Existence of Solutions of Abstract Nonlinear Second-Order Neutral Functional Integrodifferential Equations

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Abstract—Sufficient conditions for existence of mild solutions for abstract second-order neutral functional integrodifferential equations are established by using the theory of strongly continuous cosine families of operators and the Schaefer theorem. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we are concerned with the abstract Cauchy problem for the nonlinear second-order neutral functional integrodifferential equation

\[
\frac{d}{dt} [x'(t) - g(t, x_t)] = A x(t) + \int_0^t F \left( t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau)) d\tau \right) ds,
\]

\[ t \in (0, T), \] 

\[ x_0 = \phi, \quad x'(0) = y_0 \in X, \]

where \( A \) is the infinitesimal generator of the strongly continuous cosine family \( C(t) \), \( t \in R \), of bounded linear operators in a Banach space \( X \), \( f : [0, T] \times [0, T] \times C \times X \to X \), \( F : [0, T] \times [0, T] \times C \times X \times X \to X \), and \( g : [0, T] \times C \to X \) are given functions and \( \phi \in C = C([-T, 0], X) \).

Several papers have appeared for the existence of solutions of first-order neutral functional differential equations in Banach spaces [1–4]. There seems to be little known about the solvability of the nonlinear second-order neutral equations in abstract spaces. Recently, Balachandran

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and Marshal Anthoni [5,6] studied the existence problem for both Volterra integrodifferential equations and neutral differential equations in Banach spaces. Ntouyas [7] and Ntouyas and Tsamatos [8] established the existence of solutions for semilinear second-order delay differential equations. In many cases, it is advantageous to treat the second-order abstract differential equations directly rather than to convert them to first-order equations. A useful tool for the study of abstract second-order differential equations is the theory of strongly continuous cosine families. We refer to the papers [9,10] for a detailed discussion of cosine family theory. Second-order equations which appear in a variety of physical problems can be found in [11,12]. The purpose of this paper is to study the existence of mild solutions for second-order neutral functional integrodifferential equations in Banach spaces using the Schaefer fixed-point theorem.

2. PRELIMINARIES

Let $X$ be a Banach space with norm $|\cdot|$ and let $T > 0$ be a real number. By $C$ we denote the Banach space of all continuous functions $\phi : [-r, 0] \to X$ endowed with the sup-norm

$$||\phi|| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$ 

Also for $x \in C([-r, T], X)$, we have $x_t \in C$ for $t \in J = [0, T]$, and $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

**Definition 2.1.** (See [9,10].) A one-parameter family $C(t), t \in R$, of bounded linear operators in the Banach space $X$ is called a strongly continuous cosine family iff

(i) $C(s + t) + C(s - t) = 2C(s)C(t)$ for all $s, t \in R$;

(ii) $C(0) = I$;

(iii) $C(t)x$ is continuous in $t$ on $R$ for each fixed $x \in X$.

Define the associated sine family $S(t), t \in R,$ by

$$S(t)x = \int_0^t C(s)x \, ds, \quad x \in X, \quad t \in R.$$ 

Assume the following conditions on $A$.

(H$_1$) $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in R,$ of bounded linear operators from $X$ into itself, and the adjoint operator $A^*$ is densely defined; i.e., $D(A^*) = X^*$ (see [13]).

The infinitesimal generator of a strongly continuous cosine family $C(t), t \in R,$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x \bigg|_{t=0}, \quad x \in D(A),$$ 

where $D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$.

Define $E = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$.

To establish our main theorem, we need the following lemmas.

**Lemma 2.1.** (See [9,10].) Let (H$_1$) hold. Then

(i) there exist constants $M \geq 1$ and $\omega \geq 0$ such that $|C(t)| \leq Me^{\omega|t|}$ and $|S(t) - S(t^*)| \leq M \int_t^{t^*} e^{\omega|s|} \, ds$ for $t, t^* \in R$;

(ii) $S(t)X \subseteq E$ and $S(t)E \subseteq D(A)$ for $t \in R$;

(iii) $\frac{d}{dt} C(t)x = AS(t)x$ for $x \in E$ and $t \in R$;

(iv) $\frac{d^2}{dt^2} C(t)x = AC(t)x$ for $x \in D(A)$ and $t \in E$.

