem are satisfied. Hence, equation (6) has at least one mild solution on $[-r,T]$.

5. APPLICATION

As an application of Theorem 3.1, we shall consider the system with a control variable such as

$$\begin{aligned} \frac{d}{dt}\left[x'(t) - g(t,x_t)\right] &= Ax(t) + Bu(t) \\ &\quad + \int_0^t F\left(t,s,x_s,x'(s),\int_0^s f(s,\tau,x_\tau,x'(\tau))\,d\tau\right)ds, \qquad t \in (0,T), \\ x_0 = \phi, \qquad & x'(0) = y_0, \end{aligned} \tag{7}$$

where B is a bounded linear operator from a Banach space U to X and $u \in L^2(J,U)$.

A continuous function $x : [-r,T] \to X$, $T > 0$, is called a mild solution of (7) if $x_0 = \phi$, and if it satisfies the integral equation

$$\begin{aligned} x(t) = C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,x_s)\,ds \\ + \int_0^t S(t-s)\left[Bu(s) + \int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f(\tau,\theta,x_\theta,x'(\theta))\,d\theta\right)d\tau\right]ds, \qquad t \in J. \end{aligned} \tag{8}$$

DEFINITION 5.1. *System (7) is said to be controllable on J if for every $\phi \in C$ with $\phi(0) \in D(A)$, $y_0 \in E$, and $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (7) satisfies $x(T) = x_1$.*

For the controllability of second-order systems, one can refer to paper [15] and the references cited therein. To establish the controllability result, we need the following additional assumptions.

(H_{12}) $Bu(t)$ is continuous in t and $\|B\| \le M_1$ for some constant $M_1 > 0$.

(H_{13}) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)\,ds$$

induces a bounded invertible operator $\tilde{W} : L^2(J,U)/\ker W \to X$ such that $\|\tilde{W}^{-1}\| \le M_2$ for some constant $M_2 > 0$.

(H_{14})

$$\int_0^T m^*(s)\,ds < \int_a^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)} < \infty,$$

where

$$m^*(t) = \max\left\{c_1\left[Mc_1 + M + M^*\right], M(c_1T + T + 1)\int_0^t l(t,s)\,ds, h(t,t)\right\},$$

$$a = (M + M^* + c_1)\,\|\phi\| + (1+T)M\{|y_0| + c_1\|\phi\| + c_2\}$$
$$+ (M + M^*)\,c_2T + \left(T^2 + T\right)MN + c_2,$$

$$N = M_1M_2\left[|x_1| + M\|\phi\| + MT\{|y_0| + c_1\|\phi\| + 2c_2\} + Mc_1\int_0^T \|x_\tau\|\,d\tau\right.$$
$$\left. + MT\int_0^T\int_0^s l(s,\tau)\Omega\left(\|x_\tau\| + |x'(\tau)| + \int_0^\tau h(\tau,\theta)\Omega_0\left(\|x_\theta\| + |x'(\theta)|\right)d\theta\right)d\tau\,ds\right].$$

THEOREM 5.1. *If the hypotheses (H_1)–(H_{14}) hold, then system (7) is controllable on J.*

PROOF. Using (H_{13}), for an arbitrary function $x(\cdot)$, we define the control

$$u(t) = \tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,\phi)] - \int_0^T C(T-s)g(s,x_s)\,ds\right.$$
$$\left. - \int_0^T S(T-s)f\left(s,x_s,x'(s)\right)ds\right](t).$$

Using this control, we will show that the operator $\Psi : Z \to Z$ defined by

$$(\Psi x)(t) = \phi(t), \qquad -r \le t \le 0,$$

$$(\Psi x)(t) = C(t)\phi(0) + S(t)[y_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,x_s)\,ds$$
$$+ \int_0^t S(t-s)B\tilde{W}^{-1}\left[x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,\phi)] - \int_0^T C(T-\tau)g(\tau,x_\tau)\,d\tau\right.$$
$$\left. - \int_0^T S(T-a)\int_0^a F\left(a,\tau,x_\tau,x'(\tau),\int_0^\tau f\left(\tau,\theta,x_\theta,x'(\theta)\right)d\theta\right)d\tau\,da\right](s)\,ds$$
$$+ \int_0^t S(t-s)\int_0^s F\left(s,\tau,x_\tau,x'(\tau),\int_0^\tau f\left(\tau,\theta,x_\theta,x'(\theta)\right)d\theta\right)d\tau\,ds, \qquad t \in J,$$

has a fixed point. This fixed point is then a solution of equation (8).

Clearly, $(\Psi x)(T) = x_1$, which means that the control u steers the system from the initial state x_0 to x_1 in time T, provided we obtain a fixed point of the nonlinear operator Ψ. The remaining part of the proof is similar to Theorem 3.1, and hence it is omitted.

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