**Lemma 2.2.** (See [9,10].) Let (H$_1$) hold, let $v : R \to X$ such that $v$ is continuously differentiable, and let $q(t) = \int_0^t S(t - s)v(s) \, ds$. Then $q$ is twice continuously differentiable and for $t \in R$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t - s)v(s) \, ds \quad \text{and} \quad q''(t) = Aq(t) + v(t).$$
Neutral Functional Integrodifferential Equations

Schaefer's Fixed-Point Theorem. (See [14].) Let $S$ be a normed linear space. Let $\Phi : S \to S$ be a completely continuous operator; that is, it is continuous and the image of any bounded set is contained in a compact set, and let

$$\xi(\Phi) = \{ x \in S : x = \lambda \Phi x \text{ for some } 0 < \lambda < 1 \}.$$  

Then either $\xi(\Phi)$ is unbounded or $\Phi$ has a fixed point.

Definition 2.2. A continuous function $x : [-r, T] \to X$, $T > 0$, is called a mild solution of problem (1) if $x_0 = \phi$, and if it satisfies the integral equation

$$x(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, x_s) \, ds$$
$$+ \int_0^t S(t - s) \int_0^s F(s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta(x_\theta)) \, d\theta) \, d\tau \, ds, \quad t \in J.$$  

We make the following assumptions.

$(H_2)$ $C(t), t > 0$ is compact.

$(H_3)$ $g : J \times C \to X$ is completely continuous and for any bounded set $K$ in $C([-r, T], X)$, the set $\{ t \to g(t, x_t) : x \in K \}$ is equicontinuous in $C([0, T], X)$.

$(H_4)$ There exist constants $c_1$ and $c_2$ such that

$$|g(t, \phi)| \leq c_1 \| \phi \| + c_2, \quad t \in J, \quad \phi \in C.$$  

$(H_5)$ The function $f(t, s, x, y) : C \times X \to X$ is continuous for each $t, s \in J$.

$(H_6)$ The function $f(\cdot, x, y) : J \times C \times X \to X$ is strongly measurable for each $x \in C$ and $y \in X$.

$(H_7)$ There exists a continuous function $h : J \times J \to [0, \infty)$ such that

$$|f(t, s, x, y)| \leq h(t, s)\Omega_0(\| x \| + | y |), \quad t, s \in J, \quad x \in C, \quad y \in X,$$

where $\Omega_0 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

$(H_8)$ The function $F(t, s, x, y) : C \times X \times X \to X$ is continuous for each $t, s \in J$.

$(H_9)$ The function $F(t, x, y, z) : J \times C \times X \times X \to X$ is strongly measurable for each $x \in C$, $y \in X$, and $z \in X$.

$(H_{10})$ For every positive constant $k$, there exists $\alpha_k \in L^1(\Omega)$ such that

$$\sup_{\| x \|, \| y \|, \| z \| \leq k} \left| \int_0^t F(t, s, x, y, z) \, ds \right| \leq \alpha_k(t), \quad \text{for } t \in J \text{ a.e.}$$  

$(H_{11})$ There exists a continuous function $l : J \times J \to [0, \infty)$ such that

$$|F(t, s, x, y, z)| \leq l(t, s)\Omega(\| x \| + | y | + | z |), \quad t \in J, \quad x \in C, \quad y, z \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function and

$$\int_0^T m(s) \, ds < \int_0^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)} < \infty,$$

where

$$m(t) = \max \left\{ c_1 \| M_1 + M + M^* \|, M(c_1 T + T + 1) \int_0^t l(t, s) \, ds, h(t, t) \right\},$$

$$M = \sup \{ \| C(t) \| : t \in J \}, \quad M^* = \sup \{ \| A S(t) \| : t \in J \},$$

$$c = (M + M^* + c_1) \| \phi \| + (1 + T)M \{ | y_0 | + c_1 \| \phi \| + c_2 \} + (M + M^*)c_2 T + c_2.$$
3. MAIN RESULT

THEOREM 3.1. Suppose $(H_I)-(H_{II})$ hold. Then the IVP (1) has at least one mild solution on $[-r,T]$.

PROOF. Consider the space $Z = C([-r,T],X) \cap C^1(J,X)$ with the norm

$$||x|| = \max \{||x||_r, ||x'||_0\},$$

where

$$||x||_r = \sup\{|x(t)| : -r \leq t \leq T\}, \quad ||x'||_0 = \sup\{|x'(t)| : 0 \leq t \leq T\}.$$ 

To prove the existence of a mild solution of the IVP (1), we have to apply the Schaefer fixed-point theorem for the nonlinear operator equation

$$x(t) = \lambda \Phi x(t), \quad 0 < \lambda < 1$$

where the operator $\Phi : Z \to Z$ is defined by

$$\Phi x(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^t C(t-s)g(s,x_s) \, ds$$

$$+ \int_0^t S(t-s) \int_0^s F\left(s, \tau, x_{\tau}, x'_{\tau}, \int_0^\tau f(\tau, \theta, x_{\theta}, x'_{\theta}) \, d\theta\right) \, d\tau \, ds, \quad t \in J.$$ 

Then we have, for $t \in J$,

$$|x(t)| \leq M||\phi|| + MT\{|y_0| + c_1||\phi|| + 2c_2\} + M c_1 \int_0^t ||x_s|| \, ds$$

$$+ MT \int_0^t \int_0^s l(s, \tau) \Omega \left(|x_{\tau}| + |x'_{\tau}| + \int_0^\tau h(\tau, \theta) \Omega_0 (||x_{\theta}| + |x'_{\theta}|) \, d\theta\right) \, d\tau \, ds.$$ 

Consider the function $q$ defined by

$$q(t) = \sup\{|x(s)| : -r \leq s \leq t\}, \quad t \in J.$$ 

Let $t^* \in [-r,t]$ be such that $q(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have, for $t \in J$,

$$q(t) \leq M||\phi|| + MT\{|y_0| + c_1||\phi|| + 2c_2\} + M c_1 \int_0^t q(s) \, ds$$

$$+ MT \int_0^t \int_0^s l(s, \tau) \Omega \left(q(\tau) + |x'_{\tau}| + \int_0^\tau h(\tau, \theta) \Omega_0 (q(\theta) + |x'_{\theta}|) \, d\theta\right) \, d\tau \, ds.$$ 

If $t^* \in [-r,0]$, then $q(t) = ||\phi||$ and the previous inequality holds since $M \geq 1$.

Denoting by $v(t)$ the right-hand side of the above inequality, we have

$$q(t) \leq v(t), \quad t \in J, \quad v(0) = M||\phi|| + MT\{|y_0| + c_1||\phi|| + 2c_2\}$$

and for $t \in J$,

$$v'(t) = M c_1 v(t) + MT \int_0^t l(t, s) \Omega \left(v(s) + |x'_{s}| + \int_0^s h(s, \tau) \Omega_0 (v(\tau) + |x'_{\tau}|) \, d\tau\right) \, ds$$

$$\leq M c_1 v(t) + MT \int_0^t l(t, s) \Omega \left(v(s) + |x'_{s}| + \int_0^s h(s, \tau) \Omega_0 (v(\tau) + |x'_{\tau}|) \, d\tau\right) \, ds.$$
By
\[ x'(t) = \lambda AS(t)\phi(0) + \lambda C(t)[y_0 - g(0,\phi)] + \lambda g(t,x_t) + \lambda \int_0^t AS(t-s)g(s,x_s)\, ds \\
\quad + \lambda \int_0^t C(t-s) \int_0^s F(s,\tau,x',\tau,\int_0^\tau f(\tau,\theta,x_\theta, x'_\theta)\, d\theta)\, d\tau\, ds, \quad t \in J, \]
we obtain
\[ |x'(t)| \leq M^* \|\phi\| + M\{\|y_0\| + c_1\|\phi\| + c_2\} + c_1\|x_t\| + c_2 + M^* \left\{ c_2 T + c_1 \int_0^t \|x_s\|\, ds \right\} \\
\quad + M \int_0^t \int_0^s \{ l(s,\tau) \Omega \left( q(\tau) + |x'(\tau)| + \int_0^\tau h(\tau,\theta)\Omega_0(q(\theta) + |x'(\theta)|)\, d\theta \right) \, d\tau\, ds, \]
Denoting by \( r(t) \) the right-hand side of the above inequality, we have for \( t \in J \),
\[ |x'(t)| \leq r(t), \quad r(0) = M^* \|\phi\| + M\{\|y_0\| + c_1\|\phi\| + c_2\} + c_1\|\phi\| + c_2 + M^* c_2 T, \]
and
\[ r'(t) \leq c_1 v'(t) + M^* c_1 v(t) \\
\quad + M \int_0^t l(t,s)\Omega \left( v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\, d\tau \right) \, ds \\
\leq c_1 \left\{ M c_1 v(t) + M T \int_0^t l(t,s)\Omega \left( v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\, d\tau \right) \, ds \right\} \\
\quad + M^* c_1 v(t) + M \int_0^t l(t,s)\Omega \left( v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\, d\tau \right) \, ds. \]
Let \( u(t) = v(t) + r(t), \ t \in J \). Then \( u(0) = c, \) and
\[ u'(t) = v'(t) + r'(t) \\
\leq c_1 \left[ M c_1 + M + M^* \right] v(t) \\
\quad + M(c_1 T + T + 1) \int_0^t l(t,s)\Omega \left( v(s) + r(s) + \int_0^s h(s,\tau)\Omega_0(v(\tau) + r(\tau))\, d\tau \right) \, ds \\
\leq c_1 \left[ M c_1 + M + M^* \right] u(t) \\
\quad + M(c_1 T + T + 1) \int_0^t l(t,s)\Omega \left( u(s) + \int_0^s h(s,\tau)\Omega_0(u(\tau))\, d\tau \right) \, ds, \quad t \in J. \]
Let \( w(t) = u(t) + \int_0^t l(h(t,s)\Omega_0(u(s))\, ds, \ t \in J \). Then \( w(0) = c, \) and for \( t \in J \),
\[ w'(t) = u'(t) + h(t,t)\Omega_0(u(t)) \\
\leq c_1 \left[ M c_1 + M + M^* \right] w(t) \\
\quad + M(c_1 T + T + 1) \int_0^t l(t,s)\Omega(w(s))\, ds + h(t,t)\Omega_0(w(t)) \\
\leq c_1 \left[ M c_1 + M + M^* \right] w(t) \\
\quad + M(c_1 T + T + 1)\Omega(w(t)) \int_0^t l(t,s)\, ds + h(t,t)\Omega_0(w(t)). \]
This implies
\[ \int_{w(0)}^{w(t)} \frac{ds}{s + \Omega(s) + \Omega_0(s)} \leq \int_0^T \frac{ds}{s + \Omega(s) + \Omega_0(s)} \leq \int_c^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)}, \quad t \in J. \]
This inequality implies that there is a constant $K$ such that $w(t) \leq K$, $t \in J$. Then

$$|x(t)| \leq v(t) \leq K, \quad t \in J,$$

$$|x'(t)| \leq r(t) \leq K, \quad t \in J,$$

and hence,

$$\|x\|^* = \max \{\|x\|_r, \|x'\|_0\} \leq K,$$

where $K$ depends on the constants $T$, $N$ and on the functions $\Omega_0$ and $\Omega$.

We shall now prove that the operator $\Phi : Z \rightarrow Z$ defined by

$$(\Phi x)(t) = \phi(t), \quad -r \leq t \leq 0,$$

$$(\Phi x)(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^t C(t-s)g(s, x_s) \, ds$$

$$+ \int_0^t S(t-s) \int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \, ds, \quad t \in J,$$

is a completely continuous operator.

Let $B_k = \{x \in Z : \|x\|^* \leq k\}$ for some $k \geq 1$. We first show that $\Phi$ maps $B_k$ into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$|(\Phi x)(t_1) - (\Phi x)(t_2)|$$

$$\leq \|C(t_1) - C(t_2)\|\phi(0)\| + \|S(t_1) - S(t_2)\|\|y_0 - g(0, \phi)\|$$

$$+ \int_0^{t_1} |C(t_1-s) - C(t_2-s)|g(s, x_s) \, ds + \int_{t_1}^{t_2} C(t_2-s)g(s, x_s) \, ds$$

$$+ \int_0^{t_1} |S(t_1-s) - S(t_2-s)| \int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \, ds$$

$$+ \int_{t_1}^{t_2} S(t_2-s) \int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \, ds$$

$$\leq \|C(t_1) - C(t_2)\|\phi(0)\| + \|S(t_1) - S(t_2)\|\|y_0 + c_1\phi(0) + c_2\|$$

$$+ \int_0^{t_1} |C(t_1-s) - C(t_2-s)|\|x_s\| + c_2 \|x_s\| \, ds$$

$$+ \int_{t_1}^{t_2} |C(t_2-s)|\|x_s\| + c_2 \|x_s\| \, ds + \int_0^{t_1} |S(t_1-s) - S(t_2-s)|\alpha_k(s) \, ds$$

$$+ \int_{t_1}^{t_2} |S(t_2-s)|\alpha_k(s) \, ds,$$

and similarly,

$$|(\Phi x)'(t_1) - (\Phi x)'(t_2)|$$

$$\leq \|A(S(t_1) - S(t_2))\phi(0)\| + \|[C(t_1) - C(t_2)]\|y_0 - g(0, \phi)\|$$

$$+ \|g(t_1, x_{t_1}) - g(t_2, x_{t_2})\| + \int_0^{t_1} A(S(t_1-s) - S(t_2-s))g(s, x_s) \, ds$$

$$+ \int_{t_1}^{t_2} AS(t_2-s)g(s, x_s) \, ds + \int_0^{t_1} |C(t_1-s) - C(t_2-s)|$$

$$\int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \, ds$$

$$+ \int_{t_1}^{t_2} C(t_2-s) \int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \, ds.$$
\[ \begin{aligned} &\leq \|A(S(t_1) - S(t_2))\|\phi + \|A(t_1) - C(t_2)\|\{y_0 + c_1\phi + c_2\} \\
 &+ g(t_1, x_{t_1}) - g(t_2, x_{t_2}) + \int_{t_1}^{t_2} \|A(S(t_1) - S(t_2))\|\{c_1\|x_s\| + c_2\} ds \\
 &+ \int_{t_1}^{t_2} |A_s(t_2 - s)\{c_1\|x_s\| + c_2\} ds + \int_{t_1}^{t_2} |C(t_1 - s) - C(t_2 - s)\{c_k(s)\} ds \\
 &+ \int_{t_1}^{t_2} |C(t_2 - s)\{c_k(s)\} ds. \end{aligned} \]

The right-hand sides of (4) and (5) are independent of \( y \in B_k \) and tend to zero as \( t_2 - t_1 \to 0 \), since \( C(t), S(t) \) are uniformly continuous for \( t \in J \) and the compactness of \( C(t), S(t) \) for \( t > 0 \) imply the continuity in the uniform operator topology. The compactness of \( S(t) \) follows from that of \( C(t) \) (see [10]).

Thus, \( \Phi \) maps \( B_k \) into an equicontinuous family of functions. It is easy to see that the family \( \Phi B_k \) is uniformly bounded.

Next we show \( \Phi B_k \) is compact. Since we have shown \( \Phi B_k \) is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that \( \Phi \) maps \( B_k \) into a precompact set in \( X \).

Let \( 0 < t \leq T \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( x \in B_k \), we define

\[ (\Phi_\epsilon x)(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_{t-\epsilon}^{t} C(t-s)g(s, x_s) ds \\
+ \int_{t-\epsilon}^{t} S(t-s) \int_0^s F(s, \tau, x_\tau, x'_\tau(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'_\theta(\theta)) d\theta) d\tau ds, \quad t \in J. \]

Since \( C(t), S(t) \) are compact operators, the set \( Y_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in B_k\} \) is precompact in \( X \) for every \( \epsilon, 0 < \epsilon < t \). Moreover, for every \( x \in B_k \), we have

\[ \begin{aligned} |(\Phi x)(t) - (\Phi_\epsilon x)(t)| \leq &\int_{t-\epsilon}^{t} |C(t-s)g(s, x_s)| ds \\
+ &\int_{t-\epsilon}^{t} |S(t-s)\int_0^s F(s, \tau, x_\tau, x'_\tau(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'_\theta(\theta)) d\theta) d\tau| ds \\
\leq &\int_{t-\epsilon}^{t} |C(t-s)|\{c_1\|x_s\| + c_2\} ds \\
+ &\int_{t-\epsilon}^{t} |S(t-s)|\alpha_k(s) ds \to 0, \quad \text{as } \epsilon \to 0, \end{aligned} \]

and

\[ \begin{aligned} |(\Phi x)'(t) - (\Phi_\epsilon x)'(t)| \leq &\int_{t-\epsilon}^{t} |g(t, x_t) - C(\epsilon)g(t-\epsilon, x_{t-\epsilon})| + \int_{t-\epsilon}^{t} |AS(t-s)g(s, x_s)| ds \\
+ &\int_{t-\epsilon}^{t} |C(t-s)\int_0^s F(s, \tau, x_\tau, x'_\tau(\tau), \int_0^\tau f(\tau, \theta, x_\theta, x'_\theta(\theta)) d\theta) d\tau| ds \\
\leq &\int_{t-\epsilon}^{t} |g(t, x_t) - C(\epsilon)g(t-\epsilon, x_{t-\epsilon})| + \int_{t-\epsilon}^{t} |AS(t-s)|\{c_1\|x_s\| + c_2\} ds \\
&\int_{t-\epsilon}^{t} |C(t-s)|\alpha_k(s) ds \to 0, \quad \text{as } \epsilon \to 0. \end{aligned} \]

Therefore, there are precompact sets arbitrarily close to the set \( \{(\Phi x)(t) : x \in B_k\} \). Hence, the set \( \{(\Phi x)(t) : x \in B_k\} \) is precompact in \( X \).

It remains to show that \( \Phi : Z \to Z \) is continuous. Let \( \{x_n\}_0^\infty \subseteq Z \) with \( x_n \to x \) in \( Z \). Then there is an integer \( q \) such that \( |x_n(t)| \leq q, |x'_n(t)| \leq q \) for all \( n \) and \( t \in J \), so \( |x(t)| \leq q, |x'(t)| \leq q, \)
and $x, x' \in Z$. By (H5) and (H8),

$$
\int_{0}^{t} F \left( t, s, x_{n_{s}}, x'_{n_{s}}(s), \int_{0}^{s} f(s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau)) \, d\tau \right) \, ds
$$

$$
= \int_{0}^{t} F \left( t, s, x_{s}, x'(s), \int_{0}^{s} f(s, \tau, x_{\tau}, x'_{\tau}(\tau)) \, d\tau \right) \, ds
$$

for each $t \in J$ and since

$$
\left| \int_{0}^{t} F \left( t, s, x_{n_{s}}, x'_{n_{s}}(s), \int_{0}^{s} f(s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau)) \, d\tau \right) \, ds
$$

$$
- \int_{0}^{t} F \left( t, s, x_{s}, x'(s), \int_{0}^{s} f(s, \tau, x_{\tau}, x'_{\tau}(\tau)) \, d\tau \right) \, ds \right| \leq 2\alpha_{q}(t),
$$

we have by the dominated convergence theorem,

$$
\| \Phi x_{n} - \Phi x \| = \sup_{t \in J} \int_{0}^{t} \left[ C(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \right] \, ds
$$

$$
+ \int_{0}^{t} S(t-s) \left[ \int_{0}^{s} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{s} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds - \int_{0}^{t} S(t-s) \left[ \int_{0}^{s} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{s} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\leq \int_{0}^{t} \left| C(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \right| \, ds
$$

$$
+ \int_{0}^{t} S(t-s) \left[ \int_{0}^{s} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{s} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\leq |g(t, x_{n_{t}}) - g(t, x_{t})| + \int_{0}^{t} AS(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \, ds
$$

$$
+ \int_{0}^{t} C(t-s) \left[ \int_{0}^{t} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{t} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\leq |g(t, x_{n_{t}}) - g(t, x_{t})| + \int_{0}^{t} \left| AS(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \right| \, ds
$$

$$
+ \int_{0}^{t} \left| C(t-s) \left[ \int_{0}^{t} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{t} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\to 0, \quad \text{as } n \to \infty,
$$

and

$$
\| (\Phi x_{n})' - (\Phi x)' \|
$$

$$
= \sup_{t \in J} \left| g(t, x_{n_{t}}) - g(t, x_{t}) \right| + \int_{0}^{t} AS(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \, ds
$$

$$
+ \int_{0}^{t} C(t-s) \left[ \int_{0}^{t} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{t} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\leq |g(t, x_{n_{t}}) - g(t, x_{t})| + \int_{0}^{t} \left| AS(t-s) \left[ g(s, x_{n_{s}}) - g(s, x_{s}) \right] \right| \, ds
$$

$$
+ \int_{0}^{t} \left| C(t-s) \left[ \int_{0}^{t} F \left( s, \tau, x_{n_{\tau}}, x'_{n_{\tau}}(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{n_{\theta}}, x'_{n_{\theta}}(\theta)) \, d\theta \right) \, d\tau
$$

$$
- \int_{0}^{t} F \left( s, \tau, x_{\tau}, x'(\tau), \int_{0}^{\tau} f(\tau, \theta, x_{\theta}, x'_{\theta}(\theta)) \, d\theta \right) \, d\tau \right] \, ds \right| 
$$

$$
\to 0, \quad \text{as } n \to \infty.
$$

Thus, $\Phi$ is continuous. This completes the proof that $\Phi$ is completely continuous.

Obviously, the set $\xi(\Phi) = \{ x \in Z : x = \lambda \Phi x, \lambda \in (0, 1) \}$ is bounded, as we proved in the first step. Consequently, by Schaefer's theorem the operator $\Phi$ has a fixed point in $Z$. This means that any fixed point of $\Phi$ is a mild solution of (1) on $[-r, T]$ satisfying $(\Phi x)(t) = x(t)$. Thus, IVP (1) has at least one mild solution on $[-r, T]$.
4. EXAMPLE

Consider the following partial differential equation:

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial t} z(y, t) - \mu(t, z(y, t - \tau)) \right) = \frac{\partial^2}{\partial y^2} z(y, t) + \int_0^t \sigma \left( t, s, z(y, s - \tau), \frac{\partial}{\partial s} z(y, s) \right) ds \eta \left( s, \tau, z(y, \tau - \tau), \frac{\partial}{\partial \tau} z(y, \tau) \right) d\tau, \quad t > 0,
\]

\[
z(0, t) = z(\pi, t) = 0, \quad \text{for } t > 0,
\]

\[
z_y(t, 0) = z_1(y), \quad \text{for } 0 < y < \pi,
\]

where \( \phi \) is continuous and the functions \( \mu, \sigma, \eta \) are defined below.

Let \( X = L^2[0, \pi] \) and let \( A : X \to X \) be defined by

\[
Aw = w'', \quad w \in D(A),
\]

where \( D(A) = \{ w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0 \} \). Then,

\[
Aw = \sum_{n=1}^{\infty} -\lambda_n^2 (w, w_n)w_n, \quad w \in D(A),
\]

where \( \lambda_n = \sqrt{2/\pi} \sin ns, n = 1, 2, 3, \ldots \) is the orthogonal set of eigenvalues of \( A \).

It can be easily shown that \( A \) is the infinitesimal generator of a strongly continuous cosine family \( C(t), t \in \mathbb{R}, \) in \( X \) and is given by

\[
C(t)w = \sum_{n=1}^{\infty} \cos nt(w, w_n)w_n, \quad w \in X.
\]

The associated sine family is given by

\[
S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(w, w_n)w_n, \quad w \in X.
\]

Let \( g : J \times C \to X \) be defined by

\[
g(t, u)(y) = \mu(t, u(y)), \quad u \in C, \quad y \in [0, \pi],
\]

where \( \mu : J \times [0, \pi] \to [0, \pi] \) is continuous and strongly measurable. Also there exist positive constants \( c_1 \) and \( c_2 \) such that

\[
\| \mu(t, \phi) \| \leq c_1 \| \phi \| + c_2.
\]

Let \( f : J \times J \times C \times X \times X \to X \) be defined by

\[
f(t, s, u, v)(y) = \eta(t, s, u(y), v(y)), \quad u \in C, \quad v \in X, \quad y \in [0, \pi],
\]

where \( \eta : J \times J \times [0, \pi] \times [0, \pi] \to [0, \pi] \) is continuous and strongly measurable. Also, the function \( \eta \) satisfies the following condition: there exists a continuous function \( \tilde{\eta} : J \times J \to [0, \infty) \) such that

\[
\| \eta(t, s, x, y) \| \leq \tilde{\eta}(t, s) \Omega_1(\| x \| + \| y \|), \quad t \in J, \quad x \in C, \quad y \in X,
\]

where \( \Omega_1 : [0, \infty) \to (0, \infty) \) is a continuous nondecreasing function.

Let \( F : J \times J \times C \times X \times X \to X \) be defined by

\[
F(t, s, u, v, w)(y) = \sigma(t, s, u(y), v(y), w(y)), \quad u \in C, \quad v \in X, \quad w \in X, \quad y \in [0, \pi],
\]

where \( \sigma : J \times J \times [0, \pi] \times [0, \pi] \times [0, \pi] \to [0, \pi] \) is continuous and strongly measurable.
Further, the function $\sigma$ satisfies the following condition: there exists a continuous function $\hat{p} : J \times J \to [0, \infty)$ such that
\[
\|\sigma(t, s, x, y, z)\| \leq \hat{p}(t, s)\Omega(\|x\| + |y| + |z|), \quad t \in J, \quad x \in C, \quad y, z \in X,
\]
where $\Omega_2 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function such that
\[
\int_0^T \hat{m}(s) \, ds < \int_0^\infty \frac{ds}{s + \Omega_2(s) + \Omega_1(s)} < \infty,
\]
where
\[
\hat{m}(t) = \max \left\{ c_1 [MC_1 + M + M^*], M(c_1T + T + 1) \int_0^t \hat{p}(t, s) \, ds, \hat{q}(t, t) \right\},
\]
and $c$ is a known constant.

With this choice of $A$, $g$, $f$, and $F$, (1) is an abstract formulation of (6). Furthermore, all the conditions stated in the above theorem are satisfied. Hence, equation (6) has at least one mild solution on $[-r, T]$.

5. APPLICATION

As an application of Theorem 3.1, we shall consider the system with a control variable such as
\[
\frac{d}{dt} [x'(t) - g(t, x_t)] = Ax(t) + Bu(t)
\]
\[
+ \int_0^t F \left( t, s, x_s, x'(s), \int_0^s f(s, \tau, x_\tau, x'(\tau)) \, d\tau \right) \, ds, \quad t \in (0, T),
\]
\[
x_0 = \phi, \quad x'(0) = y_0,
\]
where $B$ is a bounded linear operator from a Banach space $U$ to $X$ and $u \in L^2(J, U)$.

A continuous function $x : [-r, T] \to X$, $T > 0$, is called a mild solution of (7) if $x_0 = \phi$, and if it satisfies the integral equation
\[
x(t) = C(t)\phi(0) + S(t)[y_0 - g(0, \phi)] + \int_0^t C(t - s)g(s, x_s) \, ds
\]
\[
+ \int_0^t S(t - s) \left[ Bu(s) + \int_0^s F \left( s, \tau, x_\tau, x'(\tau), \int_0^T f(\tau, \theta, x_\theta, x'(\theta)) \, d\theta \right) \, d\tau \right] \, ds, \quad t \in J.
\]

DEFINITION 5.1. System (7) is said to be controllable on $J$ if for every $f \in C$ with $\phi(0) \in D(A)$, $y_0 \in E$, and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (7) satisfies $x(T) = x_1$.

For the controllability of second-order systems, one can refer to paper [15] and the references cited therein. To establish the controllability result, we need the following additional assumptions.

(H12) $Bu(t)$ is continuous in $t$ and $\|B\| \leq M_1$ for some constant $M_1 > 0$.

(H13) The linear operator $W : L^2(J, U) \to X$ defined by
\[
Wu = \int_0^T S(T - s)Bu(s) \, ds
\]
induces a bounded invertible operator $\tilde{W} : L^2(J, U)/\ker W \to X$ such that $\|\tilde{W}^{-1}\| < M_2$ for some constant $M_2 > 0$.

(H14)
\[
\int_0^T m^*(s) \, ds \leq \int_0^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)} < \infty,
\]
where
\[
m^*(t) = \max \left\{ c_1 [M_{c_1} + M + M^*], M(t) + T + 1 \int_0^t l(t,s)ds, h(t,t) \right\},
\]
\[
a = (M + M^* + c_1) ||\phi|| + (1 + T)M(y_0 + c_1 ||\phi|| + c_2)
+ (M + M^*) c_2 T + (T^2 + T) MN + c_2,
\]
\[
N = M_1 M_2 \left[ |x_1| + M ||\phi|| + MT(|y_0| + c_1 ||\phi|| + 2c_2) + M c_1 \int_0^T ||x_1|| d\tau
+ MT \int_0^T \int_0^s l(s,\tau) \Omega \left( ||x_1|| + |x'(\tau)| + \int_\tau^T h(\tau,\theta)\Omega_0 \left( ||x_1|| + |x'(\theta)| \right) d\theta \right) d\tau ds \right].
\]

**THEOREM 5.1.** If the hypotheses (H1)-(H14) hold, then system (7) is controllable on J.

**PROOF.** Using (H13), for an arbitrary function \( x(\cdot) \), we define the control
\[
u(t) = W^{-1} \left[ x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,0)] - \int_0^T C(T - s)g(s,x_s)ds
- \int_0^T S(T - s)f(s,x_s,x'(s))ds \right](t).
\]
Using this control, we will show that the operator \( \Psi : Z \rightarrow Z \) defined by
\[
(\Psi x)(t) = \phi(t), \quad -r \leq t \leq 0,
\]
\[
(\Psi x)(t) = C(t)\phi(0) + S(t)[y_0 - g(0,0)] + \int_0^t C(t - s)g(s,x_s)ds
+ \int_0^t S(t - s)BW^{-1} \left[ x_1 - C(T)\phi(0) - S(T)[y_0 - g(0,0)] - \int_0^T C(T - \tau)g(\tau,x_{\tau})d\tau
- \int_0^T S(T - \alpha) \int_0^\alpha f \left( a,\tau,x_{\tau},x'(\tau),\int_0^\tau f(\tau,\theta,x_{\theta},x'(\theta))d\theta \right) d\tau da \right](s)ds
+ \int_0^t S(t - s) \int_0^s F \left( s,\tau,x_{\tau},x'(\tau),\int_0^\tau f(\tau,\theta,x_{\theta},x'(\theta))d\theta \right) d\tau ds, \quad t \in J,
\]
has a fixed point. This fixed point is then a solution of equation (8).

Clearly, \((\Psi x)(T) = x_1\), which means that the control \( u \) steers the system from the initial state \( x_0 \) to \( x_1 \) in time \( T \), provided we obtain a fixed point of the nonlinear operator \( \Psi \). The remaining part of the proof is similar to Theorem 3.1, and hence it is omitted.

**REFERENCES**